CALCULI INTEGRALIS

LIBER PRIOR.

PARS PRIMA,

SEU

METHODUS INVESTIGANDI FUNCTIONES UNIUS VARIABILIS EX DATA RELATIONE QUACUNQVE DIFFERENTIALIUM PRIMI GRADUS.

SECTIO TERTIA.

DE

RESOLUTIONE AEQUATIONAM DIFFERENTIALIUM MAGIS COMPLICATARUM.

**

DE
RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM IN QUIBUS DIFFERENTIALIA AD PLURES DIMENSIONES ASSURGUNT, VEL ADEO TRANSCENDENTER IMPLICANTUR.

Problema 88.

668.

Posita differentialium relatione $\frac{\partial y}{\partial x} = p$, si proponatur aequatio quaecunque inter binas quantitates x et p, relationem inter ipsas variabiles x et y investigare.

Solutio.

Cum detur aequatio inter p et x, concessa aequationum resolutione, ex ea quaeratur p per x, ac reperietur functio ipsius x, quae ipsi p erit aequalis. Pervenietur ergo ad hujusmodi aequationem $p = X$, existente X functione quapiam ipsius x tantum. Quare cum sit $p = \frac{\partial y}{\partial x}$, habebimus $\partial y = X \partial x$, sicque quaestio ad sectionem primam est reducta, unde formulae $X \partial x$ integrale investigari oportet, quo facto integrale quaesitum erit $y = \int X \partial x$.

Si aequatio inter x et p data ita fuerit comparata, ut inde facilius x per p definiri possit, quaeratur x, prodeatque $x = P$, existente P functione quadam ipsius p. Hac igitur aequatione differentiata erit $\partial x = \partial P$, hincque $\partial y = p \partial x = p \partial P$, unde integrando elicitur $y = \int p \partial P$, seu $y = pP - \int P \partial p$. Hinc ergo ambae variabiles x et y per tertiam p ita determinantur, ut sit

$$x = P \text{ et } y = pP - \int P \partial p,$$

unde relatio inter x et y est manifesta.

Si neque p commode per x, neque x per p definiri queat, saepe effici potest, ut utraque commode per novam quantitatem u definiatur; ponamus ergo inveniri $x = U$ et $p = V$, ut U et V sint functiones ejusdem variabilis u. Hinc ergo erit $\partial y = p \partial x = V \partial U$, et $y = \int V \partial U$, sicque x et y per eandem novam variabilem u exprimuntur.

Corollarium 1.

669. Simili modo resolvetur casus, quo aequatio quaecunque inter p et alteram variabilem y proponitur, quoniam binas variabiles x et y inter se permutare licet. Tum autem sive p per y, sive y per p, sive utraque per novam variabilem u definiatur, notari oportet esse $\partial x = \frac{\partial y}{p}$.

Corollarium 2.

670. Cum $\sqrt{(\partial x^2 + \partial y^2)}$ exprimat elementum arcus curvae, cujus coordinatae rectangulae sunt x et y, si ratio

$$\frac{\sqrt{(\partial x^2 + \partial y^2)}}{\partial x} = \sqrt{(1 + pp)}, \text{ seu } \frac{\sqrt{(\partial x^2 + \partial y^2)}}{\partial y} = \frac{\sqrt{(1 + pp)}}{p},$$

aequetur functioni vel ipsius x vel ipsius y, hinc relatio inter x et y inveniri poterit.

Corollarium 3.

671. Quoniam hoc modo relatio inter x et y per integrationem invenitur, simul nova quantitas constans introducitur, quocirca illa relatio pro integrali completo erit habenda.

Scholion 1.

672. Hactenus ejusmodi tantum aequationes differentiales exa-

mini subjicimus, quibus posito $\frac{\partial y}{\partial x} = p$, ejusmodi relatio inter ternas quantitates x, y et p proponitur, unde valor ipsius p commode per x et y exprimi potest, ita ut $p = \frac{\partial y}{\partial x}$ aequetur functioni cuipiam ipsarum x et y. Nunc igitur ejusmodi relationes inter x, y et p considerandae veniunt, ex quibus valorem ipsius p vel minus commode, vel plane non, per x et y definire liceat; atque hic simplicissimus casus sine dubio est, quando in relatione proposita altera variabilis x seu y plane deest, ita ut tantum relatio inter p et x vel p et y proponatur; quem casum in hoc problemate expedivimus. Solutionis autem vis in eo versatur, ut proposita aequatione inter x et p, non littera p per x, nisi forte hoc facile praestari queat, sed potius x per p, vel etiam utraque per novam variabilem u definiatur. Veluti si proponatur haec aequatio

$$x\,\partial x + a\,\partial y = b\sqrt{(\partial x^2 + \partial y^2)},$$

quae posito $\frac{\partial y}{\partial x} = p$, abit in hanc

$$x + ap = b\sqrt{(1 + pp)},$$

hinc minus commode definiretur p per x. Cum autem sit

$$x = b\sqrt{(1 + pp)} - ap, \text{ ob } y = \int p\,\partial x = px - \int x\,\partial p,$$

erit

$$y = bp\sqrt{(1 + pp)} - app - b\int \partial p\sqrt{(1 + pp)} + \tfrac{1}{2}app;$$

sicque relatio inter x et y constat. Sin autem perventum fuerit ad talem aequationem

$$x^3\,\partial x^3 + \partial y^3 = ax\,\partial x^2\,\partial y \text{ seu } x^3 + p^3 = apx;$$

hic neque x per p neque p per x commode definire licet; ex quo pono $p = ux$, unde fit $x + u^3 x = au$, hincque $x = \frac{au}{1 + u^3}$ et $p = \frac{auu}{1 + u^3}$. Jam ob $\partial x = \frac{a\,\partial u(1 - 2u^3)}{(1 + u^3)^2}$, colligitur $y = aa\int \frac{uu\,\partial u(1 - 2u^3)}{(1 + u^3)^3}$, ac reducendo hanc formam ad simpliciorem

$$y = \tfrac{1}{6}aa.\frac{2u^3 - 1}{(1 + u^3)^2} - aa\int \frac{uu\,\partial u}{(1 + u^3)^2} \text{ seu}$$

$$y = \frac{1}{6}aa.\frac{2u^3 - 1}{(1+u^3)^2} + \frac{1}{3}aa.\frac{1}{1+u^3} + \text{Const.}$$

Scholion 2.

673. Cum igitur hunc casum, quo aequatio vel inter x et p vel inter y et p proponitur, generatim expedire licuerit, videndum est quibus casibus evolutio succedat, quando omnes tres quantitates x, y et p in aequatione proposita insunt. Ac primo quidem observo, dummodo binae variabiles x et y ubique eundem dimensionum numerum adimpleant, quomodocunque praeterea quantitas p ingrediatur, resolutionem semper ad casus ante tractatos revocari posse; tales scilicet aequationes perinde tractare licet, atque aequationes homogeneas, ad quod genus etiam merito referuntur, cum dimensiones a differentialibus natae ubique debeant esse pares, et indicium ex solis quantitatibus finitis x et y peti oporteat. Quae ergo dummodo ubique eundem dimensionum numerum constituant, aequatio pro homogenea erit habenda, veluti est

$$xx\partial y - yy\sqrt{(\partial x^2 + \partial y^2)} = 0 \text{ seu}$$

$$pxx - yy\sqrt{(1+pp)} = 0.$$

Deinde etiam ejusmodi aequationes evolutionem admittunt, in quibus altera variabilis x vel y plus una dimensione nusquam habet, utcunque praeterea differentialium ratio $p = \frac{\partial y}{\partial x}$ ingrediatur. Hos ergo casus hic accuratius explicemus.

Problema 89.

674. Posito $p = \frac{\partial y}{\partial x}$, si in aequatione inter x, y et p proposita binae variabiles x et y ubique eundem dimensionum numerum compleant, invenire relationem inter x et y, quae illius aequationis sit integrale completum.

Solutio.

