CAPUT X.

DE CONSTRUCTIONE AEQUATIONUM DIFFERENTIO-DIFFERENTIALIUM PER QUADRATURAS CURVARUM.

Problema 125.

1017.

Si fuerit $y = \int V \partial x$, denotante V functionem quamcunque binarum quantitatum x et u, quarum autem haec u in integratione ut constans spectatur, post integrationem vero statuatur $x = a$, ut y aequetur functioni cuidam ipsius u; quodsi jam u variabilis sumatur, investigare valorem ipsius $\frac{\partial y}{\partial u}$.

Solutio.

Cum $\int V \partial x$ exhibeat functionem quandam binarum quantitatum x et u, cujus differentiale, sumta u constante, est $= V \partial x$, si tam u quam x ut variabiles tractentur, differentiale aequationis $y = \int V \partial x$ talem habebit formam, $\partial y = V \partial x + U \partial u$, quae, quia est differentiale verum, necesse est sit $\left(\frac{\partial V}{\partial u}\right) = \left(\frac{\partial U}{\partial x}\right)$. At cum V sit functio data ipsarum x et u, ponatur $\partial V = P \partial x + Q \partial u$, eritque $\left(\frac{\partial V}{\partial u}\right) = Q$, ideoque $\left(\frac{\partial U}{\partial x}\right) = Q$. Hinc considerata iterum u ut constante, erit $\partial U = Q \partial x$, et $U = \int Q \partial x$, in qua integratione sola x pro variabili habetur. Quocirca si hunc valorem $\int Q \partial x$ ut cognitum spectemus, quippe quem per quadraturas assignare licet, erit $\partial y = V \partial x + \partial u \int Q \partial x$. Quaerimus autem id ipsius y differentiale, quod ex variabilitate ipsius u tantum nascitur; quod cum sit $\partial y = \partial u \int Q \partial x$ erit valor quaesitus $\frac{\partial y}{\partial u} = \int Q \partial x$, si nempe post integrationem itidem ponatur $x = a$.

Corollarium 1.

1018. Cum sit $y = \int V \partial x$ functio ipsarum x et u, per integrationem autem formulae $V \partial x$, in qua u constans spectatur, functio quaecunque ipsius u loco constantis accedere possit, functio y per se erit indeterminata, determinabitur autem statim, atque integrale $\int V \partial x$ ita accipiatur, ut evanescat posito $x = 0$.

Corollarium 2.

1019. Hac conditione observata evanescet y posito $x = 0$, quicunque valor alteri quantitati u tribuatur, erit ergo etiam $y + \partial u \left(\frac{\partial y}{\partial u}\right) = 0$ facto $x = 0$, ergo etiam $\left(\frac{\partial y}{\partial u}\right) = 0$. Unde patet $\int Q \partial x = \frac{\partial y}{\partial u}$ ita quoque accipi debere, ut posito $x = 0$ evanescat.

Corollarium 3.

1020. Cum $y = \int V \partial x$ erit $\left(\frac{\partial y}{\partial x}\right) = V$, hinc

$$\left(\frac{\partial \partial y}{\partial u \partial x}\right) = \left(\frac{\partial V}{\partial u}\right).$$

At si ponatur $\left(\frac{\partial y}{\partial u}\right) = Z$, erit quoque

$$\left(\frac{\partial \partial y}{\partial u \partial x}\right) = \left(\frac{\partial Z}{\partial x}\right), \text{ ergo } \left(\frac{\partial Z}{\partial x}\right) = \left(\frac{\partial V}{\partial u}\right).$$

Quare spectata u ut constante, erit

$$\partial Z = \partial x \left(\frac{\partial V}{\partial u}\right), \text{ et } Z = \int \partial x \left(\frac{\partial V}{\partial u}\right)$$

ideoque

$$\left(\frac{\partial y}{\partial u}\right) = \int \partial x \left(\frac{\partial V}{\partial u}\right).$$

Corollarium 4.

1021. Quodsi ergo post integrationes ita absolutas, ut integralia evanescant posito $x = 0$, ponatur $x = a$, tam valor $y = \int V \partial x$, quam $\frac{\partial y}{\partial u} - \int \partial x \left(\frac{\partial V}{\partial x}\right)$ erit functio determinata ipsius u.

Corollarium 5.

1022. Simili modo ulterius progrediendo erit

$$\frac{\partial \partial y}{\partial u^2} = \int \partial x \left(\frac{\partial \partial V}{\partial u^2}\right).$$

Quare si L, M et N denotent functiones quascunque ipsius u, erit

$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = \int \partial x \left[L \left(\frac{\partial \partial V}{\partial u^2}\right) + M \left(\frac{\partial V}{\partial u}\right) + N V\right]$$

totumque negotium huc redit, ut ista formula integrationem admittat.

Scholion.

1023. Datis scilicet ipsius u functionibus L, M, N, quaeri debet functio V binarum variabilium x et u, ita ut spectata u constante formula

$$\left[L \left(\frac{\partial \partial V}{\partial u^2}\right) + M \left(\frac{\partial V}{\partial u}\right) + N V\right] \partial x$$

absolute fiat integrabilis, cujus integrale, ut sit determinatum ita capiatur, ut posito $x = 0$, evanescat. Tum vero statuatur $x = a$, ac si illud integrale etiam hoc casu evanescat, erit

$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = 0,$$

hincque aequationi satisfacit valor $y = \int V \partial x$, lege indicata sumtus. Problema autem, datis functionibus L, M et N, investigandi functionem V maxime est indeterminatum, neque methodis adhuc cognitis in genere resolvi potest; ex quo conveniet id inverso modo tractari, ut sumta functione V, alterae L, M et N indagentur. Hinc aequationes differentio-differentiales consequemur, quarum integralia hoc modo assignare valebimus, quae si aliis methodis tractari nequeant, insigne lucrum suppeditant. Quodsi integrale illud

$$\int [L\left(\frac{\partial\partial V}{\partial u^2}\right) + M\left(\frac{\partial V}{\partial u}\right) + N V]\,\partial x$$

posito $x = a$ non evanescat, sed datam ipsius u functionem U exhibeat, valor $y = \int V\,\partial x$ conveniet huic aequationi

$$\frac{L\,\partial\partial y}{\partial u^2} + \frac{M\,\partial y}{\partial u} + N y = U,$$

quae cum infinitis modis in alias formas transmutari possit, etiam earum integralia innotescent, ubi simul hoc commode evenit, ut etiamsi integrale tantum particulare obtineatur, inde tamen plerumque integrale completum haud difficulter colligi queat.

Problema 130.

1024. Invenire aequationes differentio-differentiales formae $\frac{L\,\partial\partial y}{\partial u^2} + \frac{M\,\partial y}{\partial u} + N y = U$, ut L, M, N et U sint functiones ipsius u, cujus elementum ∂u hic pro constante accipitur, quarum integrale ope constructionis per quadraturas exhiberi possit.

Solutio.

