CAPUT III.

DE

AEQUATIONIBUS DIFFERENTIO-DIFFERENTIALIBUS HOMOGENEIS, ET QUAE AD EAM FORMAM REDUCI POSSUNT.

Problema 97.

790.

Aequationum differentio-differentialium homogenearum naturam explicare, atque ad formam finitam ponendo $\partial y = p\partial x$ et $\partial p = q\partial x$ accommodare.

Solutio.

Sumto elemento ∂x constante, aequatio differentio-differentialis vulgari modo expressa dicitur homogenea, si non solum ipsis variabilibus x et y, sed etiam earum differentialibus ∂x et ∂y, itemque ipsi $\partial\partial y$, singulis unam dimensionem tribuendo, omnes aequationis termini eundem dimensionum numerum contineant, veluti in hac aequatione,

$$xx\partial\partial y + x\partial x^2 + y\partial y^2 = 0,$$

ubi in singulis terminis ternae insunt dimensiones. Quodsi ergo ponamus $\frac{\partial y}{\partial x} = p$, ac $\frac{\partial p}{\partial x} = \frac{\partial\partial y}{\partial x^2} = q$, littera p nullam dimensionem, littera vero q unam dimensionem negativam continere erit censenda. Hinc aequatio differentio-differentialis ad formam hic receptam reducta, ut nonnisi quantitates finitas x, y, p et q contineat, erit homogenea, si litteris x et y unam dimensionem tribuendo, litterae p vero nullam, at litterae q unam dimensionem negativam, in singulis aequationis terminis idem oriatur dimensionum numerus. Vicissim ergo quoties haec proprietas in aequatione inter quaternas quantitates x, y, p et q proposita deprehenditur, ea aequatio erit homogenea et forma vulgari expressa manifesto homogeneitatem prae se feret.

Corollarium 1.

791. Si ergo in aequatione tali homogenea inter x, y, p et q statuatur $y = ux$ et $q = \frac{v}{x}$, omnes termini eandem potestatem ipsius x continebunt, qua ergo per divisionem sublata, aequatio prodibit tres tantum variabiles u, v et p involvens.

Corollarium 2.

792. Criterium igitur aequationis homogeneae inter quatuor quantitates x, y, p et q propositae in hoc consistit, ut posito $y = ux$ et $q = \frac{v}{x}$, quantitas x prorsus ex calculo exterminetur.

Corollarium 3.

793. Facta itaque hac substitutione, qua obtinetur aequatio inter ternas quantitates u, v et p, ex ea pro lubitu vel p per u et v, vel v per u et p, vel u per v et p definiri poterit.

Scholion.

794. Simili modo ideam homogeneitatis in aequationibus differentio-differentialibus constituimus, quo in aequationibus differentialibus primi gradus sumus usi. In his quidem, cum differentialia sponte eundem dimensionum numerum constituere debeant, homogeneitas ex solis ipsis variabilibus x et y dijudicatur. At in aequationibus differentio - differentialibus praeter ipsas variabiles x et y etiam litterae q ratio in computo dimensionum haberi debet, ita ta

men ut ipsi una dimensio negativa sit tribuenda; littera autem p in hunc computum plane non ingreditur, quae ergo utcunque aequationi implicetur, homogeneitatem non turbat. Plurimum autem interest probe nosse indolem differentio-differentialium aequationum homogenearum, cum earum resolutio ad resolutionem aequationum differentialium primi gradus reduci possit, ita ut si haec successerit, etiam ipsarum aequationum differentio-differentialium integratio habeatur, id quod in sequenti problemate luculentius ostendemus.

Problema 98.

795. Proposita aequatione differentio-differentiali homogenea, ejus resolutionem ad integrationem aequationis differentialis primi gradus reducere.

Solutio.

Reducta aequatione ponendo $\partial y = p \partial x$ et $\partial p = q \partial x$ ad formam hic receptam, ut habeatur aequatio inter quatuor quantitates finitas x, y, p et q, ponatur $y = u x$ et $q = \frac{v}{x}$, ac cum aequatio sit homogenea, hoc modo quantitas x penitus ex calculo elidetur, ita ut proditura sit aequatio inter ternas quantitates u, v et p, ex qua unam per binas reliquas definire liceat. Nunc igitur cum sit $\partial y = p \partial x$, erit $u \partial x + x \partial u = p \partial x$, hincque $\frac{\partial x}{x} = \frac{\partial u}{p-u}$. Deinde ob $\partial p = q \partial x$, erit $\partial p = \frac{v \partial x}{x}$, ideoque $\frac{\partial x}{x} = \frac{\partial p}{v}$, ex quo duplici ipsius $\frac{\partial x}{x}$ valore colligitur $\frac{\partial u}{p-u} = \frac{\partial p}{v}$, seu $v \partial u = p \partial p - u \partial p$. Quodsi ergo ex illa aequatione quantitas v definiatur per binas p et u, habebitur aequatio differentialis primi gradus inter binas variabiles p et u, cujus integratio si fuerit in potestate, ut p per u innotescat, aequatio $\frac{\partial x}{x} = \frac{\partial u}{p-u}$, in qua variabiles x et u sunt separatae, integretur, sicque x per u definietur; unde fit $y = u x$; seu statim in hoc integrali loco u scribatur $\frac{y}{x}$, et habebitur aequatio inter x et y quaesita.

Corollarium 1.

796. Totum ergo negotium reducitur ad integrationem hujus aequationis differentialis simplicis $v\partial u = p\partial p - u\partial p$, quae si ope regularum supra traditarum expediri queat, simul aequationis differentio-differentialis integratio habetur.

Corollarium 2.

797. Simul autem patet resolutionem hujusmodi aequationum duplicem integrationem requirere, unde duae quantitates arbitrariae constantes ingredientur, quibus integrale completum constituitur.

Corollarium 3.

798. Etiamsi autem integratio $v\partial u = p\partial p - u\partial p$ non succedat, tamen ingens lucrum est rem eo perduxisse, cum supra methodus generalis sit tradita integralia omnium aequationum differentialium primi gradus proxime assignandi.

Scholion.

