CAPUT VI.

DE

COMPARATIONE QUANTITATUM TRANSCENDENTIUM CONTENTARUM IN FORMA

$$\int \frac{P \partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}.$$

Problema 78.

606.

Proposita relatione inter x et y hac

$$\alpha + \gamma(xx + yy) + 2\delta xy + \zeta xxyy = 0,$$

inde elicere functiones transcendentes formae praescriptae, quas inter se comparare liceat.

Solutio.

Ex proposita aequatione definiatur utraque variabilis

$$y = \frac{-\delta x + \sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)xx - \gamma\zeta x^4]}}{\gamma + \zeta xx} \text{ et}$$

$$x = \frac{-\delta y + \sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)yy - \gamma\zeta y^4]}}{\gamma + \zeta yy},$$

quae radicalia ad formam praescriptam revocentur ponendo

$$-\alpha\gamma = Am, \delta\delta - \gamma\gamma - \alpha\zeta = Cm \text{ et } -\gamma\zeta = Em;$$

unde fit

$$\alpha = -\frac{Am}{\gamma}, \zeta = -\frac{Em}{\gamma} \text{ et } \delta\delta = Cm + \gamma\gamma + \frac{AEmm}{\gamma\gamma}.$$

Erit ergo.

$$\gamma y + \delta x + \zeta xxy = \surd m(A + Cxx + Ex^4)$$
$$\gamma x + \delta y + \zeta xyy = \surd m(A + Cyy + Ey^4).$$

Ipsa autem aequatio proposita, si differentietur, dat

$$\partial x(\gamma x + \delta y + \zeta xyy) + \partial y(\gamma y + \delta x + \zeta xxy) = 0$$

ubi illi valores substituti praebent

$$\frac{\partial x}{\surd(A + Cxx + Ex^4)} + \frac{\partial y}{\surd(A + Cyy + Ey^4)} = 0.$$

Vicissim ergo proposita hac aequatione differentiali, ei satisfaciet haec aequatio finita

$$-Am + \gamma\gamma(xx + yy) + 2xy\surd(\gamma^4 + Cm\gamma\gamma + AEmm)$$
$$- Emxxyy = 0,$$

seu ponendo $\frac{\gamma\gamma}{m} = k$, haec

$$-A + k(xx + yy) + 2xy\surd(kk + kC + AE) - Exxyy = 0,$$

quae cum involvat constantem k, in aequatione differentiali non contentam, simul erit integrale completum. Hinc autem fit

$$ky + x\surd(kk + kC + AE) - Exxy = \surd k(A + Cxx + Ex^4) \text{ et}$$
$$kx + y\surd(kk + kC + AE) - Exyy = \surd k(A + Cyy + Ey^4).$$

Corollarium 1.

607. Constans k ita assumi potest, ut posito $x = 0$, fiat $y = b$, oritur autem

$$kk = \surd Ak \text{ et } b\surd(kk + kC + AE) = \surd k(A + Cbb + Eb^4),$$

ergo

$$k = \frac{A}{bb} \text{ et } \surd(kk + kC + AE) = \frac{1}{bb}\surd A(A + Cbb + Eb^4),$$

ideoque habebimus

$$Ay + x\surd A(A + Cbb + Eb^4) - Ebbxxy = b\surd A(A + Cxx + Ex^4)$$

et

$$Ax + y\surd A(A + Cbb + Eb^4) - Ebbxyy = b\surd A(A + Cyy + Ey^4).$$

Corollarium 2.

608. Haec igitur relatio finita inter x et y erit integrale completum aequationis differentialis

$$\frac{\partial x}{\sqrt{(A+Cxx+Ex^4)}}+\frac{\partial y}{\sqrt{(A+Cyy+Ey^4)}}=0,$$

quod rationaliter inter x et y expressum erit

$$A(xx+yy-bb)+2xy\sqrt{A(A+Cbb+Eb^4)}-Ebbxxyy=0.$$

Corollarium 3.

609. Hinc ergo y ita per x exprimetur, ut sit

$$y=\frac{b\sqrt{A(A+Cxx+Ex^4)}-x\sqrt{A(A+Cbb+Eb^4)}}{A-Ebbxx},$$

atque ex hoc valore elicitur

$$\sqrt{\frac{A+Cyy+Ey^4}{A}}$$
$$=\frac{(A+Ebbxx)\sqrt{(A+Cbb+Eb^4)(A+Cxx+Ex^4)}-2AEbx(bb+xx)-Cbx(A+Ebbxx)}{(A-Ebbxx)^2}.$$

Corollarium 4.

610. Hinc constantem b pro lubitu determinando infinita integralia particularia exhiberi possunt, quorum praecipua sunt: 1) sumendo $b=0$, unde fit $y=-x$; 2) sumendo $b=\infty$, unde fit $y=\frac{\sqrt{A}}{x\sqrt{E}}$. 3) Si $A+Cbb+Eb^4=0$, hincque $bb=\frac{-C+\sqrt{(CC-4AE)}}{2E}$, unde fit $y=\frac{b\sqrt{A(A+Cxx+Ex^4)}}{A-Ebbxx}$.

Scholion.

611. Hic jam usus istius methodi, qua retrogrediendo ab aequatione finita ad aequationem differentialem pervenimus, luculenter perspicitur. Cum enim integratio formulae $\int\frac{\partial x}{\sqrt{(A+Cxx+Ex^4)}}$ nullo modo neque per logarithmos neque per arcus circulares perfici posset, mirum sane est talem aequationem differentialem adeo algebraice integrari posse; quae quidem in praecedente capite ope ejusdem methodi sunt tradita, etiam methodo ordinaria erui possunt, dum singulae formulae differentiales vel per logarithmos vel arcus circulares exprimuntur, quorum deinceps comparatio ad ae-

quationem algebraicam reducitur. Verum quia hic talis integratio plane non locum invenit, nulla certe alia methodus patet, qua idem integrale, quod hic exhibuimus, investigari posset. Quare hoc argumentum diligentius evolvamus.

