EVOLVTIO GENERALIOR FORMVLARVM COMPARATIONI CVRVARVM INSERVIENTIVM.

Auctore

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1.

Quae de comparatione arcuum circularium ex elementis sunt cognita, et quae Illustr. Comes Fagnanus de simili comparatione arcuum curvae Lemniscatae mira sagacitate elicuit, ea, vti iam aliquoties ostendi, ita generalius enunciari possunt, vt si cuiuspiam lineae curuae arcus indefinite per hanc formulam integralem exprimatur: $\int \frac{\mathfrak{A}dz}{\sqrt{(A+Czz+Ez^4)}}$, tum in ea curua, sumto arcu quocunque, ab alio quouis puncto arcum geometrice abscindi posse illi arcui aequalem. Atque hinc etiam proposito arcu quocunque ab alio quouis puncto arcus abscindi poterit, qui illius arcus sit duplus seu triplus, seu qui in genere ad eum rationem quamcunque rationalem teneat. Vnde consequitur omnium curuarum, quarum quidem rectificatio ista formula contineatur, arcus perinde atque arcus circulares inter se comparari posse.

2. Deinde quae de comparatione arcuum parabolicorum iam pridem sunt inuenta, et quae simili modo Ill. Comes Fagnanus circa arcus ellipticos et hyperbolicos summo acumine praestitit, ea deinceps tam late patere demonstraui, vt pari successu ad omnes curuas, quarum arcus indefinite per hanc formulam integralem: $\int \frac{dz(\mathfrak{A}+\mathfrak{B}zz+\mathfrak{C}z^4+\mathfrak{D}z^6+\text{etc.})}{\sqrt{(A+Czz+Dz^4)}}$ exprimatur, extendi queant. Sumto scilicet in tali curua arcu quocunque, ab alio quouis puncto arcus abscindi poterit, qui ab illo arcu differat quantitate geometrice assignabili. Tum vero etiam abscindi poterunt eiusmodi arcus, qui ab arcus propositi duplo, triplo, vel quouis multiplo differant quantitate geometrice assignabili. Quin etiam illud punctum, vnde arcus abscindi oportet, ita capi poterit, vt haec differentia plane in nihilum abeat.

3. Quaecunque ergo circa arcus parabolicos iam olim sunt praestita, eadem quoque in omnibus curuis, quarum rectificatio ad istam formulam integralem est reductibilis, pari successu expediri poterunt. Cum autem Comes Fagnanus ad has mirabiles comparationes per substitutiones admodum molestas, et quarum ratio inuentionis ne quidem perspiciatur, peruenerit; ego methodum planam aperui, quae quasi sponte ad easdem comparationes manuducat. Atque ista methodus etiam multo vberius hoc negotium conficit, quod generalissime omnes comparationes in se complectitur; aequiualet enim integrationi completae, quae simul constantem arbitrariam inuoluit, dum illae substitutiones tantum

tum integrationes particulares referre ſunt cenſendae, quam ob cauſam mihi quidem huius methodi beneficio multo longius progredi licuit, vti ex aliquot ſpeciminibus, quae iam dedi, luculenter apparet.

4. Quemadmodum autem in his formulis, quas pertractaui, iſta expreſſio ſurda $\sqrt{(A+Czz+Ez^4)}$ implicatur, quae quidem iam caſus ſolutu difficillimos complectitur, ita eadem ad expreſſionem ſurdam magis complicatam hanc: $\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}$ extendi poſſe obſeruaui; qua multo amplior campus aperitur ſimiles comparationes in pluribus aliis lineis curuis inſtituendi. Neque vero haec inueſtigatio tantum in lineis curuis tam eximium praeſtat vſum, ſed etiam in Analyſi et calculo integrali grauiſſima incrementa largiri videtur, ad quae plenius excolenda vt viam ſternam, euolutiones ad hanc formulam generaliorem pertinentes diligentius expoπam. Hunc in finem propoſita ſit ſequens aequatio relationem inter binas variabiles x et y exprimens.

Aequatio Canonica expendenda

$$0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy+2\varepsilon xy(x+y)+\zeta xxyy.$$

5. Haec aequatio praeter binas variabiles x et y continet ſex quantitates conſtantes, quae autem cum tantum earum ratio ſpectetur, ad quinque reducuntur, ita vt quinque determinationes ab arbitrio

trio

trio noſtro pendentes recipere ſit cenſenda. Deinde etſi haec aequatio ratione variabilium ad quatuor dimenſiones exſurgit, tamen vtraque ſeorſim nusquam vltra duas aſcendit, ita vt vtriusque valor per reſolutionem aequationis quadraticae exhiberi queat: id quod praeſens inſtitutum neceſſario poſtulat. Denique ambae variabiles x et y in hanc aequationem aequaliter ingrediuntur, et etiamſi permutentur nullam mutationem inducunt, vt vtraque per alteram formula omnino ſimili exprimatur. Atque ob has rationes membra x^3+y^3, x^4+y^4; et $xy(xx+yy)$, vti et altiores dimenſiones omitti oportuit.

6. Quodſi iam ex hac aequatione tam valorem ipſius x quam ipſius y extrahamus, reperiemus:

$$x=\frac{-\beta-\delta y-\varepsilon yy\pm\sqrt{((\beta+\delta y+\varepsilon yy)^2-(\alpha+2\beta y+\gamma yy)(\gamma+2\varepsilon y+\zeta yy))}}{\gamma+2\varepsilon y+\zeta yy}$$

$$y=\frac{-\beta-\delta x-\varepsilon xx\pm\sqrt{((\beta+\delta x+\varepsilon xx)^2-(\alpha+2\beta x+\gamma xx)(\gamma+2\varepsilon x+\zeta xx))}}{\gamma+2\varepsilon x+\zeta xx}.$$

Ponamus breuitatis gratia:

$$\pm\sqrt{((\beta+\delta y+\varepsilon yy)^2-(\alpha+2\beta y+\gamma yy)(\gamma+2\varepsilon y+\zeta yy))}=Y$$

$$\pm\sqrt{((\beta+\delta x+\varepsilon xx)^2-(\alpha+2\beta x+\gamma xx)(\gamma+2\varepsilon x+\zeta xx))}=X$$

vt habeamus:

$$x=\frac{-\beta-\delta y-\varepsilon yy+Y}{\gamma+2\varepsilon y+\zeta yy} \text{ et } y=\frac{-\beta-\delta x-\varepsilon xx+X}{\gamma+2\varepsilon x+\zeta xx}$$

ideoque:

$$Y=\beta+\delta y+\varepsilon yy+x(\gamma+2\varepsilon y+\zeta yy)$$
$$X=\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)$$

7. Hunc

7. Hanc aequationem canonicam differentiemus, ac prodibit aequatio differentialis per binarium diuisa:

$$0 = \begin{aligned} &+\beta dx + \gamma x dx + \delta y dx + 2\varepsilon xy dx + \varepsilon yy dx + \zeta xyy dx \\ &+\beta dy + \gamma y dy + \delta x dy + 2\varepsilon xy dy + \varepsilon xx dy + \zeta xxy dy \end{aligned}$$

quae cum reducatur ad hanc formam

$$0 = \begin{aligned} &+dx(\beta + \delta y + \varepsilon yy) + x dx(\gamma + 2\varepsilon y + \zeta yy) \\ &+dy(\beta + \delta x + \varepsilon xx) + y dy(\gamma + 2\varepsilon x + \zeta xx) \end{aligned}$$

quoniam coefficientes ipſorum dx et dy ſunt eae ipſae quantitates, quas modo pro formulis radicalibus X et Y exhibuimus, iſta aequatio differentialis erit

$$0 = Y dx + X dy \text{ ſeu } \frac{dx}{X} + \frac{dy}{Y} = 0$$

in qua cum variabiles x et y ſint ſeparatae, ſi quidem pro X et Y valores illos ſurdos ſubſtituamus, per integrationem inde hanc aequationem finitam obtinebimus

$$\int \frac{dx}{X} + \int \frac{dy}{Y} = \text{Conſt.}$$

8. Cum igitur haec aequatio integralis certam quandam relationem inter variabiles x et y exprimat, ea a relatione in aequatione contenta diverſa eſſe non poteſt, ſicque ipſa aequatio canonica continebit iſtam aequationem integralem. Etſi ergo in aequatione differentiali $\frac{dx}{X} + \frac{dy}{Y} = 0$, neutra pars eſt integrabilis, atque adeo neque per circuli quadraturam neque logarithmos expediri poteſt, tamen integratio algebraicam relationem inter ambas variabiles x et y praebet, propterea quod haec aequatio integrata cum ipſa aequatione canonica conuenit. Quin etiam dico, aequationem canonicam non ſolum caſum particularem integralis praebere, cuiusmodi caſus ſaepe aequationibus maxime complicatis ſatisfaciunt, ſed eam adeo integrale completum ſecundum omnem extenſionem exhibere.

9. Ad hoc oſtendendum, in quo ſine dubio ſumma vis huius integrationis agnoſci debet, notaſſe ſufficit in aequatione canonica vna conſtante plus contineri quam in aequatione differentiali. Vidimus enim aequationem canonicam quinque inuoluere conſtantes arbitrarias; vnde examinemus, quot huiusmodi conſtantes aequatio differentialis complectatur. Manifeſtum autem eſt eam huiusmodi habere formam:

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} + \frac{dy}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}} = 0$$

in qua quidem etiam quinque conſtantes A, B, C, D, E ineſſe videntur: verum euidens eſt, vnamquamque per diuiſionem tolli poſſe, ita vt re vera quatuor tantum ineſſe ſint cenſendae. Quare cum aequatio integralis quinque contineat, vna arbitrario noſtro relinquitur, quod eſt manifeſtum indicium integralis completi.