799. Operae igitur pretium erit eos casus perpendere, quibus aequatio $v\partial u = p\partial p - u\partial p$ integrationem admittit; quamobrem examinemus, qualis functio v debeat esse ipsarum p et u, ut hoc eveniat. Primum autem patet hoc fieri, si v fuerit functio homogenea unius dimensionis ipsarum p et u, quoniam tum ipsa haec aequatio fit homogenea ac per regulas supra expositas ad integrationem perduci potest. Deinde etiam integratio succedit si fuerit v functio quaecunque ipsius p, quoniam tum altera variabilis u unam dimensionem non superat, et aequationis $\partial u + \frac{u\partial p}{v} = \frac{p\partial p}{v}$ integrale est

$$e^{\int\frac{\partial p}{v}}u = \int\frac{e^{\int\frac{\partial p}{v}}p\partial p}{v}.$$

Tertio integrationem absolvere licebit, si v fuerit functio quaecunque quantitatis $p - u$. Posito enim $p - u = s$, ut sit v functio ipsius s, ob $p = s + u$, nostra aequatio erit $v \partial u = s \partial s + s \partial u$, adeoque $\partial u = \frac{s \partial s}{v - s}$ et $u = \int \frac{s \partial s}{v - s}$, quae integratio adeo ad formulas simplices est referenda. Quarto manente $s = p - u$, si P, Q, R denotent functiones quascunque ipsius s, aequatio nostra $v \partial u = s \partial s + s \partial u$ tractari poterit, si fuerit $v = s + \frac{Ps}{Qu + Ru^n}$, tum enim fit $P \partial u = Qu \partial s + Ru^n \partial s$. Quinto etiam patet, si denotantibus V et U functiones quascunque ipsius u, fuerit $v = s + Vss + Us^n$, integrationem quoque fore in potestate, fit enim aequatio nostra $Vs \partial u + Us^{n-1} \partial u = \partial s$. Atque in genere si aequatio differentialis $\partial s = Z \partial u$ fuerit integrabilis, existente Z functione binarum variabilium s et u, cum nostra aequatio sit $s \partial s = (v - s) \partial u$, habebimus $v = s + Zs$ pro omnibus casibus integrationem admittentibus.

Exemplum 1.

800. *Sumto elemento ∂x constante, si proponatur haec aequatio $xx \partial \partial y = x \partial x \partial y + ny \partial x^2$, ejus integrale invenire.*

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, erit $qxx = px + ny$, unde facto $y = ux$, prodit $qx = p + nu = v$, ita ut v sit functio unius dimensionis ipsarum p et u, et aequatio nostra $(p + nu) \partial u = p \partial p - u \partial p$ fiat homogenea. Cum ergo sit $nu \partial u + p \partial u + u \partial p = p \partial p$, erit integrando

$$C + nuu + 2pu = pp \text{ et}$$

$$p = u + \sqrt{[C + (n + 1) uu]}.$$

Habebimus ergo $\frac{\partial x}{x} = \frac{\partial u}{\sqrt{[C + (n + 1) uu]}}$, quae denuo integrata dat

$$lx = \frac{1}{\sqrt{(n + 1)}} l \frac{u \sqrt{(n + 1)} + \sqrt{[C + (n + 1) uu]}}{D}, \text{ seu}$$

$$D x^{\sqrt{(n+1)}} = u \sqrt{(n+1)} + \sqrt{[C + (n+1) u u]}$$

hincque

$$D^2 x^{2\sqrt{(n+1)}} - 2 D x^{\sqrt{(n+1)}} u \sqrt{(n+1)} = C:$$

Sit $D = f \sqrt{(n+1)}$ et $C = g (n+1)$

ut habeatur

$$ff x^{2\sqrt{(n+1)}} - 2 f x^{\sqrt{(n+1)}} u = g, \text{ existente } u = \frac{y}{x}.$$

Casu quo $u = -1$, ob $\frac{\partial x}{x} = \frac{\partial u}{\alpha}$, erit $\alpha l \frac{x}{a} = u = \frac{y}{x}$, idec que $y = \alpha x l \frac{x}{a}$. At si $n + 1$ sit numerus negativus integrati etiam angulos implicabit.

Corollarium 1.

801. Si sit $n = 0$, hujus aequationis $xx \partial\partial y = x \partial x \partial$ integrale completum erit $ff x^2 - 2 f y = g$, qui casus per se e: perspicuus. Cum ex $\frac{\partial\partial y}{\partial y} = \frac{\partial x}{x}$ fluat

$$\frac{\partial y}{\partial x} = f x \text{ et } 2 y = f x x - \frac{g}{f}.$$

Corollarium 2.

802. Si sit $n = 3$, aequationis $x x \partial \partial y = x \partial x \partial y -$ $3 y \partial x^2$ integralum completum est $ff x^4 - 2 f x y = g$. Ide evenit si loco $\sqrt{(n+1)}$ scribatur -2, fit enim

$$\frac{ff}{x^4} - \frac{2 f y}{x^3} = g \text{ et } ff - 2 f x y = g x^4,$$

utraque redit ad $y = \frac{\alpha}{x} + \beta x^3$.

Exemplum 2.

803. *Sumto elemento ∂x constante, si proponatur ha aequatio $\frac{x x \partial \partial y}{\partial x} = \sqrt{(m x x \partial y^2 + n y y \partial x^2)}$, ejus integra completum invenire.*

Ob $\partial y = p \partial x$ et $\partial p = q \partial x$, habebimus $q x x = \surd (m p p x x + n y y)$, quae posito $y = u x$ abit in

$$q x = \surd (m p p + n u u) = v, \text{ ob } q = \frac{v}{x}.$$

Quia ergo est $\frac{\partial x}{x} = \frac{\partial u}{p - u} = \frac{\partial p}{v}$, erit

$$\partial u \surd (m p p + n u u) = (p - u) \partial p,$$

quae est aequatio homogenea. Ponatur ergo $p = s u$, et prodit

$$\partial u \surd (m s s + n) = (s - 1)(s \partial u + u \partial s);$$

hincque

$$\frac{\partial u}{u} = \frac{(s - 1) \partial s}{\surd (m s s + n) - s s + s},$$

ex qua fit

$$\frac{\partial x}{x} = \frac{\partial u}{(s - 1) u} = \frac{\partial s}{\surd (m s s + n) - s s + s},$$

unde tam $u = \frac{y}{x}$ quam x per eandem variabilem s determinatur.

Exemplum 3.

804. *Sumto elemento ∂x constante, si proponatur haec aequatio $n x^3 \partial \partial y = (y \partial x - x \partial y)^2$, ejus integrale invenire.*

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, erit $n x^3 q = (y - p x)^2 = n x x v$ ob $q = \frac{v}{x}$. Si jam statuatur $y = u x$, fiet

$$n v = (u - p)^2 \text{ et } \frac{\partial x}{x} = \frac{\partial u}{p - u} = \frac{\partial p}{v} = \frac{n \partial p}{(p - u)^2},$$

unde habetur $n \partial p = p \partial u - u \partial u$: quae facto $p - u = s$, abit in $n \partial u + n \partial s = s \partial u$ seu $\partial u = \frac{n \partial s}{s - n}$, hinc $u = n l \frac{s - n}{\alpha}$. Tum vero ob $p - u = s$, erit

$$\frac{\partial x}{x} = \frac{\partial u}{s} = \frac{n \partial s}{s (s - n)} \text{ et } l x = l \frac{s - n}{\beta s}, \text{ ideoque}$$

$$x = \frac{s - n}{\beta s}, \text{ at } y = n x l \frac{s - n}{\alpha}.$$

Cum ergo sit

$s = \frac{n}{1-\beta x}$, erit $y = nx\, l \frac{n\beta x}{\alpha(1-\beta x)}$.

Corollarium.