Problema 79.

612. Si $\Pi : z$ denotet ejusmodi functionem ipsius z, ut sit $\Pi : z = \int \frac{\partial z}{\sqrt{(A + Czz + Ez^4)}}$, integrali ita sumto ut evanescat posito $z = 0$, comparationem inter hujusmodi functiones investigare.

Solutio.

Posita inter binas variabiles x et y relatione supra definita, vidimus fore

$$\frac{\partial x}{\sqrt{(A + Cxx + Ex^4)}} + \frac{\partial y}{\sqrt{(A + Cyy + Ey^4)}} = 0.$$

Hinc cum posito $x = 0$ fiat $y = b$, elicitur integrando

$$\Pi : x + \Pi : y = \Pi : b.$$

Cum jam nullum amplius discrimen inter variabiles x, y et constantem b intercedat, statuamus $x = p, y = q$, et $b = -r$, ut sit $\Pi : b = -\Pi : r$, atque haec relatio inter functiones transcendentes

$$\Pi : p + \Pi : q + \Pi : r = 0$$

per sequentes formulas algebraicas exprimetur,

$$(A - Epprr)q + p\sqrt{A(A + Crr + Er^4)} + r\sqrt{A(A + Cpp + Ep^4)} = 0$$

seu

$$(A - Eppqq)r + q\sqrt{A(A + Cpp + Ep^4)} + p\sqrt{A(A + Cqq + Eq^4)} = 0$$

seu

$$(A - Eqqrr)p + r\sqrt{A(A + Cqq + Eq^4)} + q\sqrt{A(A + Crr + Er^4)} = 0$$

quae oriuntur ex hac aequatione

$$A(pp + qq - rr) - Eppqqrr + 2pq\sqrt{A(A + Crr + Er^4)} = 0.$$

Haec vero ad rationalitatem perducta sit

$$AA(p^4+q^4+r^4-2ppqq-2pprr-2qqrr)$$
$$-2AEppqqrr(pp+qq+rr)-4ACppqqrr$$
$$+EEp^4q^4r^4=0,$$

quae autem ob pluralitatem radicum satisfacit omnibus signorum variationibus in superiori aequatione transcendente.

Corollarium 1.

613. Sumamus r negative, ut fiat

$$\Pi:r=\Pi:p+\Pi:q,$$

eritque

$$y=\frac{p\sqrt{A(A+Cqq+Eq^4)}+q\sqrt{A(A+Cpp+Ep^4)}}{A-Eppqq};$$

unde colligitur fore

$$\sqrt{\frac{A+Crr+Er^4}{A}}$$
$$=\frac{(A+Eppqq)\sqrt{(A+Cpp+Ep^4)(A+Cqq+Eq^4)}+2AEpq(pp+qq)+Cpq(A+Eppqq)}{(A-Eppqq)^2},$$

Corollarium 2.

614. Quodsi ergo ponamus $q=p$, ut sit

$$\Pi:r=2\,\Pi:p,$$

erit

$$r=\frac{2p\sqrt{A(A+Cpp+Ep^4)}}{A-Ep^4},$$

atque

$$\sqrt{\frac{A+Crr+Er^4}{A}}=\frac{AA+2ACpp+6AEp^4+2CEp^6+EEp^8}{(A-Ep^4)^2}.$$

Hoc igitur modo functio assignari potest aequalis duplo similis functionis.

Corollarium 3.

615. Si ponatur $q=\frac{2p\sqrt{A(A+Cpp+Ep^4)}}{A-Ep^4}$ et

$$\sqrt{A(A+Cqq+Eq^4)}=\frac{A(AA+2ACpp+6AEp^4+2CEp^6+EEp^8)}{(A-Ep^4)^2},$$

ut sit $\Pi:q=2\Pi:p$, fiet ex primo Coroll. $\Pi:r=3\Pi:p$.

Tum igitur erit

$$r = \frac{p(3AA + 4ACpp + 6AEp^4 - EEp^8)}{AA - 6AEp^4 - 4CEp^6 - 3EEp^8}.$$

Scholion 1.

616. Nimis operosum est hanc functionum multiplicationem ulterius continuare, multoque minus legem in earum progressione deprehendere licet. Quodsi ponamus brevitatis gratia

$$\sqrt{A(A + Cpp + Ep^4)} = AP \text{ et } A - Ep^4 = A\mathfrak{P},$$

ut sit

$$Cpp = APP - A - Ep^4 \text{ et } Ep^4 = A(1 - \mathfrak{P}),$$

hae multiplicationes usque ad quadruplum ita se habebunt, scilicet si statuamus

$$\Pi : r = 2\Pi : p; \ \Pi : s = 3\Pi : p \text{ et } \Pi : t = 4\Pi : p$$

reperietur:

$$r = \frac{2Pp}{\mathfrak{P}}, \ s = \frac{p(4PP - \mathfrak{P}\mathfrak{P})}{\mathfrak{P}\mathfrak{P} - 4PP(1 - \mathfrak{P})}, \ t = \frac{4pP\mathfrak{P}[2PP(2 - \mathfrak{P}) - \mathfrak{P}\mathfrak{P}]}{\mathfrak{P}^4 - 16P^4(1 - \mathfrak{P})}.$$

Quodsi simili modo ponamus

$$\sqrt{A(A + Crr + Er^4)} = AR \text{ et } A - Er^4 = A\mathfrak{R},$$

erit

$$R = \frac{2PP(2 - \mathfrak{P}) - \mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P}} \text{ et } \mathfrak{R} = \frac{\mathfrak{P}^4 - 16P^4(1 - \mathfrak{P})}{\mathfrak{P}^4};$$

unde pro quadruplicatione fit

$$t = \frac{2Rr}{\mathfrak{R}}, \ T = \frac{2RR(2 - \mathfrak{R}) - \mathfrak{R}\mathfrak{R}}{\mathfrak{R}\mathfrak{R}}, \ \mathfrak{T} = \frac{\mathfrak{R}^4 - 16R^4(1 - \mathfrak{R})}{\mathfrak{R}^4}.$$

