CAPUT II.

DE

AEQUATIONIBUS DIFFERENTIO-DIFFERENTIALIBUS IN QUIBUS ALTERA VARIABILIUM IPSA DEEST.

Problema 95.

750.

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, si detur aequatio quaecunque inter tres quantitates x, p et q, in quam altera variabilis y non ingrediatur, investigare relationem inter ipsas variabiles x et y.

Solutio.

Cum aequatio proposita has tres quantitates x, p et q contineat, loco q scribatur ejus valor $\frac{\partial p}{\partial x}$, atque habebitur aequatio differentialis primi gradus duas tantum quantitates variabiles x et p involvens, quam secundum praecepta prioris partis tractari, ejusque integrale investigari oportet. Integrali autem invento, quod si fuerit completum constantem arbitrariam complectetur, inde vel p per x, vel x per p determinari poterit. Priori casu quo p per x definire licet, ut p aequatur functioni cuidam ipsius x, quae sit $= X$, ob $p = X$ fiet $p \partial x = \partial y = X \partial x$, unde reperitur $y = \int X \partial x +$ Const. quae aequatio relationem desideratam inter x et y definit. Posteriori casu quo x per p detur, et functioni cuidam P ipsius p aequatur, ut sit $x = P$, erit $y = \int p \partial x = \int p \partial P$, seu $y = Pp - \int P \partial p$.

Sin autem neque x per p, neque p per x definiri queat, videndum est, ut ambas utramque per novam variabilem u exprimere liceat, unde fiat $x = V$ et $p = U$; tum enim habebitur $y = \int U \partial V$.

Corollarium 1.

751. Hujusmodi ergo aequationum differentio - differentialium resolutio ita instituitur, ut revocetur ad aequationem differentialem primi gradus inter binas variabiles x et p; quae si integrari queat, simul illius aequationis integratio habebitur, accedente quadam nova constante.

Corollarium 2.

752. Si aequatio inter x, p et q proposita ita fuerit comparata, ut q unicam dimensionem non excedat, vel si ad talem formam reduci patiatur, orietur aequatio differentialis simplex, differentialia unius tantum dimensionis involvens, ubi praecepta ante tradita in usum sunt vocanda.

Corollarium 3.

753. Sin autem quantitas q plures obtineat dimensiones, vel adeo transcendenter ingrediatur, tentanda sunt ea arficia, quae in fine superioris partis circa resolutionem hujusmodi aequationum sunt tradita.

Scholion.

754. Quando in aequatione inter x, p et q littera q unicam habet dimensionem, indeque posito $q = \frac{\partial p}{\partial x}$ aequatio differentialis simplex nascitur, praecipui casus, quibus integratio succedit, sunt: 1) si aequatio haec differentialis separationem admittat, 2) si alterutra variabilium p et x, differentialium quoque ratione habita, unam dimensionem non superet, ac 3) si ambae variabiles x et p ubique eundem dimensionum numerum constituant, quo casu aequatio ho-

mogenea appellatur. Casus minus late patentes, cujusmodi supra evolvimus, hic non commemoramus. Deinde si quantitas q vel pluribus dimensionibus sit implicata, vel adeo transcendenter ingrediatur, casus praecipui resolutionem admittentes, quemadmodum supra docuimus, sunt: 1) si proponatur aequatio quaecunque inter x et q deficiente p, 2) si aequatio tantum p et q contineat, quos binos quidem casus jam capite praecedente tractavimus; 3) si in aequatione proposita binae variabiles p et x ubique eundem dimensionum numerum constituant, 4) si in aequatione inter x, p et q altera binarum litterarum x vel p unicam dimensionem obtineat, denique 5) si aequatio ita fuerit comparata, ut posito $x = v^{\mu}$, $p = z^{\mu+\nu}$ et $q = t^{\nu}$, aequatio oriatur homogenea inter v, z et t, quae scilicet ubique eundem dimensionum numerum constituant. Secundum hos ergo casus exempla proferamus.

Exemplum 1.

755. *Investigare aequationem inter x et y, ut posita ∂x constante, haec formula $\frac{(\partial x^2 + \partial y^2)^{\frac{3}{2}}}{\partial x \partial \partial y}$ aequetur datae functioni ipsius x, quae sit $= X$.*

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, erit

$$\frac{(1+pp)^{\frac{3}{2}}}{q} = X = \frac{(1+pp)^{\frac{3}{2}} \partial x}{\partial p}, \text{ ideoque}$$

$$\frac{\partial x}{X} = \frac{\partial p}{(1+pp)^{\frac{3}{2}}},$$

ubi cum variabiles x et p sint a se invicem separatae, integratio dat

$$\frac{p}{\sqrt{(1+pp)}} = \int \frac{\partial x}{X}.$$

Ponatur $\int \frac{\partial x}{X} = V$, integrali completo sumto, erit V functio ipsius x, hinc

$$p = V\sqrt{(1+pp)} \text{ et } p = \frac{V}{\sqrt{(1-VV)}}.$$

Quare

$$\partial y = p\,\partial x = \frac{V\partial x}{\sqrt{(1-VV)}},$$

unde obtinetur

$$y = \int \frac{V\partial x}{\sqrt{(1-VV)}}.$$

Tum vero praeterea elicitur elementum

$$\sqrt{(\partial x^2 + \partial y^2)} = \partial x\sqrt{(1+pp)} = \frac{\partial x}{\sqrt{(1-VV)}},$$

cujus integrale praebet

$$\int \partial x\sqrt{(1+pp)} = \int \frac{\partial x}{\sqrt{(1-VV)}}.$$

Corollarium 1.

756. Si x et y sint coordinatae orthogonales curvae, erit formula $\frac{(1+pp)^{\frac{3}{2}}}{q}$ ejus radius curvedinis, unde hinc curva definitur, cujus radius curvedinis aequetur functioni cuicunque abscissae x.

Corollarium 2.

757. Si ergo radius curvedinis debeat esse reciproce proportionalis abscissae x, sumatur $X = \frac{aa}{2x}$, eritque

$$V = \int \frac{2x\partial x}{aa} = \frac{xx+ab}{aa}, \text{ hinc}$$

$$y = \int \frac{(xx+ab)\partial x}{\sqrt{[a^4-(xx+ab)^2]}},$$

quae conditio praebet curvas a lamina elastica formatas.

Corollarium 3.