Sumatur functio quaecunque binarum variabilium u et x, quae sit V, capiaturque integrale $\int V\,\partial x$ spectata quantitate u ut constante, ita ut posito $x = 0$, evanescat, tum vero fiat $x = a$, denotante a quantitatem quamcunque constantem, ut jam $\int V\,\partial x$ exprimat functionem quandam ipsius u tantum, cui quantitas y aequetur, ut sit $y = \int V\,\partial x$. Cum jam sit

$$\frac{\partial y}{\partial u} = \int \partial x\left(\frac{\partial V}{\partial u}\right), \text{ et } \frac{\partial\partial y}{\partial u^2} = \int \partial x\left(\frac{\partial\partial V}{\partial u^2}\right),$$

his integralibus pariter ita sumtis, ut posito $x = 0$ evanescant, tum vero statuatur $x = a$, quaerantur functiones L, M, N ipsius u, ut haec formula

$$\int \partial x\,[L\left(\frac{\partial\partial V}{\partial u^2}\right) + M\left(\frac{\partial V}{\partial u}\right) + N V]$$

fiat absolute integrabilis, ejusque integrale ita determinetur, ut posito $x = a$, fiat id $= U$. Quod si fuerit praestitum, evidens est, aequationi differentio-differentiali

$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = U$$

satisfacere formulam assumtam $y = \int V \partial x$.

Corollarium 1.

1025. Assumtio ergo functionis V non penitus arbitrio nostro permittitur, sed ad hoc potissimum est spectandum, ut talis forma

$$\int \partial x \left[L \left(\frac{\partial \partial V}{\partial u^2}\right) + M \left(\frac{\partial V}{\partial u}\right) + N V\right],$$

per se fiat integrabilis.

Corollarium 2.

1026. Infinitae ergo hinc statim excluduntur formae ad hunc scopum ineptae, cujusmodi sunt $V = UP$, existente U functione ipsius u et P ipsius x tantum; quia tum foret

$$y = U \int P \partial y, \quad \frac{\partial y}{\partial u} = \frac{\partial U}{\partial u} \int P \partial x \quad \text{et} \quad \frac{\partial \partial y}{\partial u^2} = \frac{\partial \partial U}{\partial u^2} \int P \partial x,$$

quippe quae idem integrale complectuntur, ita ut ex earum conjunctione formula absolute integrabilis confici nequeat.

Exemplum 1.

1027. *Sit* $V = x^n \sqrt{\frac{uu + xx}{cc - xx}}$ *et* $y = \int x^n \partial x \sqrt{\frac{uu + xx}{cc - xx}}$ *integrali evanescente posito* $x = 0$, *tum vero facto* $x = a$.

Erit ergo

$$\left(\frac{\partial V}{\partial u}\right) = x^n \frac{u}{\sqrt{(uu + xx)(cc - xx)}} \text{ et } \left(\frac{\partial \partial V}{\partial u^2}\right) = x^n \frac{xx}{(uu + xx)^{\frac{3}{2}} \sqrt{(cc - xx)}},$$

integrabilis reddi debet haec formula

$$x^n \partial x \left(\frac{L x x}{(uu+xx)^{\frac{3}{2}} \sqrt{(cc-xx)}} + \frac{M u}{\sqrt{(uu+xx)(cc-xx)}} + N \sqrt{\frac{uu+xx}{cc-xx}} \right),$$

seu

$$\frac{x^n \partial x}{(uu+xx)^{\frac{3}{2}} \sqrt{(cc-xx)}} [Lxx + Mu(uu+xx) + N(uu+xx)^2].$$

Statuatur integrale $= \frac{x^{n+1} \sqrt{(cc-xx)}}{\sqrt{(uu+xx)}}$, cujus differentiale cum sit

$$\frac{(n+1)x^n(cc-xx)(uu+xx) - x^{n+2}(uu+xx) - x^{n+2}(cc-xx)}{(uu+xx)^{\frac{3}{2}} \sqrt{(cc-xx)}} . \partial x,$$

seu

$$\frac{x^n \partial x}{(uu+xx)^{\frac{3}{2}} \sqrt{(cc-xx)}} \left\{ \begin{matrix} (n+1)ccuu + (n+1)ccxx - (n+1)uuxx - (n+1)x^4 \\ - ccxx \quad - uuxx \end{matrix} \right\},$$

cum qua si proposita comparetur, fiet

$$M u^3 + N u^4 = (n+1) c c u u,$$

$$L + M u + 2 N u u = n c c - (n+2) u u \text{ et}$$

$$N = -(n+1).$$

Hinc elicitur

$$M u = (n+1)(cc+uu), \quad \text{seu } M = \frac{(n+1)(cc+uu)}{u}, \text{ et}$$

$$L = -(n+1)(cc+uu) + 2(n+1)uu + ncc - (n+2)uu,$$

seu $L = -cc - uu$.

Quamobrem habebimus

$$\frac{(cc+uu)\partial\partial y}{\partial u^2} + \frac{(n+1)(cc+uu)\partial y}{u \partial u} - (n+1) y = \frac{a^{n+1} \sqrt{(cc-aa)}}{\sqrt{(aa+uu)}},$$

cui aequationi satisfacit $y = \int x^n \partial x \sqrt{\frac{uu + xx}{cc - xx}}$, integratione absoluta ut est indicatum.

Corollarium 1.

1028. Sumto ergo $a = c$, formula integralis

$$y = \int x^n \partial x \sqrt{\frac{uu + xx}{cc - xx}}$$

posito post integrationem $x = c$, exhibebit integrale hujus aequationis

$$u(cc + uu)\partial\partial y - (n+1)(cc + uu)\partial u \partial y + (n+1) u y \partial u^2 = 0,$$

seu

$$\partial\partial y - \frac{(n+1)\partial u \partial y}{u} + \frac{(n+1) y \partial u^2}{cc + uu} = 0.$$

Corollarium 2.

1029. Si sit $n = 1$, per integrationem invenitur

$$\int x \partial x \sqrt{\frac{uu + xx}{cc - xx}} = \tfrac{1}{4}(cc + uu) \text{ Ang. sin. } \frac{2xx - cc + uu}{cc + uu}$$
$$- \tfrac{1}{2}\sqrt{(ccuu + ccxx - uuxx - x^4)},$$
$$- \tfrac{1}{4}(cc + uu) \text{ Ang. sin. } \frac{-cc + uu}{cc + uu} + \tfrac{1}{2} cu,$$

et posito $x = c$ fit

$$y = \tfrac{1}{4}(cc + uu) \text{ Ang. cos. } \frac{uu - cc}{cc + uu} + \tfrac{1}{2} cu, \text{ hincque}$$
$$\frac{\partial y}{\partial u} = \tfrac{1}{2} u \text{ Ang. cos. } \frac{uu - cc}{cc + uu} \text{ et}$$
$$\frac{\partial\partial y}{\partial u^2} = \tfrac{1}{2} \text{ Ang. cos. } \frac{uu - cc}{cc + uu} - \frac{cu}{cc + uu},$$

quae formulae evidenter satisfaciunt aequationi

$$\partial\partial y - \frac{2\partial u \partial y}{u} + \frac{2 y \partial u^2}{cc + uu} = 0.$$

Corollarium 3.

1030. Hoc casu integrale etiam hoc modo exprimi potest

$$y = \tfrac{1}{4}(cc + uu) \text{ Ang. sin. } \frac{2cu}{cc + uu} + \tfrac{1}{2} cu,$$

seu cum ejus multiplum quodvis aeque satisfaciat

$$y = (cc + uu) \text{ Ang. sin. } \frac{2cu}{cc+uu} + 2cu,$$

satisfacit vero etiam $y = cc + uu$, unde integrale completum est

$$y = \alpha(cc + uu) \text{ Ang. sin. } \frac{2cu}{cc+uu} + 2\alpha cu + \beta(cc+uu).$$

Scholion.