805. Aequatio $nx^3 q = (y - px)^2$ facilius resolvitur, ponendo $y - px = z$, unde fit $-x\partial p = \partial z$; quare ob $q\partial x = \partial p$, erit

$$nx^3 \partial p = zz\partial x = -nxx\partial z,$$

ideoque

$$\frac{1}{x} = \frac{1}{a} - \frac{n}{z} \text{ seu } \frac{x-a}{ax} = \frac{n}{y-px}, \text{ ergo}$$
$$y - px = \frac{nax}{x-a} = \frac{y\partial x - x\partial y}{\partial x}.$$

Quare

$$\frac{y\partial x - x\partial y}{xx} = \frac{na\partial x}{x(x-a)} \text{ et}$$
$$\frac{y}{x} = n\, l \frac{x}{x-a} + C \text{ ut ante.}$$

Exemplum 4.

806. *Sumto ∂x constante, si proponatur haec aequatio*

$$(\partial x^2 + \partial y^2)\sqrt{(\partial x^2 + \partial y^2)} = n\partial x\partial\partial y\sqrt{(xx+yy)},$$

integrale invenire.

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, aequatio proposita hanc induit formam

$$(1+pp)\sqrt{(1+pp)} = nq\sqrt{(xx+yy)},$$

quae facto $y = ux$ et $q = \frac{v}{x}$ transit in hanc

$$(1+pp)\sqrt{(1+pp)} = nv\sqrt{(1+uu)}.$$

Cum igitur sit $\frac{\partial x}{x} = \frac{\partial u}{p-u} = \frac{\partial p}{v}$, ob

$$v = \frac{(1+pp)\sqrt{(1+pp)}}{n\sqrt{(1+uu)}} \text{ erit}$$

$$(1+pp)^{\frac{3}{2}}\partial u = n(p-u)\partial p\sqrt{(1+uu)}$$

cujus resolutio non sponte patet. Calculum autem ad angulos revocando, sit $p = \text{tang.}\,\phi$ et $u = \text{tang.}\,\omega$, erit

$$\partial p = \frac{\partial \phi}{\cos. \phi^2}, \ \partial u = \frac{\partial \omega}{\cos. \omega^2},$$

$$\sqrt{(1 + pp)} = \frac{1}{\cos. \phi}, \ \sqrt{(1 + uu)} = \frac{1}{\cos. \omega}, \text{ et}$$

$$p - u = \frac{\sin. (\phi - \omega)}{\cos. \phi \cos. \omega}, \text{ hincque}$$

$$\frac{1}{\cos. \phi^3} \cdot \frac{\partial \omega}{\cos. \omega^2} = \frac{n \sin. (\phi - \omega)}{\cos. \phi \cos. \omega} \cdot \frac{1}{\cos. \omega} \cdot \frac{\partial \phi}{\cos. \phi^2} \text{ sive}$$

$$\partial \omega = n \partial \phi \sin. (\phi - \omega) = \partial \phi - (\partial \phi - \partial \omega) \text{ ergo}$$

$$\partial \phi = \frac{\partial \phi - \partial \omega}{1 - n \sin. (\phi - \omega)}.$$

Posito ergo $\phi - \omega = \psi$, erit

$$\phi = \int \frac{\partial \psi}{1 - n \sin. \psi} \text{ et } \omega = \int \frac{\partial \psi}{1 - n \sin. \psi} - \psi;$$

hinc ob $p = \text{tang.} \phi$, et $u = \text{tang.} \omega$, obtinetur

$$\frac{\partial x}{x} = \frac{\partial u \cos. \phi \cos. \omega}{\sin. \psi} = \frac{\partial \omega \cos. \phi}{\sin. \psi \cos. \omega} = \frac{n \partial \psi \cos. \phi}{\cos. \omega (1 - n \sin. \psi)}.$$

Casu $n = 1$, fit

$$\partial \phi = \frac{\partial \psi}{1 - \sin. \psi} = \frac{\partial \psi (1 + \sin. \psi)}{\cos. \psi^2},$$

$$\phi = \text{tang.} \psi + \frac{1}{\cos. \psi} = \frac{1 + \sin. \psi}{\cos. \psi} + \alpha,$$

$$\omega = \frac{1 + \sin. \psi}{\cos. \psi} + \alpha - \psi, \text{ et}$$

$$\frac{\partial x}{x} = \frac{\partial \phi \cos. \phi}{\cos. \omega} = \frac{\partial \phi \cos. \phi}{\cos. \phi \cos \psi + \sin. \phi \sin. \psi}.$$

Cum autem sit $\phi - \alpha = \sqrt{\frac{1 + \sin. \psi}{1 - \sin. \psi}}$, erit

$$\sin \psi = \frac{(\phi - \alpha)^2 - 1}{(\phi - \alpha)^2 + 1}, \ \cos. \psi = \frac{2(\phi - \alpha)}{(\phi - \alpha)^2 + 1}. \quad \text{Ergo}$$

$$\frac{\partial x}{x} = \frac{\partial \phi \cos. \phi [(\phi - \alpha)^2 + 1]}{2 (\phi - \alpha) \cos. \phi + (\phi - \alpha)^2 \sin. \phi - \sin. \phi}.$$

Corollarium 3.

807. Si casu $n = 1$ sumatur constans α infinita, erit $\sin \psi = 1$, hinc $\psi = 90^\circ$ et $\omega = \phi - 90^\circ$, tum vero $\frac{\partial x}{x} = \frac{\partial \phi \cos. \phi}{\sin. \phi}$, ideoque $x = a \sin. \phi$, et $u = -\cot. \phi$, ergo $y = -a \cos. \phi$, et $xx + yy = aa$.

Corollarium 2.

808. Eodem autem casu $n = 1$, quo constans α non sumitur infinita, fractionis, cui $\frac{\partial x}{x}$ aequatur, numerator commode est differentiale denominatoris; unde fit

$$x = a\,[(\Phi - \alpha)^2 \sin.\Phi - \sin.\Phi + 2(\Phi - \alpha)\cos.\Phi].$$

Tum vero est

$$\omega = \Phi - \text{Ang. tang.}\ \frac{(\Phi - \alpha)^2 - 1}{2(\Phi - \alpha)},$$

ideoque

$$u = \frac{y}{x} = \text{tang.}\ \omega = \frac{\text{tang.}\,\Phi - \frac{(\Phi - \alpha)^2 + 1}{2(\Phi - \alpha)}}{1 + \frac{(\Phi - \alpha)^2 - 1}{2(\Phi - \alpha)}\,\text{tang.}\,\Phi}, \text{ seu}$$

$$\frac{y}{x} = \frac{2(\Phi - \alpha)\sin.\Phi - (\Phi - \alpha)^2\cos.\Phi + \cos.\Phi}{(\Phi - \alpha)^2\sin.\Phi - \sin.\Phi + 2(\Phi - \alpha)\cos.\Phi};$$

consequenter

$$y = -a\,[(\Phi - \alpha)^2\cos.\Phi - \cos.\Phi - 2(\Phi - \alpha)\sin.\Phi] \text{ et}$$

$$\sqrt{(xx + yy)} = [(\Phi - \alpha)^2 + 1].$$

Scholion 1.