758. Si fit $Y = x^n$, seu $X = \frac{1}{n x^{n-1}}$ neglecta constante addenda, oritur $y = \int \frac{x^n \partial x}{\sqrt{(1 - x^{2n})}}$, quod integrale algebraice exhiberi potest casibus, quibus est vel $n = \frac{1}{2i+1}$ vel $n = \frac{-1}{2i}$, denotante i numerum integrum positivum.

Exemplum 2.

759. *Si posito ∂x constante, oporteat esse*

$$\partial x (\partial x^2 + \partial y^2) + x \partial y \partial \partial y = a \partial \partial y \sqrt{(\partial x^2 + \partial y^2)},$$

invenire aequationem inter x et y.

Posito $\partial y = p \partial x$, nostra aequatio ob $\partial \partial y = \partial p \partial x$ induit hanc formam

$$\partial x (1 + pp) + x p \partial p = a \partial p \sqrt{(1 + pp)},$$

quae per $\sqrt{(1 + pp)}$ divisa fit integrabilis, oritur enim

$$x \sqrt{(1 + pp)} = ap + b \text{ seu } x = \frac{ap + b}{\sqrt{(1 + pp)}}.$$

Cum nunc sit

$$y = \int p \partial x = px - \int x \partial p, \text{ erit}$$

$$y = \frac{app + bp}{\sqrt{(1 + pp)}} - \int \frac{\partial p (ap + b)}{\sqrt{(1 + pp)}},$$

et integratione evoluta

$$y = \frac{app + bp}{\sqrt{(1 + pp)}} - a \sqrt{(1 + pp)} - b\, l \frac{p + \sqrt{(1 + pp)}}{n}, \text{ seu}$$

$$y = \frac{bp - a}{\sqrt{(1 + pp)}} - b\, l \frac{p + \sqrt{(1 + pp)}}{n};$$

ita ut ambae variabiles x et y per p definiantur

Cum igitur ex priori eliciatur

$p = \frac{ab + x\sqrt{(aa+bb-xx)}}{xx-aa}$ et $\sqrt{(1+pp)} = \frac{bx + a\sqrt{(aa+bb-xx)}}{xx-aa}$,

erit his valoribus substituis

$$y = \frac{a(aa+bb-xx) + bx\sqrt{(aa+bb-xx)}}{bx + a\sqrt{(aa+bb-xx)}} - b\,l\,\frac{b+\sqrt{(aa+bb-xx)}}{n(x-a)}, \text{ seu}$$

$$y = \sqrt{(aa+bb-xx)} - b\,l\,\frac{b+\sqrt{(aa+bb-xx)}}{n(x-a)}.$$

Corollarium.

760. Si constans priori integratione ingressa b evanescens sumatur, aequatio inter x et y fit algebraica, erit enim $y = \sqrt{(aa-xx)}$. Sin autem b non evanescat, aequatio integralis est transcendens, et logarithmos involvit.

Exemplum 3.

761. *Posito ∂x constante, si debeat esse*

$$aa\,\partial\partial y\sqrt{(aa+xx)} + aa\,\partial x\,\partial y = xx\,\partial x^2,$$

invenire aequationem inter x et y.

Posito $\partial y = p\,\partial x$, habebimus hanc aequationem

$$aa\,\partial p\sqrt{(aa+xx)} + aap\,\partial x = xx\,\partial x, \text{ seu}$$

$$\partial p + \frac{p\,\partial x}{\sqrt{(aa+xx)}} = \frac{xx\,\partial x}{aa\sqrt{(aa+xx)}},$$

in qua variabilis p unam dimensionem non superat. Cum ergo sit

$$\int\frac{\partial x}{\sqrt{(aa+xx)}} = l\,[x+\sqrt{(aa+xx)}],$$

haec aequatio integrabilis redditur, si multiplicetur per $x+\sqrt{(aa+xx)}$, tum enim prodit

$$p\,[x+\sqrt{(aa+xx)}] = \int\frac{xx\,\partial x\,[x+\sqrt{(aa+xx)}]}{aa\sqrt{(aa+xx)}}, \text{ seu}$$

$$p\,[x+\sqrt{(aa+xx)}] = \frac{1}{aa}\int\frac{x^3\,\partial x}{\sqrt{(aa+xx)}} + \frac{x^3}{3aa}, \text{ et}$$

$$\int\frac{x^3\,\partial x}{\sqrt{(aa+xx)}} = \tfrac{1}{3}(xx-2aa)\sqrt{(aa+xx)} + C,$$

hinc

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$$p[x+\sqrt{(aa+xx)}] = \frac{(xx-2aa)\sqrt{(aa+xx)}+x^3}{3aa} + C.$$

Haec multiplicetur per $\sqrt{(aa+xx)} - x$, ut prodeat,

$$aap = \frac{-xx-2aa+2x\sqrt{(aa+xx)}}{3} + C\sqrt{(aa+xx)} - Cx,$$

et quia $\partial y = p\partial x$, erit integrando

$$aay = -\tfrac{1}{9}x^3 - \tfrac{2}{3}aax + \tfrac{2}{9}(aa+xx)\sqrt{(aa+xx)} - \tfrac{1}{2}Cxx + C\int\partial x\sqrt{(aa+xx)}.$$

Quodsi ergo constans C evanescat, aequatio inter x et y erit algebraica, scilicet

$$9aay + 6aax + x^3 = 2(aa+xx)\sqrt{(aa+xx)}.$$

Exemplum 4.

762. *Posito ∂x constante, invenire integrale hujus aequationis differentio-differentialis*

$$(aa\partial y^2 + xx\partial x^2)\partial\partial y = nx\partial x^3\partial y.$$

Fiat $\partial y = p\partial x$, et ob $\partial\partial y = \partial p\,\partial x$ habebimus

$$(aapp + xx)\partial p = npx\partial x,$$

quae aequatio cum sit homogenea, statuamus $x = pu$, eritque

$$pp(aa+uu)\partial p = nppu(p\partial u + u\partial p), \text{ seu}$$
$$\frac{\partial p}{p} = \frac{nu\partial u}{aa+(1-n)uu},$$

quae integrata dat

$$lp = \frac{n}{2(1-n)}l[aa+(1-n)uu] + \text{Const.}$$

Hinc colligitur

$$p = C[aa+(1-n)uu]^{\frac{n}{2(1-n)}}, \text{ atque}$$

$$x = Cu[aa+(1-n)uu]^{\frac{n}{2(1-n)}}.$$

Cum nunc sit

$$y = px - \int x\partial p \text{ et } \partial p = Cnu\partial u[aa+(1-n)uu]^{\frac{3n-2}{2(1-n)}},$$

erit

$$y = CCu[aa+(1-n)uu]^{\frac{n}{1-n}} - nCC\int uu\partial u[aa+(1-n)uu]^{\frac{2n-1}{2(1-n)}}.$$