Quare si pro octuplicatione statuamus $\Pi : z = 8\Pi : p$ erit

$$z = \frac{2Tt}{\mathfrak{T}} = \frac{4rR\mathfrak{R}[2RR(2 - \mathfrak{R}) - \mathfrak{R}\mathfrak{R}]}{\mathfrak{R}^4 - 16R^4(1 - \mathfrak{R})}.$$

Hinc intelligitur quomodo in continua duplicatione versari oporteat, neque tamen legem progressionis observare licet. Caeterum cognitio hujus legis ad incrementum Analyseos maxime esset optanda, ut inde generatim relatio inter z et p, pro aequalitate $\Pi : z = n\Pi : p$ definiri posset, quaemadmodum hoc in capite praecedente successit;

hinc enim eximias proprietates circa integralia formae $\int \frac{\partial z}{\sqrt{(A + Czz + Ez^4)}}$ cognoscere liceret, quibus scientia analytica haud mediocriter promoveretur.

Scholion 2.

617. Modus maxime idoneus in legem progressionis inquirendi, videtur, si ternos terminos se ordine excipientes contemplemur hoc modo

$$\Pi : x = (n - 1)\,\Pi : p,\ \Pi : y = n\,\Pi : p,\ \Pi : z = (n + 1)\,\Pi : p;$$

ubi cum sit

$$\Pi : x = \Pi : y - \Pi : p \text{ et } \Pi : z = \Pi : y + \Pi : p, \text{ erit}$$

$$x = \frac{y\sqrt{A(A + Cpp + Ep^4)} - p\sqrt{A(A + Cyy + Ey^4)}}{A - Eppyy}$$

$$z = \frac{y\sqrt{A(A + Cpp + Ep^4)} + p\sqrt{A(A + Cyy + Ey^4)}}{A - Eppyy};$$

unde concludimus

$$(A - Eppyy)(x + z) = 2y\sqrt{A(A + Cpp + Ep^4)}.$$

Ponamus ut ante

$$\sqrt{A(A + Cpp + Ep^4)} = AP \text{ et } A - Ep^4 = A\mathfrak{P},$$

et quia singulae quantitates x, y, z factorem p simpliciter involvunt, sit

$$x = pX,\ y = pY \text{ et } z = pZ;$$

erit

$$[1 - (1 - \mathfrak{P})YY](X + Z) = 2PY$$

seu

$$Z = \frac{2PY}{1 - (1 - \mathfrak{P})YY} - X,$$

cujus formulae ope ex binis terminis contiguis X et Y sequens Z haud difficulter invenitur. Quod quo facilius appareat, ponatur $2P = Q$ et $1 - \mathfrak{P} = \mathfrak{Q}$, ut sit $Z = \frac{QY}{1 - \mathfrak{Q}YY} - X$. Jam progressio quaesita ita se habebit

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1) 1;

2) $\frac{Q}{\mathfrak{P}}$;

3) $\frac{QQ - \mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P} - QQ\Omega}$;

4) $\frac{Q^3\mathfrak{P}(1+\Omega) - 2Q\mathfrak{P}^3}{\mathfrak{P}^4 - Q^4\Omega}$;

5) $\frac{\mathfrak{P}^6 - 3QQ\mathfrak{P}^4 + Q^4\mathfrak{P}^2(1+2\Omega) - Q^6\Omega\Omega}{\mathfrak{P}^6 - 3QQ\mathfrak{P}^4\Omega + Q^4\mathfrak{P}\mathfrak{P}\Omega(2+\Omega) - Q^6\Omega}$ etc.

Quaestio ergo huc redit, ut investigetur progressio, ex data relatione inter ternos terminos successivos X, Y, Z, quae sit $Z = \frac{QY}{1-\Omega YY} - X$; existente termino primo $= 1$ et secundo $= \frac{Q}{1-\Omega}$.

Problema 80.

618. Si $\Pi : z$ ejusmodi denotet functionem ipsius z, ut sit $\Pi : z = \int \frac{\partial z(L + Mzz + Nz^4)}{\sqrt{(A + Czz + Ez^4)}}$, integrali ita sumto ut evanescat posito $z = 0$, comparationem inter hujusmodi functiones transcendentes investigare.

Solutio.

Stabilita inter binas variabiles x et y hac relatione, ut sit

$Ay + \mathfrak{B}x - Ebbxxy = b\sqrt{A(A + Cxx + Ex^4)}$ seu

$Ax + \mathfrak{B}y - Ebbxyy = b\sqrt{A(A + Cyy + Ey^4)}$,

sive sublata irrationalitate

$$A(xx + yy - bb) + 2\mathfrak{B}xy - Ebbxxyy = 0,$$

existente brevitatis gratia $\mathfrak{B} = \sqrt{A(A + Cbb + Eb^4)}$, erit uti ante vidimus

$$\frac{\partial x}{\sqrt{(A + Cxx + Ex^4)}} + \frac{\partial y}{\sqrt{(A + Cyy + Ey^4)}} = 0.$$