Cum in aequatione inter x, y et p proposita binae variabiles

x et y ubique eundem dimensionum numerum constituant, si ponamus $y = ux$, quantitas x inde per divisionem tolletur, habebiturque aequatio inter duas tantum quantitates u et p, qua earum relatio ita definietur, ut vel u per p, vel p per u determinari possit. Jam ex positione $y = ux$ sequitur $\partial y = u \partial x + x \partial u$, cum igitur sit $\partial y = p \partial x$, erit $p \partial x - u \partial x = x \partial u$, ideoque $\frac{\partial x}{x} = \frac{\partial u}{p-u}$. Quia itaque p per u datur, formula differentialis $\frac{\partial u}{p-u}$ unicam variabilem complectens per regulas primae sectionis integretur, eritque $l x = \int \frac{\partial u}{p-u}$, sicque x per u determinatur; et cum sit $y = ux$, ambae variabiles x et y per eandem tertiam variabilem u determinantur, et quia illa integratio constantem arbitrariam inducit, haec relatio inter x et y erit integrale completum.

Corollarium 1.

675. Cum sit $\frac{\partial x}{x} = \frac{\partial u}{p-u}$, erit etiam $l x = - l(p-u) + \int \frac{\partial p}{p-u}$, quae formula commodior est, si forte ex aequatione inter p et u proposita, quantitas u facilius per p definitur.

Corollarium 2.

676. Quodsi integrale $\int \frac{\partial u}{p-u}$ vel $\int \frac{\partial p}{p-u}$ per logarithmos exprimi possit, ut sit $\int \frac{\partial u}{p-u} = l\,\mathrm{U}$, erit $l x = l\,\mathrm{C} + l\,\mathrm{U}$; hincque $x = \mathrm{C\,U}$, et $y = \mathrm{C\,U}\,u$; unde relatio inter x et y algebraice dabitur: et cum sit $u = \frac{y}{x}$, haec tertia variabilis u facile eliditur.

Scholion.

677. Eandem hanc resolutionem supra in aequationibus homogeneis ordinariis docuimus, quae ergo ob dimensiones differentialium non turbatur; quin etiam succedit, etiamsi ratio differentialium

$\frac{\partial y}{\partial x} = p$ transcendenter ingrediatur. Hoc modo scilicet resolutio ad integrationem aequationis differentialis separatae $\frac{\partial x}{x} = \frac{\partial u}{p - u}$ perducitur, quemadmodum etiam supra per priorem methodum negotium fuit expeditum. Altera vero methodus, qua supra usi sumus, quaerendo factorem qui aequationem differentialem reddat per se integrabilem, hic plane locum non habet, cum per differentiationem aequationis finitae nunquam differentialia ad plures dimensiones exsurgere queant. Non ergo hoc modo invenitur aequatio finita inter x et y, quae differentiata ipsam aequationem propositam reproducat, sed quae saltem cum ea conveniat, et quidem non obstante arbitraria illa constante, quae per integrationem ingressa, integrale completum reddit.

Exemplum 1.

678. *Si in aequationem propositam neutra variabilium x et y ipsa ingrediatur, sed tantum differentialium ratio $\frac{\partial y}{\partial x} = p$, integrale completum assignare.*

Posito ergo $\frac{\partial y}{\partial x} = p$, aequatio proposita solam variabilem p cum constantibus complectetur, unde ex ejus resolutione, prout plures involvat radices, orietur $p = \alpha$, $p = \beta$, $p = \gamma$ etc. Jam ob $p = \frac{\partial y}{\partial x}$, ex singulis radicibus integralia completa elicientur, quae erunt

$$y = \alpha x + a,\ y = \beta x + b,\ y = \gamma x + c,\ \text{etc.}$$

quae singula aequationi propositae aeque satisfaciunt. Quae si velimus omnia una aequatione finita complecti, erit integrale completum

$$(y - \alpha x - a)(y - \beta x - b)(y - \gamma x - c)\ \text{etc.} = 0,$$

quae uti apparet non unam novam constantem, sed plures a, b, c, etc. comprehendit, tot scilicet, quot aequatio differentialis plurium dimensionum habuerit radices.

Corollarium 1.

679. Ita aequationis differentialis $\partial y^2 - \partial x^2 = 0$ seu $pp - 1 = 0$, ob $p = +1$ et $p = -1$, duo habemus integralia $y = x + a$ et $y = -x + b$, quae in unum collecta dant $(y - x - a)(y + x - b) = 0$, seu

$$yy - xx - (a + b) y - (a - b) x + ab = 0.$$

Corollarium 2.

680. Proposita aequatione $\partial y^3 + \partial x^3 = 0$ seu $p^3 + 1 = 0$, ob radices $p = -1$, $p = \frac{1 + \sqrt{-3}}{2}$, et $p = \frac{1 - \sqrt{-3}}{2}$, erit vel $y = -x + a$, vel $y = \frac{1 + \sqrt{-3}}{2} x + b$, vel $y = \frac{1 - \sqrt{-3}}{2} x + c$, quae collecta praebent

$$y^3 + x^3 - (a + b + c) yy + \left(a - \frac{1 - \sqrt{-3}}{2} b - \frac{1 + \sqrt{-3}}{2} c\right) xy$$
$$+ \left(-a + \frac{1 - \sqrt{-3}}{2} b + \frac{1 + \sqrt{-3}}{2} c\right) xx + (ab + ac + bc) y$$
$$+ \left(bc - \frac{1 - \sqrt{-3}}{2} ac - \frac{1 + \sqrt{-3}}{2} ab\right) x - abc = 0,$$

quae aequatio etiam ita exhiberi potest

$$y^3 + x^3 - fyy - gxy - hxx + Ay + Bx + C = 0,$$

ubi constantes A, B, C, ita debent esse comparatae, ut aequatio haec resolutionem in tres simplices admittat.

Exemplum 2.

681. *Proposita aequatione differentiali*

$$y \partial x - x \sqrt{(\partial x^2 + \partial y^2)} = 0,$$

ejus integrale completum invenire.

Posito $\frac{\partial y}{\partial x} = p$, fit $y - x\sqrt{(1 + pp)} = 0$; sit ergo $y = ux$, erit $u = \sqrt{(1 + pp)}$, et $\frac{\partial x}{x} = \frac{\partial u}{p - u}$, unde per alteram formulam

$$lx = -l(p-u) + \int \frac{\partial p}{p - \sqrt{(1+pp)}} = -l(p-u) - \int \partial p\,[p + \sqrt{(1+pp)}]$$

at

$$\int \partial p \sqrt{(1+pp)} = \tfrac{1}{2} p \sqrt{(1+pp)} + \tfrac{1}{2} l\,[p + \sqrt{(1+pp)}];$$

unde colligitur

$$lx = C - \tfrac{1}{2} l\,[\sqrt{(1+pp)} - p] - \tfrac{1}{2} p \sqrt{(1+pp)} - \tfrac{1}{2} pp$$

vel

$$lx = C + \tfrac{1}{2} l\,[\sqrt{(1+pp)} + p] - \tfrac{1}{2} p \sqrt{(1+pp)} - \tfrac{1}{2} pp, \text{ et}$$
$$y = ux = x\sqrt{(pp+1)}.$$

Exemplum 3.

682. *Hujus aequationis*

$$y\,\partial x - x\,\partial y = nx\sqrt{(\partial x^2 + \partial y^2)}$$

integrale completum invenire.