Casu autem $n = 1$ erit

$$ldp = \frac{uu}{2aa} + C, \text{ et } u = a\sqrt{2l\frac{p}{c}}, \text{ hinc}$$

$$x = ap\sqrt{2l\frac{p}{c}}, \text{ et } y = app\sqrt{2l\frac{p}{c}} - a\int p\partial p\sqrt{2l\frac{p}{c}}.$$

Corollarium,

763. Si fuerit $n = \frac{1}{2}$, erit

$$x = Cu\sqrt{(aa + \tfrac{1}{2}uu)} \text{ et}$$

$$y = CCu(aa + \tfrac{1}{2}uu) - \frac{CCu^3}{6} + D = CCu(aa + \tfrac{1}{3}uu) + D;$$

sicque relatio inter x et y algebraice exprimitur, quod etiam fit, si $n = \frac{2}{3}$, vel $n = \frac{3}{4}$, vel $n = \frac{4}{5}$, etc.

Exemplum 5.

764. *Posito ∂x constante, integrare hanc aequationem differentio-differentialem*

$$a\partial x\partial y^2 + xx\partial x\partial\partial y = nx\partial y\sqrt{(\partial x^4 + aa\partial\partial y^2)},$$

Fiat $\partial y = p\partial x$ et $\partial p = q\partial x$, ut sit $\partial\partial y = q\partial x^2$, et nostra aequatio induet hanc formam

$$app + qxx = npx\sqrt{(1 + aaqq)},$$

quae est homogenea inter p et x. Statuatur ergo $p = ux$, fietque

$$auu + q = nu\sqrt{(1 + aaqq)}.$$

Jam vero est

$$\partial p = q\partial x = u\partial x + x\partial u,$$

unde fit $\frac{\partial x}{x} = \frac{\partial u}{q-u}$.

At ex illa aequatione inter q et u colligitur

$$q = \frac{auu + nu\sqrt{(1 - nnaauu + a^4u^4)}}{nnaauu - 1}, \text{ et}$$

$$g - u = \frac{u(1 + au - nnaauu) + nu\sqrt{(1 - nnaauu + a^4u^4)}}{nnaauu - 1},$$

sicque

$$\frac{\partial x}{x} = \frac{\partial u}{u} \cdot \frac{nnaauu - 1}{1 + au - nnaauu + n\sqrt{(1 - nnaauu + a^4u^4)}},$$

Dabitur ergo x per u, hincque etiam $p = ux$ per u; unde deducitur $y = \int p\,\partial x = \int ux\,\partial x$.

Corollarium 1.

765. Illa aequatio differentialis transformatur in hanc

$$\frac{\partial x}{x} = \frac{\partial u}{u} \cdot \frac{1 + au - nnaauu - n\sqrt{(1 - nnaauu + a^4u^4)}}{nn - 1 - 2au + (nn - 1)aauu},$$

unde ratio integrationis facilius perspicitur.

Corollarium 2.

766. Notatu dignus autem est casus $nn = 2$, quo fit

$$\frac{\partial x}{x} = \frac{\partial u}{u} \cdot \frac{1 + au - 2aauu - (1 - aauu)\sqrt{2}}{(1 - au)^2}, \text{ seu}$$

$$\frac{\partial x}{x} = \frac{\partial u}{u} \cdot \frac{1 + 2au - (1 + au)\sqrt{2}}{1 - au} = \frac{\partial u(1 - \sqrt{2})}{u} + \frac{a\,\partial u(3 - 2\sqrt{2})}{1 - au},$$

unde colligitur

$$lx = (1 - \sqrt{2})\,lu - (3 - 2\sqrt{2})\,l(1 - au) + \text{Const. seu}$$

$$xu^{\sqrt{2} - 1}(1 - au)^{3 - 2\sqrt{2}} = C.$$

Exemplum 6.

767. *Sumto elemento $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ constante, invenire integrale hujus aequationis*

$$\partial x^3 \partial y - x\,\partial s^2\,\partial\partial y = a\,\partial x\,\partial s\sqrt{(\partial\partial x^2 + \partial\partial y^2)}.$$

Posito $\partial y = p\,\partial x$, erit $\partial s = \partial x\sqrt{(1 + pp)}$, et ob $\partial\partial s = 0$ fit

$$\partial\partial x = -\frac{p\,\partial p\,\partial x}{1 + pp} = \frac{-pq\,\partial x^2}{1 + pp},$$

existente $\partial p = q\,\partial x$, tum vero

$$\partial\partial y = p\partial\partial x + \partial p\partial x = \frac{-ppq\partial x^2}{1+pp} + q\partial x^2 = \frac{q\partial x^2}{1+pp},$$

ideoque

$$\sqrt{(\partial\partial x^2 + \partial\partial y^2)} = \frac{q\partial x^2}{\sqrt{(1+pp)}},$$

quibus substitutis, aequatio nostra induit hanc formam $p - qx = aq$, quae differentiata praebet $-x\partial q = a\partial q$, ideoque $\partial q = 0$ et $q = \frac{1}{c}$. Hinc $p = \int q\partial x = \frac{x+a}{c}$, qui idem valor ex aequatione $p = (x+a)q$ sine integratione obtinetur. Tum vero est $y = \int p\partial x = \frac{xx+2ax}{2c} + b$, quae est aequatio integralis completa binas constantes b et c involvens.

Exemplum 7.

768. *Sumto elemento $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ constante, invenire integrale hujus aequationis differentio-differentialis*

$$\partial x^3\partial y - x\partial s^2\partial\partial y = \frac{b\partial x^4\partial s^2\partial\partial x}{\sqrt{(\partial x^8 + aa\partial s^4\partial\partial y^2)}}.$$

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, ob $\partial s = \partial x\sqrt{(1+pp)}$ et $\partial\partial s = 0$, erit

$$\partial\partial x = \frac{-pq\partial x^2}{1+pp} \text{ et } \partial\partial y = \frac{-\partial x\partial\partial x}{\partial y} = \frac{-\partial\partial x}{p} = \frac{q\partial x^2}{1+pp},$$

ergo $\partial s^2\partial\partial y = q\partial x^4$, unde aequatio nostra fit

$$p - qx = \frac{bq}{\sqrt{(1+aaqq)}},$$

quae differentiata praebet

$$-x\partial q = \frac{b\partial q}{(1+aaqq)^{\frac{3}{2}}},$$

unde concluditur vel $\partial q = 0$ vel $x = \dfrac{-b}{(1+aaqq)^{\frac{3}{2}}}$.