809. In genere etiam integrationem absolvere licet. Cum enim sit

$$\partial\Phi = \frac{\partial\psi}{1 - n\sin.\psi} \text{ et } \frac{\partial x}{x} = \frac{n\,\partial\Phi\cos.\Phi}{\cos.\omega}, \text{ erit}$$

$$\Phi + \alpha = \frac{1}{\sqrt{(1 - nn)}}\,\text{Ang. cos.}\ \frac{n - \sin.\psi}{1 - n\sin.\psi};$$

unde posito

$$(\Phi + \alpha)\sqrt{(1 - nn)} = \theta, \text{ erit } \cos.\theta = \frac{n - \sin.\psi}{1 - n\sin.\psi};$$

hincque

$$\sin.\psi = \frac{n - \cos.\theta}{1 - n\cos.\theta} \text{ et } \cos.\psi = \frac{\sin.\theta\sqrt{(1 - nn)}}{1 - n\cos.\theta}.$$

At ob $\omega = \Phi - \psi$, habebitur

$$\frac{\partial x}{x} = \frac{n\,\partial\Phi\cos.\Phi\,(1 - n\cos.\theta)}{\cos.\Phi\sin.\theta\sqrt{(1 - nn)} + \sin.\Phi\,(n - \cos.\theta)}.$$

Jam cum sit $\partial\theta = \partial\Phi\sqrt{(1-nn)}$, differentiale hujus denominatoris est

$$-\partial\Phi\sin.\Phi\sin.\theta\sqrt{(1-nn)}+\partial\Phi\cos.\Phi\cos.\theta(1-nn)$$
$$+n\partial\Phi\cos.\Phi-\partial\Phi\cos.\Phi\cos.\theta+\partial\Phi\sin.\Phi\sin.\theta\sqrt{(1-nn)},$$

quod redit ad $n\partial\Phi\cos.\Phi(1-n\cos.\theta)$ ipsum scilicet numeratorem. Ita ut sit

$$x=a[\cos.\Phi\sin\theta\sqrt{(1-nn)}+\sin.\Phi(n-\cos.\theta)]$$

seu $x=a\cos.\omega(1-n\cos.\theta)$, ideoque

$$y=ux=a\sin.\omega(1-n\cos\theta).$$

Assumto ergo angulo θ, quaeratur angulus ψ, ut sit

$$\sin.\psi=\frac{n-\cos.\theta}{1-n\cos.\theta} \text{ et } \cos.\psi=\frac{\sin.\theta}{1-n\cos.\theta}\sqrt{(1-nn)},$$

tum vero fiat

$$\omega=\frac{\theta}{\sqrt{(1-nn)}}-\alpha-\psi,$$

eritque integrale completum

$$x=a(1-n\cos.\theta)\cos.\omega \text{ et } y=a(1-n\cos.\theta)\sin.\omega.$$

Scholion 2.

810. At si numerus n sit unitate major, haec integratio fit ima inaria, quod incommodum ut tollatur, notandum est, aequationis $\partial\Phi=\frac{\partial\psi}{1-n\sin.\psi}$ integrale esse

$$\Phi+\alpha=\frac{1}{\sqrt{(nn-1)}}\,l\frac{\sqrt{(n-1)(1+\sin.\psi)}+\sqrt{(n+1)(1-\sin.\psi)}}{\sqrt{(n-1)(1+\sin.\psi)}-\sqrt{(n-1)(1-\sin.\psi)}}.$$

Quare si ponatur $(\Phi+\alpha)\sqrt{(nn-1)}=\theta$, ut sit

$$\partial\theta=\partial\Phi\sqrt{(nn-1)} \text{ et } \omega=\Phi-\psi=\frac{1}{\sqrt{(nn-1)}}-\alpha-\psi,$$

erit

$$\frac{e^\theta+1}{e^\theta-1}=\frac{\sqrt{(n-1)(1+\sin.\psi)}}{\sqrt{(n+1)(1-\sin.\psi)}}=\frac{(n-1)(1+\sin.\psi)}{\cos.\psi\sqrt{(nn-1)}};$$

unde reperitur

$$\sin.\psi = \frac{e^{\theta}+2n+e^{-\theta}}{ne^{\theta}+2+ne^{-\theta}} \text{ et } \cos.\psi = \frac{(e^{\theta}-e^{-\theta})\sqrt{(nn-1)}}{ne^{\theta}+2+ne^{-\theta}},$$

ita ut ex angulo θ definiantur anguli ψ, Φ et ω. Cum jam sit

$$\frac{\partial x}{x} = \frac{n\partial\Phi \cos.\Phi}{\cos.\omega} = \frac{n\partial\Phi\cos.\Phi}{\cos.\Phi\cos.\psi+\sin.\Phi\sin.\psi}, \text{ erit}$$

$$\frac{\partial x}{x} = \frac{n\partial\Phi\cos.\Phi(ne^{\theta}+2+ne^{-\theta})}{\cos.\Phi(e^{\theta}-e^{-\theta})\sqrt{(nn-1)}+\sin.\Phi(e^{\theta}+2n+e^{-\theta})},$$

ubi iterum commode evenit, ut numerator sit ipsum differentiale denominatoris, quemadmodum differentiationem instituenti mox patebit. Hinc ergo erit

$$x = a[\cos.\Phi(e^{\theta}-e^{-\theta})\sqrt{(nn-1)}+\sin.\Phi(e^{\theta}+2n+e^{-\theta})] \text{ seu}$$

$$x = a\cos.\omega(ne^{\theta}+2+ne^{-\theta}), \text{ et ob } u = \frac{y}{x} = \text{tang.}\,\omega \text{ fit}$$

$$y = a\sin.\omega(ne^{\theta}+2+ne^{-\theta}).$$

Quocirca ex angulo θ primo quaeratur angulus ψ, ut sit

$$\sin.\psi = \frac{e^{\theta}+2n+e^{-\theta}}{ne^{\theta}+2+ne^{-\theta}} \text{ et } \cos.\psi = \frac{(e^{\theta}-e^{-\theta})\sqrt{(nn-1)}}{ne^{\theta}+2+ne^{-\theta}},$$

quo invento capiatur angulus $\omega = \frac{\theta}{\sqrt{(nn-1)}} - \alpha - \psi$, ac formulae illae pro x et y inventae dabunt integrale completum ob duas constantes a et α introductas.

Scholion 3.

