CAPUT XI.

DE CONSTRUCTIONE AEQUATIONUM DIFFERENTIO - DIFFERENTIALIUM EX EARUM RESOLUTIONE PER SERIES INFINITAS PETITA.

Problema 133.

1059.

Proposita serie infinita

$$A + Bs + Cs^2 + \ldots + Ms^{i-1} + Ns^i + \text{etc.}$$

in qua sit $B = \frac{0m+h}{1n+k}A$, $C = \frac{1m+h}{2n+k}B$, $D = \frac{2m+h}{3n+k}C$, et in genere $N = \frac{(i-1)m+h}{in+k}M$, ejus summam per formulam integralem exprimere.

Solutio.

Ponatur summa quaesita $= z$, ita ut sit

$$z = A + Bs + Cs^2 + Ds^3 + \ldots + Ms^{i-1} + Ns^i + \text{etc.}$$

eritque differentiando

$$\frac{s\partial z}{\partial s} = 0A + 1Bs + 2Cs^2 + 3Ds^3 + \ldots + (i-1)Ms^{i-1} + iNs^i + \text{etc.}$$

ex cujus combinatione cum praecedente oritur

$$\frac{ms\partial z}{\partial s} + hz = hA + (m+h)Bs + (2m+h)Cs^2 + \ldots$$
$$+ [(i-1)m+h]Ms^{i-1} + (im+h)Ns^i + \text{etc.}$$

Deinde vero etiam simili modo est

$$\frac{ns\partial z}{\partial s} + kz = kA + (n+k)Bs + (2n+k)Cs^2 + \ldots$$
$$+ [(i-1)n+k]Ms^{i-1} + (in+k)Ns^i + \text{etc.}$$

ergo ob

$$(n+k)\,B = h\,A,\ (2n+k)\,C = (m+h)\,B,\ \text{etc. erit}$$

$$\frac{ns\,\partial z}{\partial s} + kz = kA + hAs + (m+h)\,Bs^2 + (2m+h)\,Cs^3 + \text{etc.}$$

unde manifesto conficitur

$$\frac{ns\,\partial z}{\partial s} + kz = kA + \frac{mss\,\partial z}{\partial s} + hsz, \text{ seu}$$

$$s\,\partial z\,(n - ms) + z\,\partial s\,(k - hs) = kA\,\partial s$$

hoc est

$$\partial z + \frac{z\,\partial s\,(k - hs)}{s\,(n - ms)} = \frac{kA\,\partial s}{s\,(n - ms)}.$$

Cum nunc sit

$$\frac{\partial s\,(k - hs)}{s\,(n - ms)} = \frac{k\,\partial s}{ns} + \frac{(mk - nh)\,\partial s}{n\,(n - ms)},$$

aequatio ista integrabilis fit multiplicata per

$$s^{\frac{k}{n}}\,(n - ms)^{\frac{nh - mk}{mn}}, \text{ proditque}$$

$$(n - ms)^{\frac{nh - mk}{mn}}\,s^{\frac{k}{n}}\,z = Ak\int s^{\frac{k}{n} - 1}\,\partial s\,(n - ms)^{\frac{nh - mk}{mn} - 1}$$

quod integrale ita capi oportet, ut posito $s = 0$ fiat $z = A$, quo observato habebimus

$$z = Aks^{-\frac{k}{n}}\,(n - ms)^{\frac{mk - nh}{mn}}\int s^{\frac{k}{n} - 1}\,\partial s\,(n - ms)^{\frac{nh - mk}{mn} - 1}.$$

Corollarium 1.

1060. Peculiari solutione eget casus $m = 0$, quo fit

$$\partial z + \frac{z\,\partial s\,(k - hs)}{ns} = \frac{Ak\,\partial s}{ns},$$

quae per $s^{\frac{k}{n}}\,e^{\frac{-hs}{n}}$ multiplicata praebet

$$e^{\frac{-hs}{n}}\,s^{\frac{k}{n}}\,z = \frac{Ak}{n}\int e^{\frac{-hs}{n}}\,s^{\frac{k}{n} - 1}\,\partial s$$

ideoque

$$z = \frac{Ak}{n}\, e^{\frac{hs}{n}}\, s^{\frac{-k}{n}} \int e^{\frac{-hs}{n}}\, s^{\frac{k}{n}-1}\,\partial s,$$

integrali ita sumto, ut fiat $z = A$ posito $s = 0$.

Corollarium 2.

1061. Casus etiam $n = 0$ seorsim resolvi debet, aequatio enim

$$\partial z + z\,\partial s\left(\frac{hs-k}{mss}\right) = -\frac{Ak\,\partial s}{mss}$$

multiplicari debet per $s^{\frac{h}{m}}\, e^{\frac{k}{ms}}$, et invenitur integrale

$$e^{\frac{k}{ms}}\, s^{\frac{h}{m}}\, z = -\frac{Ak}{m}\int e^{\frac{k}{ms}}\, s^{\frac{h}{m}-2}\,\partial s,$$

ideoque

$$z = -\frac{Ak}{m}\, e^{\frac{-k}{ms}}\, s^{\frac{-h}{m}} \int e^{\frac{k}{ms}}\, s^{\frac{h}{m}-2}\,\partial s.$$

Corollarium 3.

1062. Si fuerit et $m = 0$ et $n = 0$, ob $N = \frac{h}{k}M$, series nostra erit geometrica, aequatio vero nostra erit

$$z\,\partial s(k - hs) = Ak\,\partial s \text{ seu } z = \frac{Ak}{k-hs},$$

uti natura rei manifesto postulat.

Scholion.

1063. Imprimis hic casus notari meretur, quo est $k = 0$, et summa z sine signo integrali exprimi potest; erit namque

$$(n - ms)^{\frac{h}{m}}\, z = \text{Const.}$$

et quia si $s = 0$, fieri debet $z = A$, erit Const. $= A\, n^{\frac{h}{m}}$ ideoque

$$z = A\, n^{\frac{h}{m}}\,(n - ms)^{\frac{-h}{m}}, \text{ seu } z = A\left(\frac{n}{n-ms}\right)^{\frac{h}{m}}.$$

vel etiam

$$z = A\left(1 - \frac{ms}{n}\right)^{\frac{-b}{m}}.$$

At vero integratio etiam succedit casu quo $k = n$, erit enim

$$(n - ms)^{\frac{b}{m} - 1} sz = An \int \partial s\,(n - ms)^{\frac{b}{m} - 2},$$

quod integrale est $= \text{Const.} - \frac{An(n-ms)^{\frac{b}{m}-1}}{b-m}$, et quia posito $s = 0$, fit $z = A$, erit $0 = \text{Const.} - \frac{A}{b-m} \cdot n^{\frac{b}{m}}$, hincque

$$z = \frac{An}{(b-m)s}\left[\left(\frac{n}{n-ms}\right)^{\frac{b}{m}-1} - 1\right] = \frac{An}{(b-m)s}\left[\left(1 - \frac{ms}{n}\right)^{1-\frac{b}{m}} - 1\right].$$

Porro perspicuum est, integrationem expediri posse casu $k = 2n$, quo cum sit

$$(n - ms)^{\frac{b}{m} - 2} ssz = 2An \int s\,\partial s\,(n - ms)^{\frac{b}{m} - 3},$$

erit hoc integrale

$$= \text{Const.} - \frac{2Ans}{b-2m}(n-ms)^{\frac{b}{m}-2} + \frac{2An}{b-2m}\int \partial s\,(n-ms)^{\frac{b}{m}-2} \text{ vel}$$

$$= \text{Const.} - \frac{2Ans}{b-2m}(n-ms)^{\frac{b}{m}-2} - \frac{2An(n-ms)^{\frac{b}{m}-1}}{(h-m)(h-2m)};$$

ubi $\text{Const.} = \frac{2An^{\frac{b}{m}}}{(h-m)(h-2m)}$, ideoque

$$z = \frac{2Ann}{(b-m)(b-2m)ss}\left[\left(\frac{n}{n-ms}\right)^{\frac{b}{m}-2} - 1 - \frac{(b-2m)s}{n}\right];$$

similique modo etiam integratio casibus $k = 3n$, $k = 4n$ etc. absoetur.