10. Vtcunque autem iſti quinque coefficientes A, B, C, D, E ſe habeant, ſemper coefficientes aequationis canonicae iis conformiter ita definiri poſſunt,

possunt, vt vnus maneat indeterminatus. Diuidamus enim aequationem differentialem per quantitatem indefinitam p, quae iam sublata est censenda, vt re vera fuerit:

$$X = \sqrt{(Ap + 2Bpx + Cpxx + 2Dpx^3 + Epx^4)}.$$

Iam euoluamus quoque secundum potestates ipsius x valorem primitiuum ipsius X, qui erit

$$X = \sqrt{\left(\begin{matrix} \beta\beta \\ -\alpha\gamma \end{matrix} \begin{matrix} +2\beta\delta \\ -2\alpha\varepsilon \\ -2\beta\gamma \end{matrix} \Bigg\} x \begin{matrix} +2\beta\varepsilon \\ +\delta\delta \\ -\alpha\zeta \\ -4\beta\varepsilon \\ -\gamma\gamma \end{matrix} \Bigg\} x^2 \begin{matrix} +2\delta\varepsilon \\ -2\beta\zeta \\ -2\gamma\varepsilon \end{matrix} \Bigg\} x^3 \begin{matrix} +\varepsilon\varepsilon \\ -\gamma\zeta \end{matrix} \Bigg\} x^4 \right)}$$

atque istae literae α, β, γ, δ, ε, ζ ita definiantur, vt haec forma cum priori congruens reddatur, sic enim patebit vnam determinationem adhuc arbitrio nostro relinqui.

11. Satisfieri igitur oportet sequentibus quinque aequationibus:

I. $\beta\beta - \alpha\gamma = Ap$

II. $\beta\delta - \alpha\varepsilon - \beta\gamma = Bp$

III. $\delta\delta - \alpha\zeta - 2\beta\varepsilon - \gamma\gamma = Cp$

IV. $\delta\varepsilon - \beta\zeta - \gamma\varepsilon = Dp$

V. $\varepsilon\varepsilon - \gamma\zeta = Ep$.

Ponamus ad abbreuiandum $\delta - \gamma = \lambda$ seu $\delta = \gamma + \lambda$ et incipiamus a II et IV,

II. $\beta\lambda - \alpha\varepsilon = Bp$ et IV. $\varepsilon\lambda - \beta\zeta = Dp$

vnde

vnde definiemus β et ε ita vt sit:

$$\beta = \frac{D\alpha + B\lambda}{\lambda\lambda - \alpha\zeta} p \quad \text{et} \quad \varepsilon = \frac{B\zeta + D\lambda}{\lambda\lambda - \alpha\zeta} p.$$

At I et V coniunctae dant:

$$\beta\beta\zeta - \alpha\varepsilon\varepsilon = Ap\zeta - Ep\alpha = \frac{BB\zeta - DD\alpha}{\lambda\lambda - \alpha\zeta} pp$$

vnde eruitur $p = \frac{(\lambda\lambda - \alpha\zeta)(A\zeta - E\alpha)}{BB\zeta - DD\alpha}$,

qui valor in alterutra substitutus praebet

$$\gamma = \frac{(A\zeta - E\alpha)((ADD - BBE)\lambda\lambda + 2BD(A\zeta - E\alpha)\lambda + ABB\zeta\zeta - DDE\alpha\alpha)}{(BB\zeta - DD\alpha)^2}.$$

12. Superest igitur III. aequatio, quae ob $\delta = \gamma + \lambda$ transit in

$$2\gamma\lambda + \lambda\lambda - \alpha\zeta - 2\beta\varepsilon = Cp.$$

Cum nunc substituto valore ipsius p sit

$$\beta = \frac{(A\zeta - E\alpha)(D\alpha + B\lambda)}{BB\zeta - DD\alpha} \quad \text{et} \quad \varepsilon = \frac{(A\zeta - E\alpha)(B\zeta + D\lambda)}{BB\zeta - DD\alpha}$$

si isti valores pro γ, β, ε et p substituantur, tota aequatio per $\lambda\lambda - \alpha\zeta$ diuidi poterit, quo facto reperietur

$$\lambda = \frac{C(A\zeta - E\alpha)(BB\zeta - DD\alpha) - 2BD(A\zeta - E\alpha)^2 - (BB\zeta - DD\alpha)^2}{2(A\zeta - E\alpha)(ADD - BBE)}.$$

Quoniam igitur nunc omnibus conditionibus est satisfactum, arbitrio nostro adhuc relinquuntur duo coefficientes α et ζ, seu potius eorum ratio mutua, quam ergo pro lubitu definire licet. Ex quo manifestum est, in aequatione integrali seu ipsa canonica inesse constantem arbitrariam ab aequatione differentiali non pendentem.

Alia

Alia reſolutio earundem formularum.

13. Quia illorum valorum applicatio fieri nequit caſibus, quibus $ADD-BBE=0$, aliam reſolutionem huic incommodo non obnoxiam tradam. Poſito autem $\delta=\gamma+\lambda$, ſtatuo porro: $\lambda\lambda-\alpha\zeta=\mu$ ſeu $\lambda\lambda=\mu+\alpha\zeta$, atque vt ante ex aequationibus II et IV habebimus:

$$\beta=\tfrac{p}{\mu}(D\alpha+B\lambda)\,;\ \varepsilon=\tfrac{p}{\mu}(B\zeta+D\lambda).$$

Tum vero quia I et V coniunctae dant

$$A\zeta-E\alpha=(BB\zeta-DD\alpha)\tfrac{p}{\mu}$$

hinc definio rationem inter α et ζ, ſeu quoniam alterutram pro lubitu accipere licet, vtramque hoc modo vt fit:

$$\alpha=\mu A-BBp \quad \text{et} \quad \zeta=\mu E-DDp$$

hincque $\lambda\lambda=\mu+(\mu A-BBp)(\mu E-DDp)$. At alterutra I et V, valoribus hactenus inuentis ſubſtitutis, praebebit:

$$\gamma=\tfrac{pp}{\mu\mu}(2BD\lambda+(ADD+BBE)\mu)-\tfrac{2BBDDp^3}{\mu\mu}-\tfrac{p}{\mu}.$$

14. Quodſi iam hi valores in aequatione III ſubſtituantur, ea ad formam quidem admodum prolixam reducitur: verum negotium commodius abſoluetur, ſi valores pro α et ζ inuenti in formula vltima praecedentis reſolutionis ſubſtituantur, tum enim prodibit:

$$\lambda=\tfrac{\mu\mu}{2p}+BDp-\tfrac{1}{2}C\mu$$

cuius

cuius quadratum cum ſuperiori ipſius $\lambda\lambda$ valore coaequatum praebet:

$$\mu(\mu-Cp)^2+4(BD-AE)pp\mu+4(ADD-BCD+BBE)p^3=4pp$$

ad quam reſoluendam ponamus $\mu=pM$ eritque

$$p=\frac{4}{M(M-C)^2+4M(BD-AE)+4(ADD-BCD+BBE)}$$

et $\mu=\dfrac{4M}{M(M-C)^2+4M(BD-AE)+4(ADD-BCD+BBE)}$

atque iam M eſt conſtans illa arbitraria integrale reddens completum.

15. Hoc modo omnes coefficientes α, β, γ, δ etc. eodem denominatore affecti prodibunt, qui ergo ſi per eundem multiplicentur ſequenti modo ſeſe habebunt:

$$\alpha=4(AM-BB);\ \beta=2B(M-C)+4AD;\ \gamma=4AE-(M-C)^2$$
$$\zeta=4(EM-DD);\ \varepsilon=2D(M-C)+4BE;\ \delta=MM-CC+4(AE+BD)$$

ac ſi illum denominatorem breuitatis gratia ſtatuamus:

$$M(M-C)^2+4M(BD-AE)+4(ADD-BCD+BBE)=\Delta$$

aequatio noſtra canonica

$$0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy+2\varepsilon xy(x+y)+\zeta xxyy$$

reſoluta dabit:

$$\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)=\mp 2\sqrt{\Delta(A+2Bx+Cxx+2Dx^3+Ex^4)}$$
$$\beta+\delta y+\varepsilon yy+x(\gamma+2\varepsilon y+\zeta yy)=\mp 2\sqrt{\Delta(A+2By+Cyy+2Dy^3+Ey^4)}$$

ſimul-

ſimulque eſt integrale completum huius aequationis differentialis:

$$0 = \frac{dx}{\pm\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} + \frac{dy}{\pm\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}}$$

quia conſtantem arbitrariam M inuoluit, quae in aequationem differentialem non ingreditur.

Inueſtigatio caſuum, quibus formula $\frac{P\,dx}{X} + \frac{Q\,dy}{Y}$ ſit integrabilis.

16. Deſignat hic P functionem ipſius x et Q ſimilem functionem ipſius y, et quia haec formula integrabilis eſſe debet, ſit V eius integrale, vt habeamus:

$$\frac{P\,dx}{X} + \frac{Q\,dy}{Y} = dV \text{ et } \int\frac{P\,dx}{X} + \int\frac{Q\,dy}{Y} = V.$$

Cum autem ſit $\frac{dx}{X} + \frac{dy}{Y} = 0$, ideoque $\frac{dy}{Y} = \frac{-dx}{X}$, erit

$$dV = \frac{(P-Q)dx}{X} = \frac{(P-Q)dx}{\beta + \delta x + \epsilon xx + y(\gamma + 2\epsilon x + \zeta xx)}$$

Caſus ergo inueſtigari oportet, quibus haec formula integrationem admittit.