Priori casu est $q = \frac{1}{c}$, et $p = \frac{x}{c} + \frac{b}{\sqrt{(cc+aa)}}$, hincque

$$y = \int p\partial x = \frac{xx}{2c} + \frac{bx}{\sqrt{(cc+aa)}} + f.$$

Posteriori casu quo $x = \frac{-b}{(1+aapq)^{\frac{3}{2}}}$ fit

$$p = \frac{-bq}{(1+aaqq)^{\frac{3}{2}}} + \frac{bq}{\sqrt{(1+aaqq)}} = \frac{aabq^3}{(1+aaqq)^{\frac{3}{2}}}.$$

At est

$$\partial x = \frac{+3aabq\partial q}{(1+aaqq)^{\frac{5}{2}}}, \text{ hincque}$$

$$\partial y = p\,\partial x = \frac{3a^4bbq^4\partial q}{(1+aaqq)^4},$$

et ope reductionum

$$y = \frac{-\frac{1}{2}bbq - aabbq^3}{(1+aaqq)^3} + \tfrac{1}{2}bb\int\frac{\partial q}{(1+aaqq)^3}.$$

Est vero

$$\int\frac{\partial q}{(1+aaqq)^{n+1}} = \frac{q}{2n(1+aaqq)^n} + \frac{2n-1}{2n}\int\frac{\partial q}{(1+aaqq)^n}.$$

Ergo

$$\int\frac{\partial q}{(1+aaqq)^3} = \frac{q}{4(1+aaqq)^2} + \tfrac{3}{4}\int\frac{\partial q}{(1+aaqq)^2} \text{ et}$$

$$\int\frac{\partial q}{(1+aaqq)^2} = \frac{q}{2(1+aaqq)} + \tfrac{1}{2}\int\frac{\partial q}{1+aaqq}$$

$$= \frac{q}{2(1+aaqq)} + \frac{1}{2a}\text{ Ang. tang. } aq.$$

Hinc

$$\int\frac{\partial q}{(1+aaqq)^3} = \frac{q}{4(1+aaqq)^4} + \frac{3q}{8(1+aaqq)} + \frac{3}{8a}\text{ Ang. tang. } aq,$$

ideoque

$$y = \frac{-bbq(1+2aaqq)}{2(1+aaqq)^3} + \frac{bbq}{8(1+aaqq)^2} + \frac{3bbq}{16(1+aaqq)}$$
$$+ \frac{3bb}{16a}\text{ Ang. tang. } aq,$$

existente

$$x = \frac{-b}{(1+aaqq)^{\frac{3}{2}}},$$

unde fit

$$1 + aaqq = \sqrt[3]{\frac{bb}{xx}},$$

ita ut hoc modo aequatio inter x et y exhiberi possit. Hoc autem integrale, ut supra vidimus, tantum est particulare.

Problema 96.

769. Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, si detur aequatio quaecunque inter y, p et q, ita ut variabilis x ipsa in ea desit; investigare aequationem integralem inter x et y.

Solutio.

Cum sit $q = \frac{\partial p}{\partial x}$ et $\partial x = \frac{\partial y}{p}$, erit $q = \frac{p\partial p}{\partial y}$; in aequatione ergo inter y, p et q ubique loco q substituatur iste valor $\frac{p\partial p}{\partial y}$, atque habebitur aequatio differentialis primi gradus binas tantum variabiles p et y involvens, cujus resolutionem per methodos supra expositas tentari oportet. Inventa autem aequatione integrali inter p et y, inde vel p per y, vel y per p definiatur, quo facilius altera integratio institui possit. Si y per p commode definiri queat, ut y aequetur functioni cuipiam ipsius p, quae sit $= P$, ut sit $y = P$, erit $\partial x = \frac{\partial P}{p}$, hincque $x = \int \frac{\partial P}{p} = \frac{P}{p} + \int \frac{P\partial p}{pp}$. Sin autem commodius p per y definire liceat, ut sit $p = Y$ denotante Y functionem quampiam ipsius y, ob $\partial x = \frac{\partial y}{p}$, habebitur $x = \int \frac{\partial y}{Y}$. At si neutrum succedat, novam variabilem u introducendo, per eam utraque quantitas p et y definiatur, ut fiat $p = U$ et $y = V$, existentibus U et V functionibus ipsius u, atque hinc erit $\partial x = \frac{\partial V}{U}$, et $x = \int \frac{\partial V}{U}$; hocque modo per duplicem integrationem integrale completum obtinebitur.

Corollarium 1.

770. Hujusmodi ergo aequationum differentio-differentialium

resolutio quoque revocatur ad aequationem differentialem primi gradus, cujus resolutio si fuerit in potestate, simul illius integrale exhiberi poterit.

Corollarium 2.

771. Si aequatio inter y, p et q ita fuerit comparata, ut ex ea commode valor ipsius q elici queat, hincque q aequetur functioni ipsarum y et p, quae sit T, erit $p\partial p = T\partial y$, quae est aequatio differentialis primi gradus simplex.

Corollarium 3.

772. Sin autem hujusmodi evolutio non succedat, dum littera q vel ad altiores potestates exsurgit, vel signis radicalibus involvitur, vel adeo transcendenter ingreditur, aequatio differentialis quidem erit primi gradus sed complicata, quae methodis supra expositis erit tractanda.

Scholion 1.

773. Cum paucis casibus aequationes differentiales primi gradus integrari queant, eosdem etiam hic notasse et per exempla illustrasse juvabit. Interim vero et reliquos casus quasi solutos spectari convenit, quandoquidem in aequationibus differentialibus altiorum ordinum id potissimum desideratur ut earum resolutio ad ordinem inferiorem reducatur. Perpetuo enim in Analysi quae ordine tractationis praecedunt, tanquam penitus confecta spectari solent, etiamsi plurima adhuc desiderentur, ut hoc modo multitudo desideratorum diminuatur. Ita quamvis longe adhuc absit, quominus aequationes algebraicas omnium ordinum resolvere valeamus, dum adeo vires nostrae non ultra quartum extenduntur, tamen in Analysi sublimiori omnium istarum aequationum resolutionem pro cognita habemus. Quod etiam usu non caret, cum in praxi resolutio per approxima-

tionem, quam quousque luberit, extendere licet, sufficere possit. Similimodo etiam, quoniam methodum tradidimus, aequationum differentialium primi gradus integralia proxime inveniendi, merito totum negotium, ut plane confectum, est censendum, si eo resolutionem aequationum differentialium altiorum graduum reducere potuerimus. Quare in hac secunda parte statim atque aequationem differentialem secundi gradus ad primum gradum perduxerimus, totum negotium pro confecto erit habendum.