Problema 134.

1064. Proposita hujusmodi serie infinita

$$A\mathfrak{A} + B\mathfrak{B}u + C\mathfrak{C}u^2 + D\mathfrak{D}u^3 + \ldots\ldots + M\mathfrak{M}u^{i-1} + \mathfrak{N}Nu^i + \text{etc.}$$

coëfficientium lege existente

$$\mathrm{B} = \frac{0m+b}{1n-k}\mathrm{A},\ \mathrm{C} = \frac{1m+b}{2n+k}\mathrm{B},\ \mathrm{D} = \frac{2m+b}{3n+k}\mathrm{C} \ldots \mathrm{N} = \frac{(i-1)m+b}{in+k}\mathrm{M},$$

$$\mathfrak{B} = \frac{0\mu+\eta}{1\nu+\theta}\mathfrak{A},\ \mathfrak{C} = \frac{1\mu+\eta}{2\nu+\theta}\mathfrak{B},\ \mathfrak{D} = \frac{2\mu+\eta}{3\nu+\theta}\mathfrak{C} \ldots \mathfrak{N} = \frac{(i-1)\mu+\eta}{i\nu+\theta}\mathfrak{M},$$

ejus summam per formulam integralem exprimere.

Solutio.

Posita summa

$$y = \mathrm{A}\mathfrak{A} + \mathrm{B}\mathfrak{B}u + \mathrm{C}\mathfrak{C}u^2 + \mathrm{D}\mathfrak{D}u^3 + \mathrm{E}\mathfrak{E}u^4 + \text{etc.}$$

consideretur series hoc modo formata

$$z = \mathrm{A} + \mathrm{B}ux + \mathrm{C}u^2x^2 + \mathrm{D}u^3x^3 + \mathrm{E}u^4x^4 + \text{etc.}$$

cujus summa posito $ux = s$ est, ut modo invenimus

$$z = \mathrm{A}ks^{-\frac{k}{n}}(n-ms)^{\frac{mk-nb}{mn}}\int s^{\frac{k}{n}-1}\partial s\,(n-ms)^{\frac{nb-mk}{mn}-1},$$

integrali ita determinato, ut posito $s = 0$ fiat $z = \mathrm{A}$.

Formetur hujusmodi formula integralis

$$\mathrm{V} = \int \mathrm{P}z\partial x = \int \mathrm{P}\partial x(\mathrm{A} + \mathrm{B}ux + \mathrm{C}u^2x^2 + \mathrm{D}u^3x^3 + \text{etc.})$$

in qua u spectetur ut constans, pro P autem ejusmodi functio ipsius x accipiatur, ut fiat

$$\int \mathrm{P}x\partial x = \frac{\mathfrak{B}}{\mathfrak{A}}\int \mathrm{P}\partial x,\ \int \mathrm{P}x^2\partial x = \frac{\mathfrak{C}}{\mathfrak{B}}\int \mathrm{P}x\partial x,\ \int \mathrm{P}x^3\partial x = \frac{\mathfrak{D}}{\mathfrak{C}}\int \mathrm{P}x^2\partial x, \text{ etc.}$$

postquam scilicet in his integralibus data lege sumtis variabili x datus quidem valor fuerit tributus. Cum igitur hinc sit

$$\int \mathrm{P}x\partial x = \frac{\mathfrak{B}}{\mathfrak{A}}\int \mathrm{P}\partial x,\ \int \mathrm{P}x^2\partial x = \frac{\mathfrak{C}}{\mathfrak{A}}\int \mathrm{P}\partial x,\ \int \mathrm{P}x^3\partial x = \frac{\mathfrak{D}}{\mathfrak{A}}\int \mathrm{P}\partial x, \text{ etc.}$$

erit

$$\mathrm{V} = (\mathrm{A} + \frac{\mathrm{B}\mathfrak{B}}{\mathfrak{A}}u + \frac{\mathrm{C}\mathfrak{C}}{\mathfrak{A}}u^2 + \frac{\mathrm{D}\mathfrak{D}}{\mathfrak{A}}u^3 + \text{etc.})\int \mathrm{P}\partial x,$$

unde patet fore

$$y = \frac{\mathfrak{A}\mathrm{V}}{\int \mathrm{P}\partial x} = \frac{\mathfrak{A}\int \mathrm{P}z\partial x}{\int \mathrm{P}\partial x}.$$

Quare cum valor ipsius z sit cognitus, tantum superest, ut functio P ipsius x conditionibus memoratis praedita investigetur. In genere autem esse oportet

$$\int P x^i \partial x = \frac{\mathfrak{N}}{\mathfrak{M}} \int P x^{i-1} \partial x = \frac{(i-1)\mu + \eta}{i\nu + \theta} \int P x^{i-1} \partial x,$$

quae aequalitas cum sufficiat, ut tantum certo quodam casu, quo ipsi x datus tribuitur valor, subsistat, ponamus in genere esse

$$(i\nu + \theta) \int P x^i \partial x = [(i-1)\mu + \eta] \int P x^{i-1} \partial x + x^i Q,$$

ita ut pro terminis integralibus sit $Q = 0$. Differentiando ergo habebimus

$$(i\nu + \theta) P x^i \partial x = (i\mu - \mu + \eta) P x^{i-1} \partial x + x^i \partial Q + i x^{i-1} Q \partial x,$$

seu per x^{i-1} dividendo

$$(i\nu + \theta) P x \partial x = (i\mu - \mu + \eta) P \partial x + x \partial Q + i Q \partial x,$$

quae aequalitas cum pro omnibus valoribus ipsius i aeque subsistere debeat, hinc duas adipiscimur aequationes

$$\nu P x \partial x = \mu P \partial x + Q \partial x, \text{ et } \theta P x \partial x = (\eta - \mu) P \partial x + x \partial Q;$$

unde duplici modo colligimus

$$P \partial x = \frac{Q \partial x}{\nu x - \mu}, \text{ et } P \partial x = \frac{x \partial Q}{\theta x - (\eta - \mu)},$$

sicque alterum valorem per alterum dividendo

$$\frac{x \partial Q}{Q \partial x} = \frac{\theta x + \mu - \eta}{\nu x - \mu}, \text{ seu } \frac{\partial Q}{Q} = \frac{\partial x (\theta x + \mu - \eta)}{x (\nu x - \mu)},$$

quae evolvitur in

$$\frac{\partial Q}{Q} = \frac{\eta - \mu}{\mu x} \partial x + \frac{\mu \theta + \mu \nu - \eta \nu}{\mu (\nu x - \mu)} \partial x:$$

hinc integrando elicitur

$$Q = x^{\frac{\eta}{\mu} - 1} (\nu x - \mu)^{\frac{\theta\mu - \eta\nu}{\mu\nu} + 1} \text{ per Const. mult.}$$

seu

$$Q = - x^{\frac{\eta}{\mu} - 1} (\mu - \nu x)^{\frac{\theta\mu - \eta\nu}{\mu\nu} + 1};$$

unde fit

$$P\partial x = x^{\frac{\eta}{m}-1}\partial x(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}.$$