17. Quoniam vero nulla eſt ratio, cur hic differentiale dx potius inſit, quam dy, tertiam variabilem introducamus, quae ad vtramque aequaliter referatur, ſiquidem quantitas V vtramque aequaliter inuoluere debet. Statuamus ergo $x + y = s$, et in aequatione differentiali (§. 7.) pro dy ſcribamus $ds - dx$; ſicque prodibit:

$$0 = + dx(\beta + \delta y + \epsilon yy) + x\,dx(\gamma + 2\epsilon y + \zeta yy)$$
$$- dx(\beta + \delta x + \epsilon xx) - y\,dx(\gamma + 2\epsilon x + \zeta xx)$$
$$+ ds(\beta + \delta x + \epsilon xx) + y\,ds(\gamma + 2\epsilon x + \zeta xx)$$

vnde dx per ds ita definietur, vt ſit:

$$dx = \frac{ds(\beta + \delta x + \epsilon xx) + y\,ds(\gamma + 2\epsilon x + \zeta xx)}{\delta(x-y) + \epsilon(xx-yy) - \gamma(x-y) + \zeta xy(x-y)} \text{ ſiue}$$

$$dx = \frac{ds}{x-y} \cdot \frac{\beta + \delta x + \epsilon xx + y(\gamma + 2\epsilon x + \zeta xx)}{\delta - \gamma + \epsilon(x+y) + \zeta xy}$$

quo valore ſubſtituto fiet:

$$dV = \frac{(P-Q)ds}{(x-y)(\delta - \gamma + \epsilon(x+y) + \zeta xy)}.$$

18. Cum P et Q ſint ſimiles functiones ipſarum x et y, manifeſtum eſt $P-Q$ per $x-y$ fore diuiſibile, et fractionem $\frac{P-Q}{x-y}$ vtramque variabilem x et y aequaliter eſſe complexuram. Quia vero poſuimus $x + y = s$, ponamus inſuper $xy = t$, vt ſit:

$$dV = \frac{P-Q}{x-y} \cdot \frac{ds}{\delta - \gamma + \epsilon s + \zeta t}.$$

At ob $xx + yy = ss - 2t$ aequatio canonica induet hanc formam:

$$0 = \alpha + 2\beta s + \gamma ss + 2(\delta - \gamma)t + 2\epsilon st + \zeta tt$$

ex qua elicitur:

$$t = \frac{-\delta + \gamma - \epsilon s + \sqrt{((\delta-\gamma)^2 - \alpha\zeta + 2(\delta-\gamma)\epsilon s - 2\beta\zeta s + \epsilon\epsilon ss - \gamma\zeta ss)}}{\zeta}$$

ita vt ſit:

$$\delta - \gamma + \epsilon s + \zeta t = \sqrt{((\delta-\gamma)^2 - \alpha\zeta + 2((\delta-\gamma)\epsilon - \beta\zeta)s + (\epsilon\epsilon - \gamma\zeta)ss)}.$$

Statuamus hanc formulam irrationalem:

$$\sqrt{((\delta-\gamma)^2 - \alpha\zeta + 2((\delta-\gamma)\epsilon - \beta\zeta)s + (\epsilon\epsilon - \gamma\zeta)ss)} = S$$

vt fit

$$t = \frac{-(\delta-\gamma)s - \varepsilon ss + S}{\zeta} \text{ et } dV = \frac{P-Q}{x-y} \cdot \frac{ds}{S}.$$

19. Vt hinc iam casus integrabilitatis eruamus, ponamus:

$$P = a + bx + cxx + dx^3 + ex^4$$
$$Q = a + by + cyy + dy^3 + ey^4$$

eritque

$$\frac{P-Q}{x-y} = b + c(x-y) + d(xx+xy+yy) + e(x^3+xxy+xyy+y^3)$$

siue introductis nouis variabilibus s et t

$$\frac{P-Q}{x-y} = b + cs + d(ss-t) + es(ss-2t),$$

At pro t valore substituto habebimus ob $\lambda = \delta - \gamma$

$$\frac{P-Q}{x-y} = \begin{array}{l} b + cs + dss + es^3 \\ + \frac{\lambda d}{\zeta} + \frac{\varepsilon ds}{\zeta} + \frac{2\varepsilon ess}{\zeta} \\ \quad + \frac{2\lambda es}{\zeta} \end{array} - \frac{(d+2es)S}{\zeta}$$

vnde consequimur:

$$dV = \frac{\zeta b + \lambda d + (\zeta c + \varepsilon d + 2\lambda e)s + (\zeta d + 2\varepsilon e)ss + \zeta e s^3}{\zeta S} ds - \frac{(d+2es)ds}{\zeta}$$

quam formulam integrabilem esse oportet.

20. Quo hoc facilius praestemus, recordemur ex §. §. 13 et 14. esse $(\delta-\gamma)^2 - \alpha\zeta = \lambda\lambda - \alpha\zeta = \mu$; $(\delta-\gamma)\varepsilon - \beta\zeta = Dp$; et $\varepsilon\varepsilon - \gamma\zeta = Ep$ vnde fit $S = \sqrt{(\mu + 2Dps + Epss)}$; siue ex §. 14 et 15.

$$S = \frac{\varepsilon\sqrt{(M+2Ds+Ess)}}{\sqrt{\Delta}}.$$

Ponamus porro breuitatis gratia:

$$b + \frac{\lambda d}{\zeta} = b; \quad c + \frac{\varepsilon d + 2\lambda e}{\zeta} = g; \quad d + \frac{2\varepsilon e}{\zeta} = f$$

vt

vt fit

$$dV = \frac{(b + gs + fss + es^3)ds\sqrt{\Delta}}{2\sqrt{(M+2Ds+Ess)}} - \frac{(d+2es)ds}{\zeta}$$

statuatur partis prioris integrale:

$$(\mathfrak{F} + \mathfrak{G}s + \mathfrak{H}ss)\sqrt{\Delta(M+2Ds+Ess)}$$

eritque differentialium comparatione instituta:

$$b = 2\mathfrak{G}M + 2\mathfrak{F}D; \quad g = 4\mathfrak{H}M + 6\mathfrak{G}D + 2\mathfrak{F}E$$
$$f = 10\mathfrak{H}D + 4\mathfrak{G}E; \quad e = 6\mathfrak{H}E$$

vnde pro integrabilitate requiritur vt sit:

$$0 = eD(3EM - 5DD) + fE(3DD - EM) - 2gDEE + 2bE^3$$

21. Hac autem conditione impleta, erit:

$$\mathfrak{F} = \frac{b}{2D} - \frac{fM}{4DE} + \frac{5eM}{12EE}; \quad \mathfrak{G} = \frac{f}{4E} - \frac{5eD}{12EE}; \quad \mathfrak{H} = \frac{e}{6E}$$

et integrale quaesitum reperietur:

$$V = (\mathfrak{F} + \mathfrak{G}s + \mathfrak{H}ss)\sqrt{\Delta(M+2Ds+Ess)} - \frac{(d+es)s}{\zeta}, \text{ vel}$$

$$V = \frac{1}{\varepsilon}(\mathfrak{F} + \mathfrak{G}s + \mathfrak{H}ss)\Delta S - \frac{(d+es)s}{\zeta}.$$

Cum nunc sit $S = \lambda + \varepsilon(x+y) + \zeta xy$, si pro s scribamus $x+y$, valor integralis V ita per x et y exprimetur, vt fit

$$V = \frac{1}{\varepsilon}\Delta(\mathfrak{F} + \mathfrak{G}(x+y) + \mathfrak{H}(x+y)^2)(\lambda + \varepsilon(x+y) + \zeta xy) - \frac{d(x+y) - e(x+y)^2}{\zeta}.$$

Quare vt pro V prodeat quantitas algebraica, coefficientes b, c, d et e non pro lubitu assumere licet, sed certam quandam relationem inter eos statui oportet, quae vltima aequalitate §. praec. exprimitur. Ceterum hic assumsi, non esse $E = 0$, si enim esset

esset $E=0$, valor ipsius V semper algebraice exhiberi posset, vti ex elementis integrationis est manifestum.

22. Verum si coefficientes b, c, d, e etc. vtcunque assumamus, tum expressio $\int\frac{P\,dx}{X}+\int\frac{Q\,dy}{Y}$ non quidem semper algebraice exhib ri poterit; attamen eius valor altiorem quadraturam non inuoluet, quam in formula $\int\frac{ds}{\surd(M+2Ds+Ess)}$ contentam, quae propter ea semper vel per logarithmos vel per arcus circulares exhiberi poterit. Cum igitur sit

$$X=\surd p(A+2Bx+Cxx+2Dx^3+Ex^4)$$

et $\surd p=\frac{2}{\surd\Delta}$ erit $X=\frac{2}{\surd\Delta}\surd(A+2Bx+Cxx+2Dx^3+Ex^4)$ vnde inuento valore ipsius V habebitur sequens integratio:

$$\int\frac{dx(a+bx+cxx+dx^3+ex^4)}{\surd(A+2Bx+Cx^2+2Dx^3+Ex^4)}+\int\frac{dy(a+by+cyy+dy^3+ey^4)}{\surd(A+2By+Cyy+2Dy^3+Ey^4)}=\frac{2V}{\surd\Delta}.$$

At substitutis superioribus valoribus erit:

$$\frac{2V}{\surd\Delta}=\int\frac{\zeta b+\lambda d+(\zeta c+\varepsilon d+2\lambda e)s+(\zeta d+2\varepsilon e)ss+\zeta es^3}{\zeta\surd(M+2Ds+Ess)}ds-\frac{(d+es)s\surd\Delta}{2(EM-DD)}.$$

Existente $s=x+y$. Atque hinc sequentia problemata resolui poterunt.

Problema I.

23. Inuenire integrale completum huius aequationis differentialis:

$$\frac{dy}{\surd(A+2By+Cyy+2Dy^3+Ey^4)}=\frac{dx}{\surd(A+2Bx+Cxx+2Dx^3+Ex^4)}.$$

Solutio.

Solutio.

Statim apparet huic aequationi differentiali satisfacere casum $y=x$, qui autem non nisi integrale particulare largitur. Verum ad integrale completum inueniendum, quod praeter constantes A, B, C, D, E nouam constantem arbitrariam M inuoluat, ponamus secundum §. 15 breuitatis gratia:

$\alpha=4(AM-BB)$; $\beta=2B(M-C)+4AD$; $\gamma=4AE-(M-C)^2$
$\zeta=4(EM-DD)$; $\varepsilon=2D(M-C)+4BE$; $\delta=MM-CC+4(AE-BD)$

atque aequatio integralis completa erit

$$0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy+2\varepsilon xy(x+y)+\zeta xxyy$$

quae ergo est algebraica. Hinc autem siue y per x, siue vicissim x per y sequenti modo definietur, posito item breuitatis ergo:

$$\Delta=M(M-C)^2+4M(BD-AE)+4(ADD+BBE)-4BCD$$

vt sit

$$\text{vel } y=\frac{-\beta-\delta x-\varepsilon xx\pm2\surd\Delta(A+2Bx+Cxx+2Dx^3+Ex^4)}{\gamma+2\varepsilon x+\zeta xx}$$
$$\text{vel } x=\frac{-\beta-\delta y-\varepsilon yy\pm2\surd\Delta(A+2By+Cyy+2Dy^3+Ey^4)}{\gamma+2\varepsilon y+\zeta yy}$$

scilicet ratione signorum ambiguorum in vtraque expressione vel signa superiora vel inferiora capi debent, ita vt si in altera formulae surdae tribuatur signum +, in altera formulae surdae signum − tribui debeat. Quae ratio ex §. 15. intelligitur, vbi in aequatione differentiali formulis surdis signa ambigua sunt adiuncta.

Coroll. 1.

