CAPUT I.

DE

INTEGRATIONE FORMULARUM DIFFERENTIALIUM SECUNDI GRADUS SIMPLICIUM.

Definitio.

706.

Positis binis variabilibus x et y, si vocetur $\partial y = p \partial x$ et $\partial p = q \partial x$, aequatio quaecunque, relationem inter quantitates x, y, p et q definiens, vocatur aequatio differentialis secundi gradus inter binas variabiles x et y.

Corollarium 1.

707. Quemadmodum ergo littera p implicat rationem differentialium primi gradus, dum est $p = \frac{\partial y}{\partial x}$, ita littera $q = \frac{\partial p}{\partial x}$ implicat rationem differentialium secundi gradus. Sumto enim ut vulgo fieri solet, elemento ∂x constante, erit $\partial p = \frac{\partial \partial y}{\partial x}$, ideoque $q = \frac{\partial \partial y}{\partial x^2}$.

Corollarium 2.

708. Quatenus ergo in aequatione proposita littera q inest, eatenus ea est differentialis secundi gradus. Si enim q abesset, ob solam p esset tantum differentialis primi gradus; ac si neque p neque q inesset, aequatio foret inter x et y, neque quicquam praeterea quaereretur.

Corollarium 3.

709. Methodus ergo desideratur, proposita aequatione quacunque praeter binas variabiles x et y, etiam quantitates $p = \frac{\partial y}{\partial x}$ et $q = \frac{\partial p}{\partial x}$, involvente, inveniendi relationem inter ipsas x et y, unde pateat, qualis y sit functio ipsius x, seu vicissim.

Scholion 1.

710. Hoc modo litteram q introducendo aequationes differentio-differentiales a conditione illa, qua quodpiam differentiale primi gradus pro constante assumi solet, liberantur. Cum enim ad meras quantitates finitas revocentur, quae rationem differentialium primi gradus exprimunt, consideratio differentialis constantis ne locum quidem habere potest. Quando ergo aequationes differentio-differentiales more solito ita exhibentur, ut quodpiam differentiale constans sit assumtum, introducendo litteras $p = \frac{\partial y}{\partial x}$ et $q = \frac{\partial p}{\partial x}$, species differentialium penitus tollitur, dum aequatio tantum quantitates finitas complectitur. Atque etiam vicissim proposita aequatione inter quantitates finitas x, y, p, q, ea ad formam vulgarem infinitis modis reduci potest, prout aliud atque aliud differentiale pro constante assumitur, quae tamen omnes formae specie diversae inter se perfecte conveniunt, quin etiam nullo differentiali constante assumto evolutio in formam solitam fieri potest.

Scholion 2.

711. Primum igitur breviter exponi conveniet, quomodo aequatio more solito per differentialia secundi gradus expressa ad formam nostram reduci queat, quodcunque differentiale constans fuerit assumtum. Sit ∂s hoc differentiale pro constante sumtum, cujus ergo ratio ad ∂x, ob $\frac{\partial y}{\partial x} = p$, per p et forte ipsas variabiles x et y datur; ponatur ergo $\partial s = v \partial x$, ut v fiat quantitas finita.

Jam cum in aequatione occurrant $\partial\partial x$ et $\partial\partial y$, vel alterutrum saltem, loco $\partial\partial x$ scribatur $\partial s\partial.\frac{\partial x}{\partial s}$, quia ob ∂s constans fit utique $\partial s.\partial.\frac{\partial x}{\partial s}=\partial\partial x$. Erit ergo $\partial\partial x = \partial s\partial.\frac{1}{v} = -\frac{\partial s\partial v}{vv}$. Simili modo loco $\partial\partial y$ scribendo $\partial s.\partial.\frac{\partial y}{\partial s}=\partial s\partial.\frac{p}{v}$, fiet $\partial\partial y = \frac{\partial s(v\partial p - p\partial v)}{vv}$. Cum igitur v per p, x et y detur, erit

$$\partial v = M\partial x + N\partial y + P\partial p = \partial x(M + Np + Pq),$$

ob $\partial p = q\partial x$, sicque fiet

$$\partial\partial x = -\frac{\partial x^2}{v}(M + Np + Pq) \text{ et}$$

$$\partial\partial y = \frac{\partial x^2}{v}(qv - Mp - N'p^2 - Ppq),$$

hique valores loco $\partial\partial x$ et $\partial\partial y$ substituti in aequatione tantum differentialia primi gradus relinquent, quibus omnibus ad ∂x reductis, aequatio per divisionem prorsus a differentialibus liberabitur. Deinde vicissim hujusmodi aequatio inter x, y, p et r proposita in formam solitam, sumto quopiam elemento ∂s constante, evolvetur, si primo pro p ubique scribatur $\frac{\partial y}{\partial x}$, loco q autem $\frac{1}{\partial x}\partial.\frac{\partial y}{\partial x}=\frac{\partial x\partial\partial y - \partial y\partial\partial x}{\partial x^3}$ ubi quidem nullius adhuc elementi constantis ratio est habita. At ob $\partial s = v\partial x$ constans, insuper erit