Cum igitur sit

$$(i\nu+\theta)\int Px^i\partial x = [(i-1)\mu+\eta]\int Px^{i-1}\partial x - x^{i+\frac{\eta}{\mu}-1}(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}+1}$$

si haec integralia ita capiantur, ut evanescant posito $x=0$, tum vero statuatur $x=\frac{\mu}{\nu}$, fiet uti hypothesis nostra postulat

$$\int Px^i\partial x = \frac{(i-1)\mu+\eta}{i\nu+\theta}\int Px^{i-1}\partial x;$$

at vero in hunc finem necesse est, ut sit

$$i+\frac{\eta}{\mu}-1>0 \text{ et } \frac{\theta\mu-\eta\nu}{\mu\nu}+1>0.$$

Quae conditio si locum habeat, seriei propositae summa ita exprimetur, ut sit

$$y\int x^{\frac{\eta}{\mu}-1}\partial x(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}} = \mathfrak{A}\int x^{\frac{\eta}{\mu}-1}z\partial x(\mu-\nu x)^{\frac{\theta\mu-\eta\nu}{\mu\nu}};$$

existente

$$z = Aks^{\frac{-k}{n}}(n-ms)^{\frac{mk-nb}{mn}}\int s^{\frac{k}{n}-1}\partial s(n-ms)^{\frac{nb-mk}{mn}-1},$$

integrali hoc ita sumto, ut fiat $z=A$ posito $s=0$. Hoc autem integrali invento, pro s scribatur ux, et hoc valore ipsius z in illa formula substituto, quantitatem u tanquam constantem tractari oportet, quoad illae integrationes lege praescripta fuerint absolutae, tum enim pro y prodibit functio ipsius u, summam seriei propositae exprimens.

Corollarium 1.

1065. Quia in geminatis coëfficientibus nostrae seriei similis lex progressionis assumitur, singulas series A, B, C, D etc. et $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. inter se permutare licet, unde hac methodo duplex formula summam seriei exprimens obtinetur.

Corollarium 2.

1066. Etsi functio Q non in calculum ingreditur, eam tamen nosse oportet, quoniam ex ejus indole termini integrationis constitui debent, ita ut pro utroque fiat $Q = 0$. Hi scilicet termini sunt $x = 0$ et $x = \frac{\mu}{\nu}$, dum fuerit

$$i + \frac{\eta}{\mu} - 1 > 0 \text{ et } \frac{\theta\mu - \eta\nu}{\mu\nu} + 1 > 0,$$

ubi i est numerus integer positivus.

Corollarium 3.

1067. Pro functione Q casus quo vel $\mu = 0$ vel $\nu = 0$ seorsim sunt evolvendi. Priore quo $\mu = 0$, est

$$\frac{\partial Q}{Q} = \frac{\partial x(\theta x - \eta)}{\nu xx} = \frac{\theta \partial x}{\nu x} - \frac{\eta \partial x}{\nu xx}, \text{ unde fit}$$

$$Q = e^{\frac{\eta}{\nu x}} x^{\frac{\theta}{\nu}}.$$

Posteriori quo $\nu = 0$, est

$$\frac{\partial Q}{Q} = \frac{\partial x(\theta x + \mu - \eta)}{-\mu x} = -\frac{\theta \partial x}{\mu} + \frac{\eta - \mu}{\mu} \cdot \frac{\partial x}{x};$$

ideoque

$$Q = e^{\frac{-\theta x}{\mu}} x^{\frac{\eta}{\mu} - 1}.$$

Scholion.

1068. Constructiones hoc modo odornandae prorsus similes sunt iis, quas capite praecedente tradidimus, cum res etiam ad formulam integralem hujusmodi $\int V \partial x$ reducatur, in qua V est functio binarum variabilium u et x, quarum illa autem u in ipsa integratione constans reputatur, post integrationem vero ipsi x datus quidam valor assignatur. Verum tamen haec constructio ad casus in superiori methodo non contentos extenditur, quandoquidem fieri potest, ut quantitas z functiones maxime transcendentes involvat.

Vicissim autem vidimus, methodum praecedentem ad ejusmodi aequationes applicari posse, quae per series, quales hic tractamus, evolvi nequeant, unde in Analysin ex hoc fonte haud contemnenda incrementa hauriri posse videntur.

Problema 135.

1069. Proposita aequatione differentio-differentiali

$$xx(a+bx^n)\partial\partial y+(c+ex^n)\partial x\partial y+(f+gx^n)y\partial x^2=0,$$

valorem ipsius y per formulam integralem construere.

Solutio 1.

Hanc aequationem supra (967.) ita in seriem evolvimus, ut posito

$$y=x^\lambda(A+Bx^n+Cx^{2n}+Dx^{3n}+Ex^{4n}+\text{etc.}),$$

primo exponenti λ tribui debeat radix hujus aequationis

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

tum vero posito brevitatis gratia

$$\lambda(\lambda-1)b+\lambda e+g=h \text{ fit}$$

$$B=\frac{-b}{n[na+(2\lambda-1)a+c]}A,$$

$$C=\frac{-nnb-(2\lambda-1)nb-ne-b}{2n[2na+(2\lambda-1)a+c]}B,$$

$$D=\frac{-4nnb-2(2\lambda-1)nb-2ne-b}{3n[3na+(2\lambda-1)a+c]}C,$$

ideoque, illius seriei positis binis terminis contiguis indefinite $Mx^{(i-1)n}+Nx^{in}$, generaliter

$$N=\frac{-(i-1)^2nnb-(2\lambda-1)(i-1)nb-(i-1)ne-b}{in[ina+(2\lambda-1)a+c]}M,$$

ubi cum denominator jam habeat factores, quales ante assumsimus, numeratorem quoque in factores resolvamus, quo nihilo aequali posito invenitur

$$(i-1)n=-\tfrac{1}{2}(2\lambda-1)-\frac{e}{2b}\pm\sqrt{[\tfrac{1}{4}(2\lambda-1)^2+\frac{(\lambda-1)e}{2b}+\frac{ee}{4bb}-\frac{b}{b}]},$$

seu

$$(i-1)n = -\tfrac{1}{2}(2\lambda-1) - \tfrac{e}{2b} \pm \tfrac{1}{2b}\sqrt{[(b-e)^2 - 4bg]}.$$

Ponatur brevitatis gratia

$$\sqrt{[(b-e)^2 - 4bg]} = q,$$

ut sit

$$(i-1)n = \frac{-(2\lambda-1)b - e \pm q}{2b},$$

et nostra relatio fiet

$$N = \frac{-[(i-1)nb + \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e - \tfrac{1}{2}q][(i-1)nb + \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e + \tfrac{1}{2}q]}{inb\,[ina + (2\lambda-1)a + c]}M.$$

Ponamus jam $x^n = u$, et seriem inventam ita repraesentemus

$$\frac{y}{x^\lambda} = A\mathfrak{A} + B\mathfrak{B}u + C\mathfrak{C}u^2 + \ldots\ldots + M\mathfrak{M}u^{i-1} + N\mathfrak{N}u^i + \text{etc.}$$

horumque duplicium coëfficientium lex ita se habebit

$$N = \frac{(i-1)nb + \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e - \tfrac{1}{2}q}{-inb} M \text{ et}$$

$$\mathfrak{N} = \frac{(i-1)nb + \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e + \tfrac{1}{2}q}{ina + (2\lambda-1)a + c} \mathfrak{M}.$$

Cum igitur haec series similis sit ei, quam ante construximus, comparationem instituamus, et habebimus

$m = nb$, $h = \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e - \tfrac{1}{2}q$,

$n = -nb$, et $k = 0$:

$\mu = nb$, $\eta = \tfrac{1}{2}(2\lambda-1)b + \tfrac{1}{2}e + \tfrac{1}{2}q$,

$\nu = na$, et $\theta = (2\lambda-1)a + c$.