24. Quanquam igitur aequationis differentialis propofitae, in qua ambae variabiles x et y a fe inuicem funt feparatae, neutrum membrum integrationem abfolutam admittit, atque adeo neque per logarithmos neque arcus circulares in genere exprimi poteft, tamen vera relatio inter variabiles x et y aequatione algebraica exhiberi poteft.

Coroll. 2.

25. Quemadmodum fcilicet fi duo arcus quantitate conftante differunt, etfi neuter algebraice exprimitur, tamen eorum finus inter fe algebraicam tenent rationem, quae fatisfacit aequationi differentiali $\frac{dy}{\sqrt{(1-yy)}} = \frac{dx}{\sqrt{(1-xx)}}$, ita quoque aequationis differentialis propofitae multoque latius patentis integrale completum algebraice exhiberi poteft.

Scholion.

26. Vis huius folutionis facilius percipietur, fi eam ad cafus magis reftrictos applicemus, inter quos ii praecipue funt notatu digni, vbi fignum radicale vel vnico vel duobus tantum terminis praefigitur ac fi vnicus tantum terminus reperiatur, ratio per fe eft manifefta.

I. Sit enim $B=0$; $C=0$; $D=0$, et $E=0$ vt integranda fit aequatio:

$\frac{dy}{\sqrt{A}} = \frac{dx}{\sqrt{A}}$ fiue $dy = dx$ erit

$\alpha = 4AM$; $\beta = 0$; $\gamma = -MM$; $\delta = MM$; $\varepsilon = 0$; $\zeta = 0$;

ideoque

ideoque aequatio integralis:

$$0 = 4AM - MM(xx+yy) + 2MMxy$$

feu $x - y = 2\sqrt{\frac{A}{M}}$ vel $y = x \pm$ Conft.

II. Sit $A=0$; $C=0$; $D=0$; et $E=0$, vt integranda fit aequatio:

$\frac{dy}{\sqrt{2By}} = \frac{dx}{\sqrt{2Bx}}$ feu $\frac{dy}{\sqrt{y}} = \frac{dx}{\sqrt{x}}$ erit

$\alpha = -4BB$; $\beta = 2BM$; $\gamma = -MM$; $\delta = MM$; $\varepsilon = 0$ et $\zeta = 0$

ideoque aequatio integralis, ob $\Delta = M^4$

$$0 = -4BB + 4BM(x+y) - MM(xx+yy) + 2MMxy$$

feu $y = \frac{-2BM - MMx \pm 2\sqrt{2BM^3x}}{-MM} = x + \frac{2B}{M} \pm 2\sqrt{\frac{2B}{M}x}$

hincque $\sqrt{y} = \sqrt{x} +$ Conft. vti eft perfpicuum.

III. Sit $A=0$; $B=0$; $D=0$; $E=0$, vt integranda fit haec aequatio:

$\frac{dy}{\sqrt{Cyy}} = \frac{dx}{\sqrt{Cxx}}$ feu $\frac{dy}{y} = \frac{dx}{x}$; erit

$\alpha = 0$; $\beta = 0$; $\gamma = -(M-C)^2$; $\delta = MM - CC$; $\varepsilon = 0$, et $\zeta = 0$

ideoque aequatio integralis

$$0 = -(M-C)^2(xx+yy) + 2(MM-CC)xy \text{ feu } y = nx.$$

IV. Sit $A=0$; $B=0$; $C=0$; et $E=0$ vt integranda fit haec aequatio:

$\frac{dy}{\sqrt{2Dy^3}} = \frac{dx}{\sqrt{2Dx^3}}$ feu $\frac{dy}{y\sqrt{y}} = \frac{dx}{x\sqrt{x}}$; erit

$\alpha = 0$; $\beta = 0$; $\gamma = -MM$; $\delta = MM$; $\varepsilon = 2DM$; $\zeta = -4DD$

ideo-

ideoque aequatio integralis:

$$0 = -MM(xx+yy) + 2MMxy + 4DMxy(x+y) - 4DDxxyy$$

quae ob $\Delta = M^3$ dat

$$y = \frac{-MMx - 2DMxx \pm 2\sqrt{2DM^3x^3}}{-MM + 4DMx - 4DDxx}$$

seu $\sqrt{y} = \frac{M \pm \sqrt{2DMx}}{M - 2Dx}\sqrt{x} = \frac{\sqrt{Mx}}{\sqrt{M} \mp \sqrt{2Dx}}$

vel $\frac{1}{\sqrt{y}} = \frac{1}{\sqrt{x}} \pm \sqrt{\frac{2D}{M}}$, vti rei natura postulat.

V. Sit $A = 0$; $B = 0$; $C = 0$; et $D = 0$ vt integranda sit haec aequatio:

$$\frac{dy}{\sqrt{Ey^4}} = \frac{dx}{\sqrt{Ex^4}} \text{ seu } \frac{dy}{yy} = \frac{dx}{xx}; \text{ erit}$$

$\alpha = 0$; $\beta = 0$; $\gamma = -MM$; $\delta = MM$; $\varepsilon = 0$; et $\zeta = 4EM$ ideoque aequatio integralis:

$$0 = -MM(xx+yy) + 2MMxy + 4EMxxyy$$

hincque $y - x = 2xy\sqrt{\frac{E}{M}}$ seu $\frac{1}{y} = \frac{1}{x} \pm 2\sqrt{\frac{E}{M}}$.

Quando autem signum radicale complectitur duos terminos, varios casus, qui huc pertinent, sequentibus exemplis euoluemus.

Exemplum 1.

27. Si sit $C = 0$; $D = 0$ et $E = 0$, vt integranda sit aequatio:

$$\frac{dy}{\sqrt{(A + 2By)}} = \frac{dx}{\sqrt{(A + 2Bx)}}$$

inuenire aequationem integralem completam.

Erit

Erit ergo $\alpha = 4(AM - BB)$; $\beta = 2BM$; $\gamma = -MM$; $\delta = MM$; $\varepsilon = 0$; $\zeta = 0$; vnde aequatio integralis:

$$0 = 4(AM - BB) + 4BM(x+y) - MM(xx+yy) + 2MMxy$$

et ob $\Delta = M^3$

$$y = \frac{-2BM - MMx \pm 2\sqrt{M^3(\Delta + 2Bx)}}{-MM} = \frac{2B + Mx}{M} \mp 2\sqrt{\frac{A + 2Bx}{M}}.$$

Vnde ponendo $A = f$; $2B = g$; et $M = c$ sequitur

Theorema 1.

28. Huius aequationis differentialis $\frac{dy}{\sqrt{(f + gy)}} = \frac{dx}{\sqrt{(f + gx)}}$; Integrale completum est:

$$0 = 4cf - gg + 2cg(x+y) - cc(xx+yy) + 2ccxy$$

vnde fit:

$$y = x + \frac{g}{c} \mp 2\sqrt{\frac{f + gx}{c}} \text{ et } x = y + \frac{g}{c} \pm 2\sqrt{\frac{f + gy}{c}}.$$

Exemplum 2.

29. Si sit $B = 0$, $D = 0$ et $E = 0$, vt integranda sit aequatio:

$$\frac{dy}{\sqrt{(A + Cyy)}} = \frac{dx}{\sqrt{(A + Cxx)}}$$

inuenire aequationem integralem completam.

Erit ergo: $\alpha = 4AM$, $\beta = 0$, $\gamma = -(M-C)^2$, $\delta = MM - CC$, $\varepsilon = 0$ et $\zeta = 0$, vnde aequatio integralis quaesita erit:

$$0 = 4AM - (M-C)^2(xx+yy) + 2(MM - CC)xy$$

et

et ob $\Delta = M(M-C)^2$ erit

$$y = \frac{-(MM-CC)x \pm 2(M-C)\sqrt{M(A+Cxx)}}{-(M-C)^2} = \frac{(M+C)x \mp 2\sqrt{M(A+Cxx)}}{M-C}.$$

Quare ponendo $A=f$; $C=g$; et $M=c$, sequitur

Theorema 2.

30. Huius aequationis differentialis $\frac{dy}{\sqrt{(f+gyy)}} = \frac{dx}{\sqrt{(f+gxx)}}$ integrale completum est:

$$0 = 4cf - (c-g)^2(xx+yy) + 2(cc-gg)xy$$

vnde fit:

$$y = \frac{(c+g)x \pm 2\sqrt{c(f+gxx)}}{c-g} \text{ et } x = \frac{(c+g)y \mp 2\sqrt{c(f+gyy)}}{c-g}.$$

Exemplum 3.

31. Si fit $B=0$; $C=0$; et $E=0$, vt integranda fit haec aequatio:

$$\frac{dy}{\sqrt{(A+2Dy^3)}} = \frac{dx}{\sqrt{(A+2Dx^3)}}$$

inuenire aequationem integralem completam.

Erit ergo $\alpha = 4AM$; $\beta = 4AD$; $\gamma = -M^2$; $\delta = M^2$; $\varepsilon = 2DM$ et $\zeta = -4DD$; vnde aequatio integralis quaesita est

$$0 = 4AM + 8AD(x+y) - M^2(xx+yy) + 2M^2xy + 4DMxy(x+y) - 4DDxxyy$$

et cum fit $\Delta = M^3 + 4ADD$ erit

$$y = \frac{-4AD - MMx - 2DMxx \pm 2\sqrt{(M^3+4ADD)(A+2Dx^3)}}{-MM+4DMx-4DDxx}$$

siue:

$$y = \frac{4AD + MMx + 2DMxx \pm 2\sqrt{(M^3+4ADD)(A+2Dx^3)}}{(M-2Dx)^2}.$$

Quare si ponatur $A=f$; $2D=g$ et $M=c$ sequitur

Theorema 3.

32. Huius aequationis differentialis $\frac{dy}{\sqrt{(f+gy^3)}} = \frac{dx}{\sqrt{(f+gx^3)}}$ integrale completum est:

$$0 = 4cf + 4fg(x+y) - cc(xx+yy) + 2ccxy + 2cgxy(x+y) - ggxxyy$$

vnde fit:

$$y = \frac{2fg + ccx + cgxx \pm 2\sqrt{(c^3+fgg)(f+gx^3)}}{(c-gx)^2} \text{ et}$$

$$x = \frac{2fg + ccy + cgyy \mp 2\sqrt{(c^3+fgg)(f+gy^3)}}{(c-gy)^2}.$$

Exemplum 4.