1031. Quod valor $y = cc + uu$ satisfaciat, ex integrali invento concludere licet, quia enim Ang. sin. $\frac{2cu}{cc+uu}$ est functio multiplex et termino 2π augeri potest, integrale ipsium augeri potest termino $2\pi(cc + uu)$. At in genere differentia binorum integralium quoque satisfacit, ergo etiam satisfacere debet $y = 2\pi(cc+uu)$ et generatim $y = \beta(cc + uu)$. Ex hoc casu facilius perspicitur, quomodo valor assumtus aequationi generali satisfaciat, etiamsi is per integrationem evolvi nequeat. Patet autem $n + 1$ esse debere numerum positivum, quia alioquin conditio integralis, ut posito $x = 0$ evanescat, impleri nequit.

Exemplum 2.

1032. *Sumatur*

$$V = x^{n-1}(uu + xx)^{\mu}(cc - xx)^{\nu}, \text{ erit}$$

$$\left(\frac{\partial V}{\partial u}\right) = 2\mu u x^{n-1}(uu + xx)^{\mu-1}(cc - xx)^{\nu} \text{ et}$$

$$\left(\frac{\partial\partial V}{\partial u^2}\right) = 2\mu x^{n-1}(cc-xx)^{\nu}[(uu+xx)^{\mu-1} + 2(\mu-1)uu(uu+xx)^{\mu-2}]$$

$$\text{seu } = 2\mu x^{n-1}(cc - xx)^{\nu}(uu+xx)^{\mu-2}[(2\mu-1)uu + xx].$$

Integrabilis igitur reddi debet absolute haec formula

$$\int x^{n-1}\partial x(cc - xx)^{\nu}(uu + xx)^{\mu-2} \times$$

$$(2\mu[(2\mu-1)uu+xx]L + 2\mu u(uu+xx)M + (uu+xx)^2 N)$$

seu

$$\int x^{n-1}\partial x(cc-xx)^{\nu}(uu+xx)^{\mu-2}\left\{\begin{matrix}2\mu(2\mu-1)Luu\mp 2\mu Lxx & +Nx^4 \\ +2\mu Mu^3 & +2\mu Muxx \\ +Nu^4 & +2Nuuxx\end{matrix}\right\}$$

Statuatur integrale $x^n(uu+xx)^{\mu-1}(cc-xx)^{\nu+1}$, cujus differentiale cum sit

$$x^{n-1}\partial x(uu\mp xx)^{\mu-2}(cc-xx)^{\nu}\times$$
$$[n(uu+xx)(cc-xx)+2(\mu-1)xx(cc-xx)-2(\nu+1)xx(uu+xx)],$$

erit

$$2\mu(2\mu-1)Luu+2\mu Mu^3+Nu^4=nccuu,$$
$$2\mu L+2\mu Mu+2Nuu=ncc-nuu+2(\mu-1)cc-2(\nu+1)uu,$$
$$N=-n-2(\mu-1)-2(\nu+1)=-n-2\mu-2\nu.$$

At prima

$$2\mu(2\mu-1)L+2\mu Mu+Nuu=ncc$$

demta secunda dat

$$4\mu(\mu-1)L-Nuu=(n+2\nu+2)uu-2(\mu-1)cc, \text{ seu}$$
$$4\mu(\mu-1)L=-2(\mu-1)(uu+cc), \text{ hinc } L=\frac{-cc-uu}{2\mu};$$

qui valor in prima sustitutus dat

$$-(2\mu-1)(cc+uu)+2\mu Mu-(n+2\mu+2\nu)uu=ncc,$$

seu

$$2\mu Mu=(n+2\mu-1)cc+(n+4\mu+2\nu-1)uu.$$

Ergo

$$M=\frac{(n+2\mu-1)(cc+uu)}{2\mu u}+\frac{(\mu+\nu)}{\mu}u.$$

Si $n>0$, superius integrale evanescit posito $x=0$, quare si ponamus $x=a$, orietur haec aequatio

$$-\frac{(cc+uu)\partial\partial y}{2\mu\partial u^2}+\frac{(n+2\mu-1)(cc+uu)\partial y}{2\mu u\partial u}+\frac{(\mu+\nu)u\partial y}{\mu\partial u}$$
$$-(n+2\mu+2\nu)y=a^n(aa+uu)^{\mu-1}(cc-aa)^{\nu+1},$$

cujus integrale est

$$y=\int x^{n-1}\partial x(uu+xx)^{\mu}(cc-xx)^{\nu},$$

integrali hoc ita sumto ut evanescat posito $x=0$; tum vero posito $x=a$.

Corollarium 1.

1033. Si capiatur $a = c$, ut postrema pars fiat $= 0$, siquidem exponens $\nu + 1$ sit nihilo major, formula

$$y = \int x^{n-1} \partial x (uu + xx)^{\mu} (cc - xx)^{\nu}$$

posito $x = c$, post integrationem ita peractam ut casu $x = 0$ fiat $y = 0$, erit integrale hujus aequationis

$$u(cc + uu)\partial\partial y - (n + 2\mu - 1)(cc + uu)\partial u \partial y - 2(\mu + \nu) uu \partial u \partial y$$
$$+ 2\mu(n + 2\mu + 2\nu) u y \partial u^2 = 0.$$

Corollarium 2.

1034. Sit $n + 2\mu - 1 = \alpha$, et $n + 4\mu + 2\nu - 1 = \beta$, fiet $2\mu = \alpha + 1 - n$ et $2\nu = \beta + 1 - n - 2\alpha - 2 + 2n = \beta - 1 + n - 2\alpha$, et aequationis

$$u(cc + uu)\partial\partial y - (\alpha cc + \beta uu)\partial u \partial y + (\alpha + 1 - n)(\beta - \alpha + n) u y \partial u^2 = 0$$

integrale erit

$$y = \int x^{n-1} \partial x (uu + xx)^{\frac{\alpha + 1 - n}{2}} (cc - xx)^{\frac{\beta - 1 + n - 2\alpha}{2}},$$

posito $x = c$, si modo sit $n > 0$ et $\beta - 1 + n > 2\alpha$.

Scholion.

1035. Haec constructio latissime ad hanc aequationem patet

$$xx(a + bx^n)\partial\partial z + x(c + ex^n)\partial x \partial z + (f + gx^n) z \partial x^2 = 0,$$

primo enim hic sine detrimento amplitudinis sumi potest $n = 2$, ponendo $x^n = uu$. Tum vero uti supra §. 997. vidimus, ponendo

$$z = x^{\frac{a - c}{a} + h} (a + bx)^{\frac{bc - ae}{nab} + 1} y,$$

aequatio abit in hanc

$$xx(a + bx^n)\partial\partial y + x[2a - c + 2ah + (2b - e + 2nb + 2bh)x^n]\partial x \partial y$$
$$+ [f + ah - ch + ahh + (g + (b - e + nb + bh)(n + h))x^n] y \partial x^2 = 0,$$

ubi si h ita accipiatur, ut sit $ahh + (a - c)h + f = 0$, prodit aequatio formae, cujus constructionem dedimus. In casibus autem specialibus difficultates occurrere possunt, quibus superandis sequentia exempla inserviunt.

Exemplum 3.