$$v\partial\partial x + \partial v\partial x = 0, \text{ seu ob } \partial v = M\partial x + N\partial y + P\partial.\frac{\partial y}{\partial x},$$

$$v\partial\partial x + M\partial x^2 + N\partial x\partial y + \frac{P(\partial x\partial\partial y - \partial y\partial\partial x)}{\partial x} = 0,$$

unde pro lubitu vel $\partial\partial x$ vel $\partial\partial y$ elidi potest, neutro autem eliso infinitae formae aequivalentes exhiberi possunt.

Scholion 3.

712. Hinc ergo praestantia formae finitae, ad quam hic aequationes differentio-differentiales revocamus, prae more solito eas exhibendi luculenter perspicitur; cum eadem aequatio more solito infinitis modis, prout aliud atque aliud elementum constans assumitur, repraesentari possit, dum nostro more eadem aequatio semper ad unicam formam reducitur. Quodsi ergo nostro more aequatio-

nes prodeant diversae, certum est iis quoque diversas relationes, inter variabiles x et y exprimi, cum contra solito more diversissimae aequationes differentio-differentiales eandem relationem indicare queant, ex quibus plerumque difficile est eam eligere, quae ad resolutionem maxime sit accommodata. Cum igitur hic ejusmodi methodus requiratur, cujus ope proposita quacunque aequatione inter quaternas quantitates x, y, p et q, relatio inter binas variabiles x et y definiri queat, quoniam haec quaestio vires humanas superare videtur, a casibus simplicissimis erit exordiendum. Casus autem simplicissimi sine dubio sunt, quando in aequatione proposita duae tantum insunt quantitates, scilicet vel x et q tantum, vel y et q, vel p et q, hoc est si q aequetur functioni vel ipsius x, vel ipsius y, vel ipsius p tantum; quos casus in hoc capite evolvere constituimus.

Definitio.

713. Formula differentio-differentialis simplex est, quando posito $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, quantitas q aequatur functioni vel ipsius x, vel ipsius y; vel ipsius p tantum.

Corollarium 1.

714. Triplices ergo habemus formulas differentio-differentiales simplices, quarum resolutionem in hoc capite doceri convenit, prout quantitas q vel per functionem ipsius p, vel ipsius x, vel ipsius y tantum determinatur.

Corollarium 2.

715. Si ergo X denotet functionem ipsius x, Y ipsius y, et P ipsius p tantum, terna genera harum formularum simplicium sunt 1) $q = X$, 2) $q = Y$, 3) $q = P$; in quibus continetur casus simplicissimus $q =$ Const.

Corollarium 3.

716. Si has formulas more solito exprimere velimus, ob $q = \frac{\partial p}{\partial x} = \frac{1}{\partial x} \partial . \frac{\partial y}{\partial x}$, sumto elemento ∂x constante, erit $q = \frac{\partial \partial y}{\partial x^2}$ sumto elemento ∂y constante, erit $q = -\frac{\partial y \partial \partial x}{\partial x^3}$; nullo autem sumto constante, erit $q = \frac{\partial x \partial \partial y - \partial y \partial \partial x}{\partial x^3}$, quibus simplicitas earum formularum haud mediocriter offuscatur.

Corollarium 4.

717. Si elementum $\sqrt{(\partial x^2 + \partial y^2)}$, quod saepe fit, constans accipiatur, erit $\partial x \partial \partial x + \partial y \partial \partial y = 0$; unde postremus valor ipsius q, vel ob $\partial \partial y = -\frac{\partial x \partial \partial x}{\partial y}$ abit in $q = -\frac{(\partial x^2 + \partial y^2) \partial \partial x}{\partial x^3 \partial y}$, vel ob $\partial \partial x = -\frac{\partial y \partial \partial y}{\partial x}$ abit in $q = \frac{(\partial x^2 + \partial y^2) \partial \partial y}{\partial x^4}$.

Scholion.

718. Repudiata ergo penitus vulgari ratione aequationes differentio-differentiales exprimendi, quippe qua formulae in se satis simplices vehementer complicatae evadere possent, ratione hic stabilita utamur, indeque resolutionem hujusmodi formularum simplicium doceamus.