33. Si fit $B=0$; $C=0$; $D=0$ vt aequatio integranda fit:

$$\frac{dy}{\sqrt{(A+Ey^4)}} = \frac{dx}{\sqrt{(A+Ex^4)}}$$

inuenire aequationem integralem completam.

Erit ergo: $\alpha = 4AM$; $\beta = 0$; $\gamma = 4AE - MM$; $\delta = MM + 4AE$; $\varepsilon = 0$ et $\zeta = 4EM$, vnde aequatio integralis quaesita est:

$$0 = 4AM + (4AE - MM)(xx+yy) + 2(4AE+MM)xy + 4EMxxyy$$

et cum fit $\Delta = M^3 - 4AEM$ erit

$$y = \frac{-(MM+4AE)x \pm 2\sqrt{M(MM-4AE)(A+Ex^4)}}{4AE - MM + 4EMxx}.$$

Quare

Quare si ponatur $A=f$; $E=g$; et $M=2c$ sequitur

Theorema 4.

34. Huius aequationis differentialis $\frac{dy}{\sqrt{(f+gy^4)}} = \frac{dx}{\sqrt{(f+gx^4)}}$ integrale completum est:

$$o = 2cf - (cc-fg)(xx+yy) + 2(cc+fg)xy + 2cgxxyy$$

vnde fit

$$y = \frac{+(cc+fg)x \pm \sqrt{2c(cc-fg)(f+gx^4)}}{cc-fg-2cgxx}$$

$$x = \frac{\pm(cc+fg)y \mp \sqrt{2c(cc-fg)(f+gx^4)}}{cc-fg-2cgxx}.$$

Exemplum 5.

35. Si sit $A=o$, $C=o$ et $D=o$ vt integranda sit haec aequatio:

$$\frac{dy}{\sqrt{(2By+Ey^4)}} = \frac{dx}{\sqrt{(2Bx+Ex^4)}}$$

inuenire aequationem integralem completam.

Erit ergo: $\alpha=-4BB$; $\beta=2BM$; $\gamma=-MM$; $\delta=MM$, $\varepsilon=4BE$ et $\zeta=4EM$, hincque aequatio integralis quaesita:

$$o = -4BB + 4BM(x+y) - MM(xx+yy) + 2MMxy + 8BExy(x+y) + 4EMxxyy$$

et cum sit $\Delta = M^4 + 4BBE$ erit

$$y = \frac{2BM + MMx + 4BExx \pm 2\sqrt{(M^4+4BBE)(2Bx+Ex^4)}}{MM - 8BEx - 4EMxx}.$$

Quare si ponatur $2B=f$; $E=g$; $M=c$; $x=xx$ et $y=yy$ sequitur

Theorema 5.

36. Huius aequationis differentialis $\frac{dy}{\sqrt{(f+gy^8)}} = \frac{dx}{\sqrt{(f+gx^8)}}$ integrale completum est:

$$o = -ff + 2cf(xx+yy) - cc(x^4+y^4) + 2ccxxyy + 4fgxxyy(xx+yy) + 4cgx^4y^4$$

vnde fit

$$yy = \frac{cf + ccxx + 2fgx^6 \pm 2x\sqrt{(c^3+ffg)(f+gx^8)}}{cc - 4fgxx - 4cgx^4}$$

et $xx = \frac{cf + ccyy + 2fgy^6 \pm 2y\sqrt{(c^3+ffg)(f+gy^8)}}{cc - 4fgyy - 4cgy^4}$.

Scholion 1.

37. Probabile hinc videtur etiam huius aequationis differentialis:

$$\frac{dy}{\sqrt{(f+gy^n)}} = \frac{dx}{\sqrt{(f+gx^n)}}$$

atque adeo huius latissime patentis:

$$\frac{dy}{\sqrt{(a+by+cy^2+dy^3+ey^4+fy^5+etc.)}} = \frac{dx}{\sqrt{(a+bx+cx^2+dx^3+ex^4+fx^5+etc.)}}$$

ad quotcunque dimensiones variabiles x et y in vinculis radicalibus assurgant, aequationem dari integralem completam algebraicam. Hoc enim assertum non solum verum est ostensum, quando potestates ipsarum x et y quartum ordinem non superant, sed etiam casu $n=6$, vti vidimus, priorum formularum integratio completa algebraice succedit. Interim tamen nullus adhuc modus patet pro casu $n=5$ integrale completum aequationis $\frac{dy}{\sqrt{(f+gy^5)}} = \frac{dx}{\sqrt{(f+gx^5)}}$ exhibendi, multo minus id ad casus, quibus

quibus n senarium superat extendere licet, etiamsi pro casibus $n=1$, $n=2$, $n=3$, $n=4$, et $n=6$ sit in promtu. Etsi autem de successu in reliquis casibus vix dubitare licet, tamen restrictio necessaria videtur, vt exponens n sit numerus integer, nisi forte et eos casus fractionum adiicere lubuerit, quibus vtraque formula per se sit integrabilis, vti euenit si n sit fractio vnitatem pro numeratore habens. Praeterea vero certum est, veritatem nonnisi pro signo radicali quadrato subsistere posse; neque enim haec aequatio $\frac{dy}{\sqrt[7]{(f+gx^7)}} = \frac{dx}{\sqrt[7]{(f+gx^7)}}$ neque haec $\frac{dy}{\sqrt[8]{(f+gy^8)}} = \frac{dx}{\sqrt[8]{(f+gx^8)}}$ aliaeque harum similes integralia completa algebraica admittunt; quia hae formulae, ad rationalitatem perductae, tam logarithmos quam quadraturam circuli mixtim inuoluunt, atque ex talium quantitatum heterogenearum comparatione aequatio algebraica resultare nequit. Haec eadem vero ratio dubitationem superiorem quoque decidit; ac iam audacter pronunciare possumus hanc aequationem differentialem:

$$\frac{dy}{\sqrt{(a+by+cy^2+dy^3+ey^4+fy^5+gy^6)}} = \frac{dx}{\sqrt{(a+bx+cx^2+dx^3+ex^4+fx^5+gx^6)}}$$

generaliter per aequationem algebraicam integrari non posse; inde enim sequeretur integratio algebraica huius aequationis

$$\frac{dy}{A+By+Cyy+Dy^3} = \frac{dx}{A+Bx+Cxx+Dx^3}$$

quod

quod vtique esset absurdum; multo minus igitur integratio in aequationibus magis compositis succedet. Verum nequidem integrabilitas ad potestatem quintam vsque extendi potest; nam posito $g=0$ si etiam statuatur $a=0$, et pro y et x scribatur yy et xx, prodit haec aequatio differentialis:

$$\frac{dy}{\sqrt{(b+cy^2+dy^4+ey^6+fy^8)}} \mp \frac{dx}{\sqrt{(b+cx^2+dx^4+ex^6+fx^8)}}$$

in qua si radicis extractio succedat, continebitur haec:

$$\frac{dy}{A+Byy+Cy^4} = \frac{dx}{A+Bx^2+Cx^4}$$

quam in genere integrationem algebraicam non admittere est manifestum.

Scholion 2.

38 Nunc igitur pro certo affirmare licet, ex hoc genere aequationem differentialem latissime patentem, quae quidem generaliter algebraice integrari queat, esse eam ipsam, quam hactenus tractavimus:

$$\frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}} = \frac{dx}{\sqrt{(A+2Bx+Cx^2+2Dx^3+Ex^4)}}$$

et cuius aequationem integralem completam assignavimus. Quam ob causam haec aequatio multo magis est notatu digna, quod in hoc genere est generalissima, quae integrationem algebraicam admittat. Quoniam igitur eius integrationem iam exposui, operae pretium erit eius vsum in comparatione li-

 nearum

nearum curuarum, quarum elementa per huiusmodi formulas exprimuntur, vberius ostendere, si quidem in iis omnia continentur, quae in hoc genere praestari possunt. Atque hac ipsa consideratio nos quoque integrationem huiusmodi aequationum

$$\frac{n\,dy}{\sqrt{(A+2By+Cy^2+2Dy^3+Ey^4)}}=\frac{m\,dx}{\sqrt{(A+2Bx+Cx^2+2Dx^3+Ex^4)}}$$

manuducet, si quidem m et n fuerint numeri integri.

Problema 2.

Tab. I. Fig. 4. **39.** Si linea curua habeatur, cuius arcus siue abscissae, siue applicatae, siue cordae, siue alii cuicunque rectae variabili z ad curuam relatae respondens sit $=\int\frac{\mathfrak{M}\,dz}{\sqrt{A+2Bz+Cz^2+2Dz^3+Ez^4)}}$; deturque in hac curua arcus quicunque AB, ab alio quouis puncto P arcum abscindere PQ, qui aequalis sit illi arcui AB.

Solutio.

Ex coefficientibus datis A, B, C, D, E quaerantur hi alii:

$\alpha=4(AM-BB)$; $\beta=2B(M-C)+4AD$; $\gamma=4AE-(M-C)^2$
$\zeta=4(EM-DD)$; $\varepsilon=2D(M-C)+4BE$; $\delta=MM-CC+4(AE+BD)$

vbi M denotat nouam constantem arbitrariam, atque vidimus hanc aequationem algebraicam:

$$0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy+2\varepsilon xy(x+y)+\zeta xxyy$$

con-

congruere cum hac transcendente:

$$\int\frac{\mathfrak{M}\,dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}-\int\frac{\mathfrak{M}\,dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}=\text{Const.}$$

vbi quantitas constans ita definiri debet, vt illi M sit consentanea. Si iam ponamus in curua proposita variabilem z puncto Z respondere, curuaeque initium in puncto Δ statui, atque ad abbreuiandum hunc arcum ΔZ ita indicemus $\Pi:z$ vt sit

$$\int\frac{\mathfrak{M}\,dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}=\Pi:z$$

erit ex aequatione superiori

$$\Pi:y-\Pi:x=\text{Const.}$$

Respondeant nunc punctis A et B rectae a et b, punctis vero P et Q rectae p et q, vt sint arcus

$$\Delta A=\Pi:a;\ \Delta B=\Pi:b;\ \Delta P=\Pi:p \text{ et } \Delta Q=\Pi:q$$

ideoque

arcus $AB=\Pi:b-\Pi:a$ et arcus $PQ=\Pi:q-\Pi:p$

ac loco x et y scribamus p et q vt fit

$$0=\alpha+2\beta(p+q)+\gamma(pp+qq)+2\delta pq+2\varepsilon pq(p+q)+\zeta ppqq$$