1036. *Sit* $V = e^{mux} x^n (c - x)^\nu$, *erit*

$$\left(\frac{\partial V}{\partial u}\right) = m e^{mux} x^{n+1} (c - x)^\nu \text{ et } \left(\frac{\partial\partial V}{\partial u^2}\right) = mm e^{mux} x^{n+2} (c - x)^\nu.$$

Integrabilem ergo reddi oportet hanc formulam

$$e^{mux} x^n \partial x (c - x)^\nu (mmLxx + mMx + N),$$

cujus integrale ponatur $= e^{mux} x^{n+1} (c - x)^{\nu+1}$, cujus propterea differentiale illi formulae aequari debet: quod cum sit

$$e^{mux} x^n \partial x (c - x)^\nu [mux(c - x) + (n + 1)(c - x) - (\nu + 1)x],$$

erit

$$N = (n + 1)c, \quad mM = mcu - (n + \nu + 2), \quad mmL = -mu.$$

Statuatur nunc $x = a$, et formula

$$y = \int e^{mux} x^n \partial x (c - x)^\nu$$

erit integrale hujus aequationis

$$-\frac{u\partial\partial y}{m\partial u^2} + \frac{cu\partial y}{\partial u} - \frac{(n + \nu + 2)\partial y}{m\partial u} + (n + 1)cy = e^{mau} a^{n+1} (c - a)^{\nu+1}.$$

Hic poni potest $m = 1$, ac sumto $c = a$, aequationis

$$u\partial\partial y - au\partial u\partial y + (n + \nu + 2)\partial u\partial y - (n + 1)ay\partial u^2 = 0,$$

integrale est $y = \int e^{ux} x^n \partial x (a - x)^\nu$, posito post integrationem $x = a$, dum sit $\nu + 1 > 0$, et $n + 1 > 0$, ut integrale evanescens reddi possit posito $x = 0$.

Corollarium 1.

1037. Si hic ponatur $y = e^{\int z\partial u}$, erit

$$u\partial z + uzz\partial u - auz\partial u + (n + \nu + 2)z\partial u - (n + 1)a\partial u = 0,$$

cujus integrale est

$$z = \frac{\partial y}{y \partial u} = \frac{\int e^{ux} x^{n+1} \partial x (a-x)^{\nu}}{\int e^{ux} x^{n} \partial x (a-x)^{\nu}}.$$

Illa aequatio autem posito $z = \frac{1}{2} a + v$, transmutatur in hanc

$$u \partial v + uvv \partial u + (n+\nu+2) v \partial u - \tfrac{1}{4} aau \partial u - \tfrac{1}{2}(n-\nu) a \partial u = 0,$$

quae ponendo $v = u^{-n-\nu-2} s$ abit in hanc

$$u^{-n-\nu-1} \partial s + u^{-2n-2\nu-3} ss \partial u - \tfrac{1}{4} aau \partial u - \tfrac{1}{2}(n-\nu) a \partial u = 0.$$

Corollarium 2.

1038. Sit porro

$$u^{-n-\nu-2} \partial u = \partial t \text{ seu } u^{-n-\nu-1} = -(n+\nu+1) t,$$

ut fiat

$$\partial s + ss \partial t - \tfrac{1}{4} aa u^{2n+2\nu+4} \partial t - \tfrac{1}{2}(n-\nu) u^{2n+2\nu+1} \partial t = 0,$$

quae ergo aequatio etiam construi potest. Vel sit

$-(n+\nu+1) t = r$, erit

$$\partial s - \frac{ss \partial r}{n+\nu+1} + \frac{aar^{\frac{-2n-2\nu-4}{n+\nu+1}} \partial r}{4(n+\nu+1)} + \frac{(n-\nu) r^{\frac{-2n-2\nu-3}{n+\nu+1}} \partial r}{2(n+\nu+1)} = 0,$$

quae posito $s = -(n+\nu+1) q$ abit in

$$\partial q + qq \partial r - \frac{aar^{\frac{-2n-2\nu-4}{n+\nu+1}} \partial r - 2(n-\nu) r^{\frac{-2n-2\nu-3}{n+\nu+1}} \partial r}{4(n+\nu+1)^2} = 0,$$

hicque est

$$u = r^{\frac{-1}{n+\nu+1}} \text{ et } z = \tfrac{1}{2} a - (n+\nu+1) r^{\frac{n+\nu+2}{n+\nu+1}} q.$$

Scholion.

1039. Cum aequationis differentio-differentialis

$$\frac{\partial \partial y}{\partial u} - a \partial y + \frac{(n+\nu+2) \partial u}{u} - \frac{(n+1) ay \partial u}{u} = 0,$$

integrale sit $y = \int e^{ux} x^n \partial x (a-x)^\nu$, videamus, quomodo haec ipsa aequatio in alias formas transfundi possit. Sit primo $u = \alpha t^\lambda$, ideoque $\partial u = \alpha \lambda t^{\lambda-1} \partial t$, unde fit

$$\frac{1}{\alpha\lambda}\partial \cdot \frac{\partial y}{t^{\lambda-1}\partial t} - a\partial y + \frac{(n+\nu+2)\partial y}{\alpha t^\lambda} - \frac{\lambda(n+1) a y \partial t}{t} = 0;$$

sumatur jam elementum ∂t constans, eritque

$$\frac{\partial \partial y}{\alpha\lambda t^{\lambda-1}\partial t} - \frac{(\lambda-1)\partial y}{\alpha\lambda t^\lambda} - a\partial y + \frac{(n+\nu+2)\partial y}{\alpha t^\lambda} - \frac{\lambda(n+1)ay\partial t}{t} = 0$$

seu

$$\partial\partial y - \alpha\lambda a t^{\lambda-1}\partial t \partial y + \frac{(\lambda n + \lambda\nu + \lambda + 1)\partial t \partial y}{t} - \alpha\lambda\lambda(n+1) a t^{\lambda-2} y \partial t^2 = 0,$$

cujus integrale est

$$y = \int e^{\alpha t^\lambda x} x^n \partial x (a-x)^\nu.$$

Ponatur porro

$$\frac{\partial y}{y} = \mathrm{P}\partial t + \frac{\partial z}{z}, \text{ ut sit } z = e^{-\int \mathrm{P}\partial t} y, \text{ erit}$$

$$\partial\partial z + 2\mathrm{P}\partial t \partial z - \alpha\lambda a t^{\lambda-1}\partial t\partial z + (\lambda n + \lambda\nu + \lambda + 1)\frac{\partial t \partial z}{t} + z\partial t \partial \mathrm{P}$$
$$+ z\partial t^2 \left[\mathrm{PP} - \alpha\lambda a t^{\lambda-1}\mathrm{P} + \frac{(\lambda n + \lambda\nu + \lambda + 1)\mathrm{P}}{t} - \alpha\lambda\lambda(n+1) a t^{\lambda-2}\right] = 0.$$

Ad terminos elemento ∂z affectos tollendos statuatur

$$\mathrm{P} = \tfrac{1}{2}\alpha\lambda a t^{\lambda-1} - \frac{(\lambda n + \lambda\nu + \lambda + 1)}{2t},$$

fietque ac prodibit haec aequatio

$$\partial\partial z - z\partial t^2 \left[\frac{(\lambda n + \lambda\nu + \lambda)^2 - 1}{4tt} + \tfrac{1}{2}\alpha\lambda\lambda(n-\nu) a t^{\lambda-2} + \tfrac{1}{4}\alpha^2\lambda^2 a^2 t^{2\lambda-1}\right] = 0,$$

cujus propterea integrale est

$$z = e^{-\frac{1}{2}\alpha a t^\lambda} t^{\frac{\lambda n + \lambda\nu + \lambda + 1}{2}} \int e^{\alpha t^\lambda x} x^n \partial x (a-x)^\nu.$$