Primum ergo quaeramus quantitatem z, et ne littera x ambiguitatem creet, loco litterae x in praecedente problemate usurpatae utamur littera t, sitque $ut = s$, et quia est $k = 0$, erit per §. 1063.

$$z = A(1+s)^{\frac{-(2\lambda-1)b-e+q}{2nb}} = A(1+ut)^{\frac{-(2\lambda-1)b-e+q}{2nb}}.$$

Hoc valore invento tractetur in sequentibus integrationibus quantitas u ut constans, et cum quod supra erat y hic sit $\frac{y}{x^\lambda}$, et t quod supra erat x, erit

$$\frac{y}{x^\lambda}\int t^{\frac{\eta}{\mu}-1}\partial t(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}}=\mathfrak{A}\int t^{\frac{\eta}{\mu}-1}z\partial t(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}},$$

ubi cum sit $u=x^n$, hic valor statim pro u scribi potest, ut sit

$$z=A(1+x^n t)^{\frac{-(2\lambda-1)b-e+q}{2nb}},$$

et in his integrationibus littera x ut constans spectari debet. Quodsi autem fuerit

$$i+\frac{\eta}{\mu}-1>0 \text{ et } \frac{\theta\mu-\eta\nu}{\mu\nu}+1>0,$$

haec integralia ita capi debent, ut evanescant posito $t=0$, quo facto ipsi t tribui debet valor $t=\frac{\mu}{\nu}=\frac{b}{a}$. Cum unitas sit minimus valor ipsius i, sufficit ut sit

$$\frac{(2\lambda-1)b+e-q}{2nb}>0, \text{ tum vero } \frac{(2\lambda+1)ab+2bc-ae-aq}{2nab}+1>0.$$

Nunc vero cum

$$\int t^{\frac{\eta}{\mu}-1}\partial t(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}},$$

fiat quantitas constans, pro y autem ejus multiplum quodvis nostrae aequationi aeque satisfaciat, ejus integrale ita exprimetur

$$y=Cx^\lambda\int t^{\frac{\eta}{\mu}-1}z\partial t(\mu-\nu t)^{\frac{\theta\mu-\eta\nu}{\mu\nu}},$$

existente

$$z=(1+x^n t)^{\frac{-(2\lambda-1)b-e+q}{2nb}}.$$

Solutio II.

1071. Si coëfficientes geminatos inter se permutemus, ut sit

$$m = n b, \quad h = \tfrac{1}{2}(2\lambda - 1) b + \tfrac{1}{2} e + \tfrac{1}{2} q,$$
$$n = n a, \quad k = (2\lambda - 1) a + c,$$
$$\mu = n b, \quad \eta = \tfrac{1}{2}(2\lambda - 1) b + \tfrac{1}{2} e - \tfrac{1}{2} q,$$
$$\nu = - n b, \; \theta = 0,$$

sumaturque λ ex aequatione

$$\lambda(\lambda - 1) a + \lambda c + f = 0.$$

Primo ponatur $x^n t = s$, et quaeratur z ut sit

$$z = A k s^{\frac{-k}{n}} (n - m s)^{\frac{m k - n b}{m n}} \int s^{\frac{k}{n} - 1} \partial s (n - m s)^{\frac{n b - m k}{m n} - 1},$$

integrali ita definito, ut posito $s = 0$ fiat $z = A$, qui quidem valor A est arbitrarius, tum spectata x ut constante erit

$$y = C x^{\lambda} \int t^{\frac{\eta}{\mu} - 1} z \partial t (\mu - \nu t)^{\frac{\theta \mu - \eta \nu}{\mu \nu}},$$

integrali ita sumto ut evanescat posito $t = 0$; tum vero facto $t = \frac{\mu}{\nu}$, si modo fuerit $\frac{\eta}{\mu} > 0$ et $1 - \frac{\eta}{\mu} > 0$, ob $\theta = 0$; ubi notetur z per hanc aequationem differentialem definiri

$$\frac{\partial z}{\partial s} = \frac{A k - z(k - b s)}{s(n - m s)}.$$

Solutio III.

1072. Per seriem descendentem aequationem propositam ita resolvimus, ut posito

$$y = x^{\lambda} (A + B x^{-n} + C x^{-2n} + D x^{-3n} + \text{etc.})$$

exponens λ definiri debeat ex hac aequatione

$$\lambda(\lambda - 1) b + \lambda e + g = 0,$$

tum vero posito $\lambda(\lambda - 1) a + \lambda c + f = h$, sit

$$B = \frac{-b}{n[n b - (2\lambda - 1) b - e]} A,$$
$$C = \frac{-n n a + (2\lambda - 1) n a - n c - h}{2n[2 n b - (2\lambda - 1) b - e} B,$$

et generatim

$$N = \frac{-(i-1)^2 nna + (2\lambda - 1)(i-1)na + (i-1)nc - b}{in[inb - (2\lambda - 1)b - e]} M,$$

quae aequalitas posito $\sqrt{[(a-c)^2 - 4af]} = p$, ita per factores exhibetur

$$N = \frac{-[(i-1)na - \frac{1}{2}(2\lambda - 1)a - \frac{1}{2}c - \frac{1}{2}p][(i-1)na - \frac{1}{2}(2\lambda - 1)a - \frac{1}{2}c + \frac{1}{2}p]}{ina[inb - (2\lambda - 1)b - e]} M.$$

Quodsi jam ponamus $x^{-n} = u$, et talem seriem constituamus

$$\frac{y}{x^\lambda} = A\mathfrak{A} + B\mathfrak{B}u + C\mathfrak{C}u^2 + \ldots\ldots + M\mathfrak{M}u^{i-1} + N\mathfrak{N}u^i + \text{etc.}$$

erit

$$N = \frac{(i-1)na - \frac{1}{2}(2\lambda - 1)a - \frac{1}{2}c - \frac{1}{2}p}{-ina} M, \text{ et}$$

$$\mathfrak{N} = \frac{(i-1)na - \frac{1}{2}(2\lambda - 1)a - \frac{1}{2}c + \frac{1}{2}p}{inb - (2\lambda - 1)b - e} \mathfrak{M},$$

et habebimus comparatione instituta cum constructione generali

$$m = na, \; h = -\tfrac{1}{2}(2\lambda - 1)a - \tfrac{1}{2}c - \tfrac{1}{2}p,$$
$$n = -na, \text{ et } k = 0,$$
$$\mu = na, \; \eta = -\tfrac{1}{2}(2\lambda - 1)a - \tfrac{1}{2}c + \tfrac{1}{2}p,$$
$$\nu = nb, \text{ et } \theta = -(2\lambda - 1)b - e.$$

Hinc posito $s = ut = x^{-n}t$, erit

$$z = A(1+s)^{\frac{-h}{m}} = A(1 + x^{-n}t)^{\frac{-h}{m}},$$

quo valore invento, si jam sola quantitas t pro variabili habeatur, orietur haec constructio

$$y = Cx^\lambda \int t^{\frac{\eta}{\mu} - 1} z\,\partial t\,(\mu - \nu t)^{\frac{\theta\mu - \eta\nu}{\mu\nu}},$$

ubi termini integrationis ita sunt constituendi, ut utroque fiat

$$t^{\frac{\eta}{\mu}}(\mu - \nu t)^{\frac{\theta\mu - \eta\nu}{\mu\nu} + 1} = 0.$$

Solutio. IV.