Scholion 2.

774. Aequationes ergo differentio-differentiales, quae hoc modo ad differentiales primi gradus reducuntur, ita sunt comparatae, ut posito $\partial y = p \partial x$ et $\partial p = q \partial x$, variabilis x ipsa inde tollatur, et aequatio inter solas tres variabiles y, p et q oriatur. Casus ergo quibus talis aequatio resolutionem admittit, duplicis sunt generis, ad quorum prius referendi sunt ii, quibus q unicam obtinet dimensionem, unde q functioni cuipiam ipsarum y et p aequari potest. Cum igitur sit $q = \frac{p \partial p}{\partial y} = f:(y \text{ et } p)$ quam ponamus $= \mathrm{T}$, resolutio succedet. 1) Si T sit functio homogenea unius dimensionis ipsarum y et p. 2) Si fuerit $\mathrm{T} = \frac{\mathrm{P}}{y + \mathrm{Q}}$, designantibus P et Q functiones quascunque ipsius p tantum, hinc enim fit

$$\mathrm{P} \partial y = y p \partial p + \mathrm{Q} p \partial p;$$

quorsum etiam refertur casus,

$$\mathrm{T} = \frac{\mathrm{P}}{y + \mathrm{Q} y^n}.$$

3) Si fuerit $\mathrm{T} = p(\mathrm{Y} p + \mathrm{Z})$, si quidem Y et Z sint functiones quaecunque ipsius y, quia tum aequatio

$$\partial p = \mathrm{Y} p \partial y + \mathrm{Y} \partial y,$$

ob unicam dimensionem ipsius p est integrabilis, quorsum etiam referendus est casus $\mathrm{T} = p(\mathrm{Y} p + \mathrm{Z} p^n)$. Pro altero genere si quan-

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titas q plures habeat dimensiones, vel signis radicalibus sit implicata, vel adeo transcendenter ingrediatur, aequatio inter y, p et q, resolutionem admittet. 1) Si posito $q = pu$, ut sit $u = \frac{\partial p}{\partial y}$, aequatio resultet homogenea inter y et p, in qua scilicet y et p ubique eundem dimensionum numerum compleant, utcunque caeterum u in eam ingrediatur. 2) Si in aequatione post substitutionem $q = pu$ inter y, p et u orta, altera quantitas y vel p unicam obtineat dimensionem. 3) Si posito $y = v^{\mu}$, $p = z^{\mu + \nu}$ et $u = t^{\nu}$ aequatio oriatur homogenea inter ternas quantitas v, z et t, hujusmodi enim aequationes supra resolvere docuimus.

Exemplum 1.

775. *Posito elemento ∂x constante, si habeatur haec aequatio differentio-differentialis*

$$\partial \partial y + A \partial x \partial y + B y \partial x^2 = 0,$$

ejus integrale completum invenire.

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, aequatio nostra erit

$$q + A p + B y = 0, \text{ seu } p \partial p + A p \partial y + B y \partial y = 0,$$

quae cum sit homogenea, posito $p = v y$, abit in

$$v v y \partial y + v y y \partial v + A v y \partial y + B y \partial y = 0,$$

unde fit

$$\frac{\partial y}{y} + \frac{v \partial v}{v v + A v + B} = 0.$$

Sit $v v + A v + B = (v + \alpha)(v + \beta)$,

ut sit $\alpha + \beta = A$ et $\alpha \beta = B$ erit

$$\frac{\partial y}{y} + \frac{\alpha \partial v}{(\alpha - \beta)(v + \alpha)} - \frac{\beta \partial v}{(\alpha - \beta)(v + \beta)} = 0,$$

hincque integrando

$$l y + \frac{\alpha}{\alpha - \beta} l (v + \alpha) - \frac{\beta}{\alpha - \beta} l (v + \beta) = C \text{ seu}$$

$$y = a(v+\beta)^{\frac{\beta}{\alpha-\beta}}(v+\alpha)^{\frac{-\alpha}{\alpha-\beta}}, \text{ ideoque}$$

$$p = vy = av(v+\beta)^{\frac{\beta}{\alpha-\beta}}(v+\alpha)^{\frac{-\alpha}{\alpha-\beta}}.$$

Tum vero est

$$\partial x = \frac{\partial y}{p} = \frac{\partial y}{vy}, \text{ unde ob}$$

$$\frac{\partial y}{y} = \frac{-v\partial v}{vv+Av+B}, \text{ erit}$$

$$\partial x = \frac{-\partial v}{vv+Av+B} = \frac{\partial v}{(\alpha-\beta)(v+\alpha)} - \frac{\partial v}{(\alpha-\beta)(v+\beta)}, \text{ et}$$

$$x = \frac{1}{\alpha-\beta}\, l\, \frac{v+\alpha}{v+\beta} + \text{Const.}$$

Verum haec resolutio fit facilior sequenti modo:

Cum sit

$$\frac{\partial y}{y} = \frac{-v\partial v}{(v+\alpha)(v+\beta)} \text{ et } \partial x = \frac{-\partial v}{(v+\alpha)(v+\beta)}, \text{ erit}$$

$$\frac{\partial y}{y} + \alpha\partial x = \frac{-\partial v}{v+\beta} \text{ et } \frac{\partial y}{y} + \beta\partial x = \frac{-\partial v}{v+\alpha}, \text{ hinc}$$

$$ly + \alpha x = la - l(v+\beta) \text{ et } ly + \beta x = lb - l(v+\alpha).$$

Ergo

$$v + \beta = \frac{a}{y}e^{-\alpha x} \text{ et } v + \alpha = \frac{b}{y}e^{-\beta x},$$

unde fit

$$\alpha - \beta = \frac{1}{y}(be^{-\beta x} - ae^{-\alpha x}),$$

ideoque mutatis constantibus

$$y = \mathfrak{A}e^{-\alpha x} + \mathfrak{B}e^{-\beta x},$$

quae integratio locum habet, si α et β sint quantitates reales et inaequales. Cum igitur posuerimus

$$vv + Av + B = (v+\alpha)(v+\beta) \text{ erit}$$

$$\alpha = \tfrac{1}{2}A + \sqrt{(\tfrac{1}{4}AA - B)} \text{ et } \beta = \tfrac{1}{2}A - \sqrt{(\tfrac{1}{4}AA - B)},$$

hinc prout expressio $\frac{1}{4}AA - B$ fuerit vel positiva, vel negativa, vel evanescens, tres habebimus casus evolvendos:

1) Sit $\frac{1}{2}A=m$ et $\sqrt{(\frac{1}{4}AA-B)}=n$, erit aequationis propositae integrale completum

$$y=\mathfrak{A}e^{-(m+n)x}+\mathfrak{B}e^{-(m-n)x}=e^{-mx}(\mathfrak{A}e^{-nx}+\mathfrak{B}e^{nx}).$$

2) Sit $\frac{1}{2}A=m$ et $\sqrt{(\frac{1}{4}AA-B)}=n\sqrt{-1}$, ob

$$e^{nx\sqrt{-1}}=\cos.nx+\sqrt{-1}.\sin.nx \text{ et}$$

$$e^{-nx\sqrt{-1}}=\cos.nx-\sqrt{-1}.\sin.nx,$$

erit constantibus mutandis

$$y=e^{-mx}(\mathfrak{C}\cos.nx+\mathfrak{D}\sin.nx)=\mathfrak{E}e^{-mx}\cos.(nx+\mathfrak{e}).$$

3) Sit $\frac{1}{2}A=m$ et $\sqrt{(\frac{1}{4}AA-B)}=0$, seu in casu primo $n=0$, ob $e^{-nx}=1-nx$ et $e^{nx}=1+nx$ fiet

$$y=e^{-mx}(\mathfrak{C}+\mathfrak{D}x).$$

Corollarium 1.

776. Ad aequationis ergo propositae integrale inveniendum, aequationis $vv+Av+B=0$ radices investigari oportet, quibus inventis facile erit integrale completum assignare.

Corollarium 2.

777. Haec autem aequatio quadratica $vv+Av+B=0$ insignem habet analogiam cum ipsa aequatione proposita

$$\partial\partial y+A\partial y\partial x+By\partial x^2=0,$$

ex qua quippe oritur scribendo 1, v, v^2 loco y; $\frac{\partial y}{\partial x}$ et $\frac{\partial\partial y}{\partial x^2}$.

Corollarium 3.

778. Formata autem aequatione hac algebraica $vv+Av+B=0$, si ejus factor sit $v+\alpha$, ex eo statim integrale particulare deducitur $y=\mathfrak{A}e^{-\alpha x}$, similiterque alter factor $v+\beta$ integrale parti-

culare dabit $y = \mathfrak{B} e^{-\beta x}$, quibus conjunctis obtinetur integrale completum $y = \mathfrak{A} e^{-\alpha x} + \mathfrak{B} e^{-\beta x}$.

Scholion.

779. Infra methodus facilior tradetur hujusmodi aequationes differentio-differentiales tractandi, quae adeo ad talem formam

$$\partial \partial y + P \partial y \partial x + Q y \partial x^2 = 0$$

patet, ubi P et Q sint functiones quaecunque ipsius x, quae etiam extendetur ad formam

$$\partial \partial y + P \partial y \partial x + Q y \partial x^2 = X \partial x^2,$$

sumendo pro X functionem quamcunque ipsius x. Methodus scilicet ea inde haurietur, quod in hujusmodi aequationibus variabilis y cum suis differentialibus ∂y et $\partial \partial y$ ubique unicam dimensionem constituat vel etiam nullam, ejusque ope resolutio ad aequationem differentialem primi gradus reducetur, quo ipso negotium pro confecto erit habendum. Quando autem hoc modo aequatio differentio-differentialis ad aequationem differentialem primi gradus reducitur, probe cavendum est, ne haec reductio pro integratione habeatur, quippe ad quam tantum ope idoneae substitutionis est perventum; nihilo enim minus duae adhuc integrationes supersunt absolvendae, quibus totidem constantes arbitrariae introducantur, si quidem integrale completum desideretur, quemadmodum in hoc exemplo et praecedentibus clare videmus.

Exemplum 2.

780. *Proposita aequatione differentio-differentiali*

$$ab \partial \partial y = \partial x \sqrt{(yy \partial x^2 + aa \partial y^2)},$$

ejus integrale investigare.

Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, haec aequatio abit in hanc

$$ab q = \sqrt{(yy + aapp)} = \frac{abp\partial p}{y}, \text{ ob } q = \frac{p\partial p}{\partial y},$$

quae cum sit homogenea ponatur $p = \frac{y}{u}$, erit

$$y\partial y\sqrt{(1 + \frac{aa}{uu})} = \frac{aby}{u^3}(u\partial y - y\partial u), \text{ seu}$$

$$uu\partial y\sqrt{(aa + uu)} = abu\partial y - aby\partial u, \text{ unde fit}$$

$$\frac{\partial y}{y} = \frac{ab\partial u}{abu - uu\sqrt{(aa + uu)}}.$$

Ponatur $\sqrt{(aa + uu)} = su$, erit $uu = \frac{aa}{ss - 1}$,

$$\frac{\partial u}{u} = \frac{-s\partial s}{ss - 1}, \text{ et } \frac{\partial y}{y} = \frac{-bs\partial s}{bss - as - b} = \frac{-s\partial s}{ss - 2ns - s},$$

posito $\frac{a}{b} = 2n$. Ergo

$$\frac{2\partial y\sqrt{(nn + 1)}}{y} = \frac{-\partial s[n + \sqrt{(nn + 1)}]}{s - n - \sqrt{(nn + 1)}} + \frac{\partial s[n - \sqrt{(nn + 1)}]}{s - n + \sqrt{(nn + 1)}},$$

ideoque

$$y^{2\sqrt{(nn+1)}} = \frac{C[s - n + \sqrt{(nn + 1)}]^{n - \sqrt{(nn + 1)}}}{[s - n - \sqrt{(nn + 1)}]^{n + \sqrt{(nn + 1)}}}.$$

Datur igitur y per s, ut sit $y = S$, hincque

$$u = \frac{a}{\sqrt{(ss - 1)}} \text{ et } p = \frac{S\sqrt{(ss - 1)}}{a}, \text{ atque}$$

$$\partial x = \frac{a\partial S}{S\sqrt{(ss - 1)}}, \text{ seu } \partial x = \frac{-as\partial s}{(ss - 2ns - 1)\sqrt{(ss - 1)}},$$

quae formula ad rationalitatem perduci et per logarithmos seu arcus circulares integrare potest.

Exemplum 3.

781. *Posito* $\partial y = p\partial x$ et $\partial p = q\partial x$, *invenire integrale hujus aequationis* $\frac{(pp + yy)\sqrt{(pp + yy)}}{2pp + yy - qy} = ny$.