Quodsi ergo sit $\nu = n$, $\lambda\lambda(2n+1)^2 - 1 = 0$, seu

$$\lambda = \frac{\pm 1}{2n+1}, \text{ et } \alpha = \pm \tfrac{2}{\lambda} = \pm 2(2n+1),$$

habebitur haec aequatio

$$\partial\partial z - aaz\, t^{\frac{\pm 2}{2n+1} - 2}\, \partial t^2 = 0,$$

cujus integrale est

$$z = e^{\pm(2n+1)a t^{\frac{\pm 1}{2n+1}}}\, t^{\pm\frac{1}{2}+\frac{1}{2}} \int e^{\pm 2(2n+1) t^{\frac{\pm 1}{2n+1}} x}\, x^n \partial x (a-x)^n.$$

Vel hujus aequationis

$$\partial\partial z - aa t^{2\lambda - 2} z\, \partial t^2 = 0$$

integrale est

$$z = e^{-\frac{a}{\lambda} t^\lambda}\, t \int e^{\frac{2x}{\lambda} t^\lambda}\, x^{\pm\frac{1}{2\lambda} - \frac{1}{2}}\, \partial x\, (a - x)^{\pm\frac{1}{2\lambda} - \frac{1}{2}}:$$

unde occasionem arripimus hujusmodi integrationes generalius investigandi.

Exemplum 4.

1040. *Si sint P et Q functiones quaecunque ipsius u, et capiatur*

$$y = P \int e^{Qx} x^{n-1} \partial x (a-x)^{\nu-1},$$

posito scilicet post integrationem $x = a$, erit hic valor ipsius y integrale cujuspiam aequationis differentio-differentialis

$$\frac{L\,\partial\partial y}{\partial u^2} + \frac{M\,\partial y}{\partial u} + N y = 0$$

quae quaeritur.

Ad calculum contrahendum ponamus $\partial P = P'\partial u$ et $\partial P' = P''\partial u$, item $\partial Q = Q'\partial u$ et $\partial Q' = Q''\partial u$. Hinc erit

$$\frac{\partial y}{\partial u} = P' \int e^{Qx} x^{n-1} \partial x (a-x)^{\nu-1} + PQ' \int e^{Qx} x^n \partial x (a-x)^{\nu-1} \text{ et}$$

$$\frac{\partial\partial y}{\partial u^2} = P'' \int e^{Qx} x^{n-1} \partial x (a-x)^{\nu-1} + 2P'Q' \int e^{Qx} x^n \partial x (a-x)^{\nu-1}$$
$$+ PQ'' \int e^{Qx} x^n \partial x (a-x)^{\nu-1} + PQ'Q' \int e^{Qx} x^{n+1} \partial x (a-x)^{\nu-1}$$

unde colligitur

$$\frac{L\partial\partial y}{\partial u^2}+\frac{M\partial y}{\partial u}+Ny=$$
$$\int e^{Qx}x^{n-1}\partial x(a-x)^{\nu-1}\left\{\begin{array}{l}LP''+2LP'Q'x+LPQ''x+LPQ'Q'xx\\+MP'+MPQ'x+NP\end{array}\right\}$$

quod integrale statuatur $=e^{Qx}x^n(a-x)^\nu$, ita ut evanescat posito $x=a$, dum sit $\nu>0$, uti etiam evanescit casu $x=0$ si modo $\nu>0$. Cum igitur hujus formulae differentiale sit

$$e^{Qx}x^{n-1}\partial x(a-x)^{\nu-1}[Qx(a-x)+na-(n+\nu)x],$$

ejus comparatio cum forma inventa praebet

$LP''+MP'+NP=na$,
$2LP'Q'+LPQ'+MPQ'=aQ-(n+\nu)$, et
$LPQ'Q'=-Q$, ergo $L=\frac{-Q}{PQ'Q'}$, hinc
$M=\frac{aQ}{PQ'}-\frac{(n+\nu)}{PQ'}+\frac{2P'Q}{PPQ'Q'}+\frac{QQ''}{PQ'Q'Q'}$ et
$N=\frac{na}{P}+\frac{P''Q}{PPQ'Q'}-\frac{MP'}{P}$:

sicque aequatio differentio-differentialis erit cognita.

Corollarium 1.

1041. Si velimus ut sit $M=0$, erit

$$aQ-(n+\nu)+\frac{2P'Q}{PQ'}+\frac{QQ''}{Q'Q'}=0,$$

quae per $\frac{Q'\partial u}{Q}$ multiplicata abit in hanc

$$\frac{2\partial P}{P}+a\partial Q-\frac{(n+\nu)\partial Q}{Q}+\frac{\partial Q'}{Q'}=0,$$

cujus integrale est

$$\frac{e^{aQ}P^2Q'}{Q^{n+\nu}}=\text{Const. sive}$$

$$P=Ce^{-\frac{1}{2}aQ}Q^{\frac{n+\nu}{2}}\sqrt{\frac{\partial u}{\partial Q}}.$$

Corollarium 2.

1042. Sit $Q = 2\alpha u^\lambda$, erit $Q' = 2\alpha\lambda u^{\lambda-1}$, et

$$P = C e^{-\alpha a u^\lambda} u^{\frac{\lambda(n+\nu-1)+1}{2}}, \text{ hinc}$$

$$L = -\frac{1}{2\alpha\lambda\lambda} e^{\alpha a u^\lambda} u^{\frac{-\lambda(n+\nu+1)+3}{2}}, \text{ et}$$

$$N = \frac{na}{P} + \frac{Q\partial\partial P}{PP\partial Q^2}, \text{ at est}$$

$$\frac{Q}{\partial Q^2} = \frac{u^{-\lambda+2}}{2\alpha\lambda\lambda\partial u^2}, \text{ et ob}$$

$$\frac{\partial P}{P} = -\alpha\lambda a u^{\lambda-1}\partial u + \frac{\lambda(n+\nu-1)+1}{2}\cdot\frac{\partial u}{u}, \text{ erit}$$

$$\frac{\partial\partial P}{P} = -\alpha\lambda(\lambda-1)au^{\lambda-2}\partial u^2 - \frac{\lambda(n+\nu-1)-1}{2}\cdot\frac{\partial u^2}{uu} + \alpha\alpha\lambda\lambda aau^{2\lambda-2}\partial u^2$$
$$-\alpha\lambda a[\lambda(n+\nu-1)+1] + \frac{[\lambda(n+\nu-1)+1]^2}{4}$$

seu

$$\frac{\partial\partial P}{P} = \alpha\alpha\lambda\lambda aau^{2\lambda-2}\partial u^2 - \alpha\lambda\lambda(n+\nu)au^{\lambda-2}\partial u^2 + \frac{\lambda\lambda(n+\nu-1)^2-1}{4}\cdot\frac{\partial u^2}{uu},$$

hinc

$$na + \frac{Q}{\partial Q^2}\cdot\frac{\partial\partial P}{P} = \tfrac{1}{2}\alpha aau^\lambda + \tfrac{1}{2}(n-\nu)a + \frac{\lambda\lambda(n+\nu-1)^2-1}{8\alpha\lambda\lambda}u^{-\lambda}, \text{ et}$$

$$N = e^{\alpha a u^\lambda} u^{\frac{-\lambda(n+\nu-1)-1}{2}} [\tfrac{1}{2}\alpha aau^\lambda + \tfrac{1}{2}(n-\nu)a + \frac{\lambda\lambda(n+\nu-1)^2-1}{8\alpha\lambda\lambda}u^{-\lambda}].$$

Corollarium. 3.