1703. Hic etiam coëfficientes geminati permutari possunt, ut sit

$$m = na,\ h = -\tfrac{1}{2}(2\lambda - 1)a - \tfrac{1}{2}c + \tfrac{1}{2}p,$$
$$n = nb,\ k = -(2\lambda - 1)b - e,$$
$$\mu = na,\ \eta = -\tfrac{1}{2}(2\lambda - 1)a - \tfrac{1}{2}c - \tfrac{1}{2}p,$$
$$\nu = -na, \text{ et } \theta = 0,$$

sumtoque ut ante λ ex aequatione

$$\lambda(\lambda - 1)b + \lambda e + g = 0,$$

et posito $\surd[(a - c)^2 - 4af) = p$, sit $s = x^{-n}t$, et quaeratur z ex hac aequatione

$$\frac{\partial z}{\partial s} = \frac{Ak - z(k - bs)}{s(n - ms)},$$

ita ut posito $s = 0$ fiat $z = A$, unde fit

$$z = Aks^{\frac{-k}{n}}(n - ms)^{\frac{mk - nb}{mn}} \int s^{\frac{k}{n} - 1} \partial s (n - ms)^{\frac{nb - mk}{mn} - 1},$$

tum vero spectata x ut constante, erit

$$y = Cx^{\lambda} \int t^{\frac{\eta}{\mu} - 1} z \partial t (\mu - \nu t)^{\frac{\theta\mu - \eta\nu}{\mu\nu}},$$

ubi bini termini integrationis ita sumi debent, ut utroque fiat

$$t^{\frac{\eta}{\mu}} (\mu - \nu t)^{\frac{\theta\mu - \eta\nu}{\mu\nu} + 1} = 0.$$

Scholion.

1074. Singulae hae constructiones plerumque pluribus modis exhiberi possunt, cum non solum λ duplicem valorem habere queat, sed etiam formulae radicales p et q signo ambiguo gaudeant. At hae constructiones alio modo negotium conficiunt atque praecedentes, quod quo clarius appareat, consideremus aequationem

$$xx(1-xx)\partial\partial y - x(1+xx)\partial x\partial y + xxy\partial x^2 = 0,$$

ita ut sit $a=1$, $b=-1$, $c=-1$, $e=-1$, $f=0$ et $g=1$, atque $n=2$; unde pro duabus prioribus constructionibus habetur $\lambda(\lambda-1)-\lambda=0$, ergo vel $\lambda=0$ vel $\lambda=2$, tum vero $q=\pm 2$. Constructio ergo prima dat $\lambda=0$, $m=-2$, $h=\mp 1$, $n=2$, $k=0$, $\mu=-2$, $\eta=\pm 1$, $\nu=2$, $\theta=-2$; unde colligitur

$$z=(1+xxt)^{\mp\frac{1}{2}}, \text{ et } y=C\int t^{\mp\frac{1}{2}-1} z\partial t(1+t)^{\pm\frac{1}{2}-1}.$$

Valeant signa inferiora, ut sit

$$z=\sqrt{(1+xxt)} \text{ et } y=C\int\frac{z\partial t}{(1+t)\sqrt{t(1+t)}}.$$

Quae formulae quomodo satisfaciant, videamus. Sumta nempe sola x variabili, fit

$$\frac{\partial z}{\partial x}=\frac{xt}{\sqrt{(1+xxt)}} \text{ et } \frac{\partial\partial z}{\partial x^2}=\frac{t}{(1+xxt)^{\frac{3}{2}}},$$

hinc

$$\frac{\partial y}{\partial x}=C\int\frac{xt\partial t}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{1}{2}}} \text{ et } \frac{\partial\partial y}{\partial x}=C\int\frac{t\partial t}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{3}{2}}}$$

unde conficitur

$$xx(1-xx)\frac{\partial\partial y}{\partial x^2}-x(1+xx)\frac{\partial y}{\partial x}+xxy=C\int\frac{xx\partial t(1-xxtt)}{t^{\frac{1}{2}}(1+t)^{\frac{3}{2}}(1+xxt)^{\frac{3}{2}}},$$

cujus formulae integrale est $\frac{2Cxx\sqrt{t}}{\sqrt{(1+t)(1+xxt)}}$; quod cum evanescat tam casu $t=0$ quam casu $t=\infty$, constructio nostrae aequationis

$$y=C\int\frac{z\partial t}{(1+t)\sqrt{t(1+t)}}=C\int\frac{\partial t\sqrt{(1+xxt)}}{(1+t)\sqrt{t(1+t)}},$$

ita confici debet; sumto x pro constante, integratio ita instituatur, ut integrale evanescat posito $t=0$, quo facto statuatur $t=\infty$, et functio ipsius x, quae pro y prodibit, satisfaciet aequationi propositae.

Sin autem secundam constructionem eligamus, sumta $\lambda = 0$, erit $m = -2$, $h = \pm 1$, $n = 2$, $k = -2$, $\mu = -2$, $\eta = \mp 1$, $\nu = 2$, $\theta = 0$, atque z ita definiri debet ex hac aequatione

$$\frac{\partial z}{\partial s} = \frac{-2A + z(2 \pm s)}{s(2 + 2s)},$$

existente $s = xxt$, ut posito $s = 0$ fiat $z = A$.

Valeat signum superius, et cum sit

$$\partial z - \frac{z \partial s (2 + s)}{2s(1 + s)} = \frac{-A \partial s}{s(1 + s)}:$$

multiplicetur per $\frac{\sqrt{(1+s)}}{s}$, eritque integrale

$$\frac{z\sqrt{(1+s)}}{s} = \text{Const.} - A \int \frac{\partial s}{ss\sqrt{(1+s)}}, \text{ seu}$$

$$z = A - \frac{As}{\sqrt{(1+s)}} l \frac{1 + \sqrt{(1+s)}}{\sqrt{s}} + \frac{Bs}{\sqrt{(1+s)}},$$

quae posito $s = 0$ dat $z = A$, quicquid sit B. Deinde est

$$y = C \int t^{\frac{1}{2} - 1} z \partial t (1 + t)^{-\frac{1}{2}} \text{ seu } y = C \int \frac{z \partial t}{\sqrt{t(1+t)}},$$

qui valor quomodo satisfaciat haud facile ostendi potest; hocque magis ista methodus excoli meretur.

Exemplum.

1075. *Constructiones aequationis differentio-differentialis*

$$xx(1 - xx)\partial\partial y - x(1 + xx)\partial x \partial y + xxy\partial x^2 = 0$$

ex præcedente problemate oriundas exhibere.

Ob $n = 2$, $a = 1$, $b = -1$, $c = -1$, $e = -1$, $f = 0$, et $g = 1$, pro prima constructione habemus vel $\lambda = 0$ vel $\lambda = 2$, unde obtinemus.

1.) Si $\lambda = 0$, ut modo invenimus

$m = -2$, $h = \mp 1$, $n = 2$, $k = 0$, $\mu = -2$, $\eta = \pm 1$; $\nu = 2$, $\theta = -2$, unde fit

$$z = (1 \mp xxt)^{\mp\frac{1}{2}} \text{ et } y = C\int t^{\mp\frac{1}{2}-1} z\partial t(1+t)^{\pm\frac{1}{2}-1},$$

sicque duplex oritur constructio

altera $z = \sqrt{(1+xxt)}$ et $y = C\int \frac{z\partial t}{(1+t)\sqrt{t(1+t)}}$,

altera $z = \frac{1}{\sqrt{(1+xxt)}}$ et $y = C\int \frac{z\partial t}{t\sqrt{t(1+t)}}$.