Cum sit $q = \frac{p\partial p}{\partial y}$, erit

$$\partial y(pp + yy)\sqrt{(pp + yy)} = 2nppy\partial y + ny^3\partial y - nyyp\partial p,$$

ob cujus homogeneitatem ponatur $p = uy$, fietque

$$y^3\partial y(uu + 1)^{\frac{3}{2}} = 2nuuy^3\partial y + ny^3\partial y - nu^2y^3\partial y - nuy^4\partial u$$

unde colligitur

$$\frac{\partial y}{y} = \frac{-nu\partial u}{(uu+1)\sqrt{(uu+1)} - nuu - n} = \frac{nu\partial u}{(uu+1)[n-\sqrt{(uu+1)}]},$$

et y per u definitur; ex quo erit $p = uy$ et

$$\partial x = \frac{\partial y}{uy} = \frac{n\partial u}{(uu+1)[n-\sqrt{(uu+1)}]}.$$

Casu quo $n = 1$ erit

$$\frac{\partial y}{y} = \frac{-u\partial u}{(uu+1)[\sqrt{(uu+1)}-1]} = \frac{-\partial u[1+\sqrt{(uu+1)}]}{u(uu+1)}, \text{ et}$$

$$\partial x = \frac{-\partial u[1+\sqrt{(uu+1)}]}{uu(uu+1)}. \quad \text{Est vero}$$

$$\int \frac{\partial u}{u(uu+1)} = l\frac{u}{\sqrt{(uu+1)}}, \int \frac{\partial u}{uu(uu+1)} = -\frac{1}{u} - \text{Ang. tang. } u,$$

$$\int \frac{\partial u}{u\sqrt{(uu+1)}} = l\frac{\sqrt{(uu+1)}-1}{u}, \int \frac{\partial u}{uu\sqrt{(uu+1)}} = -\frac{\sqrt{(uu+1)}}{u};$$

unde colligitur

$$y = \frac{C\sqrt{(uu+1)}}{\sqrt{(uu+1)}-1} = C\left(1 + \frac{1}{\sqrt{(uu+1)}-1}\right) \text{ et}$$

$$x = D + \frac{1+\sqrt{(uu+1)}}{u} + \text{Ang. tang. } u.$$

Inde est

$$\sqrt{(uu+1)} = \frac{y}{y-a}; \text{ et } u = \frac{\sqrt{(2ay-aa)}}{y-a},$$

ideoque

$$x = D + \sqrt{\frac{2y-a}{a}} + \text{Ang. cos. } \frac{y-a}{y},$$

quae formulae introducendo angulum Φ, cujus cosinus est $\frac{y-a}{a}$, ita commodius exhibentur

$$y = \frac{a}{1-\cos.\Phi} \text{ et } x = \zeta + \Phi + \cot.\tfrac{1}{2}\Phi.$$

Corollarium 1.

782. Ex aequatione separata primum inventa solutio particularis eruitur, tribuendo ipsi u ejusmodi valorem constantem, ut

denominator evanescat, qui est $u = \sqrt{(nn-1)}$; hinc $p = y\sqrt{(nn-1)}$ et $\partial x \sqrt{(nn-1)} = \frac{\partial y}{y}$, unde fit

$$ly = la + x\sqrt{(nn-1)}.$$

Corollarium 2.

783. Casu quo $n = 1$, hic casus particularis praebet $y = a$, pro valore quocunque alterius variabilis; fit enim $u = 0$, ideoque et $p = 0$, ita ut ex aequatione $\partial y = p\,\partial x$ quantitas x non determinetur.

Scholion.

784. Si y designet radium vectorem ex puncto fixo ad curvam quampiam ductum, et x angulum, quam iste radius cum recta quadam positione data constituit, formula

$$\frac{(pp+yy)\sqrt{(pp+yy)}}{2pp+yy-qy}$$

exprimit hujus curvae radium curvedinis. In exemplo ergo proposito ejusmodi quaeritur curva, cujus radius curvedinis aequetur ipsi ny, cui quaestioni casu $n = 1$ utique satisfacit valor $y = a$, qui praebet circulum, qui etiam ex aequatione integrali colligitur $y = \frac{C\sqrt{(uu+1)}}{\sqrt{(uu+1)}-1}$, sumendo constantem C nihilo aequalem, tum enim necesse est sit $u = 0$ et $p = 0$, sicque angulus x non determinatur. Praeter circulum autem infinitae aliae lineae curvae satisfaciunt. At si $n > 1$, solutio particularis $ly = la + x\sqrt{(nn-1)}$ praebet spiralem logarithmicam, praeter quam autem etiam infinitae aliae curvae satisfaciunt; casibus autem $n < 1$ nulla hujusmodi solutio particularis locum habet, sed formulas pro $\frac{\partial y}{y}$ et ∂x inventas revera integrari oportet.

Exemplum 4.

785. *Posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, invenire relationem inter x et y, ut fiat* $\frac{(pp+yy)\sqrt{(pp+yy)}}{2pp+yy-qy} = a$.

Cum sit $q = \frac{p \partial p}{\partial y}$, ponatur $pp + yy = zz$, ob $p \partial p = q \partial y$ erit $q \partial y + y \partial y = z \partial z$, seu $q + y = \frac{z \partial z}{\partial y}$. Aequatio autem proposita induit hanc formam:

$$z^3 = a(2zz - yy - qy) = a\left(2zz - \frac{yz\partial z}{\partial y}\right), \text{ seu}$$

$$zz\partial y = 2az\partial y - ay\partial z,$$

unde fit

$$\frac{\partial y}{y} = \frac{a\partial z}{2az - zz}, \text{ seu } \frac{2\partial y}{y} = \frac{\partial y}{z} + \frac{\partial z}{2a - z},$$

quare integrando colligitur

$$yy = \frac{Cz}{2a - z}, \text{ et } pp = zz - \frac{Cz}{2a - z} = \frac{-Cz + 2azz - z^3}{2a - z}.$$

At est $z = \frac{2ayy}{C + yy}$, ergo

$$pp = \frac{4aay^4}{(C + yy)^2} - yy = \frac{yy[4aayy - (C + yy)^2]}{(C + yy)^2}.$$