1043. Hinc erit

$$\frac{N}{L} = -2\alpha\lambda\lambda u^{\lambda-2}[\tfrac{1}{2}\alpha aau^\lambda + \tfrac{1}{2}(n-\nu)a + \frac{\lambda\lambda(n+\nu-1)^2-1}{8\alpha\lambda\lambda}u^{-\lambda}],$$

et hujus aequationis

$$\frac{\partial\partial y}{\partial u^2} = y[\alpha\alpha\lambda\lambda aau^{2\lambda-2} + \alpha\lambda\lambda(n-\nu)au^{\lambda-2} + \frac{\lambda\lambda(n+\nu-1)^2-1}{4uu}]$$

integrale est

$$y = e^{-\alpha a u^\lambda} u^{\frac{\lambda(n+\nu-1)+1}{2}} \int e^{2\alpha u^\lambda x} x^{n-1}\partial x(a-x)^{\nu-1}.$$

Ponamus $\alpha = \frac{1}{\lambda}$,

$$\lambda(n-\nu) = f, \text{ seu } \nu = n - \frac{f}{\lambda}, \text{ et } \frac{\lambda\lambda(n+\nu-1)^2 - 1}{4} = g,$$

unde fit

$$n = \frac{f+\lambda+\sqrt{(1+4g)}}{2\lambda} \text{ et } \nu = \frac{-f+\lambda+\sqrt{(1+4g)}}{2\lambda},$$

et hujus aequationis

$$\partial\partial y = y\,\partial u^2 (a a u^{2\lambda-2} + a f u^{\lambda-2} + g u^{-2})$$

integrale est

$$y = e^{\frac{-a}{\lambda}u^\lambda} u^{\frac{\lambda(n+\nu-1)+1}{2}} \int e^{\frac{2x}{\lambda}u^\lambda} x^{n-1} \partial x (a-x)^{\nu-1}, \text{ seu}$$

$$y = e^{\frac{-a}{\lambda}u^\lambda} u^{\frac{1+\sqrt{(1+4g)}}{2}} \int e^{\frac{2x}{\lambda}u^\lambda} x^{\frac{f-\lambda+\sqrt{(1+4g)}}{2\lambda}} \partial x (a-x)^{\frac{-f-\lambda+\sqrt{(1+4g)}}{2\lambda}}$$

Corollarium 4.

1044. Si ponamus $\alpha = \frac{-1}{\lambda}$,

$$\lambda(n-\nu) = -f, \text{ et } \frac{\lambda\lambda(n+\nu-1)^2 - 1}{4} = g \text{ erit}$$

$$n = \frac{-f+\lambda+\sqrt{(1+4g)}}{2\lambda} \text{ et } \nu = \frac{f+\lambda+\sqrt{(1+4g)}}{2\lambda},$$

unde hujus aequationis, quae cum praecedente convenit,

$$\partial\partial y = y\,\partial u^2 (a a u^{2\lambda-2} + a f u^{\lambda-2} + g u^{-2})$$

integrale erit

$$y = e^{\frac{a}{\lambda}u^\lambda} u^{\frac{1+\sqrt{(1+4g)}}{2}} \int e^{\frac{-2x}{\lambda}u^\lambda} x^{\frac{-f-\lambda+\sqrt{(1+4g)}}{2\lambda}} \partial x (a-x)^{\frac{+f-\lambda+\sqrt{(1+4g)}}{2\lambda}}$$

ubi necesse est sit $n > 0$ et $\nu > 0$.

Exemplum 5.

1045. *Si ponamus* $y = \int \partial x\,(aa - xx)^{\nu-1} \cos. a u^\lambda x$, *posito post integrationem* $x = a$, *ut* y *aequetur certae functioni ipsius* u, *invenire aequationem differentio-differentialem, cui ea satisfaciat.*

Cum sit

$\frac{\partial y}{\partial u} = -\alpha\lambda u^{\lambda-1}\int x\,\partial x\,(aa-xx)^{\nu-1}\sin.\alpha u^{\lambda}x$, et

$\frac{\partial\partial y}{\partial u^2} = \int x\partial x(aa-xx)^{\nu-1}[-\alpha\lambda(\lambda-1)u^{\lambda-2}\sin.\alpha u^{\lambda}x - \alpha\alpha\lambda\lambda u^{2\lambda-2}x\cos.\alpha u^{\lambda}x]$.

Hinc erit $\frac{L\partial\partial y}{\partial u^2} + \frac{M\partial y}{\partial u} + Ny =$

$$\int(aa-xx)^{\nu-1}\partial x\left\{\begin{array}{l}N\cos.\alpha u^{\lambda}x - \alpha\lambda M u^{\lambda-1}x\sin.\alpha u^{\lambda}x - \alpha\lambda(\lambda-1)Lu^{\lambda-2}x\sin.\alpha u^{\lambda}x\\ -\alpha\alpha\lambda\lambda L u^{2\lambda-2}xx\cos\alpha u^{\lambda}x.\end{array}\right.$$

Fingatur integrale $= (aa-xx)^{\nu}\sin.\alpha u^{\lambda}x$, quod evanescit posito tam $x=0$ quam $x=a$, reperiturque comparatione instituta

$$L = \frac{u^{-\lambda+2}}{\alpha\lambda\lambda},\quad M = \frac{2\lambda\nu-\lambda+1}{\alpha\lambda\lambda}u^{-\lambda+1},\quad N = \alpha aau^{\lambda}.$$

Quare hujus aequationis

$$\frac{\partial\partial y}{\partial u^2} + (2\lambda\nu-\lambda+1)\frac{\partial y}{u\partial u} + \alpha\alpha\lambda\lambda aau^{2\lambda-2}y = 0$$

integrale est

$$y = \int\partial x\,(aa-xx)^{\nu-1}\cos.\alpha u^{\lambda}x.$$

Corollarium 1.

1046. Si ergo sit $\nu = \frac{\lambda-1}{2\lambda}$ et $\alpha = \frac{1}{\lambda}$, hujus aequationis

$$\frac{\partial\partial y}{\partial u^2} + aau^{2\lambda-2}y = 0$$

integrale est

$$y = \int\partial x\,(aa-xx)^{\frac{-\lambda-1}{2\lambda}}\cos.\tfrac{1}{\lambda}u^{\lambda}x,$$

si quidem post integrationem statuatur $x=a$, integrali ita sumto ut evanescat posito $x=0$.

Corollarium 2.

1047. Si igitur sit $\frac{-\lambda-1}{2\lambda} = i$ numero integro, seu $\lambda = \frac{-1}{2i+1}$, hujus aequationis

$$\partial\partial y + aau^{\frac{-4i-4}{2i+1}} y \partial u^2 = u$$

integrale est

$$y = \int \partial x (aa - xx)^i \cos. \tfrac{1}{\lambda} u^\lambda x$$

quod revera exhiberi potest. Prodeunt scilicet casus integrabiles supra indicati.

Scholion.