2.) Si $\lambda = 2$, erit ob $q = \pm 2$,
$m = -2$, $h = -2 \mp 1$, $n = 2$, $k = 0$, $\mu = -2$,
$\eta = -2 \pm 1$, $\nu = 2$, $\theta = 2$,

unde fit

$$z = (1+xxt)^{\frac{-2\mp 1}{2}} \text{ et } y = C x^2 \int t^{\mp 1 \mp\frac{1}{2}-1} z\partial t(1+t)^{\pm\frac{1}{2}},$$

sicque duplex habetur constructio

altera $z = (1+xxt)^{-\frac{3}{2}}$ et $y = C x^2 \int \frac{z\partial t\sqrt{(1+t)}}{\sqrt{t}}$,

altera $z = (1+xxt)^{-\frac{1}{2}}$ et $y = C x^2 \int \frac{z\partial t\sqrt{t}}{\sqrt{(1+t)}}$,

Pro secunda constructione generali habemus:

3.) Si $\lambda = 0$, permutando illos indices,
$m = -2$, $h = \pm 1$, $n = 2$, $k = -2$; $\mu = -2$, $\eta = \mp 1$,
$\nu = 2$, $\theta = 0$,

unde posito $xxt = s$, primo quaeratur z ex hac aequatione $\frac{\partial z}{\partial s} = \frac{-2A + z(2 \pm s)}{2s(1+s)}$, ut posito $s = 0$ fiat $z = A$; tum vero erit

$$y = C\int t^{\pm\frac{1}{2}-1} z\partial t(1+t)^{\mp\frac{1}{2}}.$$

Hinc ergo nascitur duplex constructio

altera $\frac{\partial z}{\partial s} = \frac{-2A + z(2+s)}{2s(1+s)}$ et $y = C\int \frac{z\partial t}{\sqrt{t(1+t)}}$,

altera $\frac{\partial z}{\partial s} = \frac{-2A + z(2-s)}{2s(1+s)}$ et $y = C\int \frac{z\partial t\sqrt{(1+t)}}{t\sqrt{t}}$,

4.) Si $\lambda = 2$ habebimus
$m = -2$, $h = -2 \pm 1$, $n = 2$, $k = 2$, $\mu = -2$,
$\eta = -2 \mp 1$, $\nu = 2$, $\theta = 0$,

positoque $xxt = s$, ut ante, quaeratur z ex aequatione

$$\frac{\partial z}{\partial s} = \frac{2A - z[2 + (2 \mp 1)s]}{2s(1+s)}, \text{ eritque}$$

$$y = Cx^2 \int t^{\pm\frac{1}{2}} z \partial t (1+t)^{\frac{-2 \mp 1}{2}};$$

ideoque etiam duplex constructio elicitur

altera $\frac{\partial z}{\partial s} = \frac{2A - z(2+s)}{2s(1+s)}$ et $y = Cx^2 \int \frac{z \partial t \sqrt{t}}{(1+t)\sqrt{(1+t)}}$,

altera $\frac{\partial z}{\partial s} = \frac{2A - z(2+3s)}{2s(1+s)}$ et $y = Cx^2 \int \frac{z \partial t}{\sqrt{t(1+t)}}$.

Ex solutione tertia colligimus primo

$$-\lambda(\lambda - 1) - \lambda + 1 = 0 \text{ seu } \lambda\lambda = 1,$$

ideoque

$$\lambda = \pm 1 \text{ et } p = \sqrt{4} = \pm 2; \text{ quare}$$

5.) si capiatur $\lambda = +1$, erit

$$m = 2,\ h = \mp 1,\ n = -2,\ k = 0,\ \mu = 2,\ \eta = \pm 1,$$
$$\nu = -2;\ \theta = 2,$$

hincque

$$z = \left(1 + \frac{t}{xx}\right)^{\pm\frac{1}{2}} \text{ et } y = Cx \int t^{\pm\frac{1}{2} - 1} z \partial t (1+t)^{-1 \mp \frac{1}{2}},$$

ita ut iterum duplex habeatur constructio

altera $z = \frac{1}{x}\sqrt{(xx+t)}$ et $y = Cx \int \frac{z \partial t}{(1+t)\sqrt{t(1+t)}}$

altera $z = \frac{x}{\sqrt{(xx+t)}}$ et $y = Cx \int \frac{z \partial t}{t\sqrt{t(1+t)}}$.

6.) Si capiatur $\lambda = -1$ erit

$$m = 2,\ h = 2 \mp 1,\ n = -2,\ k = 0,\ \mu = 2,\ \eta = 2 \pm 1,$$
$$\nu = -2,\ \theta = -2,$$

hincque

$$z = \left(1 + \frac{t}{xx}\right)^{-1 \pm \frac{1}{2}} \text{ et } y = \frac{C}{x} \int t^{\pm\frac{1}{2}} z \partial t (1+t)^{\mp\frac{1}{2}},$$

unde binae constructiones fluunt

$$\text{altera } z = \frac{x}{\sqrt{(xx+t)}} \text{ et } y = \frac{C}{x}\int \frac{z\,\partial t\sqrt{t}}{\sqrt{(1+t)}}$$

$$\text{altera } z = \frac{x^3}{(xx+t)^{\frac{3}{2}}} \text{ et } y = \frac{C}{x}\int \frac{z\,\partial t\sqrt{(1+t)}}{\sqrt{t}}.$$

Ex solutione quarta denique concludimus

7.) Si $\lambda = +1$,

$$m = 2,\ h = \pm 1,\ n = -2,\ k = 2,\ \mu = 2,\ \eta = \mp 1,\ \nu = -2,\ \theta = 0.$$

Posito nunc $s = \frac{t}{xx}$, quaeratur z ex hac aequatione

$$\frac{\partial z}{\partial s} = \frac{-2A + z(2 \mp s)}{2s(1+s)},$$

ut posito $s = 0$ fiat $z = A$, tumque erit

$$y = C\,x\int t^{\mp\frac{1}{2}-1}\, z\,\partial t\,(1+t)^{\pm\frac{1}{2}},$$

unde duplex constructio

$$\text{altera } \frac{\partial z}{\partial s} = \frac{-2A + z(2-s)}{2s(1+s)} \text{ et } y = C\,x\int \frac{z\,\partial t\sqrt{(1+t)}}{t\sqrt{t}}$$

$$\text{altera } \frac{\partial z}{\partial s} = \frac{-2A + z(2+s)}{2s(1+s)} \text{ et } y = C\,x\int \frac{z\,\partial t}{\sqrt{t(1+t)}}.$$

8.) Si $\lambda = -1$, habebitur

$$m = 2,\ h = 2 \pm 1,\ n = -2,\ k = -2,\ \mu = 2,\ \eta = 2 \mp 1,\ \nu = -2,\ \theta = 0,$$

at posito $\frac{t}{xx} = s$, quaeri debet z ex hac aequatione

$$\frac{\partial z}{\partial s} = \frac{+2A - z\,[2 + (2 \pm 1)s]}{2s(1 \mp s)},$$

ut posito $s = 0$ fiat $z = A$, quo facto erit

$$y = \frac{C}{x}\int t^{\mp\frac{1}{2}}\, z\,\partial t\,(1+t)^{-1\pm\frac{1}{2}};$$

sicque duplex oritur constructio

altera $\frac{\partial z}{\partial s} = \frac{2A - z(2+3s)}{2s(1+s)}$ et $y = \frac{C}{x}\int\frac{z\,\partial t}{\sqrt{t}\,(1+t)}$

altera $\frac{\partial z}{\partial s} = \frac{2A - z(2+s)}{2s(1+s)}$ et $y = \frac{C}{x}\int\frac{z\,\partial t\sqrt{t}}{(1+t)\sqrt{(1+t)}}$.