Problema 92.

719. Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, si q aequetur functioni cuicunque ipsius p, invenire relationem inter ipsas variabiles x et y.

Solutio.

Sit ergo $q = P$, denotante P functionem quamcunque ipsius p; quoniam igitur est $q = \frac{\partial p}{\partial x}$, erit $\partial p = P \partial x$, hincque

$$\partial x = \frac{\partial p}{P}, \text{ et } \partial y = p \partial x = \frac{p \partial p}{P}.$$

Ex quo consequimur integrando

$$x = a + \int \frac{\partial p}{P}, \text{ et } y = b + \int \frac{p \partial p}{P};$$

ita ut tam x, quam y per eandem novam variabilem p determinetur. Atque cum duae novae constantes a et b per duplicem integrationem sint introductae, hoc integrale pro completo erit habendum.

Corollarium 1.

720. Aequatio $q = P$, cujus integrationem hic tradimus, si in formam consuetam, sumto ∂x constante, resolvatur, ob $q = \frac{\partial \partial y}{\partial x^2}$, transmutabitur in $\partial \partial y = \partial x^2 f : \frac{\partial y}{\partial x}$; quae est aequatio differentio-differentialis, in qua ipsae variabiles x et y non occurrunt.

Corollarium 2.

721. Talis quoque forma prodit, si elementum ∂y vel alia expressio differentialis, in quam ipsae x et y non ingrediuntur, veluti $\sqrt{(\partial x^2 + \partial y^2)}$ pro constante sumatur. Hoc ergo modo omnis aequatio differentio-differentialis in quam ipsae variabiles x et y non ingrediuntur, integrari poterit.

Corollarium 3.

722. Sin autem hujusmodi elementum $y \partial x - x \partial y$ constans assumatur, ut $y \partial \partial x - x \partial \partial y = 0$, ob

$$q = \frac{1}{\partial x} \partial \cdot \frac{\partial y}{\partial x} = \frac{\partial x \partial \partial y - \partial y \partial \partial x}{\partial x^3},$$

$$q = \frac{(y \partial x - x \partial y) \partial \partial x}{x \partial x^3} = \frac{(y \partial x - x \partial y) \partial \partial y}{y \partial x^3},$$

quae expressio, si aequetur functioni ipsius $p = \frac{\partial x}{\partial y}$, integrari poterit.

Corollarium 4.

723. Si fuerit P quantitas constans, ut sit $q = f$, erit

$$x = a + \frac{p}{f} \text{ et } y = b + \frac{pp}{2f},$$

unde fit

$$y = b + \frac{f}{2}(x - a)^2, \text{ seu } y = \tfrac{1}{2}fxx - afx + \tfrac{1}{2}aaf + b,$$

seu mutata forma constantium $y = \frac{1}{2}fxx + Cx + D$.

Scholion.

724. Cum scilicet aequatio differentio-differentialis duplici integratione indigeat, si utraque omni extensione instituatur, duae novae constantes arbitrariae introducuntur; in quo criterium, num hujusmodi integrale sit completum, consistit. Quemadmodum enim aequationum differentialium primi gradus integratio completa unam constantem arbitrariam implicat, ita si aequatio differentialis fuerit secundi gradus, binae constantes novae in integrale completum ingredientur, ternae autem ac plures, si aequatio differentialis fuerit tertii altiorisve gradus. Problemata autem, quorum resolutio ad hujusmodi aequationes differentiales altiorum graduum deducunt, natura sua ita sunt comparata, ut solutionis determinatio totidem constantes requirat. Ita in aequatione $q = f$, seu sumto ∂x constante, $\partial\partial y = f\partial x^2$, aequatio integralis completa $y = \frac{1}{2}fxx + Cx + D$ duas constantes novas C et D involvit, quod etiam in subjunctis exemplis patebit.

Exemplum 1.

725. *Aequationis differentio-differentialis* $a\partial\partial y = \partial x\partial y$, *in qua elementum* ∂x *constans est sumtum, integrale completum invenire.*

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, erit $\partial\partial y = q\partial x^2$, hincque $aq = p$, et $P = \frac{p}{a}$. Quocirca integratio praebet

$$x = \int \frac{a \partial p}{p} = C + a l p \text{ et } y = \int a \partial p = D + a p.$$

Cum igitur sit

$$p = \frac{y - D}{a}, \text{ erit } x = C + a l \frac{y - D}{a},$$

quae est aequatio integralis completa binas constantes C et D involvens.

Exemplum 2.