Ob $\frac{\partial y}{\partial x} = p$, nostra aequatio est $y - px = nx\sqrt{(1+pp)}$, quae posito $y = ux$, abit in $u - p = n\sqrt{(1+pp)}$. Cum ergo sit

$$lx = -l(p-u) + \int \frac{\partial p}{p-u}, \text{ erit}$$
$$lx = -ln\sqrt{(1+pp)} - \int \frac{\partial p}{n\sqrt{(1+pp)}},$$

hincque

$$lx = C - ln\sqrt{(1+pp)} + \frac{1}{n} l\,[p + \sqrt{(1+pp)}].$$

Quare habetur

$$x = \frac{a}{\sqrt{(1+pp)}}\,[\sqrt{(1+pp)} - p]^{\frac{1}{n}}, \text{ et}$$

$$y = \frac{a\,[p + n\sqrt{(1+pp)}]}{\sqrt{(1+pp)}}\,[\sqrt{(1+pp)} - p]^{\frac{1}{n}}.$$

Cum nunc sit $uu - 2up + pp = nn + nnpp$, erit

$$p = \frac{u - n\sqrt{(uu + 1 - nn)}}{1 - nn} \text{ et } \sqrt{(1+pp)} = \frac{-nu + \sqrt{(uu + 1 - nn)}}{1 - nn}$$

atque

$$\sqrt{(1+pp)} - p = \frac{-u + \sqrt{(uu + 1 - nn)}}{1 - n},$$

unde fit

$$\frac{x[-nu + \sqrt{(uu + 1 - nn)}]}{a(1 - nn)} = \left(\frac{-u + \sqrt{(uu + 1 - nn)}}{1 - n}\right)^{\frac{1}{n}}, \text{ ubi } u = \frac{y}{x}.$$

At si $n = 1$, erit $p = \frac{uu - 1}{2u}$, $\sqrt{(1 + pp)} = \frac{uu + 1}{2u}$, atque $x = \frac{2au}{uu + 1} \cdot \frac{1}{u} = \frac{2axx}{xx + yy}$, seu $yy + xx = 2ax$.

Si $n = -1$, est quidem ut ante

$$p = \frac{uu - 1}{2u}, \text{ et } \sqrt{(1 + pp)} = \frac{-uu - 1}{2u},$$

unde

$$x = \frac{a}{\sqrt{(1 + pp)}}[\sqrt{(1 + pp)} + p] = \frac{-2a}{1 + uu} = \frac{-2axx}{xx + yy}.$$

Ergo et $x = 0$, et $xx + yy + 2ax = 0$.

Scholion.

683. Haec aequatio sumendis utrinque quadratis et radice $p = \frac{\partial y}{\partial x}$ extrahenda, ad aequationem homogeneam ordinariam reducitur. Fit enim primo

$$yy - 2pxy + ppxx = nnxx + nnppxx,$$

tum vero

$$px = \frac{x\partial y}{\partial x} = \frac{y \pm n\sqrt{(yy + xx - nnxx)}}{1 - nn},$$

quae posito $y = ux$ separabilis redditur. Ubi imprimis casus quo $nn = 1$ notari meretur, quo fit $yy - 2pxy = xx$, seu $p = \frac{\partial y}{\partial x} = \frac{yy - xx}{2xy}$, ideoque

$$2xy\partial y + xx\partial x - yy\partial x = 0:$$

quae etiam per partes integrari potest, cum $2xy\partial y - yy\partial x$ integrabile fiat per factorem $\frac{1}{xyy} f : \frac{yy}{x}$, quo ut etiam pars $xx\partial x$ integrabilis reddatur, illa forma abit in $\frac{1}{xx}$, sicque habebitur

$$\frac{2xy\partial y - yy\partial x}{xx} + \partial x = 0,$$

cujus integrale est $\frac{yy}{x} + x = 2a$, ut ante, nisi quod altera solutio $x = 0$ hinc non eliciatur. Verum cum aequatio illa quadrata posito $n = 1$, subito abeat in simplicem, altera radix perit, quae reperitur ponendo $n = 1 - \alpha$, quo fit

$$yy - 2pxy = xx - 2\alpha xx - 2\alpha ppxx,$$

ideoque px infinitum, rejectis ergo terminis prae reliquis evanescentibus est $-pxy = xx - 2\alpha ppxx$, quae divisibilis per x, alteram praebet solutionem $x = 0$. Talis quidem resolutio succedit, quando valorem p per radicis extractionem elicere licet; sed si aequatio ad plures dimensiones ascendat, vel adeo transcendens fiat, methodo hic exposita carere non possumus.

Exemplum 4.

684. *Proposita aequatione*

$$x\partial y^3 + y\partial x^3 = \partial y\partial x\sqrt{xy(\partial x^2 + \partial y^2)},$$

ejus integrale completum investigare.

Posito $\frac{\partial y}{\partial x} = p$, et $y = ux$, nostra aequatio induet hanc formam $p^3 + u = p\sqrt{u(1 + pp)}$, unde conficitur

$$\frac{\partial x}{x} = \frac{\partial u}{p - u}, \text{ seu } lx = \int\frac{\partial u}{p - u} = -l(p - u) + \int\frac{\partial p}{p - u}.$$

Inde autem est

$$\sqrt{u} = \tfrac{1}{2}p\sqrt{(1 + pp)} + \tfrac{1}{2}p\sqrt{(1 - 4p + pp)},$$

et quadrando

$$u = \tfrac{1}{2}pp - p^3 + \tfrac{1}{2}p^4 + \tfrac{1}{2}pp\sqrt{(1 + pp)(1 - 4p + pp)},$$

hincque

$$p - u = \tfrac{1}{2}p(1 + pp)(2 - p) - \tfrac{1}{2}pp\sqrt{(1 + pp)(1 - 4p + pp)},$$

unde colligimus

$$\frac{\partial p}{p - u} = \frac{\partial p(2 - p)}{2p(1 - p + pp)} + \frac{\partial p\sqrt{(1 - 4p + pp)}}{2(1 - p + pp)\sqrt{(1 + pp)}}.$$

In quorum membrorum posteriore, si ponatur $\sqrt{\frac{1 - 4p + pp}{1 + pp}} = q$, ob

$p = \frac{2 + \sqrt{[4 - (1 - qq)^2]}}{1 - qq}$, $\partial p = \frac{4q\partial q\,[2 + \sqrt{(4 - (1 - qq)^2)}]}{(1 - qq)^2 \sqrt{[4 - (1 - qq)^2]}}$, et

$$1 - p + pp = \frac{(3 + qq)\,[2 + \sqrt{(4 - (1 - qq)^2)}]}{(1 - qq)^2}$$

obtinebitur

$$\int \frac{\partial p}{p - u} = \tfrac{1}{2} \int \frac{\partial p\,(2 - p)}{p\,(1 - p + pp)} + 2 \int \frac{qq\partial q}{(3 + qq)\sqrt{[4 - (1 - qq)^2]}};$$

ubi membrum posterius neque per logarithmos, neque arcus circulares integrari potest.

Exemplum 5.

685. *Invenire relationem inter x et y, ut posito $s = \int \sqrt{(\partial x^2 + \partial y^2)}$, fiat $ss = 2xy$.*

Cum sit $s = \sqrt{2xy}$, erit

$$\partial s = \sqrt{(\partial x^2 + \partial y^2)} = \frac{x\partial y + y\partial x}{\sqrt{2xy}},$$

hincque posito $\frac{\partial y}{\partial x} = p$ et $y = ux$, fiet $\sqrt{(1 + pp)} = \frac{p + u}{\sqrt{2u}}$, seu $u = \sqrt{2u(1 + pp)} - p$, et radice extracta

$$\sqrt{u} = \sqrt{\frac{1 + pp}{2}} + \frac{1 - p}{\sqrt{2}} = \frac{1 - p + \sqrt{(1 + pp)}}{\sqrt{2}},$$

quare

$$u = 1 - p + pp + (1 - p)\sqrt{(1 + pp)}, \text{ et}$$

$$p - u = -(1 - p)[1 - p + \sqrt{(1 + pp)}].$$

Ergo

$$\int \frac{\partial p}{p - u} = \int \frac{\partial p}{2p(1 - p)}[1 - p - \sqrt{(1 + pp)}] = \tfrac{1}{2} lp - \tfrac{1}{2} \int \frac{\partial p \sqrt{(1 + pp)}}{p(1 - p)}.$$