Omnino ergo sedecim constructiones sumus consecuti.

Scholion.

1076. Periculum faciamus ostendendi, quomodo hae constructiones, quae magis arduae videntur, aequationi propositae satisfaciant; atque in hunc finem eligamus constructionem posteriorem No. 4. quae habet

$$\partial z + \frac{z\,\partial s\,(2+3s)}{2s(1+s)} = \frac{A\,\partial s}{s(1+s)},$$

haec per $s\sqrt{(1+s)}$ multiplicata praebet integrale

$$sz\sqrt{(1+s)} = A\int\frac{\partial s}{\sqrt{(1+s)}} = 2A\sqrt{(1+s)} + B, \text{ seu}$$
$$z = \frac{2A}{s} + \frac{B}{s\sqrt{(1+s)}}.$$

Jam ut posito $s = 0$ fiat $z = A$, debet esse $B = -2A$, ut sit

$$z = \frac{2A[\sqrt{(1+s)} - 1]}{s\sqrt{(1+s)}} = \frac{2A}{txx} - \frac{2A}{txx\sqrt{(1+txx)}}.$$

Hinc fit

$$\left(\frac{\partial z}{\partial x}\right) = \frac{-4A}{tx^3} + \frac{2A(2+3txx)}{tx^3(1+txx)^{\frac{3}{2}}} \text{ et}$$

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{12A}{tx^4} - \frac{6A(2+5txx+4ttx^4)}{tx^4(1+txx)^{\frac{5}{2}}}.$$

Cum nunc sit $y = C\int\frac{xxz\,\partial t}{\sqrt{t}(1+t)}$, erit

$$\left(\frac{\partial y}{\partial x}\right) = 2C\int\frac{xz\,\partial t}{\sqrt{t}(1+t)} + C\int\frac{xx\,\partial t}{\sqrt{t}(1+t)}\cdot\left(\frac{\partial z}{\partial x}\right) \text{ et}$$

$$\left(\frac{\partial\partial y}{\partial x^2}\right) = 2C\int\frac{z\,\partial t}{\sqrt{t}(1+t)} + 4C\int\frac{x\,\partial t}{\sqrt{t}(1+t)}\left(\frac{\partial z}{\partial x}\right) + C\int\frac{xx\,\partial t}{\sqrt{t}(1+t)}\left(\frac{\partial\partial z}{\partial x^2}\right);$$

hincque

$$x(1-xx)\frac{\partial\partial y}{\partial x^2} - x(1+xx)\frac{\partial y}{\partial x} + xxy =$$

$$C\int\frac{\partial t}{\sqrt{t(1+t)}}\cdot\left(\frac{2Axx}{t} + \frac{2Axx(1+4txx+3ttxx)}{t(1+txx)^{\frac{5}{2}}}\right),$$

quod integrale est

$$\frac{-4ACxx\sqrt{(1+t)}}{\sqrt{t}} + \frac{4ACxx\sqrt{(1+t)}}{(1+txx)^{\frac{3}{2}}\sqrt{t}}$$

$$= \frac{4ACxx\sqrt{(1+t)}}{\sqrt{t}}\left(\frac{1}{(1+txx)^{\frac{3}{2}}} - 1\right),$$

et hac forma exprimi potest

$$-2Cx^4[3z + x(\tfrac{\partial z}{\partial x})]\sqrt{t(1+t)}$$

vel etiam hoc modo

$$-2Cx^4\left(\tfrac{2A+z}{1+txx}\right)\sqrt{t(1+t)}.$$

Expressio autem ista fit $=0$, primo si $t=-1$, deinde etiam si $t=0$, unde valor pro y inventus

$$y = D\int\tfrac{\partial t}{t\sqrt{t(1+t)}}\left(1 - \tfrac{1}{\sqrt{(1+txx)}}\right)$$

ita per integrationem definiri debet, ut evanescat posito $t=0$; tum vero ponatur $t=-1$. Vel posito $t=-v$ erit

$$y = D\int\tfrac{\partial v}{v\sqrt{v(1-v)}}\left(1 - \tfrac{1}{\sqrt{(1-vxx)}}\right)$$

integrali ita sumto ut evanescat posito $v=0$, tum vero facto $v=1$.

Exemplum hoc sufficit ad ostendendum, quomodo constructiones exhibitae aequationi differentio-differentiali satisfaciant; interim vero si quantitas z transcendenter, per logarithmos scilicet exprimitur, consensum nonnisi per calculos nimium taediosos declarare licet.

Problema 136.

1077. Posito $y = C\int (1+t)^{\nu-1} (a+tx)^{\lambda}\, \partial t$, in qua integratione quantitas x ut constans spectatur, integrali per terminos deinceps investigandos definito, ut y aequetur certae functioni ipsius x, invenire aequationes differentio-differentiales formae

$$Lxx \cdot \frac{\partial\partial y}{\partial x^2} + Mx \cdot \frac{\partial y}{\partial x} + Ny = 0,$$

cui ea functio satisfaciat.

Solutio.

Cum sit ex principiis ante stabilitis

$$\frac{\partial y}{\partial x} = C\int \lambda t (1+t)^{\nu-1} (a+tx)^{\lambda-1}\, \partial t \text{ et}$$

$$\frac{\partial\partial y}{\partial x^2} = C\int \lambda(\lambda-1)\, tt (1+t)^{\nu-1} (a+tx)^{\lambda-2}\, \partial t, \text{ erit}$$

$$Lxx \cdot \frac{\partial\partial y}{\partial x^2} + Mx \cdot \frac{\partial y}{\partial x} + Ny = C\int (1+t)^{\nu-1} (a+tx)^{\lambda-2}\, \partial t \times$$

$$[\lambda(\lambda-1) Lttxx + \lambda M tx (a+tx) + N(a+tx)^2] =$$

$$C\int (1+t)^{\nu-1} (a+tx)^{\lambda-2} \partial t \begin{Bmatrix} Naa + 2Natx + Nttxx \\ \quad + \lambda Matx + \lambda Mttxx \\ \qquad + \lambda(\lambda-1) Lttxx \end{Bmatrix}$$

quae formula sumta x constante absolute integrabilis esse debet. Ponatur ergo integrale

$$C (1+t)^{\nu} (a+tx)^{\lambda-1} (Paa + Qatx)$$

denotantibus P et Q functionibus quibuscunque ipsius x, erit ejus differentiale

$$C (1+t)^{\nu-1} (a+tx)^{\lambda-2} \partial t \times$$

$$[\nu(a+tx)(Paa+Qatx) + (\lambda-1)x(1+t)(Paa+Qatx) + Qax(1+t)(a+tx)]$$

$$= C(1+t)^{\nu-1}(a+tx)^{\lambda-2}\partial t \begin{Bmatrix} +\nu Pa^3 & +\nu Qaatx & +\nu Qattxx \\ +(\lambda-1)Paax & +\nu Paatx & +(\lambda-1)Qattxx \\ +Qaax & +(\lambda-1)Paatx & +Qattxx \\ & +(\lambda-1)Qatxx & \\ & +Qaatx & \\ & +Qatxx & \end{Bmatrix}$$

qua forma cum illa comparata, adipiscimur

$$N = \nu P a + (\lambda - 1) P x + Q x$$
$$2 N + \lambda M = (\nu + 1) Q a + (\lambda + \nu - 1) P a + \lambda Q x$$
$$N + \lambda M + \lambda(\lambda - 1) L = (\lambda + \nu) Q a,$$

quarum aequationum extremae demta media praebent

$$\lambda(\lambda-1) L = -(\lambda-1) P a + (\lambda-1) P x + (\lambda-1) Q a - (\lambda-1) Q x,$$

hincque

$$\lambda L = (a - x)(Q - P) \text{ seu } L = \tfrac{1}{\lambda}(a - x)(Q - P),$$

secunda autem demto primae duplo, dat

$$\lambda M = (\lambda - \nu - 1) P a - 2(\lambda - 1) P x + (\nu + 1) Q a + (\lambda - 2) Q x, \text{ seu}$$
$$\lambda M = [(\nu + 1) a + (\lambda - 2) x](Q - P) + \lambda(a - x) P.$$