726. *Posito ∂x constante, invenire aequationem inter x et y, ut fiat $\frac{(\partial x^2 + \partial y^2) \sqrt{(\partial x^2 + \partial y^2)}}{-\partial x \partial \partial y} = a$.*

Posito $\partial y = p \partial x$, ob ∂x constans, erit $\partial \partial y = \partial p \partial x$, sicque nostra aequatio est $\frac{(1 + pp) \sqrt{(1 + pp)}}{-\partial p} \partial x = a$, unde fit

$$\partial x = \frac{-a \partial p}{(1 + pp)^{\frac{3}{2}}} \text{ et } \partial y = \frac{-a p \partial p}{(1 + pp)^{\frac{3}{2}}}.$$

Per integrationem ergo nanciscimur

$$x = A - \frac{a p}{\sqrt{(1 + pp)}}, \text{ et } y = B + \frac{a}{\sqrt{(1 + pp)}};$$

unde concludimus

$$(A - x)^2 + (y - B)^2 = a a.$$

Corollarium.

727. Si x et y denotent coordinatas rectangulas lineae curvae, formula $\frac{(\partial x^2 + \partial y^2) \sqrt{(\partial x^2 + \partial y^2)}}{-\partial x \partial \partial y}$ exprimit ejus radium osculi, qui ergo ut sit constans $= a$, aequatio integralis inventa circulum radio a describendum indicat.

Exemplum 3.

728. *Posito $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ eoque sumto constante, invenire aequationem inter x et y, ut fiat $\frac{\partial s \partial y}{\partial \partial x} = \frac{a \partial x}{\partial y}$.*

Ponatur $\partial y = p\,\partial x$, erit $\partial s = \partial x \sqrt{(1+pp)}$, et ob ∂s constans

$$\partial\partial x \sqrt{(1+pp)} + \frac{p\,\partial x\,\partial p}{\sqrt{(1+pp)}} = 0, \text{ seu}$$

$$\partial\partial x = \frac{-p\,\partial x\,\partial p}{1+pp},$$

unde aequatio proposita abit in

$$\frac{p\,\partial x^2 \sqrt{(1+pp)}}{-p\,\partial x\,\partial p}(1+pp) = \frac{a}{p}, \text{ seu}$$

$$\partial x = \frac{-a\,\partial p}{p(1+pp)^{\frac{3}{2}}}, \text{ et } \partial y = \frac{-a\,\partial p}{(1+pp)^{\frac{3}{2}}}, \text{ ergo}$$

$$y = D - \frac{ap}{\sqrt{(1+pp)}}.$$

At pro illa formula statuatur $p = \frac{1}{r}$, eritque

$$\partial x = \frac{arr\,\partial r}{(1+rr)^{\frac{3}{2}}} = \frac{a\,\partial r}{\sqrt{(1+rr)}} - \frac{a\,\partial r}{(1+rr)^{\frac{3}{2}}};$$

unde fit integrando

$$x = C - \frac{ar}{\sqrt{(1+rr)}} + a\,l\,[r + \sqrt{(1+rr)}], \text{ seu}$$

$$x = C - \frac{a}{\sqrt{(1+pp)}} + a\,l\,\frac{1+\sqrt{(1+pp)}}{p}.$$

Exemplum 4.

729. *Posito $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$, hocque elemento sumto constante, fieri oporteat $\frac{\partial s\,\partial y}{\partial\partial x} = a$ Ang. tang. $\frac{\partial y}{\partial x}$.*

Si fiat ut ante $\partial y = p\,\partial x$, orietur haec aequatio integranda

$$\frac{-\partial x(1+pp)^{\frac{3}{2}}}{\partial p} = a \text{ Ang. tang. } p, \text{ seu}$$

$$\partial x = \frac{-a\,\partial p}{(1+pp)^{\frac{3}{2}}} \text{ Ang. tang. } p, \text{ et}$$

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$$\partial y = \frac{-a p \partial p}{(1+pp)^{\frac{3}{2}}} \text{Ang. tang. } p.$$

Cum nunc sit $\partial . \text{Ang. tang. } p = \frac{\partial p}{1+pp}$, erit

$$x = \frac{-a p}{\sqrt{(1+pp)}} \text{Ang. tang. } p - a \int \frac{p \partial p}{(1+pp)^{\frac{3}{2}}}, \text{ et}$$

$$y = \frac{a}{\sqrt{(1+pp)}} \text{Ang. tang. } p - a \int \frac{\partial p}{(1+pp)^{\frac{3}{2}}}.$$

Quamobrem colligimus

$$x = C - \frac{ap}{\sqrt{(1+pp)}} - \frac{ap}{\sqrt{(1+pp)}} \text{Ang. tang. } p, \text{ et}$$

$$y = D - \frac{a}{\sqrt{(1+pp)}} + \frac{a}{\sqrt{(1+pp)}} \text{Ang. tang. } p.$$

Corollarium 1.