Ponamus igitur

$$\frac{\partial x(L + Mxx + Nx^4)}{\sqrt{(A + Cxx + Ex^4)}} + \frac{\partial y(L + Myy + Ny^4)}{\sqrt{(A + Cyy + Ey^4)}} = b\,\partial V\sqrt{A},$$

ut sit nostro signandi more

$$\Pi : x + \Pi : y = \text{Const.} + b\,V\sqrt{A},$$

ubi constans ita definiri debet, ut posito $x=0$ fiat $y=b$. Quaestio ergo ad inventionem functionis V revocatur; quem in finem loco ∂y valore ex priori aequatione substituto, erit

$$b\,\partial V\sqrt{A} = \frac{\partial x\,[M(xx-yy)+N(x^4-y^4)]}{\sqrt{(A+Cxx+Ex^4)}};$$

verum quia

$$b\sqrt{A(A+Cxx+Ex^4)} = Ay + \mathfrak{B}x - Ebbxxy,$$

habebimus

$$\partial V = \frac{\partial x\,(xx-yy)\,[M+N(xx+yy)]}{Ay+\mathfrak{B}x-Ebbxxy}.$$

Sumamus jam aequationem rationalem

$$A(xx+yy-bb)+2\mathfrak{B}xy-Ebbxxyy=0,$$

et ponamus

$$xx+yy=tt \text{ et } xy=u,$$

ut sit

$$A(tt-bb)+2\mathfrak{B}u-Ebbuu=0,$$

ideoque

$$At\partial t = -\mathfrak{B}\partial u + Ebbu\partial u.$$

Cum porro sit

$$x\partial x + y\partial y = t\partial t \text{ et } x\partial y + y\partial x = \partial u,$$

erit

$$(xx-yy)\,\partial x = xt\partial t - y\partial u$$

seu

$$A(xx-yy)\partial x = -\partial u\,(Ay+\mathfrak{B}x-Ebbxxy),$$

ita ut sit

$$\frac{\partial x\,(xx-yy)}{Ay+\mathfrak{B}x-Ebbxxy} = -\frac{\partial u}{A},$$

ex quo deducitur

$$\partial V = -\frac{\partial u}{A}(M+Ntt),$$

et ob

$$tt = bb - \frac{2\mathfrak{B}u}{A} + \frac{Ebbuu}{A}, \text{ erit}$$

$$\partial V = -\frac{\partial u}{AA}(AM+ANbb-2\mathfrak{B}Nu+ENbbuu):$$

unde integrando elicitur

$$V = -\frac{Mu}{A} - \frac{Nbbu}{A} + \frac{\mathfrak{B}Nuu}{AA} - \frac{ENbbu^3}{3AA}.$$

Hoc ergo valore substituto, ob $u = xy$, habebimus

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}} - \frac{Nb^3xy}{\sqrt{A}} + \frac{\mathfrak{B}Nbx^2y^2}{A\sqrt{A}} - \frac{ENb^3x^3y^3}{3A\sqrt{A}}.$$

Cum autem sit

$$\mathfrak{B}xy = \tfrac{1}{2}Abb - \tfrac{1}{2}A(xx+yy) + Ebbxxyy$$

erit

$$\Pi x + \Pi y = \Pi b - \frac{Mbxy}{\sqrt{A}} - \frac{Nbxy}{2A\sqrt{A}}[A(bb+xx+yy) - \tfrac{1}{3}Ebbxxyy]$$

cui ergo aequationi satisfit per formulas algebraicas supra exhibitas, quibus relatio inter x, y et b exprimitur. Quodsi ergo statuatur haec aequatio

$$\Pi : p + \Pi : q + \Pi : r$$
$$= \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2A\sqrt{A}}[A(pp+qq+rr) - \tfrac{1}{3}Eppqqrr]$$

ea efficitur sequenti relatione inter p, q, r constituta

$$(A - Eppqq)r + p\sqrt{A(A+Cqq+Eq^4)} + q\sqrt{A(A+Cpp+Ep^4)} = 0$$

seu

$$(A - Epprr)q + p\sqrt{A(A+Crr+Er^4)} + r\sqrt{A(A+Cpp+Ep^4)} = 0$$

seu

$$(A - Eqqrr)p + q\sqrt{A(A+Crr+Er^4)} + r\sqrt{A(A+Cqq+Eq^4)} = 0$$

sive per simplicem irrationalitatem

$$A(pp+qq+rr) + 2pq\sqrt{A(A+Crr+Er^4)} - Eppqqrr = 0$$

seu

$$A(pp+rr-qq) + 2pr\sqrt{A(A+Cqq+Eq^4)} - Eppqqrr = 0$$

seu

$$A(qq+rr-pp) + 2qr\sqrt{A(A+Cpp+Ep^4)} - Eppqqrr = 0$$

penitusque irrationalitate sublata

$$EEp^4q^4r^4 - 2AEppqqrr(pp+qq+rr) - 4ACppqqrr$$
$$+ AA(p^4+q^4+r^4 - 2ppqq - 2pprr - 2qqrr) = 0.$$

Corollarium 1.

619. Sit $q = r = s$, ut habeamus hanc aequationem

$$\Pi : p + 2\,\Pi : s = \frac{Mpss}{\sqrt{A}} + \frac{Npss}{2A\sqrt{A}}\left[A(pp + 2ss) - \tfrac{1}{3}Epps^4\right]$$

cui satisfacit haec relatio

$$(A - Es^4)p + 2s\sqrt{A(A + Css + Es^4)} = 0.$$

Corollarium 2.

620. Sumamus s negative, et loco p sustituamus ibi hunc valorem, ut habeamus

$$2\,\Pi : s + \Pi : q + \Pi : r + \frac{Mpss}{\sqrt{A}} + \frac{Npss}{2A\sqrt{A}}\left[A(pp + 2ss) - \tfrac{1}{3}Epps^4\right]$$
$$= \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2A\sqrt{A}}\left[A(pp + qq + rr) - \tfrac{1}{3}Eppqqrr\right]$$

existente

$$p = \frac{2s\sqrt{A(A + Css + Es^4)}}{A - Es^4},$$

unde fit

$$\sqrt{A(A + Cpp + Ep^4)} = \frac{A(A + Css + Es^4)^2 + A(4AE - CC)s^4}{(AE - s^4)^2}$$

qui valores in superioribus formulis substitui debent.