Quare sumtis pro P et Q functionibus quibuscunque ipsius x, si functiones L, M, N ita definiantur, ut sit

$$L = \tfrac{1}{\lambda}(a - x)(Q - P)$$
$$M = \tfrac{1}{\lambda}[(\nu + 1) a + (\lambda + 2) x](Q - P) + (a - x) P$$
$$N = x(Q - P) + (\nu a + \lambda x) P,$$

aequationi differentio-differentiali

$$L x x \partial\partial y + M x \partial x \partial y + N y \partial x^2 = 0$$

satisfaciet formula integralis

$$y = C \int (1 + t)^{\nu - 1} (a + t x)^{\lambda} \partial t,$$

tractata x ut constante, dummodo integrationis termini ita constituantur, ut utroque haec expressio

$$(1 + t)^{\nu} (a + t x)^{\lambda - 1} (P a + Q t x) \text{ evanescat.}$$

Notari autem oportet, hos terminos non ab x pendere debere. Primo autem patet hanc expressionem fieri $= 0$ casu $t = -1$, si modo sit $\nu > 0$. Deinde posito $t = \infty$ etiam evanescet, si modo sit $\nu + \lambda - 1 + 1$ numerus negativus, seu $\nu + \lambda < 0$. Quocirca si sit $\nu > 0$ et $\nu + \lambda < 0$, integrale

$$y = C \int (1 + t)^{\nu - 1} (a + t x)^{\lambda} \partial t$$

ita capi debet, ut posito $t = -1$ evanescat, tum vero statuatur $t = \infty$, functioque ipsius x pro y resultans satisfaciet aequationi propositae.

Corollarium 1.

1078. Quoniam functiones P et Q in formulam integralem pro y assumtam non ingrediuntur, manifestum est eandem formulam satisfacere omnibus aequationibus differentio-differentialibus, quicunque valores litteris P et Q tribuantur.

Corollarium 2.

1079. Sumto ergo $Q = P$ eadem formula integralis

$$y = C \int (1 + t)^{\nu - 1} (a + t)^{\lambda} \partial t$$

satisfacit etiam huic aequationi differentiali primi gradus

$$(a - x) x \partial y + (\nu a + \lambda x) y \partial x = 0.$$

Hujus vero integrale est

$$y = \frac{D (a - x)^{\lambda + n}}{x^{\nu}},$$

qui ergo valor quoque in generae nostrae aequationi differentio-differentiali satisfacit, id quod tentanti mox patebit.

Corollarium 3.

1080. Hic ergo valor integralis

$$y = C \int (1 + t)^{\nu - 1} (a + t x)^{\lambda} \partial t$$

secundum terminos definitos sumtus, congruere debet cum formula algebraica $y = \frac{D (a - x)^{\lambda + \nu}}{x^{\nu}}$, si modo sit $\nu > 0$ et $\lambda + \nu < 0$.

Scholion.

1081. Parum ergo integratio hoc problemate exhibita habet in recessu. Verum reductio formulae integralis

$$y = C\int (1+t)^{\nu-1}(a+tx)^{\lambda}\partial t \text{ ad } y = \frac{D(a-x)^{\lambda+\nu}}{x^{\nu}}$$

eo magis est notatu digna, ad quam illa reducitur, si, integrali ita sumto ut evanescat posito $t = -1$, ponatur $t = \infty$. Posito ergo $\lambda + \nu = -\mu$, ut μ et ν sint numeri positivi, erit

$$C\int \frac{(1+t)^{\nu-1}\partial t}{(a+tx)^{\mu+\nu}} = \frac{D}{x^{\nu}(a-x)^{\mu}}.$$

Vel ponatur $1 + t = z$, erit

$$C\int \frac{z^{\nu-1}\partial z}{(a-x+xz)^{\mu+\nu}} = \frac{D}{x^{\nu}(a-x)^{\mu}},$$

terminis illius integrationis existentibus $z = 0$ et $z = \infty$. Verum etiam haec observatio non magni est momenti, nam posito $a - x = ux$, fit

$$\frac{C}{x^{\mu+\nu}}\int \frac{z^{\nu-1}\partial z}{(u+z)^{\mu+\nu}} = \frac{D}{x^{\mu+\nu}u^{\mu}},$$

ideoque haec formula $\int \frac{z^{\nu-1}\partial z}{(u+z)^{\mu+\nu}}$ ita integrata ut evanescat posito $z = 0$, si tum ponatur $z = \infty$, hanc induet formam $\frac{A}{u^{\mu}}$, in qua A quantitatem constantem denotat ab u non pendentem. Pendet autem ab exponentibus μ et ν, lege ex casibus facile observanda. Scilicet posito

$$\int \frac{z^{\nu-1}\partial z}{(u+z)^{\mu+\nu}} = \frac{A}{u^{\mu}},$$

si sit $\nu = 1$, integrale illud praebet $-\frac{1}{\mu(u+z)^{\mu}} + \frac{1}{\mu u^{\mu}}$, et posito $z = \infty$, prodit $\frac{1}{\mu u^{\mu}}$ ita ut hoc casu sit $A = \frac{1}{\mu}$. Si sit $\nu = 2$, integratio quoque succedit, reperiturque $A = \frac{1}{\mu(\mu+1)}$ si $\nu = 3$ fit $A = \frac{1.2}{\mu(\mu+1)(\mu+2)}$, et si $\nu = 4$ fit

$$A = \frac{1.2.3}{\mu(\mu+1)(\mu+2)(\mu+3)},$$

unde in genere concludimus fore

$$A = \frac{1.2.3. \ldots \ldots (\nu-1)}{\mu(\mu+1)(\mu+2) \ldots \ldots (\mu+\nu-1)}.$$

Quare integratione secundum regulam praescriptam instituta, erit

$$\frac{1}{\mu+1} \cdot \frac{2}{\mu+2} \cdot \frac{3}{\mu+3} \cdots \cdots \frac{\nu-1}{\mu+\nu-1} = \mu u^{\mu} \int \frac{z^{\nu-1} \partial z}{(u+z)^{\mu+\nu}}.$$

Quod si exponens ν non fuerit integer, valor ipsius A ope interpolationis hujus formulae per factores procedentis definietur. Quadratura scilicet circuli ingredietur si exponens ν fractionem $\frac{1}{2}$ involvat, de hujusmodi autem interpolationibus alibi fusius egimus, neque hic locus est hoc argumentum uberius prosequendi. Restat ultimum hujus sectionis caput, quo aequationum differentio-differentialium integratio per approximationes docebitur.