Hinc igitur oritur

$$\partial x = \frac{(C + yy)\partial y}{y\sqrt{[4aayy - (C + yy)^2]}},$$

sit $yy = u$, erit

$$\partial x = \frac{(C + u)\partial u}{2u\sqrt{[4aau - (C + u)^2]}}.$$

Haec aequatio tractabilior redditur ponendo

$$u = 2aa - C + 2a\cos.\phi\sqrt{(aa - C)},$$

fit enim

$$\partial x = \frac{-a\partial\phi[a + \cos.\phi\sqrt{(aa - C)}]}{2aa - C + 2a\cos.\phi\sqrt{(aa - C)}}, \text{ seu}$$

$$2\partial x = -\partial\phi - \frac{C\partial\phi}{2aa - C + 2a\cos.\phi\sqrt{(aa - C)}},$$

quae integrata dat

$$2x = \zeta - \phi - \text{Ang. cos. } \frac{m + \cos.\phi}{1 + m\cos.\phi},$$

posito $m = \frac{2a\sqrt{(aa - C)}}{2aa - C}$, ut sit

$$C = \frac{2aa\sqrt{(1-mm)}}{1+\sqrt{(1-mm)}} \text{ et } \sqrt{(aa - C)} = \frac{ma}{1+\sqrt{(1-mm)}},$$

hincque $yy = \frac{2aa(1+m\cos.\Phi)}{1+\sqrt{(1-mm)}}$,

unde fit

$$\cos.\Phi = \frac{yy[1+\sqrt{(1-mm)}] - 2aa}{2maa} \text{ et}$$

$$\frac{m+\cos.\Phi}{1+m\cos.\Phi} = \frac{yy[1+\sqrt{(1-mm)}] - 2aa(1-mm)}{myy[1+\sqrt{(1-mm)}]}$$

Corollarium 1.

786. Cum sit $yy = \frac{2aa(1+m\cos.\Phi)}{1+\sqrt{(1-mm)}}$, erit

$$yy = aa + bb + 2ab\cos.\Phi,$$

si ponatur $b = \frac{a[1-\sqrt{(1-mm)}]}{m}$, unde fit

$$m = \frac{2ab}{aa+bb} \text{ et } \sqrt{(1-mm)} = \frac{aa-bb}{aa+bb},$$

hincque

$$2x = \zeta - \Phi - \text{Ang.}\cos.\frac{2ab+(aa+bb)\cos.\Phi}{aa+bb+2ab\cos.\Phi}, \text{ seu}$$

$$2x = \zeta - \Phi - \text{Ang.}\sin.\frac{(aa-bb)\sin.\Phi}{yy}.$$

Corollarium 2.

787. Si ut supra radius vector y cum angulo x referatur ad lineam curvam, hanc curvam circulum esse oportet radio $= a$ descriptum. Fit autem $\partial x = \frac{\partial\Phi(aa-ab\cos.\Phi)}{aa+bb-2ab\cos.\Phi}$ sumto $yy = aa+bb-2ab\cos.\Phi$, hincque

$$x = \zeta + \text{Ang.tang.}\frac{a\sin.\Phi}{a\cos.\Phi - b},$$

cujus applicatio ad Geometriam rem facit perspicuam.

Exemplum 5.

788. *Sumto elemento ∂x constante, si proponatur haec aequatio $\partial\partial y(y\partial y + a\partial x) = \partial y(\partial x^2 + \partial y^2)$, ejus integrale invenire.*

Posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$ habebimus

$$q(py + a) = p(1 + pp), \text{ et ob } q = \frac{p\,\partial p}{\partial y},$$
$$\partial p\,(py + a) = \partial y\,(1 + pp), \text{ sive}$$
$$\partial y - \frac{py\,\partial p}{1 + pp} = \frac{a\,\partial p}{1 + pp},$$

quae integrata dat

$$\frac{y}{\sqrt{(1 + pp)}} = \frac{ap}{\sqrt{(1 + pp)}} + b, \text{ ideoque}$$
$$y = ap + b\sqrt{(1 + pp)} \text{ et}$$
$$x = \int \frac{\partial y}{p} = a\,lp + b\,l\,[p + \sqrt{(1 + pp)}] + c,$$

ita ut x et y per eandem variabilem p exprimantur. Si constans b sumatur $= 0$, obtinetur integrale particulare

$$y = ap \text{ et } x = a\,l\,p + c = a\,l\,\frac{y}{a} + c,$$

seu in exponentialibus $y = c\,e^{x:a}$. Sin autem sumatur $b = a$, ob

$$p + \sqrt{(1 + pp)} = \frac{y}{a} \text{ et } p = \frac{yy - aa}{2ay}, \text{ erit}$$
$$x = a\,l\,\frac{yy - aa}{2aa} + c \text{ seu } yy = aa + c\,e^{x:a}.$$

Exemplum 6.

789. *Sumto ∂x constante, hujus aequationis differentio-differentialis*

$$\partial y^2 - y\,\partial\partial y = n\sqrt{(\partial x^2\,\partial y^2 + aa\,\partial\partial y^2)},$$

integrale invenire.

Posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, erit

$$pp - qy = n\sqrt{(pp + aaqq)},$$

quae facto $q = pu$, ut sit $\frac{p\partial p}{\partial y} = pu$ ideoque $\partial p = u\,\partial y$, abit in

$$pp - puy = np\sqrt{(1 + aauu)} \text{ seu}$$
$$p - uy = n\sqrt{(1 + aauu)}.$$

Jam quia $\partial p = u\,\partial y$ differentietur haec aequatio, prodibitque

$$-y\,\partial u = \frac{naau\,\partial u}{\sqrt{(1 + aauu)}},$$

hinc vel $\partial u = 0$ vel $y = \frac{-naau}{\sqrt{(1 + aauu)}}$.

1) Casu $\partial u = 0$ fit $u = \alpha$, $p = \alpha y + \beta$, et $\partial x = \frac{\partial y}{\alpha y + \beta}$, hinc $\alpha x = l(\alpha y + \beta) + C$.

2) Si $y = \frac{-naau}{\sqrt{(1 + aauu)}}$, erit

$$p = uy + n\sqrt{(1 + aauu)} = \frac{n}{\sqrt{(1 + aauu)}},$$

hincque

$$\partial x = \frac{\partial y}{p} = \frac{-aa\partial u}{1 + aauu} \text{ et } x = -a \,\text{Ang. tang.}\, au + C,$$

vel ob $u = \frac{y}{a\sqrt{(nnaa - yy)}}$, aequatio inter x et y quaesita erit

$$\frac{b - x}{a} = \text{Ang. tang.} \frac{y}{\sqrt{(nnaa - yy)}} = \text{Ang. sin.} \frac{y}{na},$$

unde fit $y = na \sin. \frac{b - x}{a}$. Haec autem relatio tantum pro integrali particulari est habenda.