Corollarium 3.

621. Hoc modo effici poterit, ut partes algebraicae evanescant, atque functiones transcendentes solae inter se comparentur. Veluti si esset $N = 0$, statui oporteret $ss = qr$, ut fieret

$$2\,\Pi : s + \Pi : q + \Pi : r = 0.$$

At posito $ss = qr$, fit

$$p = \frac{2\sqrt{Aqr(A + Cqr + Eqqrr)}}{A - Eqqrr}.$$

Est vero etiam

$$p = \frac{-q\sqrt{A(A + Crr + Er^4)} - r\sqrt{A(A + Cqq + Eq^4)}}{A - Eqqrr},$$

quibus valoribus aequatis, oritur haec aequatio

$$(AA + EEq^4r^4)(qq - 6qr + rr) - 8Cqqrr(A + Eqqrr)$$
$$- 2AEqqrr(qq + 10qr + rr) = 0.$$

Scholion.

622. Si $\Pi : z$ exprimat arcum cujuspiam lineae curvae respondentem abscissae vel cordae z, hinc plures arcus ejusdem curvae inter se comparare licet, ut vel differentia binorum arcuum fiat algebraica, vel arcus exhibeantur datam rationem inter se tenentes. Hoc modo ejusmodi insignes curvarum proprietates eruuntur, quarum ratio aliunde vix perspici queat. Comparatio quidem arcuum circularium ex elementis nota per caput praecedens, ut vidimus, facile expeditur, unde etiam comparatio arcuum parabolicorum derivatur. Ex hoc autem capite comparatio arcuum ellipticorum et hyperbolicorum simili modo institui potest; cum enim in genere arcus sectionis conicae tali formula exprimatur $\int \partial x \sqrt{\frac{a + bxx}{c + exx}}$, haec transformata in istam $\int \frac{\partial x (a + bxx)}{\sqrt{[ac + (ae + bc)xx + bex^4]}}$, per praecepta tradita tractari potest, ponendo $A = ac$, $C = ae + bc$, et $E = be$, $L = a$, $M = b$ atque $N = 0$. Haec autem investigatio ad formulas, quarum denominator est

$$\sqrt{(A + 2Bz + Czz + Dz^3 + Ez^4)}$$

extendi potest, similisque est praecedenti, quam idcirco hic sum expositurus, unde simul patebit, hunc esse ultimum terminum, quousque progredi liceat. Formulae enim integrales magis complicatae, ubi post signum radicale altiores potestates ipsius z occurrunt, vel ipsum signum radicale altiorem dignitatem involvit, hoc modo non videntur inter se comparari posse, paucissimis casibus exceptis, qui per quampiam substitutionem ad hujusmodi formam reduci queant.

Problema 81.

623. Si $\Pi : z$ ejusmodi functionem ipsius z denotet, ut sit

$$\Pi : z = \frac{\partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}},$$

hujusmodi functiones inter se comparare.

Solutio.

Inter binas variabiles x et y statuatur relatio hae aequatione expressa

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy = 0,$$

unde cum fiat

$$yy = \frac{-2y(\beta + \delta x + \varepsilon xx) - \alpha - 2\beta x - \gamma xx}{\gamma + 2\varepsilon x + \zeta xx},$$

erit radice extracta

$$y = \frac{-\beta - \delta x - \varepsilon xx + \sqrt{[(\beta + \delta x + \varepsilon xx)^2 - (\alpha + 2\beta x + \gamma xx)(\gamma + 2\varepsilon x + \zeta xx)]}}{\gamma + 2\varepsilon x + \zeta xx}.$$

Reducatur signum radicale ad formam propositam, ponendo

$$\beta\beta - \alpha\gamma = Am, \quad \beta\delta - \alpha\varepsilon - \beta\gamma = Bm,$$
$$\delta\delta - 2\beta\varepsilon - \alpha\zeta - \gamma\gamma = Cm, \quad \delta\varepsilon - \beta\zeta - \gamma\varepsilon = Dm,$$
$$\varepsilon\varepsilon - \gamma\zeta = Em;$$

unde ex sex coëfficientibus α, β, γ, δ, ε, ζ, quinque definiuntur, atque ad sextum insuper accedit littera m, ita ut aequatio assumta adhuc constantem arbitrariam involvat. Inde ergo si brevitatis gratia ponamus

$$\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)} = X \quad \text{et}$$
$$\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)} = Y,$$

habebimus

$$\beta + \gamma y + \delta x + \varepsilon xx + 2\varepsilon xy + \zeta xxy = X\sqrt{m} \quad \text{et}$$
$$\beta + \gamma x + \delta y + \varepsilon yy + 2\varepsilon xy + \zeta xyy = Y\sqrt{m}.$$

At aequatio assumta per differentiationem dat

$$+\partial x(\beta + \gamma x + \delta y + 2\varepsilon xy + \varepsilon yy + \zeta xyy)$$
$$+\partial y(\beta + \gamma y + \delta x + \varepsilon xx + 2\varepsilon xy + \zeta xxy) = 0,$$

quae expressiones quia cum superioribus conveniunt, dant

$$Y \partial x \sqrt{m} + X \partial y \sqrt{m} = 0, \text{ seu } \frac{\partial x}{X} + \frac{\partial y}{Y} = 0:$$

unde integrando colligimus

$$\Pi : x + \Pi : y = \text{Const.}$$

quae constans, si posito $x = 0$ fiat $y = b$, erit $= \Pi : 0 + \Pi : b$; vel in genere, si posito $x = a$ fiat $y = b$, ea erit $= \Pi : a + \Pi : b$. Quodsi ergo litterae α, β, γ, δ, ε, ζ per conditiones superiores definiantur, aequatio assumta algebraica inter x et y erit integrale completum hujus aequationis differentialis

$$\frac{\partial x}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} + \frac{\partial y}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}} = 0.$$

Corollarium 1.