At posito $p = \frac{1 - qq}{2q}$, fit

$$\int \frac{\partial p \sqrt{(1 + pp)}}{p(1 - p)} = \int \frac{-\partial q(1 + qq)^2}{q(1 - qq)(qq + 2q - 1)} = + \int \frac{\partial q}{q} - 2 \int \frac{\partial q}{1 - qq} - 4 \int \frac{\partial q}{(q + 1)^2 - 2}$$

$$= + lq - l \frac{1 + q}{1 - q} + \sqrt{2}\, l \frac{\sqrt{2} + 1 + q}{\sqrt{2} - 1 - q},$$

hincque

$$\int\frac{\partial p}{p-u}=\tfrac{1}{2}lp-\tfrac{1}{2}lq+\tfrac{1}{2}l\frac{1+q}{1-q}-\frac{1}{\sqrt{2}}l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}$$
$$=l\left(\frac{1+q}{2q}\right)-\frac{1}{\sqrt{2}}l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}.$$

Jam

$$p-u=\frac{(1+q)(1-2q-qq)}{2q}=+\frac{(1+q)[2-(1+q)^2]}{2q},$$

sicque habetur

$$lx=C-l(1+q)+lq-l[2-(1+q)^2]+l\left(\frac{1+q}{q}\right)$$
$$-\frac{1}{\sqrt{2}}l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}=la-l[2-(1+q)^2]+\frac{1}{\sqrt{2}}l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}$$

ubi est $u=\frac{y}{x}=\frac{1}{2}(1+q)^2$, et $1+q=\sqrt{\frac{2y}{x}}$, unde

$$x=\frac{ax}{x-y}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}} \text{ seu } x-y=a\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}}, \text{ vel}$$
$$(\sqrt{x}+\sqrt{y})^{1+\frac{1}{\sqrt{2}}}=a(\sqrt{x}-\sqrt{y})^{\frac{1}{\sqrt{2}}-1}.$$

Est ergo aequatio inter x et y interscendens, uti vocari solet.

Scholion.

686. Facilius haec resolutio absolvitur quaerendo statim ex aequatione

$$u+p=\sqrt{2u(1+pp)}, \text{ seu } uu+2up+pp=2u+2upp$$

valorem ipsius p, qui fit

$$p=\frac{u+\sqrt{(uu-4uu+2u+2u^3-uu)}}{2u-1}, \text{ seu } p=\frac{u+(1-u)\sqrt{2u}}{2u-1}, \text{ et}$$
$$p-u=\frac{(1-u)(2u+\sqrt{2u})}{2u-1}=\frac{(1-u)\sqrt{2u}}{\sqrt{2u}-1}.$$

Quare

$$lx=\int\frac{\partial u}{p-u}=\int\frac{\partial u(\sqrt{2u}-1)}{(1-u)\sqrt{2u}}=C-l(1-u)-\int\frac{\partial u}{(1-u)\sqrt{2u}}.$$

Sit $u=vv$, eritque

$$\int\frac{\partial u}{(1-u)\sqrt{2u}}=\frac{1}{\sqrt{2}}\int\frac{2\partial v}{1-vv}=\frac{1}{\sqrt{2}}l\frac{1+v}{1-v},$$

hincque

$$lx = la - l(1 - uu) - \frac{1}{\sqrt{2}} l \frac{1 + \sqrt{u}}{1 - \sqrt{u}}$$

Unde ob $u = \frac{y}{x}$, reperitur $x = \frac{ax}{x - y} \left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)^{\frac{1}{\sqrt{2}}}$, ut ante. Quare si curva desideretur coordinatis rectangulis x et y determinanda, ut ejus arcus s sit $= \sqrt{2xy}$, erit aequatio ejus naturam definiens

$$(\sqrt{x} + \sqrt{y})^{\frac{1}{\sqrt{2}} + 1} = a(\sqrt{x} - \sqrt{y})^{\frac{1}{\sqrt{2}} - 1}.$$

Caeterum evidens est simili modo quaestionem resolvi posse, si arcus s functioni cuicunque homogeneae unius dimensionis ipsarum x et y aequetur, seu si proponatur aequatio quaecunque homogenea inter x, y et s, id quod sequenti problemate ostendisse operae erit pretium.

Problema 90.

687. Si fuerit $s = \int \sqrt{(\partial x^2 + \partial y^2)}$, atque aequatio proponatur homogenea quaecunque inter x, y et s, in qua scilicet hae tres variabiles x, y et s, ubique eundem dimensionum numerum constituant, invenire aequationem finitam inter x et y.

Solutio.

Ponatur $y = ux$ et $s = vx$, ut hac substitutione ex aequatione homogenea proposita variabilis x elidatur, et aequatio obtineatur inter binas u et v, unde v per u definiri possit. Tum vero sit $\partial y = p \partial x$, eritque

$\partial s = \partial x \sqrt{(1 + pp)}$, unde fit

$p \partial x = u \partial x + x \partial u$, et $\partial x \sqrt{(1 + pp)} = v \partial x + x \partial v$,

ergo

$$\frac{\partial x}{x} = \frac{\partial u}{p - u} = \frac{\partial v}{\sqrt{(1 + pp)} - v}$$

Quia nunc v datur per u, sit $\partial v = q \partial u$, ut habeatur

$$\sqrt{(1+pp)} = v + pq - qu,$$

et sumtis quadratis

$$1 + pp = (v - qu)^2 + 2pq(v - qu) + ppqq,$$

unde elicitur

$$p = \frac{q(v-qu) + \sqrt{[(v-qu)^2 - 1 + qq]}}{1 - qq} \text{ et}$$

$$p - u = \frac{qv - u + \sqrt{[(v-qu)^2 - 1 + qq]}}{1 - qq}.$$

Quare hinc deducimus

$$\frac{\partial x}{x} = \frac{\partial u(1-qq)}{qv - u + \sqrt{[(v-qu)^2 - 1 + qq]}} = \frac{\partial u(qv - u - \sqrt{[(v-qu)^2 - 1 + qq]})}{1 + uu - vv},$$

unde cum v et q dentur per u, inveniri potest x per eandem u: at ob $q\partial u = \partial v$ fiet

$$lx = la - l\sqrt{(1 + uu - vv)} - \int \frac{\partial u\sqrt{[(v-qu)^2 - 1 + qq]}}{1 + uu - vv},$$

tum vero est $y = ux$, seu posito $\frac{y}{x}$ loco u habebitur aequatio quaesita inter x et y.

Corollarium 1.

688. Cum s exprimat arcum curvae coordinatis rectangulis x et y respondentem, sic definitur curva, cujus arcus aequatur functioni cuicunque unius dimensionis ipsarum x et y; quae ergo erit algebraica, si integrale

$$\int \frac{\partial u\sqrt{[(v-qu)^2 - 1 + qq]}}{1 + uu - vv}$$

per logarithmos exhiberi potest.

Corollarium 2.

689. Simili modo resolvi poterit problema, si s ejusmodi formulam integralem exprimat, ut sit $\partial s = Q\partial x$, existente Q functione quacunque quantitatum p, u et v. Tum autem ex aequalitate $\frac{\partial x}{x} = \frac{\partial u}{p - u} = \frac{\partial u}{Q - v}$ valorem ipsius p elici oportet, et quia v per u datur, erit $lx = \int \frac{\partial u}{p - u}$.

Exemplum 1.