1048. Cum posuerimus $y = \int V \partial x$, existente V functione quacunque ipsarum u et x, quarum autem in hac integratione sola x ut variabilis tractatur, non opus est absolute integrale ita determinari, ut evanescat posito $x = 0$, sed sufficit ut certo quodam casu $x = b$ evanescat, quo facto si porro ponatur $x = a$, ut y aequetur functioni cuipiam ipsius u, quam per quadraturas assignare licet, quandoquidem hic integrationem formularum simplicium nobis concedi jure postulamus. Atque hic valor ipsius y per u expressus integrale exhibet cujusdam aequationis differentio-differentialis

$$L \partial\partial y + M \partial u \partial y + N y \partial u^2 = U \partial u^2,$$

ubi autem necesse est, ut haec formula

$$\int \partial x [L (\tfrac{\partial\partial V}{\partial u^2}) + M (\tfrac{\partial V}{\partial u}) + N V]$$

integrari actu possit, quod integrale itidem ita est capiendum, ut evanescat posito $x = b$, tum vero posito $x = a$, id fiat $= U$.

Problema 131.

1049. Si fuerint P et Q functiones ipsius x, at K functio ipsius u, ac ponatur

$$y = \int P \partial x (\lambda + Q)^n,$$

integrali ita sumto ut evanescat casu $x = b$, tum vero statuatur $x = a$, ut pro y prodeat functio ipsius u, invenire aequationem differentio - differentialem inter y et u, cui ille valor ipsius y satisfaciat.

Solutio.

Sit $\partial K = K' \partial u$ et $\partial K' = K'' \partial u$, et ob
$y = \int (K + Q)^n P \partial x$, erit $\frac{\partial y}{\partial u} = \int n K' (K + Q)^{n-1} P \partial x$,
ac denuo differentiando
$$\frac{\partial \partial y}{\partial u^2} = \int [n K'' (K + Q)^{n-1} + n(n-1) K' K' (K + Q)^{n-2}] P \partial x,$$
unde si L, M, N denotent functiones ipsius u, erit haec expressio
$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = \int P \partial x (K + Q)^{n-2} \times$$
$$[N (K + Q)^2 + n M K' (K + Q) + n L K'' (K + Q) + n(n-1) L K' K']$$
$$= \int P \partial x (K + Q)^{n-2} \left\{ \begin{matrix} N K K + n M K K' + n L K K'' + n(n-1) L K' K' \\ + 2 N K Q + n M K' Q + n L K'' Q + N Q Q \end{matrix} \right\},$$
quae cum debeat esse integrabilis, statuatur integrale
$$= R (K + Q)^{n-1} + \text{Const.}$$
ita ut evanescat posito ut ante $x = b$, ubi R sit functio ipsius x tantum. Cujus formae differentiale quia est
$$(K + Q)^{n-2} [K \partial R + Q \partial R + (n - 1) R \partial Q],$$
oportet sit
$$[N K K + n M K K' + n L K K'' + n(n - 1) L K' K'] P \partial x$$
$$+ (2 N K + n M K' + n L K'') P Q \partial x + N P Q Q \partial x$$
$$= K \partial R + Q \partial R + (n - 1) R \partial Q.$$
Hic ergo duplicis generis termini adesse debent, alii ab u plane liberi, alii vero functione K affecti, quos deinceps seorsim aequari conveniet. Hunc in finem ponamus
$$N K K + n M K K' + n L K K'' + n(n-1) L K' K' = A + \alpha K,$$
$$2 N K + n M K' + n L K'' = B + \beta K, \text{ et}$$
$$N = C + \gamma K.$$

Ex binis prioribus elidendo M colligitur

$$-NKK + n(n-1)LK'K' = A + \alpha K - BK - \beta KK,$$

unde ob $N = C + \gamma K$, concluditur

$$L = \frac{A + (\alpha - B)K - (\beta - C)KK + \gamma K^3}{n(n-1)K'K'},$$

hincque

$$M = \frac{B + \beta K - 2NK - nLK''}{nK'},$$

ita ut ex functione K litterae L, M et N, determinantur, dum A, α, B, β, C, γ constantes quascunque denotant. Nunc autem superest ut efficiatur

$$(A + \alpha K)P\partial x + (B + \beta K)PQ\partial x + (C + \gamma K)PQQ\partial x$$
$$= K\partial R + Q\partial R + (n-1)R\partial Q,$$

unde duplicis generis terminos seorsim aequando, fit

$$P\partial x(A + BQ + CQQ) = Q\partial R + (n-1)R\partial Q$$
$$P\partial x(\alpha + \beta Q + \gamma QQ) = \partial R,$$

ideoque

$$\frac{A + BQ + CQQ}{\alpha + \beta Q + \gamma QQ} = Q + \frac{(n-1)R\partial Q}{\partial R}, \text{ seu}$$

$$\frac{(n-1)R\partial Q}{\partial R} = \frac{A + (B-\alpha)Q + (C-\beta)QQ - \gamma Q^3}{\alpha + \beta Q + \gamma QQ}, \text{ ergo}$$

$$\frac{\partial R}{R} = \frac{(n-1)\partial Q(\alpha + \beta Q + \gamma QQ)}{A + (B-\alpha)Q + (C-\beta)QQ - \gamma Q^3},$$

unde ex functione Q functio R definitur: tum vero erit

$$P\partial x = \frac{(n-1)R\partial Q}{A + (B-\alpha)Q + (C-\beta)QQ - \gamma Q^3}.$$

Abeat jam integrale illud $R(K+Q)^{n-1}$ + Const. in functionem U, posito $x = a$, ac valor initio assumtus

$$y = \int \frac{(n-1)R\partial Q(K+Q)^n}{A + (B-\alpha)Q + (C-\beta)QQ - \gamma Q^3},$$

erit integrale hujus aequationis differentio-differentialis

$$L\partial\partial y + M\partial u\partial y + Ny\partial u^2 = U\partial u^2.$$

Corollarium 1.

1050. Cum pro Q functio quaecunque ipsius x accipi possit, nihil impedit, quo minus sumamus $Q = x$. Tum igitur quaeri oportet R ex hac aequatione

$$\frac{\partial R}{R} = \frac{(n-1)\partial x(\alpha + \beta x + \gamma x x)}{A + (B - \alpha)x + (C - \beta)x x - \gamma x^3},$$

eritque pro K functione quacunque ipsius u assumta

$$y = (n-1)\int \frac{R \partial x (K + x)^n}{A + (B - \alpha)x + (C - \beta)x x - \gamma x^3},$$

in quo, integrali ita sumto ut, posito $x = b$, evanescat, deinceps statui debet $x = a$.

Corollarium 2.

1051. Ex functione autem K aequatio differentio-differentialis ita formatur, ut sit

$$L = \frac{A - (B - \alpha)K + (C - \beta)KK + \gamma K^3}{n(n-1)\partial K^2}\partial u^2,$$

$$M = \frac{B - (2C - \beta)K - 2\gamma K^2}{n \partial K}\partial u - \frac{L \partial \partial K}{\partial u \partial K}, \text{ et } N = C + \gamma K.$$

Deinde in expressione $R(K + x)^{n-1} +$ Const. ita constituta, ut posito $x = b$ evanescat, ponatur $x = a$, et functio ipsius u inde resultans vocetur U, eritque aequatio differentio-differentialis

$$L \partial \partial y + M \partial u \partial y + N y \partial u^2 = U \partial u^2.$$

Corollarium 3.