730. Si x sit abscissa et y applicata curvae, radius osculi proportionalis esse debet angulo, quem curvae tangens cum axe constituit; unde patet hanc curvam fore quandam spiralem, circa originem abscissarum se evolventem.

Corollarium 2.

731. Si angulus ille, cujus tangens $= p$, ponatur $= \Phi$, erit $p = \text{tang. } \Phi$, hincque

$$x = C - a \cos. \Phi - a \Phi \sin. \Phi, \text{ et}$$

$$y = D - a \sin. \Phi + a \Phi \cos. \Phi;$$

unde colligitur

$$x \cos. \Phi + y \sin. \Phi = C \cos. \Phi + D \sin. \Phi - a.$$

Corollarium 3.

732. Ut sumto $\Phi = 0$, ambae x et y evanescant, sumi debet $C = a$ et $D = 0$, eritque

$$x = a - a \cos.\Phi - a\Phi \sin.\Phi, \text{ et}$$

$$y = - a \sin.\Phi + a\Phi \cos.\Phi;$$

unde quamdiu angulus Φ est minimus, erit

$$x = -\tfrac{1}{2} a\Phi\Phi + \tfrac{1}{8} a\Phi^4 \text{ et } y = -\tfrac{1}{3} a\Phi^3 + \tfrac{1}{30}\Phi^5;$$

ideoque proxime

$$\frac{x^3}{yy} = -\tfrac{9}{8} a, \text{ seu } yy = -\frac{8x^3}{9a}.$$

Problema 93.

733. Posito $\partial y = p\partial x$ et $\partial p = q\partial x$, si quantitas q aequetur functioni ipsius x, quae sit X, definire relationem inter binas variabiles x et y.

Solutio.

Cum ergo sit $q = X$, erit $q\partial x = \partial p = X\partial x$, unde integrando colligimus $p = \int X\partial x + C$, atque hinc ob $\partial y = p\partial x$ adipiscemur

$$y = \int\partial x \int X\partial x + Cx + D.$$

At est

$$\int\partial x \int X\partial x = x\int X\partial x - \int Xx\partial x,$$

uti sumendis differentialibus sponte patet. Quare aequatio integralis completa relationem inter binas variabiles x et y continens est

$$y = x\int X\partial x - \int Xx\partial x + Cx + D$$

duas constantes arbitrarias C et D involvens. Quae ergo erit algebraica, si ambae formulae differentiales $X\partial x$ et $Xx\partial x$ integrationem admittant.

Corollarium 1.

734. Quodsi ergo sit $q = 0$, seu sumto ∂x constante, $\partial\partial y = 0$, ut sit $X = 0$, erit aequatio integralis completa $y = Cx + D$.

Corollarium 2.

735. Aequationes ergo differentio-differentiales, quas hoc modo integrare licet, sumto ∂x constante, in hac forma $\partial\partial y = X\partial x^2$ continentur, unde prima integratio praebet $\partial y = \partial x \int X\partial x + C$, et altera $y = \int \partial x \int X \partial x + Cx + D$.

Corollarium 3.

736. Sin autem differentiale ∂y capiatur constans ob $p = \frac{\partial y}{\partial x}$, erit $\partial p = -\frac{\partial y \partial\partial x}{\partial x^2} = q\partial x$, et forma aequationum hoc modo integrandarum erit $-\partial y \partial\partial x = X\partial x^3$.

Corollarium 4.

737. Quodsi elementum $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ sit constans, ob $\partial x \partial\partial x + \partial y \partial\partial y = 0$, erit

$$\partial p = \frac{\partial x \partial\partial y - \partial y \partial\partial x}{\partial x^2} = \frac{-\partial s^2 \partial\partial x}{\partial x^2 \partial y} = q\partial x.$$

Hinc forma aequationum hoc modo integrandarum est

$$-\partial s^2 \partial\partial x = X\partial x^3 \partial y.$$

Vel cum etiam sit

$$\partial p = q\partial x = +\frac{\partial s^2 \partial\partial y}{\partial x^3},$$

ea erit $\partial s^2 \partial\partial y = X\partial x^4$.

Scholion.