624. Ad has litteras α, β, γ, δ, ε, ζ definiendas, sumantur primo aequationes binae ad dextram positae, quae sunt

$$(\delta - \gamma)\beta - \alpha\varepsilon = Bm \text{ et } (\delta - \gamma)\varepsilon - \zeta\beta = Dm,$$

unde quaerantur binae β et ε, reperieturque

$$\beta = \frac{(\delta - \gamma)B + \alpha D}{(\delta - \gamma)^2 - \alpha\zeta} m \text{ et } \frac{(\delta - \gamma)D + \zeta B}{(\delta - \gamma)^2 - \alpha\zeta} m.$$

Corollarium 2.

625. Sit brevitatis gratia $\delta - \gamma = \lambda$ seu $\delta = \gamma + \lambda$, erit

$$\beta = \frac{D\alpha + B\lambda}{\lambda\lambda - \alpha\zeta} m \text{ et } \varepsilon = \frac{B\zeta + D\lambda}{\lambda\lambda - \alpha\zeta} m.$$

Jam ex conditione prima et ultima oritur

$$\beta\beta\zeta - \alpha\varepsilon\varepsilon = (A\zeta - E\alpha) m,$$

ubi illi valores substituti praebent

$$\frac{BB\zeta - DD\alpha}{\lambda\lambda - \alpha\zeta} m = A\zeta - E\alpha,$$

unde fit

$$m = \frac{(\lambda\lambda - \alpha\zeta)(A\zeta - E\alpha)}{BB\zeta - DD\alpha}.$$

At ex prima et ultima sequitur

$$DD\beta\beta - BB\varepsilon\varepsilon + \gamma(BB\zeta - DD\alpha) = (ADD - BBE)\,m$$

unde colligitur

$$\gamma = \frac{[(A\zeta - E\alpha)(ADD - BBE)\lambda\lambda + 2BD(A\zeta - E\alpha)\lambda + ABB\zeta\zeta - DDE\alpha\alpha]}{(BB\zeta - DD\alpha)^2}.$$

Corollarium 3.

626. Superest tertia aequatio

$$2\gamma\lambda + \lambda\lambda - 2\beta\varepsilon - \alpha\zeta = C\,m$$

quae, cum pro m substituto valore sit

$$\beta = \frac{(A\zeta - E\alpha)(D\alpha + B\lambda)}{BB\zeta - DD\alpha} \text{ et } \varepsilon = \frac{(A\zeta - E\alpha)(B\zeta + D\lambda)}{BB\zeta - DD\alpha},$$

si isti valores substituantur, commode inde colligitur

$$\lambda = \frac{C(A\zeta - E\alpha)(BB\zeta - DD\alpha) - 2BD(A\zeta - E\alpha)^2 - (BB\zeta - DD\alpha)^2}{2(A\zeta - A\alpha)(ADD - BBE)}.$$

Scholion.

627. Quia his valoribus uti non licet, quoties fuerit $ADD - BBE = 0$, aliam resolutionem huic incommodo non obnoxiam tradam. Posito $\delta = \gamma + \lambda$, sit insuper $\lambda\lambda = \alpha\zeta + \mu$, ut primae formulae fiant

$$\beta = \frac{m}{\mu}(D\alpha + B\lambda) \text{ et } \varepsilon = \frac{m}{\mu}(B\zeta + D\lambda).$$

Jam prima et ultima junctis prodit

$$A\zeta - E\alpha = \frac{m}{\mu}(BB\zeta - DD\alpha)$$

qua aequatione ratio inter α et ζ definitur, quae cum sufficiat, erit

$$\alpha = \mu A - BBm \text{ et } \zeta = \mu E - DDm,$$

hincque

$$\lambda\lambda = \mu + (\mu A - BB\,m)(\mu E - DD\,m):$$

unde colligimus

$$\gamma = \frac{mm}{\mu\mu}[2BD\lambda + (ADD - BBE)\mu] - \frac{2BBDDm^3}{\mu\mu} - \frac{m}{\mu}.$$

Valores α et ζ in formula Corollarii 3. substituti dant

$$\lambda = \frac{\mu\mu}{2m} + BD\,m - \tfrac{1}{2}C\,\mu,$$

cujus quadratum illi valori $\alpha\zeta + \mu$ aequatum, perducit ad hanc aequationem

$$\mu(\mu - C\,m)^2 + 4(BD - AE)\,mm\,\mu$$
$$+ 4(ADD - BCD + BCE)\,m^3 = 4\,mm,$$

ad quam resolvendam ponatur $\mu = M\,m$, fietque

$$m = \frac{4}{M(M-C)^2 + 4M(BD-AE) + 4(ADD - BCD + BBE)},$$

atque hic est M constans illa arbitraria pro integrali completo requisita. Hoc modo omnes litterae α, β, γ, δ, ε, ζ eodem denominatore affecti prodibunt, quo omisso habebimus

$$\alpha = 4(AM - BB),\ \beta = 2B(M-C) + 4AD,\ \gamma = 4AE - (M-C)^2,$$
$$\zeta = 4(EM - DD),\ \varepsilon = 2D(M-C) + 4BE,$$
$$\delta = MM - CC = 4(AE + BD),$$

quibus inventis aequatio nostra canonica

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy$$
$$+ 2\varepsilon xy(x+y) + \zeta xxyy$$

si brevitatis gratia ponamus

$$M(M-C)^2 + 4M(BD - AE) + 4(ADD - BCD + BBE) = \Delta,$$

resoluta dabit

$$\beta + \delta x + \varepsilon xx + y(\gamma + 2\varepsilon x + \zeta xx) =$$
$$\pm 2\sqrt{\Delta}(A + 2Bx + Cxx + 2Dx^3 + Ex^4)$$
$$\beta + \delta y + \varepsilon yy + x(\gamma + 2\varepsilon y + \zeta yy) =$$
$$\pm 2\sqrt{\Delta}(A + 2By + Cyy + 2Dy^3 + Ey^4),$$

quae ergo est integrale completum hujus aequationis differentialis

$$0 = \frac{\partial x}{\pm\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}} + \frac{\partial y}{\pm\sqrt{A+2By+Cyy+2Dy^3+Ey^4)}}.$$

Scholion.