811. Cum hic praecipua pars integrationis fortuito successisse videatur, operae praetium erit in ejus causam inquirere, num forte ratio integrandi clarius perspici queat. Cum igitur sit

$$\Phi = \psi + \omega \text{ et } \partial\Phi = \frac{\partial\psi}{1-n\sin.\psi},$$

hincque

$$\partial\omega = \frac{n\partial\psi\sin.\psi}{1-n\sin.\psi} = n\partial\Phi\sin.\psi,$$

aequatio

$$\frac{\partial x}{x} = \frac{n\,\partial\Phi\cos.\Phi}{\cos.\omega},\ \text{ob}\ \cos.\Phi = \cos.\psi\cos.\omega - \sin.\psi\sin.\omega,$$

in hanc resolvitur

$$\frac{\partial x}{x} = n\,\partial\Phi\cos.\psi - \frac{n\,\partial\Phi\sin.\psi\sin.\omega}{\cos.\omega},$$

quae ob

$$\partial\Phi = \frac{\partial\psi}{1 - n\sin.\psi}\ \text{et}\ n\,\partial\Phi\sin.\psi = \partial\omega,$$

induit hanc formam integrabilem

$$\frac{\partial x}{x} = \frac{n\,\partial\psi\cos.\psi}{1 - n\sin.\psi} - \frac{\partial\omega\sin.\omega}{\cos.\omega},$$

ex qua elicitur

$$x = \frac{a\cos.\omega}{1 - n\sin.\psi}\ \text{et}\ y = \frac{a\sin.\omega}{1 - n\sin.\psi},\ \text{ob}\ y = ux = x\,\text{tang}.\omega.$$

En ergo in genere aequationis nostrae hanc integrationem. Anguli ω et ψ hanc inter se teneant relationem, ut sit

$$\partial\omega = \frac{n\,\partial\psi\sin.\psi}{1 - n\sin.\psi},$$

tum vero erit

$$x = \frac{a\cos.\omega}{1 - n\sin.\psi}\ \text{et}\ y = \frac{a\sin.\omega}{1 - n\sin.\psi}.$$

Quodsi ergo ponamus $\sqrt{(xx + yy)} = z$, ut sit

$$x = z\cos\omega\ \text{et}\ y = z\sin.\omega,\ \text{erit}$$

$$z = \frac{a}{1 - n\sin.\psi}\ \text{et}\ \sin.\psi = \frac{z - a}{nz},\ \text{hinc}$$

$$\partial\omega = \frac{(z - a)\,\partial\psi}{a}.\ \text{At fit}$$

$$\partial\psi = \frac{a\,\partial z}{z\sqrt{[nnzz - (z - a)^2]}}\ \text{ergo}$$

$$\partial\omega = \frac{(z - a)\,\partial z}{z\sqrt{[nnzz - (z - a)^2]}}$$

unde angulus ω per z definitur. Ad irrationalitatem tollendam, si ponamus

$$\sqrt{[nnzz - (z - a)^2]} = s\,(nz + z - a),\ \text{fit}$$

**

$$z = \frac{a(ss+1)}{(n+1)ss-n+1} \text{ et } \partial\omega = \frac{2n\partial s(ss-1)}{(ss+1)[(n+1)ss-(n-1)]}, \text{ seu}$$

$$\partial\omega = \frac{2\partial s}{ss+1} - \frac{2\partial s}{(n+1)ss-n+1}$$

quae integratio est manifesta.

Problema 99.

812. Si aequatio differentio-differentialis tum demum fiat homogenea, si alteri variabili y tribuantur n dimensiones, ejus integrationem ad aequationem differentialem primi gradus reducere.

Solutio.

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, ut oriatur aequatio inter quaternas quantitates finitas x, y, p et q, quae quomodo ratione homogeneitatis futura sit comparata, videamus. Primo ergo cum pro x unam dimensionem numerando, variabilis y habeat n dimensiones, quantitati $p = \frac{\partial y}{\partial x}$ tribuendae sunt $n-1$ dimensiones, quantitati $q = \frac{\partial p}{\partial x}$ vero $n-2$ dimensiones. Quocirca ponamus

$$y = x^n u,\ p = x^{n-1} t, \text{ et } q = x^{n-2} v,$$

et ob $\partial y = p\partial x$ et $\partial p = q\partial x$, hahebimus

$$x\partial u + nu\partial x = t\partial x \text{ et } x\partial t + (n-1)t\partial x = v\partial x,$$

unde colligimus

$$\frac{\partial x}{x} = \frac{\partial u}{t-nu} = \frac{\partial t}{v-(n-1)t},$$

ideoque

$$\partial u[v-(n-1)t] = \partial t(t-nu).$$

At factis superioribus substitutionibus in aequatione inter x, y, p et q, per hypothesin variabilis x ex calculo extruditur, ita ut prodeat aequatio inter tres tantum variabiles u, t et v, ex qua litteram v per binas t et u definire licebit. Quo valore substituto habebitur aequatio differentialis primi gradus inter binas variabiles u

et t, ex qua t per u determinari queat; ope aequationis $\frac{\partial x}{x} = \frac{\partial u}{t - nu}$ definietur x per u, hincque ob $u = \frac{y}{x^n}$ obtinebitur aequatio integralis inter x et y, eaque ob duplicem integrationem completa.

Corollarium 1.

813. Aequationum ergo inter x, y, p et q hoc modo tractabilium hoc est criterium, ut posito $y = x^n u$, $p = x^{n-1} t$ et $q = x^{n-2} v$, exponens n ejusmodi determinationem patiatur, ut variabilis x prorsus ex calculo per divisionem egrediatur.

Corollarium 2.

814. Si sit $n = 0$, aequatio ita est comparata, ut tribuendo ipsi y ejusque differentialibus nullam dimensionem fiat homogenea. Hoc scilicet casu sola variabilis x cum suis differentialibus dimensiones constituere censetur.

Corollarium 3.

815. Contra vero si dimensiones ex sola variabili y aestimentur, ita ut ea cum suis differentialibus ∂y et $\partial\partial y$ ubique eundem dimensionum numerum constituat, exponens n fiet infinitus.

Scholion.

816. Si sola variabilis x cum suis differentialibus ubique eundem dimensionum numerum complet, ob $n = 0$ fit $u = y$, et in aequatione inter x, y, p et q statui conveniet $p = \frac{t}{x}$ et $q = \frac{v}{xx}$ quo facto variabilis x ex calculo deturbabitur, prodibitque aequatio inter y, t et v, cujus ope aequatio differentialis $\partial y (v + t) = t \partial t$ ad duas tantum variabiles reducetur, qua resoluta erit $\frac{\partial x}{x} = \frac{\partial y}{t}$, ubi cum t detur per y, integratio nullam habet difficultatem. Ve-

rum altero casu, quo variabilis y sola cum suis differentialibus pares ubique dimensiones habet, ideoque exponens n infinitus capi deberet, resolutio alio modo institui debet, quem mox docebimus, nisi forte permutando variabiles x et y casum ad praecedentem reducere lubuerit.

Exemplum 1.