738. Hic manifestum est, quantum intersit aequationes differentio-differentiales a forma solita, ubi elementum quodpiam con-

stans est assumtum, ab hac conditione liberare et ad formam hic stabilitam reducere. Si enim proponatur haec aequatio $\partial s^2 \partial\partial y = X\partial x^4$, in qua elementum $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ constans sit assumtum, haud facile patet, quomodo ejus integratio sit suscipienda. Nostra autem methodo, si ponamus $\partial y = p\partial x$, ut sit

$$\partial s = \partial x \sqrt{(1 + pp)} \text{ et } \partial\partial y = p\partial\partial x + \partial x \partial p,$$

induit ista aequatio hanc formam

$$\partial x^2 (1 + pp)(p\partial\partial x + \partial x \partial p) = X \partial x^4 \text{ seu}$$

$$(p\partial\partial x + \partial x \partial p)(1 + pp) = X\partial x^2.$$

At quia ∂s, ac proinde quoque $\partial s^2 = \partial x^2 (1 + pp)$ est constans, erit

$$\partial\partial x (1 + pp) + p\partial x \partial p = 0, \text{ seu } \partial\partial x = \frac{-p\partial x\partial p}{1 + pp},$$

ideoque

$$p\partial\partial x + \partial x \partial p = \frac{\partial x \partial p}{1 + pp},$$

ita ut fiat $\partial p = X\partial x$, quae aequatio jam facillime tractatur. Hic scilicet in subsidium vocari debent ea, quae supra de integratione formularum differentialium simplicium sunt tradita.

Exemplum 1.

739. *Sumto ∂x constante, si fuerit $\partial\partial y = \alpha x^n \partial x^2$, integrale completum investigare.*

Cum sit $\frac{\partial\partial y}{\partial x} = \alpha x^n \partial x$, ob ∂x constans, erit integrando $\frac{\partial y}{\partial x} = \frac{\alpha}{n+1} x^{n+1} + C$, hincque denuo integrando

$$y = \frac{\alpha}{(n+1)(n+2)} x^{n+2} + Cx + D:$$

ubi casus $n = -1$ et $n = -2$ seorsim sunt evolvendi.

I. Ergo si $n = -1$, erit $\frac{\partial\partial x}{\partial x} = \frac{\alpha\partial x}{x}$, hincque $\frac{\partial y}{\partial x} = \alpha lx + C$, unde cum sit $\partial y = \alpha\partial x lx + C\partial x$, erit denuo integrando $y = \alpha x lx -$

$\alpha x + C x + D$, seu loco $C - \alpha$ scribendo C, habebitur $y = \alpha x l x + C x + D$.

II. Si $n = -2$ et $\frac{\partial \partial y}{\partial x} = \frac{\alpha \partial x}{x^2}$, erit $\frac{\partial y}{\partial x} = \frac{-\alpha}{x} + C$, hincque $y = -\alpha l x + C x + D$.

Exemplum 2.

740. *Posito $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ constante, si fuerit*

$$\frac{\partial s^2 \partial \partial y}{\partial x^4} = \frac{1}{a} \cos. \frac{x}{c},$$

invenire integrale completum.

Ex superioribus constat fore $\frac{\partial s^2 \partial \partial y}{\partial x^4} = q$, ita ut proposita sit haec aequatio $q = \frac{1}{a} \cos. \frac{x}{c}$, unde fit

$$q \partial x = \partial p = \frac{\partial x}{a} \cos. \frac{x}{c}$$

et integrando

$$p = \frac{c}{a} \sin. \frac{x}{c} + C = \frac{\partial y}{\partial x}.$$

Quare obtinebitur

$$y = -\frac{cc}{a} \cos. \frac{x}{c} + C x + D,$$

quae est aequatio integralis completa.

Problema 94.

741. Posito $\partial y = p \partial x$ et $\partial p = q \partial x$, si quantitas q aequetur functioni cuicunque ipsius y tantum, quae sit Y, invenire aequationem integralem completam inter x et y.

Solutio.

Cum sit $q = Y = \frac{\partial p}{\partial x}$, erit $\partial x = \frac{\partial p}{Y}$, hincque $p \partial x = \partial y = \frac{p \partial p}{Y}$; unde conficitur haec aequatio inter p et y separata $p \partial p = Y \partial y$, quae integrata praebet

$$\tfrac{1}{2}pp = \int Y\partial y + \tfrac{1}{2}C \text{ et } p = \sqrt{(C + 2\int Y\partial y)} = \frac{\partial y}{\partial x}.$$

Hinc ergo porro concluditur $x = \int \frac{\partial y}{\sqrt{(C + 2\int Y\partial y)}}$, quae integratio denuo constantem arbitrariam inducit, ita ut hoc modo aequatio integralis completa inter x et y obtineatur.

Corollarium 1.