erit $\Pi:q-\Pi:p=$ Const. Quodsi ergo constantem M ita assumamus, vt facto $p=a$ prodeat $q=b$, habebimus:

$$\Pi:q-\Pi:p=\Pi:b-\Pi:a$$

ideoque arcum $PQ=$ arcui AB vti requiritur. Constans igitur M, vel si ponamus $M-C=L$ vt

sit

sit $M = C + L$, constans L ex sequenti aequatione debet definiri:

$$0 = 4AC - 4BB + 4AL + 2(2BL + 4AD)(a+b) + (4AE - LL)(aa+bb)$$
$$+ 2(LL + 2CL + 4AE + 4BD)ab + 2(2DL + 4BF)ab(a+b) + 4(CE - DD + EL)aabb$$

vnde fit:

$$LL = \frac{4L(A + B(a+b) + Cab + Dab(a+b) + Eaabb) + 4AC + 8AD(a+b) + 8(AE+BD)ab + 4CEaabb - 4BB + 4AE(aa+bb) + 8BEab(a+b) - 4DDaabb}{(b-a)^2}$$

et radice extracta:

$$L = \frac{2(A + B(a+b) + Cab + Dab(a+b) + Eaabb) \pm 2\sqrt{(A + 2Ba + Caa + 2Da^3 + Ea^4)(A + 2Bb + Cbb + 2Db^3 + Eb^4)}}{(b-a)^2}$$

sicque erit

$$M = \frac{2A + 2B(a+b) + C(aa+bb) + 2Dab(a+b) + 2Eaabb}{(b-a)^2}$$
$$\pm \frac{2}{(b-a)^2}\sqrt{(A + 2Ba + Caa + 2Da^3 + Ea^4)(A + 2Bb + Cbb + 2Db^3 + Eb^4)}$$

Quo valore inuento si iam definiantur valores coefficientium α, β, γ, δ, ε, ζ, quoniam ex dato curvae puncto P datur variabilis p, ex ea valor idoneus variabilis q, cui curuae punctum Q respondet, determinabitur per hanc aequationem

$$0 = \alpha + \beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq$$

ex qua si breuitatis gratia ponamus:

$$\Delta = M(M-C)^2 + 4M(BD - AE) + 4(ADD + BBE) - 4BCD$$

habebitur:

$$q = \frac{-\beta - \delta p - \varepsilon pp \pm 2\sqrt{\Delta(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}}{\gamma + 2\varepsilon p + \zeta pp}$$

sicque

sicque dato arcu AB et puncto P assignabitur punctum Q vt arcus PQ aequalis fiat arcui AB. Reperientur autem ob signum ambiguum bina puncta Q, quorum alterum citra alterum vltra punctum P erit situm.

Coroll. 1.

40. Inuento valore q simili modo a puncto Q vlterius abscindi poterit arcus QR arcui AB aequalis. Posita enim variabili puncto R respondente $= r$, capiatur:

$$r = \frac{-\beta - \delta q - \varepsilon qq \pm 2\sqrt{\Delta(A + 2Bq + Cqq + 2Dq^3 + Eq^4)}}{\gamma + 2\varepsilon q + \zeta qq}$$

sicque a puncto P simul abscindetur arcus PR duplus arcus dati AB.

Coroll. 2.

41. Quoniam r hinc duplicem obtinet valorem, notandum est alterum iterum in p abire, quia ante animaduertimus esse:

$$p = \frac{-\beta - \delta q - \varepsilon qq \mp 2\sqrt{\Delta(A + 2Bq + Cqq + 2Dq^3 + Eq^4)}}{\gamma + 2\varepsilon q + \zeta qq}$$

quare vt arcus PR euadat duplus, idem signum quod in valore ipsius q fuerit electum, in valore ipsius r capi oportet.

Coroll. 3.

42. Pari modo vltra R reperietur punctum S, vt denuo arcus RS aequalis, sicque angulus PS triplus

triplus euadat arcus AB, inuenta enim variabili r valor variabilis s puncto S respondentis hac formula exprimetur:

$$s = \frac{-\beta - \delta r - \epsilon rr \pm 2\sqrt{\Delta(A + 2Br + Crr + 2Dr^3 + Er^4)}}{\gamma + 2\epsilon r + \zeta rr}$$

hocque modo quousque libuerit vlterius progredi licet.

Coroll. 4.

43. Hac ergo repetita operatione a dato puncto P arcus abscindi poterit, qui se habeat ad arcum AB, vt numerus quicunque integer m ad vnitatem. Quare si ab alio puncto abscindatur arcus, qui sit ad eundem AB vt alius numerus integer n ad vnitatem, duo habebuntur arcus rationem quacunque numeri ad numerum tenentes.

Coroll. 5.

44. Omnium igitur curuarum, quarum arcus variabili cuipiam z respondens huiusmodi formula $\int \frac{\mathfrak{A}\,dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}$ exprimitur, haec est proprietas, vt earum arcus simili modo inter se comparari possint, quo arcus circuli inter se comparare licet. Atque ob rationes supra allegatas haec similitudo cum circulo vix ad alias curuas, nisi quarum rectificatio ad hanc formulam reduci potest, extendi videtur.

Exem-

Exemplum.

45. *Proposita sit linea curua, cuius arcus ad quampiam rectam variabilem* v *relatus hac formula integrali* $\int \frac{dv}{\sqrt{(1-v^4)}}$ *exprimatur, cuiusmodi curuae algebraicae infinitae exhiberi possunt, in qua a puncto* P *arcus abscindi oporteat* PQ, PR, PS, *ad datum arcum* AB *rationem tenentes vel aequalitatis, vel duplam, vel triplam.*

Quia haec expressio in nostra forma generali non continetur, eo reducatur ponendo $vv = z$ seu $v = \sqrt{z}$; sic enim arcus huic nouae variabili z respondens erit $= \int \frac{dz}{2\sqrt{(z-z^4)}}$. Fiat ergo $\mathfrak{A} = \frac{1}{2}$ et $A = 0$; $B = \frac{1}{2}$; $C = 0$; $D = 0$ et $E = -1$, vnde obtinetur:

$\alpha = -1$; $\beta = M$; $\gamma = -MM$; $\delta = MM$; $\epsilon = -2$; $\zeta = -4M$

ideoque constituta aequatione:

$$0 = -1 + 2M(p+q) - MM(pp+qq) + 2MMpq - 4pq(p+q) - 4Mppqq$$

vnde fit:

$$q = \frac{M + MMp - 2pp \pm 2\sqrt{(M^2-1)(p-p^4)}}{MM + 4p + 4Mpp}$$

erit:

$$\int \frac{dq}{2\sqrt{(q-q^4)}} - \int \frac{dp}{2\sqrt{(p-p^4)}} = \text{Const.}$$

seu $\Pi : q - \Pi : p = \Pi : b - \Pi : a$

si quidem a, b, p, q sint valores variabilis z qui arcubus ΔA, ΔB, ΔP et ΔQ conueniunt. At

iam

iam constans M ex datis a et b ita definiri debet vt sit

$$0 = -1 + 2M(a+b) - MM(b-a)^2 - 4b(a+b) - 4Maabb$$

vnde fit:

$$M = \frac{a+b-2aabb \pm 2\sqrt{(a-a^4)(b-b^4)}}{(b-a)^2}.$$

et $$\sqrt{(M-1)} = \frac{\sqrt{a(1-a)(1+b+bb)} \pm \sqrt{b(1-b)(1+a+aa)}}{(b-a)};$$

$$\sqrt{(M^2-1)} = \frac{(a+3b-4ab^3)\sqrt{(a-a^4)} + (b+3a-4a^3b)\sqrt{(b-b^4)}}{(b-a)^2}.$$

Inuento hoc modo valore constantis M ex data quantitate p inuenitur q, atque hinc porro valor variabilis r puncto R respondens scilicet:

$$r = \frac{M+MMq-2qq \pm 2\sqrt{(M^2-1)(q-q^4)}}{MM+4q+4Mqq}$$

sicque a puncto P arcus quicunque multiplus arcus dati AB abscindi poterit.

Scholion.

46. Circa huiusmodi curuas singularis affectio notari meretur, si enim breuitatis gratia ponamus $\sqrt{(a-a^4)} = \mathfrak{a}$ et $\sqrt{(b-b^4)} = \mathfrak{b}$ vt sit:

$$M = \frac{a+b-2aabb+2\mathfrak{a}\mathfrak{b}}{(b-a)^2} \text{ et } \sqrt{(M^2-1)} = \frac{(a+3b-4ab^3)\mathfrak{a} + (b+3a-4a^3b)\mathfrak{b}}{(b-a)^2}$$

vtraque quantitas radicalis $\mathfrak{a}$ et $\mathfrak{b}$ tam affirmatiue quam negatiue capi potest; vnde pro M geminus valor habetur; ex quo pro $q = \frac{M+MMp-2pp \pm 2\sqrt{(M^2-1)(p-p^4)}}{MM+4p+4Mpp}$ ob nouam signi ambiguitatem quaterni valores resultant. Binos quidem natura rei ostendit, quia punctum Q tam ante quam post punctum P capi potest

potest, sed quia quatuor reperiuntur, id indicio est curuam duplici ramo esse praeditam, et in vtroque arcus aequales exhiberi. Consideremus casum quo punctum P in ipso puncto A capitur ita vt sit $p = a$; et $q = \frac{M+MMa-2aa \pm 2a\sqrt{(M^2-1)}}{MM+4a+4Maa}$ quae forma substituto pro M valore statim duos valores praebet aequales $q = b$; at duo reliqui diuersi continentur in

$$q = \frac{4a^3+9aab-6abb+b^3-4a^4-12a^4bb+8a^6b^3 \pm 4a(3a-b-2a^3b)\sqrt{ab}}{aa+6ab+bb+8a^5-24a^4b+16a^3bb-16aab^3+16a^4b^3-8a^6bb \pm 4(a+b-4a^3b+2a^4)\sqrt{ab}}$$

qui duo valores semper sunt diuersi, nisi sit vel $b = a$ vel $a = \frac{1}{1 \pm \sqrt{2}}$: illo casu prodit $q = a = b$, hoc vero reperitur $q = \frac{1-b}{1+2b}$. Punctum ergo curuae quod respondet quantitati $\frac{1}{1 \pm \sqrt{2}}$ singulari proprietate erit praeditum.