817. *Sumto elemento ∂x constante, si proponatur haec aequatio $xx\partial\partial y = \alpha y \partial x^2 + \beta x \partial x \partial y$, ejus integrale invenire.*

Perpendatur hic ista conditio, qua sola variabilis x cum suo differentiali ∂x ubique duas constituit dimensiones, eritque $n = 0$. Cum ergo posito $\partial y = p \partial x$ et $\partial p = q \partial x$, habeamus $q x x = \alpha y + \beta p x$, statuamus $p = \frac{t}{x}$ et $q = \frac{v}{xx}$, fietque $v = \alpha y + \beta t$, unde adipiscimur istam aequationem differentialem

$$\alpha y \partial y + (\beta + 1) t \partial y = t \partial t,$$

ob cujus homogeneitatem faciamus $t = y z$, eritque

$$\alpha \partial y + (\beta + 1) z \partial y = y z \partial z + z z \partial y \text{ seu}$$

$$\frac{\partial y}{y} = \frac{z \partial z}{\alpha + (\beta + 1) z - zz}.$$

Sit $\alpha + (\beta + 1) z - zz = (f + z)(g - z)$

ut sit

$$\alpha = fg \text{ et } \beta + 1 = g - f,$$

reperieturque

$$\frac{\partial y}{y} = \frac{\frac{-f}{f+g}\partial z}{f + z} + \frac{\frac{g}{f+g}\partial z}{g - z}:$$

unde colligitur integrando

$$l y = \text{C} - \frac{f}{f+g} l (f + z) - \frac{g}{f+g} l (g - z), \text{ seu}$$

$$y (f + z)^{\frac{f}{f+g}} (g - z)^{\frac{g}{f+g}} = a.$$

Tum vero est

$$\frac{\partial x}{x} = \frac{\partial y}{y z} = \frac{\partial z}{(f+z)(g-z)}, \text{ seu}$$

$$\frac{\partial x}{x} = \frac{1}{f+g} \cdot \frac{\partial z}{f+z} + \frac{1}{f+g} \cdot \frac{\partial z}{g-z}, \text{ hinc}$$

$$x = b \left(\frac{f+z}{g-z}\right)^{\frac{1}{f+g}}, \text{ seu } \frac{f+z}{g-z} = \left(\frac{x}{b}\right)^{f+g}.$$

Unde cum sit $z = \frac{g\, x^{f+g} - f\, b^{f+g}}{b^{f+g} + x^{f+g}}$, erit hoc valore ibi substituto

$$(f+g)\, b^g\, x^f\, y = a\left(b^{f+g} + x^{f+g}\right),$$

seu posito $\frac{a}{f+g} = c$

$$y = c\left(\frac{b^f}{x^f} + \frac{x^g}{b^g}\right):$$

est vero $g - f = \beta + 1$ et $g + f = \sqrt{((\beta+1)^2 + 4\alpha)}$.

Corollarium,

818. Quoniam in aequatione proposita etiam ambae variabiles x et y simul ubique totidem dimensiones habent, eam etiam secundum praecepta praecedentis problematis tractare licet.

Exemplum 2.

819. *Posito ∂x constante, si aequatio differentio-differentialis duobus tantum terminis constet, ut sit hujusmodi*

$$\partial\partial y = c\, x^\alpha\, y^\beta\, \partial x^{2-\gamma}\, \partial y^\gamma,$$

ejus integrale investigare.

Posito $\partial y = p\, \partial x$ et $\partial p = q\, \partial x$, habebitur haec forma $q = c\, x^\alpha\, y^\beta\, p^\gamma$, ubi exponentem n ita definire licet, ut posito $y = x^n u$, $p = x^{n-1} t$ et $q = x^{n-2} v$, variabilis x per divisionem tolli possit, capi enim oportet

$$\alpha + \beta + \gamma(n-1) - n + 2 = 0\,,\quad \text{seu}\quad n = \frac{-\alpha+\gamma-2}{\beta+\gamma-1};$$

tumque erit $v = cu^{\beta}t^{\gamma}$. Aequatio ergo differentialis primi gradus resolvenda erit

$$cu^{\beta}t^{\gamma}\partial u - (n-1)\,t\partial u = t\partial t - nu\partial t,$$

ex qua cum variabilis t per u fuerit determinata, integrari oportet hanc formulam $\frac{\partial x}{x} = \frac{\partial u}{t - nu}$, quo facto ob $u = \frac{y}{x^n}$, obtinebitur aequatio integralis quaesita inter x et y.

Casus tantum $\beta + \gamma = 1$, quo n fit infinitus, peculiarem postulat tractationem infra exponendam, nisi forte simul sit $\gamma = \alpha + 2$, tum enim exponens n prorsus arbitrio nostro relinquitur, at aequatio erit homogenea.

Exemplum 3.

820. *Sumto elemento ∂x constante, si proponatur haec aequatio*

$$x^4\partial\partial y = x^3\partial x\partial y + 2xy\partial x\partial y - 4yy\partial x^2,$$

ejus integrale invenire.

Hic evidens est, si ipsi y ejusque differentialibus ∂y et $\partial\partial y$ binae dimensiones, ipsi x vero et ∂x singulae tribuantur, in omnibus terminis obtineri sex dimensiones. Quare cum posito $\partial y = p\partial x$ et $\partial p = q\partial x$, habeamus hanc aequationem

$$x^4 q = x^3 p + 2xyp - 4yy,$$ faciamus

$y = x^2 u, p = xt$ et $q = v$, prodibitque

$$v = t + 2ut - 4uu.$$

At ob $n = 2$, aequatio differentialis nostra erit

$$\partial u(v - t) = \partial t(1 - 2u),$$

quae abit in

$$2u\partial u(t-2u)=\partial t(t-2u),$$

unde deducimus vel $t=2u$, vel $t=uu+c$, quos binos casus seorsim evolvamus.

1) Si $t=2u$, ob $\frac{\partial x}{x}=\frac{\partial u}{t-2u}$ fit $\partial u=0$, ideoque $u=C$, ac propterea $y=Cxx$, quod est integrale particulare, aequationi propositae utique satisfaciens.

2) Sit $t=uu+c$, erit $\frac{\partial x}{x}=\frac{\partial u}{uu-2u+c}$, ubi tres casus sunt considerandi:

Primo si constans $c=1$, erit

$$l\frac{x}{a}=\frac{1}{1-u}=\frac{xx}{xx-y}, \text{ seu } xx=(xx-y)\,l\frac{x}{a}.$$

Secundo si constans $c=1-ff$, erit $\frac{\partial x}{x}=\frac{\partial u}{(u-1)^2-ff}$, hincque

$$lx=\frac{-1}{2f}\,l\frac{f+u-1}{f-u+1}+C,$$

ergo ob $u=\frac{y}{xx}$ erit

$$x=a\left(\frac{(f+1)xx-y}{(f-1)xx+y}\right)^{\frac{1}{2f}}.$$

Tertio si constans $c=1+ff$, ideoque $\frac{\partial x}{x}=\frac{\partial u}{(u-1)^2+ff}$, quae integrata dat

$$l\frac{x}{a}=\frac{1}{f}\text{ Ang. tang. }\frac{u-1}{f}, \text{ seu } \frac{u-1}{f}=\frac{y-xx}{fxx}=\text{tang.}\,f\,l\frac{x}{a}.$$

Pro ratione ergo constantis arbitrariae c integratio vel algebraice succedit, vel a logarithmis, vel ab angulis pendet, unde forma generali exprimi nequit.

Scholion.