690. Si debeat esse $s = \alpha x + \beta y$, erit $v = \alpha + \beta u$, et $q = \frac{\partial v}{\partial u} = \beta$, hinc $v - qu = \alpha$, ergo

$$lx = la - l\sqrt{[1 + uu - (\alpha + \beta u)^2]} - \int \frac{\partial u \sqrt{(\alpha\alpha + \beta\beta - 1)}}{1 + uu - (\alpha + \beta u)^2},$$

quae postrema pars est

$$-\int \frac{\partial u \sqrt{(\alpha\alpha + \beta\beta - 1)}}{1 - \alpha\alpha - 2\alpha\beta u + (1 - \beta\beta)uu} = (\alpha\alpha + \beta\beta - 1)^{\frac{1}{2}} \int \frac{\partial u}{\alpha\alpha + 1 - 2\alpha\beta u + (\beta\beta - 1)uu};$$

quae transformatur in

$$\int \frac{(\beta\beta - 1)\partial u \sqrt{(\alpha\alpha + \beta\beta - 1)}}{[u(\beta\beta - 1) + \alpha\beta - \sqrt{(\alpha\alpha + \beta\beta - 1)}][u(\beta\beta - 1) + \alpha\beta + \sqrt{(\alpha\alpha + \beta\beta - 1)}]}$$
$$= \tfrac{1}{2} l \frac{(\beta\beta - 1)u + \alpha\beta - \sqrt{(\alpha\alpha + \beta\beta - 1)}}{(\beta\beta - 1)u + \alpha\beta + \sqrt{(\alpha\alpha + \beta\beta - 1)}}.$$

Quare posito $u = \frac{y}{x}$, aequatio integralis quaesita est, sumtis quadratis,

$$\frac{xx + yy - (\alpha x + \beta y)^2}{aa} = \frac{(\beta\beta - 1)y + \alpha\beta x - x\sqrt{(\alpha\alpha + \beta\beta - 1)}}{(\beta\beta - 1)y + \alpha\beta x + x\sqrt{(\alpha\alpha + \beta\beta - 1)}}.$$

At posito

$$(\beta\beta - 1)y + \alpha\beta x - x\sqrt{(\alpha\alpha + \beta\beta - 1)} = P$$
$$(\beta\beta - 1)y + \alpha\beta x + x\sqrt{(\alpha\alpha + \beta\beta - 1)} = Q$$

est

$$PQ = (\beta\beta - 1)^2 yy + 2\alpha\beta(\beta\beta - 1)xy + (\alpha\alpha - 1)(\beta\beta - 1)xx$$
$$= (\beta\beta - 1)[(\alpha x + \beta y)^2 - xx - yy],$$

unde mutata constante fit $\frac{PQ}{bb} = \frac{P}{Q}$, ergo vel $P = 0$ vel $Q = b$; solutio ergo in genere est

$$(\beta\beta - 1)y + \alpha\beta x \pm x\sqrt{(\alpha\alpha + \beta\beta - 1)} = c,$$

quae est aequatio pro linea recta.

Exemplum 2.

691. Si debeat esse $s = \frac{nyy}{x}$, erit $v = nuu$ et $q = 2nu$; unde $1 + uu - vv = 1 + uu - nnu^4$ et $v - qu = -nuu$, ergo

$$lx = la - l\sqrt{(1 + uu - nnu^4)} - \int \frac{\partial u \sqrt{(nnu^4 - 1 + 4nnuu)}}{1 + uu - nnu^4}$$

quae formula autem per logarithmos integrari nequit.

Exemplum 3.

692. Si debeat esse $ss = xx + yy$, erit $v = \sqrt{(1 + uu)}$ et $q = \frac{u}{\sqrt{(1 + uu)}}$, unde fit $1 + uu - vv = 0$, solutionem ergo ex primis formulis repeti convenit, unde fit

$$v - qu = \frac{1}{\sqrt{(1 + uu)}};$$

$$qq - 1 = \frac{-1}{1 - uu}, \text{ et } qv - u = 0;$$

ergo $p - u = 0$, seu $\frac{\partial y}{\partial x} - \frac{y}{x} = 0$, ita ut prodeat $y = nx$.

Exemplum 4.

693. Si debeat esse $ss = yy + nxx$, seu $v = \sqrt{(uu + n)}$, et $q = \frac{u}{\sqrt{(uu + n)}}$, erit $1 + uu - vv = 1 - n$, $v - qu = \frac{n}{\sqrt{(uu + n)}}$ et $qq - 1 = \frac{-n}{uu + n}$. Quare habebitur

$$lx = la - l\sqrt{(1 - n)} - \frac{1}{1 - n} \int \frac{\partial u \sqrt{(nn + n)}}{\sqrt{(uu + n)}}$$

$$= lb + \frac{\sqrt{n}}{\sqrt{(n - 1)}} l[u + \sqrt{(uu + n)}],$$

hincque

$$\frac{x}{b} = \left(\frac{y + \sqrt{(yy + nxx)}}{x}\right)^{\sqrt{\frac{n-1}{n}}}.$$

Quoties ergo $\frac{n}{n - 1}$ est numerus quadratus, aequatio inter x et y prodit algebraica. Sit $\sqrt{\frac{n}{n - 1}} = m$, erit $n = \frac{mm}{mm - 1}$ et $ss = yy + \frac{mmxx}{mm - 1}$, cui conditioni satisfit hac aequatione algebraica

$$x^{m + 1} = b[y + \sqrt{(yy + \frac{mmxx}{mm - 1})}]^m$$

quae tranformatur in

$$x^{\frac{2}{m}} - 2b^{\frac{1}{m}} x^{\frac{1 - m}{m}} y = \frac{mm}{mm - 1} b^{\frac{2}{m}}, \text{ seu}$$

$$y = \frac{(mm-1)x^m - mmb^m}{2(mm-1)b^m x^m}$$

Corollarium.

694. Ponamus $m = \frac{1}{n}$, ac si fuerit

$$y = \frac{b^{2n} + (nn-1)x^{2n}}{2(nn-1)b^n x^{n-1}}, \text{ erit}$$

$$ss = yy - \frac{xx}{nn-1}, \text{ seu } s = \sqrt{\left(yy - \frac{xx}{nn-1}\right)}.$$

Quare si

$$y = \frac{b^4 + 3x^4}{6bbx}, \text{ est } s = \sqrt{\left(yy - \frac{xx}{3}\right)}.$$

Problema 91.

695. Si posito $\frac{\partial y}{\partial x} = p$, ejusmodi detur aequatio inter x, y et p, in qua altera variabilis y unicam tantum habeat dimensionem, invenire relationem inter binas variabiles x et y.

Solutio.

Hinc ergo y aequabitur functioni cuipiam ipsarum x et p, unde differentiando fiet $\partial y = P\partial x + Q\partial p$. Cum igitur sit $\partial y = p\partial x$, habebitur haec aequatio differentialis $(P-p)\partial x + Q\partial p = 0$, quam integrari oportet. Quoniam tantum duas continet variabiles x et p, et differentialia simpliciter involvit, ejus resolutio per methodos supra expositas est tentanda.

Primo ergo resolutio succedet, si fuerit $P = p$, ideoque $\partial y = p\partial x + Q\partial p$. Quod evenit, si y per x et p ita determinetur, ut sit $y = px + \Pi$, denotante Π functionem quamcunque ipsius p. Tum ergo erit $Q = x + \frac{\partial \Pi}{\partial p}$, et cum solutio ab ista aequatione $Q\partial p = 0$ pendeat, erit vel $\partial p = 0$, hincque $p = \alpha$, seu $y = \alpha x + \beta$, ubi altera constantium α et β per ipsam aequationem propositam determinatur, dum posito $p = \alpha$ fit $\beta = \Pi$; vel erit $Q = 0$, ideoque $x = -\frac{\partial \Pi}{\partial p}$, et $y = -\frac{p\partial \Pi}{\partial p} + \Pi$, ubi ergo utraque solutio est algebraica, si modo Π fuerit functio algebraica ipsius p.