Problema 3.

47. Inuenire integrale completum huius aequationis differentialis:

$$\frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}} = \frac{2dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}.$$

Solutio.

Istud integrale quaesitum ex praecedenti problemate colligi potest. Capiatur enim punctum P in ipso puncto B, vt sit $p = b$; et consideretur tantum punctum A vt fixum, B vero seu P vt variabile, ex quo continuo assignari debeat punctum Q, vt sit

arcus

arcus AQ duplus arcus AP. Posita ergo variabili p loco b sumatur:

$$M = \frac{2A + 2B(a+p) + C(aa+pp) + 2Dap(a+p) + 2Eaapp}{(p-a)^2}$$
$$\pm \frac{2}{(p-a)^2}\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bp+Cpp+2Dp^3+Ep^4)}$$

ita vt iam M sit functio variabilis p et constantis a. Deinde posito breuitatis gratia $M - C = L$ seu

$$L = \frac{2(A+B(a+p)+Cap+Dap(a+p)+Eaapp) \pm 2\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bp+Cpp+2Dp^3+Ep^4)}}{(p-a)^2}$$

definiatur q per hanc aequationem:

$$0 = 4AC - 4BB + 4AL + 2(2BL+4AD)(p+q) + (4AE-LL)(pp+qq)$$
$$+ 2(LL+2CL+4AE+4BD)pq + 2(2DL+4BE)pq(p+q) + 4(CE-DD+EL)ppqq$$

eritque ob $b = p$:

$$\Pi : q - \Pi : p = \Pi : p - \Pi : a \text{ seu } \Pi : q = 2\Pi : p - \Pi : a$$

quae aequatio differentiata dat:

$$\frac{dq}{\sqrt{(A+2Bq+Cqq+2Dq^3+Eq^4)}} = \frac{2dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}$$

cuius propterea integralis est illa aequatio algebraica inter p et q exhibita, quam simul patet esse integralem completam, quoniam continet quantitatem constantem a, quae in aequatione differentiali non inest.

Coroll. I.

48. Si retinente L valorem exhibitum, inventaque variabili q per p, ex q simili modo quaeratur

ratur r, vt sit $\Pi : r - \Pi : q = \Pi : p - \Pi : a$ erit $\Pi : r = 3\Pi : p - 2\Pi : a$, vnde prodit aequatio differentialis:

$$\frac{dr}{\sqrt{(A+2Br+Crr+2Dr^3+Er^4)}} = \frac{3dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}$$

cuius ergo aequatio integralis completa est:

$$0 = 4(AC-BB+AL) + 2(2BL+4AD)(q+r) + (4AE-LL)(qq+rr)$$
$$+ 2(LL+2CL+4AE+4BD)qr + 2(2DL+4BE)qr(q+r) + 4(CE-DD+EL)qqrr.$$

Coroll. 2

49. Quo haec magis contrahamus, postquam ex coefficientibus datis A, B, C, D, E et variabili p vna cum constanti arbitraria a ita fuerit definita quantitas L vt sit:

$$\tfrac{1}{2}L(p-a)^2 = A+B(a+p)+Cap+Dap(a+p)+Eaapp$$
$$\pm\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bp+Cpp+2Dp^3+Ep^4)}$$

hinc determinentur sequentes coefficientes variabiles:

$\alpha = 4(AC-BB+AL)$; $\beta = 2BL+4AD$; $\gamma = 4AE-LL$
$\zeta = 4(CE-DD+EL)$; $\varepsilon = 2DL+4BE$; $\delta = LL+2CL+4AE+4BD$.

Coroll. 3.

50. His iam quantitatibus inuentis erit huius aequationis differentialis:

$$\frac{dq}{\sqrt{(A+2Bq+Cqq+2Dq^3+Eq^4)}} = \frac{2dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}$$

aequatio integralis completa:

$$0 = \alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+p) + \zeta ppqq.$$

 Coroll.

Coroll. 4.

51. Porro huius aequationis differentialis:

$$\frac{dr}{\sqrt{(A+2Br+Crr+2Dr^3+Er^4)}}=\frac{2dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}$$

aequatio integralis completa erit

$$0=\alpha+2\beta(q+r)+\gamma(qq+rr)+2\delta qr+2\varepsilon qr(q+r)+\zeta qqrr$$

poſtquam ſcilicet variabilis q ope praecedentis aequationis ex p fuerit determinata.

Coroll. 5.

52. Simili modo progrediendo huius aequationis differentialis:

$$\frac{ds}{\sqrt{(A+2Bs+Css+2Ds^3+Es^4)}}=\frac{4dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}$$

aequatio integralis completa erit

$$0=\alpha+2\beta(r+s)+\gamma(rr+ss)+2\delta rs+2\varepsilon rs(r+s)+\zeta rrss$$

poſtquam ex praecedentibus aequationibus r per q, et q per p fuerint definitae.

Coroll. 6.

53. Hoc modo quousque libuerit vlterius progredi licet, ſicque ingenere aequatio integralis inveniri poterit completa huius differentialis:

$$\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}=\frac{m\,dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}$$

quicunque numerus integer pro m aſſumatur

Proble-

Problema 4.

54. Si m et n fuerint numeri integri quicunque, inuenire aequationem integralem completam huius differentialis:

$$\frac{n\,dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=\frac{m\,dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}$$

Solutio.

Quaeratur primum ope praeced. Probl. aequatio integralis completa iſtius differentialis:

$$\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}=\frac{n\,dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}$$

quae erit algebraica ac praeter variabiles p et x conſtantem arbitrariam a inuoluens. Deinde ſimili modo quaeratur aequatio integralis completa huius differentialis:

$$\frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=\frac{m\,dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}$$

quae etiam erit algebraica inter binas variabiles y et p, inſuperque conſtantem arbitrariam b complectetur. Ex his duabus aequationibus eliminetur variabilis p, vt obtineatur aequatio algebraica inter x et y, quae erit integralis completa huius differentialis:

$$\frac{n\,dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=\frac{m\,dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}$$

Quia autem duas conſtantes arbitrarias a et b continebit, alterutri pro lubitu valorem determinatum tribuere licet, vel inter eas datam rationem ſtatuere:

re; pro integrali enim completa ſufficit vt vna conſtans arbitraria introducatur.

Scholion.

55. Si m et n ſint numeri modice magni, nemo certe aequationem algebraicam inter x et y euolutam exhibebit: cum enim tot eliminationibus ſit opus, euidens eſt, ad aequationem plurimorum terminorum, in qua variabiles x et y ad ſummas dimenſiones exſurgant, perueniri oportere. Atque adeo in caſu problematis 3. vbi eſt $m=2$ et $n=1$, nemo facile eliminationis opus perficiet. Neque vero hoc etiam opus eſt, cum ad noſtrum inſtitutum ſufficiat oſtendiſſe, aequationem integralem eſſe algebraicam, eiusque conſtructionem geometrice abſolui poſſe; tantum enim abeſt, vt alienae variabiles q, r, s etc. quae in ſubſidium ſunt vocatae, calculum turbent, ideoque eliminari debeant, vt potius ad conſtructionem commode inſtituendam abſolute ſint neceſſariae. Atque haec ſunt fere quae de curuis, quarum rectificatio hac formula:

$$\int \frac{\mathfrak{A}\, dz}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$$

exprimitur, tradi operae pretium videbatur, quae eo redeunt, vt earum arcus inter ſe perinde atque arcus circulares comparari queant; ſiquidem propoſito arcu quocunque AB, a puncto dato P arcus abſcindi poſſunt, qui ad illum rationem teneant rationalem quamcunque. Conſideremus igitur etiam curuas,

curuas, quarum rectificatio tali formula exprimitur:

$$\int \frac{dz(\mathfrak{A} + \mathfrak{B}z + \mathfrak{C}zz + \mathfrak{D}z^3 + \mathfrak{E}z^4)}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$$

de quibus curuis quoque affectiones egregiae circa comparationem arcuum notari merentur; quem in finem euolutio formularum huc pertinentium ſupra §. 16 et ſeqq. eſt inſtituta. Similis ſcilicet comparatio inter arcus huiusmodi curuarum ſuſcipi poteſt, quae iam pridem inter arcus parabolae fieri poſſe eſt oſtenſa; atque inde ſequentium problematum ſolutionem deriuare licebit.

Problema 5.

56. Propoſita curua, cuius arcus indefinite variabili cuipiam z reſpondens hac formula exprimatur: Tab. I. Fig. 4.

$$\int \frac{dz(\mathfrak{A} + \mathfrak{B}z + \mathfrak{C}zz + \mathfrak{D}z^3 + \mathfrak{E}z^4)}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$$

ſi in ea detur arcus quicunque AB, a dato puncto P arcum abſcindere PQ qui ab illo arcu AB differat linea ſiue geometrice aſſignabili, ſiue a circuli hyperbolaeue quadratura pendente.

Solutio.

Sit in curua propoſita ΔZ arcus variabili z reſpondens, qui breuitatis gratia ita exprimatur $\Pi : z$, vt ſit:

$$\Pi : z = \int \frac{dz(\mathfrak{A} + \mathfrak{B}z + \mathfrak{C}zz + \mathfrak{D}z^3 + \mathfrak{E}z^4)}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}.$$

Punctis

Punctis autem A, B, P, Q respondeant variabilis x valores a, b, p, q: vt sit

$$\Delta A = \Pi : a;\ \Delta B = \Pi : b;\ \Delta P = \Pi : p \text{ et } \Delta Q = \Pi : q$$

hincque erit

arcus datus $AB = \Pi : b - \Pi : a$

et arcus quaesitus $PQ = \Pi : q - \Pi : p$.