821. Integrale autem particulare primo inventum $y=Cxx$ in nulla harum formarum, quibus integrale completum constituitur,

contineri deprehenditur: nihilo vero minus satisfacit aequationi differentio-differentiali propositae. Hoc ergo exemplo magis illustrantur ea, quae supra circa hoc paradoxon sumus commentati, quod interdum aequationi differentiali satisfaciat aequatio finita, quae in integrali completo minime contineatur. Videmus igitur hoc idem paradoxon etiam in aequationibus differentio-differentialibus locum habere. Utrum autem illa aequatio $y = Cxx$ inter integralia sit admittenda, alia est quaestio, quae nondum penitus videtur confecta; hic quidem ipsa aequatio proposita quasi factores habens est censenda, ex quorum altero illa aequatio $y = Cxx$ nascatur, verum multum abest, ut in hac explicatione acquiescere queamus. Quin potius ipsa illa quaestio, sive geometrica fuerit sive alius disciplinae, cujus solutio ad hujusmodi aequationem perduxerit, accurate perpendi debere videtur; ubi plerumque haud difficulter indicare potest, utrum quicquid aequationi differentiali satisfaciat, id etiam ipsi quaestioni conveniat nec ne? Veluti si descensus gravis ex altitudine $= a$ labentis definiri debeat, et altitudo qua jam a terra distat sit x, erit ibi celeritas ut $\sqrt{(a - x)}$, et elementum temporis $\partial t = \frac{-\partial x}{\sqrt{(a-x)}}$. Hic quidem evidens est isti aequationi differentiali satisfieri ponendo $x = a$, ita ut tempus t maneat indefinitum, quod tamen quaestioni neutiquam convenit, quae nonisi vero integrali $t = 2\sqrt{(a - x)}$ resolvitur.

Problema 100.

822. Si in aequatione differentio-differentiali variabilis y cum suis differentialibus ∂y et $\partial\partial y$ ubique eundem dimensionum numerum adimpleat, ejus integrationem ad aequationem differentialem primi gradus reducere.

Solutio.

Posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, aequatio ita erit comparata, ut in ea ternae variabiles y, p, q, ubique eundem dimensionum numerum obtineant, altera variabili x in computum dimensionum prorsus non ingrediente. Quare si statuatur $p = uy$ et $q = vy$, in omnibus terminis inerit eadem ipsius y potestas, qua per divisionem sublata habebitur aequatio inter ternas tantum variabiles x, u et v, ex qua unam per binas reliquas definire licebit, ita ut v aequetur functioni cuipiam ipsarum x et u. Jam ob $p = uy$ erit $\partial y = uy\partial x$, et ob $\partial p = q\partial x$ fiet $u\partial y + y\partial u = vy\partial x$, unde sequitur

$$\frac{\partial y}{y} = u\partial x \text{ et } \frac{\partial y}{y} = \frac{v\partial x - \partial u}{u},$$

ideoque $\partial u + uu\partial x = v\partial x$, quae aequatio differentialis duas tantum variabiles x et u complectitur. Quam ergo si integrare liceat, ut relatio inter x et u inde innotescat, superest, ut formulae $u\partial x$ integrale investigetur, quo invento erit $ly = \int u\partial x$, sicque aequatio orietur integralis inter x et y, quae ob duplicem integrationem peractam duas constantes arbitrarias involvet, ideoque integrale completum exhibebit.

Corollarium 1.

823. Hujusmodi ergo aequationum integratio reducitur ad hujusmodi aequationem differentialem $\partial u + uu\partial x = v\partial x$, cujus resolutio si succedat, simul illarum integratio habetur, cum formulae $u\partial x$ integratio difficultate careat.

Corollarium 2.

824. Cum sit $\frac{\partial y}{y} = u\partial x$, erit $y = e^{\int u\partial x}$, qua substitutione aequatio differentio-differentialis proposita statim reducitur ad aequationem differentialem primi gradus, erit enim

$$\frac{\partial y}{\partial x} = p = e^{\int u\partial x}u, \text{ et } \frac{\partial p}{\partial x} = q = \frac{e^{\int u\partial x}(\partial u + uu\partial x)}{\partial x}.$$

ac tum formula exponentialis sponte ex aequatione egreditur.

Corollarium 3.

825. Vicissim etiam proposita aequatione differentiali primi gradus $\partial u + uu\partial x = v\partial x$, in qua v sit functio quaecunque ipsarum x et u, ea posito $u = \frac{\partial y}{y\partial x}$, in ejusmodi aequationem differentio-differentialem transformatur, in qua variabilis y cum suis differentialibus ∂y et $\partial\partial y$ ubique eundem dimensionum numerum constituat.

Scholion 1.

826. Haec reductio aequationum differentialium primi gradus ad gradum secundum legibus analyseos adversari videtur, interim tamen subinde usu non caret; quodsi enim alia methodo hujusmodi aequationes differentio-differentiales tractare liceat, dum earum integralia vel per series exhibentur vel finite, simul integralia aequationum differentialium primi gradus innotescunt, quorum ratio plerumque aliunde vix perspicitur. In sequentibus autem videbimus, ejusmodi aequationes differentio-differentiales in quibus variabilis altera y unam dimensionem non superat, per series commode integrari posse, atque adeo interdum has series abrumpi, ita ut integrale finita expressione exhibeatur. Caeterum proposita hujusmodi aequatione differentiali primi gradus $\partial u + uu\partial x = v\partial x$, substitutio $u = \frac{\partial y}{y\partial x}$ eo magis est notatu digna, quod sumto elemento ∂x constante fiat

$$\partial u = \frac{\partial\partial y}{y\partial x} - \frac{\partial y^2}{yy\partial x}, \text{ ideoque}$$

$$\partial u + uu\partial x = \frac{\partial\partial y}{y\partial x},$$

ita ut duo termini hoc modo in unum coalescant.

Scholion 2.

827. Casus hic imprimis notasse juvabit, quibus aequatio $\partial u + uu\partial x = v\partial x$ integrationem admittit. Hunc in finem sit

$\partial u = V \partial x$ forma generalis aequationum resolubilium, et V certa functio ipsarum x et u, ac manifestum est, si fuerit $v = uu + V$, integrationem succedere. Primum ergo hoc eveniet si sit $V = \frac{X}{U}$, denotante X functionem ipsius x, et U ipsius u. Secundo si V sit functio homogenea nullius dimensionis ipsarum x et u. Tertio si denotantibus X et Ξ functiones quascunque ipsius x, fuerit $V = Xu + \Xi u^n$. Quarto si denotantibus P et Q functiones quascunque ipsius u, fuerit $V = \frac{1}{Px + Qx^n}$. Similique modo ex aliis formis integrabilibus alii casus concludentur.

Exemplum 1.

828. *Sumto elemento ∂x constante, si proponatur haec aequatio*

$$\alpha y \partial\partial y + \beta \partial y^2 = \frac{y \partial x \partial y}{\sqrt{(aa+xx)}};$$

ejus integrale invenire.