1052. Si expressio $R(K + x)^{n-1} +$ Const. ita sit comparata, ut utroque casu $x = b$ et $x = a$ evanescat, seu potius hi termini integrationis ita constituantur, ut hoc eveniat, formula pro y assumta satisfaciet huic aequationi

$$L \partial \partial y + M \partial u \partial y + N y \partial u^2 = 0,$$

quae si deinceps in alias formas transmutetur, earum quoque integralia assignari poterunt.

Problema 132.

1053. Si fuerint P, Q functiones ipsius x, at K functio ipsius u, ac ponatur $y = \int e^{KQ} P \partial x$, integrali ita sumto ut evanescat casu $x = b$, tum vero ponatur $x = a$, et y aequabitur functioni ipsius u, quae satisfaciet cuipiam aequationi differentio-differentiali, quam invenire oportet.

Solutio.

Cum sit $y = \int e^{KQ} P \partial x$, erit

$$\frac{\partial y}{\partial u} = \int e^{KQ} K' P Q \partial x, \text{ et } \frac{\partial \partial y}{\partial u^2} = \int e^{NQ} P \partial x (K'' Q + K' K' Q Q),$$

unde fit

$$\frac{L \partial \partial y}{\partial u^2} + \frac{M \partial y}{\partial u} + N y = \int e^{KQ} P \partial x (N + M K' Q + L K'' Q + L K' K' Q Q),$$

cujus integrale statuatur $e^{KQ} R +$ Const. quae expressio evanescat posito $x = b$, fierique oportet

$$\partial R + K R \partial Q = P \partial x [N + (M K' + L K'') Q + L K' K' Q Q];$$

et ob rationes ante allegatas faciamus

$$L K' K' = A + \alpha K, \; M K' + L K'' = B + \beta K, \; N = C + \gamma K,$$

eritque

$$L = \frac{A + \alpha K}{K' K'} \text{ et } M = \frac{B + \beta K}{K'} - \frac{L K''}{K'},$$

atque obtinebimus has aequationes

$$\partial R = P \partial x (C + B Q + A Q Q),$$
$$R \partial Q = P \partial x (\gamma + \beta Q + \alpha Q Q),$$

unde colligitur

$$\frac{\partial R}{R} = \frac{\partial Q (C + B Q + A Q Q)}{\gamma + \beta Q + \alpha Q Q},$$

inventaque functione R, erit

$$P \partial x = \frac{R \partial Q}{\gamma + \beta Q + \alpha Q Q},$$

ita ut sit

$$y = \int e^{KQ} \frac{R \partial Q}{\gamma + \beta Q + \alpha QQ}.$$

Si jam expressio $e^{KQ}R$ + Const. posito $x = a$ abeat in functionem U, aequatio differentio-differentialis, cui hoc integrale convenit, erit

$$L \partial \partial y + M \partial u \partial y + N y \partial u^2 = U \partial u^2$$

Corollarium 1.

1054. Hic pro Q scribere licet x ut ante, unde fit

$$\frac{\partial R}{R} = \frac{\partial x (C + Bx + Axx)}{\gamma + \beta x + \alpha xx}, \text{ et } y = \int e^{Kx} \cdot \frac{R \partial x}{\gamma + \beta x + \alpha xx},$$

et U oritur ex forma $e^{Kx}R$ + Const. posito $x = a$. Valor autem ipsius R, pro ratione coëfficientium α, β, γ varias formas induere potest.

Corollarium 2.

1055. Pro K autem quaecunque functio ipsius u accipi potest, a cujus indole aequatio differentio-differentialis pendet. Erit autem

$$L = \frac{A + \alpha K}{\partial K^2} \partial u^2,$$

$$M = \frac{B + \beta K}{\partial K} \partial u - \frac{(A + \alpha K) \partial u \partial \partial K}{\partial K^3}, \text{ et } N = C + \gamma K,$$

unde aequatio differentio-differentialis est

$$\frac{(A + \alpha K) \partial \partial y}{\partial K^2} + \frac{(B + \beta K) \partial y}{\partial K} - \frac{(A + \alpha K) \partial \partial K \partial y}{\partial K^3} + (C + \gamma K) y = U.$$

Corollarium 3.

1056. Cum hic etiam u ex calculo excedat, perinde est cujusmodi functio ejus pro L assumatur, quin etiam sine detrimento amplitudinis poni potest $K = u$, dummodo ratio elementi, quod constans assumitur, habeatur.

Scholion 1.

1057. Si ergo sumatur $K = u$, atque elementum ∂u sumatur constans, ut fiat $\partial \partial K = u$, hinc ista aequatio construi potest

$$\frac{(A+\alpha u)\partial\partial y}{\partial u^2}+\frac{(B+\beta u)\partial y}{\partial u}+(C+\gamma u)\,y=U,$$

existente U ejusmodi functione ipsius u, quam descripsimus. Simili autem modo ex praecedente problemate construi potest haec aequatio,

$$[A-(B-\alpha)u+(C-\beta)uu+\gamma u^3]\frac{\partial\partial y}{\partial u^2}+(n+1)[B-(2C-\beta)u-2\gamma uu]\frac{\partial y}{\partial u}$$
$$+n(n-1)(C+\gamma u)\,y=U,$$

quae aeque late patere est censenda, ac si functionem quamcunque ipsius u loco K scripsissemus. Hinc enim loco u scribendo functionem quamcunque ipsius t, ac ∂t pro constante sumendo, omnes illae formae derivari possunt. Ex quo haec aequatio multo latius patet illa, quam supra in genere per series infinitas resolvimus. Plerumque autem hae aequationes ita sunt comparatae, ut earum integratio aliis methodis expediri haud possit, quocirca haec methodus omnino digna videtur, ad quam ulterius excolendam Geometrae omnes vires intendant.

Scholion 2.

1058. In investigatione hujusmodi constructionum ita sum versatus, ut primo quasi per conjecturam formulam quandam differentialem $\int V\partial x=y$, in qua V erat certa functio ipsarum u et x, ubi autem u ut constans spectabatur, assumseram, indeque dato ipsi x valore tributo pertigerim ad aequationem differentio-differentialem inter u et y, cui formula illa assumta satisfaceret. Hic autem observandum est illam formulam integralem non prorsus ab arbitrio nostro pendere, sed certa quadam indole praeditam esse debere, ut evolutione facta res perducatur ad aequationem differentialem secundi gradus. Quamdiu autem hanc electionem soli conjecturae permittimus, perpaucae hujusmodi formulae menti se offerunt, quae ad scopum propositum perducunt: multoque minus sperare licet, ut hoc modo unquam ad datam aequationem differentio-differentialem perveniamus, casuique potissimum tribuendae videntur constructiones, quas hic tradidimus. Cum igitur longissime adhuc mus remoti a solutione problematis, quo proposita quadam aequatione differentio-differentiali quaeritur formula illa ejus integrationem suppeditans, quod problema, an unquam solutionem sit nacturum, admodum incertum videtur; eo magis opera est adhibenda, ut saltem pro casibus particularibus investigationem formulae integrantis ex indole aequationis propositae derivare conemur, sicque quodam modo viam ad solutionem directam paremus. Ad hoc autem series infinitae, per quas hujusmodi aequationes supra resolvere docuimus, utiliter adhiberi possunt; unde in sequenti capite methodum exponam ex serie infinita solutionem cujuspiam aequationis differentio-differentialis continente, formulam illam integralem investigandi.