Iam primum ex coefficientibus A, B, C, D, E et constanti arbitraria M deinceps definienda formentur quantitates sequentes:

$\alpha = 4(AM - BB)$; $\beta = 2B(M-C) + 4AD$; $\gamma = 4AE - (M-C)^2$

$\zeta = 4(EM - DD)$; $\varepsilon = 2D(M-C) + 4BE$; $\delta = MM - CC + 4(AE + BD)$

tum vero porro statuatur:

$$\Delta = M(M-C)^2 + 4M(BD - AE) + 4(ADD + BBE) - 4BCD$$

atque inter p et q haec constituatur relatio vt sit

$$0 = \alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq$$

ex qua data variabili p altera q puncto Q respondens ita definitur vt sit

$$q = \frac{-\beta - \delta p - \varepsilon pp \pm 2\sqrt{\Delta(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}}{\gamma + 2\varepsilon p + \zeta pp}$$

vnde innotescet curuae punctum Q ita, vt differentia inter arcus AB et PQ sit vel geometrice assignabilis, vel saltem a quadratura circuli seu hyperbolae pendeat, cuius rei ratio in indole coefficientium $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ numeratoris est sita. Quomodo

modo igitur differentia ista exprimatur, videamus: quia valorem ipsius q iam inuenimus, ponamus

$p + q = s$,

et ex §. 19 colligimus fore posito $\delta - \gamma = \lambda$

$$\Pi : q - \Pi : p\ \text{Const.} - \frac{2(\mathfrak{D} + \mathfrak{E}s)s\sqrt{\Delta}}{\zeta} + \int \frac{\zeta\mathfrak{B} + \lambda\mathfrak{D} + (\zeta\mathfrak{C} + \varepsilon\mathfrak{D} + 2\lambda\mathfrak{E})s + (\zeta\mathfrak{D} + 2\varepsilon\mathfrak{E})ss + \zeta\mathfrak{E}s^3}{\zeta\sqrt{(M + 2Ds + Ess)}}\, ds$$

quod integrale manifestum est vel esse algebraicum, vel a quadratura circuli hyperbolaeue pendere. Sit istud integrale breuitatis gratia $= S$; cuius valor posito $s = a + b$ fiat $= I$, et pro constante definienda statuatur $p = a$ et $q = b$, fierique debet

$$\text{Const.} = \Pi : b - \Pi : a + \frac{2(\mathfrak{D} + \mathfrak{E}(a+b))(a+b)\sqrt{\Delta}}{\zeta} - I,$$

ex quo habebitur:

$$\text{arcus } PQ - \text{arcu } AB = \frac{2\mathfrak{D}(a+b) + 2\mathfrak{E}(a+b)^2}{\zeta}\sqrt{\Delta} - \frac{2\mathfrak{D}(p+q) - 2\mathfrak{E}(p+q)^2}{\zeta}\sqrt{\Delta} - I + \int \frac{\zeta\mathfrak{B} + \lambda\mathfrak{D} + (\zeta\mathfrak{C} + \varepsilon\mathfrak{D} + 2\lambda\mathfrak{E})s + (\zeta\mathfrak{D} + 2\varepsilon\mathfrak{E})ss + \zeta\mathfrak{E}s^3}{\zeta\sqrt{(M + 2Ds + Ess)}}\, ds.$$

At constans arbitraria M etiam ita definiri debet, vt posito $p = a$ fiat $q = b$; quocirca erit:

$$M = \frac{1}{(b-a)^2}(2A + 2B(a+b) + C(aa+bb) + 2Dab(a+b) + 2Eaabb) \pm \frac{2}{(b-a)^2}\sqrt{(A + 2Ba + Caa + 2Da^3 + Ea^4)(A + 2Bb + Cbb + 2Db^3 + Eb^4)}.$$

Hinc ergo cognita constante hac M, et ex puncto P definito puncto Q, differentia arcuum AB et PQ vel geometrice vel per quadraturam circuli hyperbolaeue assignari potest.

 Coroll.

Coroll. 1.

57. Ex datis ergo punctis A et B, seu variabilis z valoribus a et b primum constans arbitraria M ita definiatur, vt sit

$$M = \frac{1}{(b-a)^2}(2A + 2B(a+b) + C(aa+bb) + 2Dab(a+b) + 2Eaabb$$
$$\mp \frac{2}{(b-a)^2}\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bb+Cbb+2Db^3+Eb^4)}.$$

Tum hinc definitis modo praecepto coefficientibus $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ ex dato puncto P punctum Q per hanc aequationem determinetur:

$$0 = \alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq$$

atque arcuum PQ et AB differentia erit vel algebraica vel a circuli hyperbolaeue quadratura pendens.

Coroll. 2.

58. Ad istam autem arcuum differentiam assignandam capi debet posito $p+q=s$ hoc integrale, vbi $\lambda = \delta - \gamma = 2M(M-C) + 4BD$

$$S = \int \frac{\zeta\mathfrak{B} + \lambda\mathfrak{D} + (\zeta\mathfrak{C} + \varepsilon\mathfrak{D} + \lambda\mathfrak{E})s + (\zeta\mathfrak{D} + 2\varepsilon\mathfrak{E})ss + \zeta\mathfrak{E}s^3}{\zeta\sqrt{(M+2Ds+Ess)}}\, ds$$

cuius valor posito $s = a+b$ fit $= \mathrm{I}$, quo facto erit

$$\text{arc. PQ} - \text{arc. AB} = \frac{2\sqrt{\Delta}}{\zeta}(\mathfrak{D}(a+b) + \mathfrak{E}(a+b)^2 - \mathfrak{D}s - \mathfrak{E}ss) - \mathrm{I} + S$$

existente $\Delta = M(M-C)^2 + 4M(BD-AE) + 4(ADD + BBE) - 4BCD$.

Coroll.

Coroll. 3.

59. Si eueniret vt esset $\zeta = 0$, determinatio puncti Q maneret vt ante, sed pro arcuum PQ et AB differentia assignanda recurri deberet ad primas operationes. Scilicet ex $p+q=s$, quaeratur t vt sit:

$$0 = \alpha + 2\beta s + \gamma ss + 2\lambda t + 2\varepsilon st + \zeta tt$$

eritque

$$\text{arc. PQ} - \text{arc. AB} = 2\int \frac{ds(\mathfrak{B} + \mathfrak{C}s + \mathfrak{D}(ss-t) + \mathfrak{E}s(ss-2t))\sqrt{\Delta}}{\sqrt{(\lambda\lambda - \alpha\zeta + 2(\lambda\varepsilon - \beta\zeta)s + (\varepsilon\varepsilon - \gamma\zeta)ss)}}$$

integrali hoc ita accepto vt euanescat posito $s = a+b$. Vbi notandum est, esse:

$$\sqrt{(\lambda\lambda - \alpha\zeta + 2(\lambda\varepsilon - \beta\zeta)s + (\varepsilon\varepsilon - \gamma\zeta)ss)} = 2\sqrt{\Delta}(M + 2Ds + Ess) = \lambda + \varepsilon s + \zeta t.$$

Coroll. 4.

60. Hinc etiam colligere licet, quaenam sit futura differentia arcuum AB et PQ, si formulae elementum curuae exhibentis numerator ad plures terminos extendatur, vt sit arcus curuae:

$$\int \frac{dz(\mathfrak{A} + \mathfrak{B}z + \mathfrak{C}z^2 + \mathfrak{D}z^3 + \mathfrak{E}z^4 + \mathfrak{F}z^5 + \mathfrak{G}z^6 + \mathfrak{H}z^7 + \text{etc.})}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$$

reliquis enim manentibus vt ante, erit

$$\text{arc. PQ} - \text{arc. AB} = \int \frac{ds(\mathfrak{B} + \mathfrak{C}s + \mathfrak{D}(ss-t) + \mathfrak{E}(s^3-2st) + \mathfrak{F}(s^4-3sst+tt) + \text{etc.}}{\sqrt{(M + 2Ds + Ess)}}$$

sequentia scilicet numeratoris membra erunt:

$$\mathfrak{G}(s^5 - 4s^3t + 3stt) + \mathfrak{H}(s^6 - 5s^4t + 6sstt - t^3) \text{ etc.}$$

Coroll.

Coroll. 5.

61. Si a puncto Q simili modo abscindatur R, vt sit

$$o = \alpha + 2\beta(q+r) + \gamma(qq+rr) + 2\delta qr + 2\epsilon qr(q+r) + \zeta\zeta qqrr$$

ponaturque $q+r=u$ et $qr=v$, ita vt sit

$$o = \alpha + 2\beta u + \gamma uu + 2\lambda v + 2\epsilon uv + \zeta vv$$

seu $\lambda + \epsilon u + \zeta v = 2\sqrt{\Delta(M + 2Du + Euu)}$ erit

$$\text{arc. PR} - 2\,\text{arc. AB} = \int \frac{ds(\mathfrak{B} + \mathfrak{C}s + \mathfrak{D}(ss - ...) + \mathfrak{E}(s^3 - 2st) + etc.)}{\sqrt{(M + 2Ds + Ess)}}$$
$$+ \int \frac{du(\mathfrak{B} + \mathfrak{C}u + \mathfrak{D}(uu - v) + \mathfrak{E}(u^3 - 2uv) + etc.)}{\sqrt{(M + 2Du + Euu)}}$$

his integralibus ita sumtis vt euanescant posito $s=a+b$ et $u=a+b$.

Coroll. 6.

62. Simili modo a puncto P abscindi potest arcus PS, qui triplum dati arcus AB superet quantitate siue geometrice assignabili siue a circuli hyperbolaeue quadratura pendente: hisque casibus punctum P ita assumi poterit, vt iste excessus plane euanescat, quod quidem semper praestare licebit, si excessus sit algebraicus; sin autem sit transcendens, insuper alter terminus arcus dati A vel B huic scopo conformiter determinabitur.

DE

DE VSV
ALGORITHMI INFINITESIMALIS
IN ARTE CONIECTANDI SPECIMEN.

Auctore

DANIELE BERNOVLLI.

§. 1.

Cum nuper de argumento cogitarem, cuius examen in proximam differam occasionem, in quaestionem incidi, quae ad artam coniectandi pertinet, quia vero ipsa haec quaestio coniecturalis ab altero argumento videri potest prorsus aliena, non incongruum putaui eam seorsim praemittere, sicque disquisitionibus proximis veluti viam sternere, et quidem tanto libentius, vt hoc facerem, animum induxi, quod ipsa methodus nostra mox explicanda aliquid conferre posse videatur ad noua in arte coniectandi principia formanda ac stabilienda, nondum quod sciam adhibita tantoque magis geometrarum attentione digna. Quoties nempe fit, vt forte continua eaque variabili rerum status permutetur, veluti cum schedulae, diuersis numeris inscriptis distinctae, successiue ex vrna extrahuntur, vna post alteram atque leges quaeruntur pro variis inde natis mutationibus determinandis, calculi infinitesimales vtiliter adhiberi possunt ad negotium istud perficien-