742. Cum sit $q = \frac{\partial p}{\partial x}$ et $\partial x = \frac{\partial y}{p}$, erit $q = \frac{p\partial p}{\partial y}$. Quare cum aequatio proposita sit $q = Y$, erit $\frac{p\partial p}{\partial y} = Y$, hincque $p\partial p = Y\partial y$, unde praecedens integratio sponte deducitur.

Corollarium 2.

743. Sumto elemento ∂x constante, cum sit $q = \frac{\partial\partial y}{\partial x^2}$, aequationes hic integratae habebunt hanc formam $\partial\partial y = Y\partial x^2$, cujus integratio, si per ∂y multiplicetur, est manifesta, fit enim

$$\tfrac{1}{2}\partial y^2 = \partial x^2 \int Y\partial y + \tfrac{1}{2}C\partial x^2,$$

ob ∂x constans, hincque $\partial x = \frac{\partial y}{\sqrt{(C + 2\int Y\partial y)}}$, ut ante.

Scholion.

744. En ergo specimen aequationum differentialium, quae per idoneum multiplicatorem integrabiles redduntur, ex quo intelligitur hanc methodum etiam in his aequationibus usum habere posse; deinceps autem locus erit hanc methodum uberius excolendi, cujus quippe usus praecipue in aequationibus differentialibus altiorum graduum est insignis, ubi variabilium separatio nihil subsidii affert. Atque hanc ob causam jam supra hanc methodum per multiplicatores integrandi commendavimus, alterique per separationem procedenti longe antetulimus.

Exemplum 1.

645. *Posito ∂x constante, si fuerit $aa\partial\partial y = y\partial x^2$ invenire integrale completum.*

Multiplicetur aequatio proposita per $2\partial y$, ut prodeat

$$2aa\partial y\partial\partial y = 2y\partial y\partial x^2,$$

quae ob ∂x constans habebit integrale

$$aa\partial y^2 = yy\partial x^2 + C\partial x^2,$$

unde colligitur

$$\partial x = \frac{a\partial y}{\sqrt{(yy+C)}},$$

quae denuo integrata dat

$$x = al[y + \sqrt{(yy+C)}] - alb,$$

unde concludimus, sumto e pro numero cujus logarithmus est $=1$, fore

$$be^{\frac{x}{a}} = y + \sqrt{(yy+C)},$$

et irrationalitatem tollendo

$$bbe^{\frac{2x}{a}} - 2bye^{\frac{x}{a}} = C,$$

ita ut sit

$$y = \tfrac{1}{2}be^{\frac{x}{a}} - \frac{C}{2b}e^{-\frac{x}{a}}$$

at forma constantium C et b mutata, habebitur

$$y = Ce^{\frac{x}{a}} + De^{-\frac{x}{a}},$$

quae est aequatio integralis completa.

Exemplum 2.

746. *Posito ∂x constante, si fuerit $aa\partial\partial y + y\partial x^2 = 0$, invenire integrale completum.*

Multiplicatione per $2\partial y$ facta, aequationis

$$2aa\partial y\partial\partial y + 2y\partial y\partial x^2 = 0,$$

integrale est

$$aa\partial y^2 + yy\,\partial x^2 = cc\,\partial x^2,$$

unde deducimus

$$\partial x = \frac{a\,\partial y}{\sqrt{(cc-yy)}},$$

quae denuo integrata dat

$$x = a \text{ Ang. sin. } \frac{y}{c} + b.$$

Erit ergo

$$\frac{y}{c} = \sin.\frac{x-b}{a} = \cos.\frac{b}{a}\sin.\frac{x}{a} - \sin.\frac{b}{a}\cos.\frac{x}{a},$$

vel mutatis constantibus b et c, ita ut sit

$$c\cos.\frac{b}{a} = \mathrm{C} \text{ et } -c\sin.\frac{b}{a} = \mathrm{D}, \text{ erit}$$

$$y = \mathrm{C}\sin.\frac{x}{a} + \mathrm{D}\cos.\frac{x}{a}.$$

Vel retenta prima forma, habemus.

$$y = \mathrm{C}\sin.\left(\frac{x}{a} + \alpha\right).$$

Corollarium.

747. Hoc exemplum ex praecedente resolvi potuisset, cum sit

$$e^{u\sqrt{-1}} = \cos.u + \sqrt{-1}.\sin.u \text{ et } e^{-u\sqrt{-1}} = \cos.u - \sqrt{-1}.\sin.u,$$

ac vicissim

$$\cos.u\sqrt{-1} = \tfrac{1}{2}e^u + \tfrac{1}{2}e^{-u} \text{ et } \sin.u\sqrt{-1} = \frac{1}{2\sqrt{-1}}e^u - \frac{1}{2\sqrt{-1}}e^{-u}.$$

Exemplum 3.