Secundo, aequatio $(P-p)\partial x + Q\partial p = 0$, resolutionem admittet, si altera variabilis x cum suo differentiali ∂x unam dimensionem non superet. Evenit hoc si fuerit $y = Px + \Pi$, dum P et Π sunt functiones ipsius p tantum, tum enim erit $P = P$ et $Q = \frac{x\partial P}{\partial p} + \frac{\partial \Pi}{\partial p}$, hincque haec habeatur aequatio integranda

$$(P-p)\partial x + x\partial P + \partial \Pi = 0 \text{ seu } \partial x + \frac{x\partial P}{P-p} = -\frac{\partial \Pi}{P-p},$$

quae per $e^{\int \frac{\partial p}{P-p}}$ multiplicata dat

$$e^{\int \frac{\partial p}{P-p}} x = -\int e^{\int \frac{\partial P}{P-p}} \frac{\partial \Pi}{P-p}.$$

Sive ponatur $\frac{\partial P}{P-p} = \frac{\partial R}{R}$, erit aequatio integralis

$$R x = C - \int \frac{R\partial \Pi}{P-p} = C - \int \frac{\partial \Pi \partial R}{\partial P};$$

unde fit

$$x = \frac{C}{R} - \frac{1}{R}\int \frac{\partial \Pi \partial R}{\partial P}, \text{ et}$$

$$y = \frac{CP}{R} + \Pi - \frac{P}{R}\int \frac{\partial \Pi \partial R}{\partial P}.$$

Tertio resolutio nullam habebit difficultatem, si denotantibus X et V functiones quascunque ipsius x, fuerit $y = X + Vp$. Tum enim erit

$$\partial y = p\partial x = \partial X + V\partial p + p\partial V,$$

ideoque

$$\partial p + p\left(\frac{\partial V - \partial x}{V}\right) = -\frac{\partial X}{V},$$

sit $\frac{\partial x}{V} = \frac{\partial R}{R}$, ut R sit etiam functio ipsius x, erit

$$\frac{V}{R} p = C - \int \frac{\partial X}{R}, \text{ seu } p = \frac{CR}{V} - \frac{R}{V}\int \frac{\partial X}{R}, \text{ et}$$

$$y = X + CR - R\int\frac{\partial X}{R},$$

quae aequatio relationem inter x et y exprimit.

Quarto aequatio $(P - p)\partial x + Q\partial p = 0$ resolutionem admittit si fuerit homogenea. Cum ergo terminus $p\partial x$ duas contineat dimensiones, hoc evenit, si totidem dimensiones et in reliquis terminis insint. Unde perspicuum est, P et Q esse debere functiones homogeneas unius dimensionis ipsarum x et p. Quare si y ita per x et p definiatur, ut y aequetur functioni homogeneae duarum dimensionum ipsarum x et p, resolutio succedet. Quodsi enim fuerit $\partial y = P\partial x + Q\partial p$, aequatio solutionem continens $(P - p)\partial x + Q\partial p = 0$, erit homogenea, fietque per se integrabilis, si dividatur per $(P - p)x + Qp$.

Corollarium 1.

696. Pro casu quarto si ponatur $y = zz$, aequatio proposita debet esse homogenea inter tres variabiles x, z et p. Unde si proponatur aequatio homogenea quaecunque inter x, z et p, in qua hae ternae litterae x, z et p ubique eundem dimensionum numerum constituant, problema semper resolutionem admittit.

Corollarium 2.

697. Simili modo conversis variabilibus, si ponatur $x = vv$ et $\frac{\partial x}{\partial y} = q$, ut sit $p = \frac{1}{q}$; ac proponatur aequatio homogenea quaecunque inter y, v et q, problema itidem resolvi potest.

Scholion.

698. Pro casu quarto, ut aequatio $(P - p)\partial x + Q\partial p = 0$ fiat homogenea, conditiones magis amplificari possunt. Ponatur enim $x = v^{\mu}$ et $p = q^{\nu}$, sitque facta substitutione haec aequatio

$$\mu(P - q^{\nu})v^{\mu-1}\partial v + \nu Q q^{\nu-1}\partial q = 0$$

homogenea inter v et q, eritque P functio homogenea ν dimensionum, et Q functio homogenea μ dimensionum. Cum jam sit

$$\partial y = \mathrm{P}\,\partial x + \mathrm{Q}\,\partial p = \mu \mathrm{P}\, v^{\mu-1}\,\partial v + \nu \mathrm{Q}\, q^{\nu-1}\,\partial q,$$

erit y functio homogenea $\mu + \nu$ dimensionum. Quare posito $y = z^{\mu+\nu}$ problema resolutionem admittit, si inter x, y et p ejusmodi relatio proponatur, ut posito $y = z^{\mu+\nu}$, $x = v^{\mu}$ et $p = q^{\nu}$ habeatur aequatio homogenea inter ternas quantitates z, v et q, ita ut dimensionum ab iis formatarum numerus ubique sit idem. Ac si proposita fuerit hujusmodi aequatio homogenea inter z, v et q, solutio problematis ita expedietur. Cum sit $\partial y = p\,\partial x$, erit $(\mu + \nu)\, z^{\mu+\nu-1}\,\partial z = \mu\, v^{\mu-1} q^{\nu}\,\partial v$; ponatur jam $z = r q$ et $v = s q$, et aequatio proposita tantum duas litteras r et s continebit, ex qua alteram per alteram definire licet, tum autem per has substitutiones prodibit haec aequatio

$$(\mu + \nu)\, r^{\mu+\nu-1}\, q^{\mu+\nu-1}\,(r\,\partial q + q\,\partial r) = \mu\, s^{\mu-1}\, q^{\mu+\nu-1}\,(s\,\partial q + q\,\partial s),$$

ex qua oritur

$$\frac{\partial q}{q} = \frac{\mu\, s^{\mu-1}\,\partial s - (\mu+\nu)\, r^{\mu+\nu-1}\,\partial r}{(\mu+\nu)\, r^{\mu+\nu} - \mu\, s^{\mu}},$$

quae est aequatio differentialis separata, quoniam s per r datur. Quin etiam bini casus allati manifesto continentur in formulis $y = z^{\mu+\nu}$, $x = v^{\mu}$ et $p = q^{\nu}$; prior scilicet si $\mu = 1$ et $\nu = 1$, posterior vero si $\mu = 2$ et $\nu = -1$. Hos igitur casus perinde ac praecedentes exemplis illustrari conveniet, quorum primus praecipue est memorabilis, cum per differentiationem aequationis propositae $y = px + \Pi$ statim praebeat aequationem integralem quaesitam, neque integratione omnino sit opus, siquidem alteram solutionem ex $\partial p = 0$ natam excludamus.

Exemplum 1.

699. *Proposita aequatione differentiali*

$$y\,\partial x - x\,\partial y = a\sqrt{(\partial x^2 + \partial y^2)}$$

ejus integrale invenire.

Posito $\frac{\partial y}{\partial x} = p$ fit $y - p x = a \sqrt{(1 + p p)}$, quae aequatio differentiata, ob $\partial y = p \partial x$, dat $- x \partial p = \frac{a p \partial p}{\sqrt{(1 + p p)}}$, quae cum sit divisibilis per ∂p praebet primo $p = \alpha$, hincque $y = \alpha x + a \sqrt{(1 + \alpha \alpha)}$. Alter vero factor suppeditat $x = \frac{- a p}{\sqrt{(1 + p p)}}$, hincque

$$y = \frac{- a p p}{\sqrt{(1 + p p)}} + a \sqrt{(1 + p p)} = \frac{a}{\sqrt{(1 + p p)}},$$

unde fit $x x + y y = a a$, quae est etiam aequatio integralis, sed quia novam constantem non involvit, non pro completo integrali haberi potest. Integrale autem completum duas aequationes complectitur. Scilicet

$$y = \alpha x + a \sqrt{(1 + \alpha \alpha)} \text{ et } x x + y y = a a,$$

quae in hac una comprehendi possunt

$$[(y - \alpha x)^2 - a a (1 + \alpha \alpha)] (x x + y y - a a) = 0.$$

Scholion.