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, prodit

$$\alpha y q + \beta pp = \frac{yp}{\sqrt{(aa+xx)}},$$

quae facto $p = uy$ et $p = vy$, abit in hanc

$$\alpha v + \beta uu = \frac{u}{\sqrt{(aa+xx)}}, \text{ seu } v = \frac{u}{\alpha\sqrt{(aa+xx)}} - \frac{\beta uu}{\alpha},$$

unde hanc aequationem resolvi oportet

$$\partial u + uu \partial x = \frac{u \partial x}{\alpha\sqrt{(aa+xx)}} - \frac{\beta uu \partial x}{\alpha}.$$

Statuatur $u = \frac{1}{s}$, fietque

$$+\partial s + \frac{s \partial x}{\alpha\sqrt{(aa+xx)}} = \left(1 + \frac{\beta}{\alpha}\right) \partial x,$$

quae per $[x + \sqrt{(aa+xx)}]^{\frac{1}{\alpha}}$ multiplicata et integrata dat

$$s[x + \sqrt{(aa+xx)}]^{\frac{1}{\alpha}} = \left(1 + \frac{\beta}{\alpha}\right) \int \partial x [x + \sqrt{(aa+xx)}]^{\frac{1}{\alpha}}.$$

Fiat $x+\surd(aa+xx)=t^{\alpha}$, erit

$$aa=t^{2\alpha}-2t^{\alpha}x,\ \text{hinc}$$

$$x=\frac{t^{2\alpha}-aa}{2t^{\alpha}}=\tfrac{1}{2}t^{\alpha}-\tfrac{1}{2}aat^{-\alpha}\ \text{et}$$

$$\partial x=\tfrac{\alpha}{2}\partial t(t^{\alpha-1}+aat^{-\alpha-1});$$

ita ut sit

$$st=(1+\tfrac{\beta}{\alpha})\int\tfrac{\alpha}{2}\partial t\,(t^{\alpha}+aat^{-\alpha})\ \text{seu}$$

$$st=C+\frac{\alpha+\beta}{\alpha}\left(\frac{t^{\alpha+1}}{\alpha+1}+\frac{aat^{1-\alpha}}{1-\alpha}\right).$$

Porro est $\frac{\partial y}{y}=u\partial x=\frac{\partial x}{s}$; at ex aequatione differentiali est

$$(1+\tfrac{\beta}{\alpha})\frac{\partial x}{s}=\frac{\partial s}{s}+\frac{\partial x}{\alpha\surd(aa+xx)},$$

hincque

$$(1+\tfrac{\beta}{\alpha})\,ly=ls+\tfrac{1}{\alpha}\,l[x+\surd(aa+xx)]=lst+D$$

ergo $y=B(st)^{\frac{\alpha}{\alpha+\beta}}$. Quare sumto $C=\frac{\alpha+\beta}{\alpha}A$, habebitur

$$y=B\left(A+\frac{t^{\alpha+1}}{\alpha+1}+\frac{aat^{1-\alpha}}{1-\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}$$

existente

$$x=\tfrac{1}{2}(t^{\alpha}-aat^{-\alpha}),\ \text{vel}\ t=[x+\surd(aa+xx)]^{\frac{1}{\alpha}},$$

ita ut aequatio inter x et y sit

$$Cy^{\frac{\alpha+\beta}{\alpha}}=A+\tfrac{1}{\alpha+1}[x+\surd(aa+xx)]^{\frac{\alpha+1}{\alpha}}$$
$$+\tfrac{aa}{1-\alpha}[x+\surd(aa+xx)]^{\frac{1-\alpha}{\alpha}}.$$

Scholion.

829. Hoc idem exemplum ita est comparatum, ut alia ratione facillime resolvi possit; aequatio enim

$$\alpha yq+\beta pp=\frac{yp}{\surd(aa+xx)},$$

si per $\frac{\partial x}{yp}$ multiplicetur, ob $q\partial x=\partial p$ et $p\partial x=\partial y$, abit in

$$\frac{\alpha\,\partial p}{p} + \frac{\beta\,\partial y}{y} = \frac{\partial x}{\sqrt{(aa+xx)}},$$

cujus singuli termini sunt integrabiles. Prodit ergo

$$p^\alpha y^\beta = C\,[x+\sqrt{(aa+xx)}], \text{ hincque}$$

$$y^{\frac{\beta}{\alpha}}\partial y = C\,\partial x\,[x+\sqrt{(aa+xx)}]^{\frac{1}{\alpha}},$$

quae aequatio denuo integrata praebet integrale ante inventum. Forma ergo generalis aequationum hoc modo resolubilium est $P\partial p + Y\partial y + X\partial x = 0$, existente P functione ipsius p, Y ipsius y et X ipsius x, quae nostro more repraesentatur per $Pq + Yp + X = 0$. Hinc ergo perspicitur quomodo etiam aequationes differentio-differentiales ope idonei multiplicatoris ad integrationem perduci queant; quae methodus, cum in aequationibus differentialibus primi gradus insignem usum praestiterit, eo magis excolenda videtur, quod etiam ad aequationes differentiales altiorum graduum pateat, quod argumentum infra fusius pertractare conabimur.

Exemplum 2.

830. *Sumto elemento ∂x constante, si proponatur haec aequatio*

$$xy\,\partial\partial y = y\,\partial x\,\partial y + x\,\partial y^2 + \frac{bx\,\partial y^2}{\sqrt{(aa-xx)}},$$

ejus integrale invenire.

Posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, erit

$$xyq = yp + xpp + \frac{bxpp}{\sqrt{(aa-xx)}},$$

quae facto $p = uy$ et $q = vy$, abit in

$$xv = u + uux + \frac{buux}{\sqrt{(aa-xx)}}:$$

unde oritur haec aequatio differentialis

$$\partial u + uu\,\partial x = \frac{u\,\partial x}{x} + uu\,\partial x + \frac{buu\,\partial u}{\sqrt{(aa-xx)}}, \text{ seu}$$

$$\frac{x\,\partial u - u\,\partial x}{uu} = \frac{bx\,\partial x}{\sqrt{(aa-xx)}},$$

cujus integrale

$$C - \frac{x}{u} = - b\sqrt{(aa - xx)}, \text{ seu}$$

$$u = \frac{x}{C + b\sqrt{(aa + xx)}}, \text{ ergo } \frac{\partial y}{y} = \frac{x\,\partial x}{C + b\sqrt{(aa - xx)}}.$$

Statuatur $\sqrt{(aa - xx)} = t$, ut sit $x\partial x = -t\partial t$, erit

$$\frac{\partial y}{y} = \frac{-t\,\partial t}{C + bt} = -\frac{\partial t}{b} + \frac{C\,\partial t}{b(C + bt)}, \text{ et}$$

$$ly = -\frac{t}{b} + \frac{C}{bb}\,l(C + bt) + lc.$$

Sit $C = nbb$, erit

$$l\frac{y}{c} = -\frac{\sqrt{(aa - xx)}}{b} + n\,l\frac{nb + \sqrt{(aa - xx)}}{b}$$

ubi c et n sunt constantes arbitrariae.