748. *Posito ∂x constante, si fuerit $\partial\partial y\sqrt{ay} = \partial x^2$, integrale completum invenire.*

Cum ergo sit $2\partial y\partial\partial y = \frac{2\partial y}{\sqrt{ay}}.\partial x^2$,
erit integrando

$$\partial y^2 = \frac{4\,\partial x^2\sqrt{y}}{\sqrt{a}} + 4n\,\partial x^2 = \frac{4\,\partial x^2(\sqrt{y}+n\sqrt{a})}{\sqrt{a}},$$

unde colligimus

$$2\,\partial x = \frac{\partial y \sqrt[4]{a}}{\sqrt{(\sqrt{y} + n\sqrt{a})}}.$$

Sit commoditatis gratia $n\sqrt{a} = b$ et $\sqrt{y} = z$, ut fiat

$$\partial y = 2\,z\,\partial z \text{ et } \frac{\partial x \sqrt{n}}{\sqrt{b}} = \frac{z\,\partial z}{\sqrt{(b+z)}},$$

cujus integrale est

$$\frac{x\sqrt{n}}{\sqrt{b}} = \tfrac{2}{3}(z - 2\,b)\sqrt{(b+z)} + \mathrm{C},$$

seu restituendo

$$\frac{x}{\sqrt[4]{a}} = \tfrac{2}{3}(\sqrt{y} - 2\sqrt{c})\sqrt{(\sqrt{y} + \sqrt{c})} + \mathrm{C},$$

ubi c et C sunt binae constantes arbitrariae. Erit ergo

$$\frac{3(x+f)}{2\sqrt[4]{a}} = (\sqrt{y} - 2\sqrt{c})\sqrt{(\sqrt{y} + \sqrt{c})},$$

posito $\mathrm{C} = \frac{-f}{\sqrt[4]{a}}$, et sumtis quadratis

$$\frac{9(x+f)^2}{4\sqrt{a}} = y\sqrt{y} - 3\,y\sqrt{c} + 4\,c\sqrt{c}.$$

Scholion.

749. Forma ergo vulgaris aequationum hoc modo integrandarum, sumto elemento ∂x constante, est $\partial\partial y = \mathrm{Y}\partial x^2$, quae per ∂y multiplicata manifesto fit integrabilis. Sin autem elementum ∂y capiatur constans, ob $q = \frac{\partial p}{\partial x}$ et $p = \frac{\partial y}{\partial x}$, erit $q = -\frac{\partial y \partial\partial x}{\partial x^3}$, hincque forma vulgaris $\partial y \partial\partial x = -\mathrm{Y}\partial x^3$. Porro sumto elemento $\partial s = \sqrt{(\partial x^2 + \partial y^2)}$ constante, ut fit $\partial x \partial\partial x + \partial y \partial\partial y = 0$, ob $\partial p = \frac{\partial x \partial\partial y - \partial y \partial\partial x}{\partial x^2}$, erit vel $q = -\frac{\partial s^2 \partial\partial x}{\partial x^3 \partial y}$ vel $q = \frac{\partial s^2 \partial\partial y}{\partial x^4}$, unde nascitur haec forma

$$-\frac{\partial s^2 \partial\partial x}{\partial x^3 \partial y} = \mathrm{Y} \text{ vel } \frac{\partial s^2 \partial\partial y}{\partial x^4} = \mathrm{Y},$$

quae etiam per ∂y multiplicatae integrabiles evadunt, etiamsi hoc jam minus pateat. Simili modo si elementum $y\,\partial x$ sumatur constans, ut sit $y\,\partial\partial x + \partial x\,\partial y = 0$ et $\partial\partial x = -\frac{\partial x \partial y}{y}$ ob $\partial p =$

$\frac{\partial \partial y}{\partial x} + \frac{\partial y^2}{y \partial x}$, orietur haec forma $y \partial \partial y + \partial y^2 = Y y \partial x^2$, cujus membrum prius integrabile redditur, si per functionem quamcunque ipsarum $y \partial y$ et $y \partial x$ multiplicetur, ergo etiam per $\frac{y \partial y}{yy \partial x^2}$, quo multiplicatore simul alterum membrum $Y y \partial x^2$ redditur integrabile. His igitur casibus simplicissimis aequationum differentio-differentialium expeditis, qui ne ulla quidem difficultate laborant, ad difficiliores progrediamur, ac primo quidem ad eas aequationes, in quibus altera binarum variabilium x et y ipsa non inest; ita ut aequatio proposita ternas tantum contineat litteras x, p et q vel y, p et q, utriusque enim ratio fere perinde est comparata.