700. Nisi hoc modo operatio instituatur, solutio hujus quaestionis fit satis difficilis. Si enim aequationem differentialem $y \partial x - x \partial y = a \sqrt{(\partial x^2 + \partial y^2)}$ quadrando ab irrationalitate liberemus, indeque rationem $\frac{\partial y}{\partial x}$ per radicis extractionem definiamus, fit

$$(x x - a a) \partial y - x y \partial x = \pm a \partial x \sqrt{(x x + y y - a a)}$$

quae aequatio per methodos cognitas difficulter tractatur. Multiplicator quidem inveniri potest utrumque membrum per se integrabile reddens; prius enim membrum $(x x - a a) \partial y - x y \partial x$ divisum per $y (x x - a a)$ fit integrabile, integrali existente $= l \frac{y}{\sqrt{(x x - a a)}}$: unde in genere multiplicator id integrabile reddens est

$$\frac{1}{y (x x - a a)} \Phi : \frac{y}{\sqrt{(x x - a a)}}$$

quae functio ita determinari debet, ut eodem multiplicatore quoque alterum membrum $a \partial x \sqrt{(x x + y y - a a)}$ fiat integrabile. Talis autem multiplicator est:

$$\frac{1}{y(xx-aa)} \cdot \frac{y}{\sqrt{(xx+yy-aa)}} = \frac{1}{(xx-aa)\sqrt{(xx+yy-aa)}}$$

quo fit

$$\frac{(xx-aa)\partial y - xy\partial x}{(xx-aa)\sqrt{(xx+yy-aa)}} = \frac{\pm a\partial x}{xx-aa}.$$

Jam ad integrale prioris membri investigandum, spectetur x ut constans, eritque integrale

$$= l[y+\sqrt{(xx+yy-aa)}] + X,$$

denotante X functionem quampiam ipsius x, ita comparatam, ut sumta jam y constante fiat.

$$\frac{x\partial x}{[y+\sqrt{(xx+yy-aa)}]\sqrt{(xx+yy-aa)}} + \partial X = \frac{-xy\partial x}{(xx-aa)\sqrt{(xx+yy-aa)}}.$$

seu

$$\frac{-x\partial x[y-\sqrt{(xx+yy-aa)}]}{(xx-aa)\sqrt{(xx+yy-aa)}} + \partial X = \frac{-xy\partial x}{(xx-aa)\sqrt{(xx+yy-aa)}},$$

unde fit

$$\partial X = \frac{-x\partial x}{xx-aa} \text{ et } X = l\frac{C}{\sqrt{(xx-aa)}}.$$

Quare integrale quaesitum est

$$l[y+\sqrt{(xx+yy-aa)}] + l\frac{C}{\sqrt{(xx-aa)}} = \pm\tfrac{1}{2}l\frac{a+x}{a-x},$$

unde fit

$$y+\sqrt{(xx+yy-aa)} = \alpha(x \pm a), \text{ hincque}$$
$$xx-aa = \alpha\alpha(x\pm a)^2 - 2\alpha(x\pm a)y, \text{ vel}$$
$$x \mp a = \alpha\alpha(x\pm a) - 2\alpha y$$

quae autem tantum est altera binarum aequationum integralium, altera autem aequatio integralis $xx+yy=aa$ jam quasi per divisionem de calculo sublata est censenda. Caeterum eadem solutio aequationis

$$(aa-xx)\partial y + xy\partial x = \pm a\partial x\sqrt{(xx+yy-aa)}$$

facilius instituitur ponendo $y = u\sqrt{(aa-xx)}$, unde fit

$$(aa-xx)^{\frac{3}{2}}\partial u = \pm a\partial x\sqrt{(aa-xx)(uu-1)} \text{ seu}$$
$$\frac{\partial u}{\sqrt{(uu-1)}} = \frac{\pm a\partial x}{aa-xx},$$

cui quidem satisfit sumendo $u=1$, neque tamen hic casus in aequatione integrali continetur, uti supra jam ostendimus. Ex quo suspicari liceret alteram solutionem $xx + yy = aa$ adeo esse excludendam, quod tamen secus se habere deprehenditur; si ipsam aequationem primariam $\frac{y\partial x - x\partial y}{\sqrt{(\partial x^2 + \partial y^2)}} = \alpha$ perpendamus. Si enim x et y sint coordinatae rectangulae lineae curvae, formula $\frac{y\partial x - x\partial y}{\sqrt{(\partial x^2 + \partial y^2)}}$ exprimit perpendiculum ex origine coordinatarum in tangentem dimissum, quod ergo constans esse debet. Hoc autem evenire in circulo, origine in centro constituta, dum aequatio fit $xx + yy = aa$, per se est manifestum. Atque hinc realitas harum solutionum, quae minus congruae videri poterant, confirmatur, etiamsi earum ratio haud satis clare perspicitur.

Exemplum 2.

701. *Proposita aequatione differentiali*

$$y\partial x - x\partial y = \frac{a(\partial x^2 + \partial y^2)}{\partial x}$$

ejus integrale invenire.

Posito $\partial y = p\partial x$, fit $y - px = a(1 + pp)$, et differentiando $-x\partial p = 2ap\partial p$; unde concluditur vel $\partial p = 0$, et $p = \alpha$, hincque $y = \alpha x + a(1 + \alpha\alpha)$, vel $x = -2ap$ et $y = a(1 - pp)$, sicque, ob $p = \frac{-x}{2a}$, habebitur $4ay = 4aa - xx$, quae aequatio ad geometriam translata illam conditionem omnino adimplet.

Ex aequatione autem proposita radicem extrahendo reperitur

$$2a\partial y + x\partial x = \partial x\sqrt{(xx + 4ay - 4aa)},$$

quae posito $y = u(4aa - xx)$, abit in

$$2a\partial u(4aa - xx) - x\partial x(4au - 1)$$
$$= \partial x\sqrt{(4aa - xx)(4au - 1)},$$

haecque posito $4au - 1 = tt$, in

$$t\partial t(4aa - xx) - tt x\partial x = t\partial x\sqrt{(4aa - xx)},$$

quae cum sit divisibilis per t, concludere licet $t = 0$, ideoque $u = \frac{1}{4a}$, atque hinc $4ay = 4aa - xx$.

Exemplum 3.

702. *Proposita aequatione differentiali*

$$y\partial x - x\partial y = a\sqrt[3]{(\partial x^3 + \partial y^3)},$$

ejus integrale assignare.

Haec aequatio more consueto, si rationem $\frac{\partial y}{\partial x}$ inde extrahere vellemus, vix tractari posset. Posito autem $\partial y = p\partial x$ fit $y - px = a\sqrt[3]{(1+p^3)}$, et differentiando $x\partial p = \frac{-app\partial p}{\sqrt[3]{(1+p^3)^2}}$, unde duplex conclusio deducitur, vel $\partial p = 0$ et $p = \alpha$, sicque $y = \alpha x + a\sqrt[3]{(1+\alpha^3)}$, vel

$$x = \frac{-app}{\sqrt[3]{(1+p^3)^2}} \text{ et } y = \frac{a}{\sqrt[3]{(1+p^3)^2}},$$

unde fit $pp = -\frac{x}{y}$, et ob $y^3(1+p^3)^2 = a^3$, erit $p^3 = \frac{a\sqrt{a}}{y\sqrt{y}} - 1$, hincque $\frac{(a\sqrt{a} - y\sqrt{y})^2}{y^3} = -\frac{x^3}{y^3}$, seu $x^3 + (a\sqrt{a} - y\sqrt{y})^2 = 0$.

Exemplum 4.

703. *Proposita aequatione differentiali*

$$y\partial x - nx\partial y = a\sqrt{(\partial x^2 + \partial y^2)},$$

ejus integrale invenire.

Posito $\partial y = p\partial x$, habetur $y - npx = a\sqrt{(1+pp)}$, unde differentiando elicitur

$$(1-n)p\partial x - nx\partial p = \frac{ap\partial p}{\sqrt{(1+pp)}}, \text{ sive}$$

$$\partial x - \frac{nx\partial p}{(1-n)p} = \frac{a\partial p}{(1-n)\sqrt{(1+pp)}},$$

quae per $p^{\frac{n}{n-1}}$ multiplicata et integrata praebet

$$p^{\frac{n}{n-1}}x = \frac{a}{1-n}\int\frac{p^{\frac{n}{n-1}}\partial p}{\sqrt{(1+pp)}}.$$

Hinc deducimus casus sequentes, integrationem admittentes

si $n = \frac{3}{2}$; $p^3 x = C - \frac{2}{3} a (pp - \frac{2}{1}) \sqrt{(1 + pp)}$,
si $n = \frac{5}{4}$; $p^5 x = C - \frac{4}{5} a (p^3 - \frac{4}{3} p^3 + \frac{4.2}{3.1}) \sqrt{(1 + pp)}$,
si $n = \frac{7}{6}$; $p^7 x = C - \frac{6}{7} a (p^6 - \frac{6}{5} p^4 + \frac{6.4}{5.3} p^2 - \frac{6.4.2}{5.3.1}) \sqrt{(1 + pp)}$,
ac si $n = \frac{2\lambda + 1}{2\lambda}$, erit $y = px + a\sqrt{(1 + pp)} + \frac{px}{2\lambda}$, et

$$x = \frac{C}{p^{2\lambda+1}} -$$

$$\frac{2\lambda a}{(2\lambda+1)p}\left(1 - \frac{2\lambda}{(2\lambda-1)pp} + \frac{2\lambda(2\lambda-2)}{(2\lambda-1)(2\lambda-3)p^4} - \text{etc.}\right)\sqrt{(1+pp)}.$$