628. Cum hic ab idonea coëfficientium determinatione totum negotium pendeat, operae pretium erit, eam luculentius exponere. Posito igitur statim

$$\delta = \gamma + \lambda \text{ et } \lambda\lambda - \alpha\zeta = Mm,$$

quinque conditiones adimplendae sunt:

I. $\beta\beta - \alpha\gamma = Am;$
II. $\varepsilon\varepsilon - \gamma\zeta = Em;$
III. $\beta\lambda - \alpha\varepsilon = Bm;$
IV. $\varepsilon\lambda - \beta\zeta = Dm;$
V. $Mm + 2\gamma\lambda - 2\beta\varepsilon = Cm.$

Hinc ex tertia et quarta combinando deducitur

$$m(B\lambda + D\alpha) = \beta(\lambda\lambda - \alpha\zeta) = \beta Mm, \text{ ergo } \beta = \frac{B\lambda + D\alpha}{M},$$
$$m(D\lambda + B\zeta) = \varepsilon(\lambda\lambda - \alpha\zeta) = \varepsilon Mm \text{ ergo } \varepsilon = \frac{D\lambda + B\zeta}{M}.$$

Jam ex prima et secunda elidendo γ, oritur

$$m(A\zeta - E\alpha) = \beta\beta\zeta - \varepsilon\varepsilon\alpha = \frac{BB\zeta - DD\alpha}{M}.m$$

hincque

$$\zeta(AM - BB) = \alpha(EM - DD);$$

quare statuatur

$$\alpha = n(AM - BB) \text{ et } \zeta = n(EM - DD).$$

Tum vero indidem est

$$E\beta\beta - E\alpha\gamma = A\varepsilon\varepsilon - A\gamma\zeta, \text{ seu}$$
$$\gamma(A\zeta - E\alpha) = A\varepsilon\varepsilon - E\beta\beta:$$

pro qua tractanda cum sit, pro α et ζ substitutis valoribus,

$$\beta = nAD + \frac{B}{M}(\lambda - nBD) \text{ et } \varepsilon = nBE + \frac{D}{M}(\lambda - nBD),$$

sit brevitatis ergo $\lambda - nBD = nMN$, ut habeamus

$$\beta = n(AD + BN) \text{ et } \varepsilon = n(BE + DN),$$

et quia

$$A\zeta - E\alpha = n(BBE - ADD)$$

atque

$$A\varepsilon\varepsilon - E\beta\beta = nn(ABBEE + ADDNN - AADDE - BBENN),\ \text{seu}$$

$$A\varepsilon\varepsilon - E\beta\beta = nn(BBE - ADD)(AE - NN)\ \text{fiet},$$

$$\gamma = n(AE - NN).$$

Cum autem sit

$$\lambda = n(BD + MN)\ \text{et}$$

$$\lambda\lambda = nn(AM - BB)(EM - DD) + Mm,\ \text{erit}$$

$$Mm = nn[2BDMN + MMNN - AEMM + M(ADD + BBE)]$$

seu

$$m = nn(2BDN + MNN - AEM + ADD + BBE).$$

Denique aequatio quinta $\beta\varepsilon - \gamma\lambda = \frac{1}{2}m(M - C)$ evoluta praebet

$$\beta\varepsilon - \gamma\lambda = nn[(AD + BN)(BE + DN) - (AE - NN)(BD + MN)]$$
$$- nnN(2BDN + MNN - AEM + ADD + BBE) = Nm,$$

unde fit $N = \frac{1}{2}(M - C)$, ac propterea

$$m = nn[BD(M - C) + \tfrac{1}{4}M(M - C)^2 - AEM + ADD + BBE].$$

Hincque sumendo $n = 4$ superiores valores obtinentur.

Exemplum 1.

629. *Invenire integrale completum hujus aequationis differentialis*

$$\frac{\partial p}{\pm\surd(a + bp)} + \frac{\partial q}{\pm\surd(a + bq)} = 0.$$

Hic est $x = p$, $y = q$, $A = a$, $B = \frac{1}{2}b$, $C = 0$, $D = 0$, $E = 0$; unde fiunt coëfficientes

$$\alpha = 4aM - bb,\quad \beta = bM,\quad \gamma = -MM,$$
$$\zeta = 0,\quad \varepsilon = 0,\quad \delta = MM,$$

et $\Delta = M^3$, unde integrale completum erit

$$bM + MMp - MMq = \pm 2M\surd M(a + bp),\ \text{seu}$$

$$b + M(p - q) = \pm 2\sqrt{M(a + bp)}, \text{ vel}$$
$$b + M(q - p) = \pm 2\sqrt{M(a + bq)};$$

quae signa ambigua radicalium cum signis in aequatione differentiali convenire debent.

Exemplum 2.

630. *Invenire integrale completum hujus aequationis differentialis* $\frac{\partial p}{\pm\sqrt{(a+bp^2)}} + \frac{\partial q}{\pm\sqrt{(a+bq^2)}} = 0$.