Quodsi ergo sumatur $\lambda = \infty$, ut sit $n = 1$, erit

$$y = px + a\sqrt{(1 + pp)}, \text{ et } x = \frac{C}{p^{2\lambda+1}} - \frac{ap}{\sqrt{(1 + pp)}},$$

unde si constans C sit $= 0$, statim sequitur solutio superior $xx + yy = aa$. At si constans C non evanescat, minimum discrimen in quantitate p infinitam varietatem ipsi x inducit. Quantumvis ergo x varietur, quantitas p ut constans spectari potest, unde posito $p = \alpha$, altera solutio $y = \alpha x + a\sqrt{(1 + \alpha\alpha)}$ obtinetur. Hinc ergo dubium supra, circa exemplum 1. natum, non mediocriter illustratur.

Exemplum 5.

704. *Proposita aequatione differentiali*

$$A\,\partial y^n = (B x^\alpha + C y^\beta)\,\partial x^n$$

existente $n = \frac{\alpha\beta}{\alpha - \beta}$, *ejus integrale investigare.*

Posito $\frac{\partial y}{\partial x} = p$ erit $A p^n = B x^\alpha + C y^\beta$. Ponamus jam $p = q^{\alpha\beta}$, $x = v^{\beta n}$ et $y = z^{\alpha n}$, ut habeamus hanc aequationem homogeneam $A q^{\alpha\beta n} = B v^{\alpha\beta n} + C z^{\alpha\beta n}$, quae positis $z = rq$ et $v = sq$, abit in $A = B s^{\alpha\beta n} + C r^{\alpha\beta n}$. Cum vero sit

$$\partial y = \alpha n z^{\alpha n - 1}\partial z = \alpha n r^{\alpha n - 1} q^{\alpha n - 1}(r\partial q + q\partial r) \text{ et}$$
$$p\partial x = \beta n v^{\beta n - 1} q^{\alpha\beta}\partial v = \beta n s^{\beta n - 1} q^{\alpha\beta + \beta n - 1}(s\partial q + q\partial s),$$

erit

$$\alpha r^{\alpha n - 1}(r\partial q + q\partial r) = \beta s^{\beta n - 1} q^{\alpha\beta + \beta n - \alpha n}(s\partial q + q\partial s).$$

Est vero per hypothesin $\alpha\beta+\beta n-\alpha n=0$, unde oritur

$$\alpha r^{\alpha n}\partial q+\alpha r^{\alpha n-1}q\partial r=\beta s^{\beta n}\partial q+\beta s^{\beta n-1}q\partial s,$$

hincque

$$\frac{\partial q}{q}=\frac{\alpha r^{\alpha n-1}\partial r-\beta s^{\beta n-1}\partial s}{\beta s^{\beta n}-\alpha r^{\alpha n}}.$$

At est

$$s^{\beta n}=\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1}{\alpha}},\ \text{hincque}$$

$$\beta s^{\beta n-1}\partial s=-\frac{\beta C}{B}r^{\alpha\beta n-1}\partial r\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1-\alpha}{\alpha}},$$

unde fit

$$\frac{\partial q}{q}=\frac{\alpha r^{\alpha n-1}\partial r+\frac{\beta C}{B}r^{\alpha\beta n-1}\partial r\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1-\alpha}{\alpha}}}{\beta\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1}{\alpha}}-\alpha r^{\alpha n}}.$$

Facilius autem calculus hoc modo instituetur; sumto $A=1$, erit

$$p=\frac{\partial y}{\partial x}=(Bx^{\alpha}+Cy^{\beta})^{\frac{1}{n}},$$

sit $y=x^{\frac{\alpha}{\beta}}u$, fiet

$$x^{\frac{\alpha}{\beta}}\partial u+\frac{\alpha}{\beta}x^{\frac{\alpha-\beta}{\beta}}u\partial x=x^{\frac{\alpha}{n}}\partial x(B+Cu^{\beta})^{\frac{1}{n}},$$

quae aequatio, cum sit $\frac{\alpha}{n}=\frac{\alpha-\beta}{\beta}$, abit in hanc

$$\beta x\partial u+\alpha u\partial x=\beta\partial x(B+Cu^{\beta})^{\frac{1}{n}},$$

unde fit

$$\frac{\partial x}{x}=\frac{\beta\partial u}{\beta(B+Cu^{\beta})^{\frac{1}{n}}-\alpha u},$$

sicque x per u determinatur, et quia $u = x^{-\frac{\alpha}{\beta}} y$, habebitur aequatio inter x et y.

Scholion.

705. Hoc igitur modo operationem institui conveniet, quando inter binas variabiles x et y una cum differentialium ratione $\frac{\partial y}{\partial x} = p$, ejusmodi relatio proponitur, ex qua valor ipsius p commode elici non potest. Tum ergo calculum ita tractari oportet, ut per differentiationem ponendo $\partial y = p \partial x$ vel $\partial x = \frac{\partial y}{p}$, tandem perveniatur ad aequationem differentialem simplicem inter duas tantum variabiles, quem in finem etiam saepe idoneis substitutionibus uti necesse est. Atque hucusque fere Geometris in resolutione aequationum differentialium primi gradus etiamnum pertingere licuit, vix enim ulla via integralia investigandi adhuc quidem adhibita hic praetermissa videtur. Num autem multo majorem calculi integralis promotionem sperare liceat? vix equidem affirmaverim, cum plurima extent inventa, quae ante vires ingenii humani superare videbantur.

Cum igitur calculum integralem in duos libros sim partitus, quorum prior circa relationem binarum tantum variabilium, posterior vero ternarum pluriumve versatur, atque jam libri primi partem priorem in differentialibus primi ordinis constitutam hic pro viribus exposuerim, ad ejus alteram partem progredior, in qua binarum variabilium relatio ex data differentialium secundi altiorisve ordinis conditione requiritur.

Corrigenda.

pag.	*lin.*	*loco:*	*lege.*
48	7 asc.	$\sqrt{\frac{f+gx}{a-bx}}$	$\sqrt{\frac{f+gx}{a+bx}}$
81	9	$3-4xx+x^4$	$1-4xx+x^4$
104	3 asc.	E =	F =
119	*ultima*	(§. 227)	(§. 228)
179	*ultima*	A′, A′	A′, A″
180	8 asc.	a, a	a, a'
182	13	A′, A′	A′, A″
—	15	$a'-a'$	$a''-a'$
201	18	*in numeratore* $\alpha-\omega$	$a-\omega$
205	9	*in numeratore* $z^{m+\nu}$	$z^{\mu+\nu}$
208	10	=	= −
208	*ultima*	$\frac{1.3.5.}{(m+1)(m+3)(m+5)}$;	$\frac{1.3.5.}{(m+1)(m+3)(m+5)}$ M;
209	4 asc.	*in exponente* $\frac{}{2}$	$\frac{5}{2}$
210	7	$\int\frac{\partial x}{\sqrt[3]{(1-x^3)^2}}=B'$	$\int\frac{\partial x}{\sqrt[3]{(1-x^3)^2}}=A'$
221	2 asc.	252.	352.
222	7	$\int\frac{x\,\partial x}{\sqrt[3]{(1+x^3)^2}}$	$\int\frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}}$
231	11	$\int=$	$=\int$
261	6	ratio $\frac{\partial y}{\partial x}$,	ratio $\frac{\partial y}{\partial x}$
272	8	concludimis	concludimus
—	9	admissuiram	admissuram
—	10	repertur	reperitur
304	11	*in denominatore* $1-zy$	$1-xy$