Sumto $x = p$ et $y = q$, erit $A = a$, $B = 0$, $C = b$, $D = 0$, ergo

$$\alpha = 4aM, \; \beta = 0, \; \gamma = -(M-b)^2,$$
$$\zeta = 0, \quad \varepsilon = 0, \; \delta = MM - bb,$$

atque $\Delta = M(M-b)^2$;

unde integrale completum in his aequationibus continebitur:

$$(MM-bb)p - (M-b)^2 q = \pm 2(M-b)\sqrt{M(a+bpp)}, \text{ seu}$$
$$(M+b)p - (M-b)q = \pm 2\sqrt{M(a+bpp)} \text{ et}$$
$$(M+b)q - (M-b)p = \pm 2\sqrt{M(a+bqq)}.$$

Exemplum 3.

631. *Invenire integrale completum hujus aequationis differentialis* $\frac{\partial p}{\pm\sqrt{(a+bp^3)}} + \frac{\partial q}{\pm\sqrt{(a+bq^3)}} = 0$.

Sumto $x = p$, $y = q$, erit $A = a$, $B = 0$, $C = 0$, $D = \frac{1}{2}b$, $E = 0$, ergo

$$\alpha = 4aM, \; \beta = 2ab, \; \gamma = -MM;$$
$$\zeta = -bb, \; \varepsilon = bM, \; \delta = MM, \text{ et}$$
$$\Delta = M^3 + abb;$$

unde integrale completum

$$2ab + MMp + bMpp + q(-MM + 2bMp - bbpp) = \pm 2\sqrt{(M^3 + abb)(a + bp^3)}$$

sive

$$2ab + Mp(M+bp) - q(M-bp)^2 = \pm 2\sqrt{(M^3+abb)(a+bp^3)}$$

et

$$2ab + Mq(M+bq) - p(M-bq)^2 = \pm 2\sqrt{(M^3+abb)(a+bq^3)}.$$

Exemplum 4.

632. *Invenire integrale completum hujus aequationis differentialis* $\frac{\partial p}{\pm\sqrt{(a+bp^4)}} + \frac{\partial q}{\pm\sqrt{(a+bq^4)}} = 0.$

Posito $x = p$, $y = q$, erit $A = a$, $B = 0$, $C = 0$, $D = 0$, $E = b$, ergo

$$\alpha = 4aM,\ \beta = 0,\ \gamma = 4ab - MM,$$
$$\zeta = 4bM,\ \varepsilon = 0,\ \delta = MM + 4ab, \text{ et}$$
$$\Delta = M^3 - 4abM;$$

unde integrale completum

$$(MM + 4ab)p + q(4ab - MM + 4bMpp) = \pm 2\sqrt{M(MM - 4ab)(a + bp^4)}$$
$$(MM + 4ab)q + p(4ab - MM + 4bMqq) = \pm 2\sqrt{M(MM - 4ab)(a + bq^4)}.$$

Exemplum 5.

633. *Invenire integrale completum hujus aequationis differentialis* $\frac{\partial p}{\pm\sqrt{(a+bp^6)}} + \frac{\partial q}{\pm\sqrt{(a+bq^6)}} = 0.$

Ponatur $x = pp$ et $y = qq$, atque aequatio nostra generalis induet, posito $A = 0$, hanc formam

$$\frac{\partial p}{\pm\sqrt{(2B + Cpp + 2Dp^4 + Ep^6)}} + \frac{\partial q}{\pm\sqrt{(2B + Cqq + 2Dq^4 + Eq^6)}} = 0.$$

Fieri ergo oportet $B = \frac{1}{2}a$, $C = 0$, $D = 0$ et $E = b$; unde coëfficientes ita determinantur

$$\alpha = -aa,\ \beta = aM,\ \gamma = -MM,$$
$$\zeta = 4bM,\ \varepsilon = 2ab,\ \delta = MM, \text{ et}$$
$$\Delta = M^3 + aab;$$

ergo integrale completum

$$aM + MMpp + 2abp^4 + qq(-MM + 4abpp + 4bMp^4) = \pm 2p\sqrt{(M^3 + aab)(a + bp^6)}$$

sive

$$aM + MMqq + 2abq^4 + pp(-MM + 4abqq + 4bMq^4) = \pm 2q\sqrt{(M^3 + aab)(a + bq^6)}.$$

Corollarium.

634. Si sumatur constans $M = -\sqrt[3]{aab}$, ut sit $M^3 + aab = 0$, prodibit integrale particulare, quod ita se habebit

$$pp = \frac{qq\sqrt[3]{b} + \sqrt[3]{a}}{2qq\sqrt[3]{b} - \sqrt[3]{a}} \cdot \sqrt[3]{\frac{a}{b}} \text{ seu } qq = \frac{pp\sqrt[3]{b} + \sqrt[3]{a}}{2pp\sqrt[3]{b} - \sqrt[3]{a}} \cdot \sqrt[3]{\frac{a}{b}}$$

quod aequationi differentiali utique satisfacit.

Problema 82.

635. Proposita hac aequatione differentiali

$$\frac{\partial p}{\pm\sqrt{(a + bpp + cp^4 + ep^6)}} + \frac{\partial q}{\pm\sqrt{(a + bqq + cq^4 + eq^6)}} = 0$$

ejus integrale completum algebraice assignare.

Solutio.

Aequatio praecedens differentialis algebraice integrata ad hanc formam reducitur, ponendo $x = pp$ et $y = qq$, atque $A = 0$; prodibit enim

$$\frac{\partial p}{\pm\sqrt{(2B + Cpp + 2Dp^4 + Ep^6)}} + \frac{\partial q}{\pm\sqrt{(2B + Cqq + 2Dq^4 + Eq^6)}} = 0.$$

Quare tantum opus est ut fiat

$$A = 0,\ B = \tfrac{1}{2}a,\ C = b,\ D = \tfrac{1}{2}c,\ E = e,$$

unde coëfficientes α, β, γ, δ, ε, ζ ita definientur

$$\alpha = -aa, \qquad \beta = a(M-b), \qquad \gamma = -(M-b)^2,$$
$$\zeta = 4eM - cc, \; \varepsilon = c(M-b) + 2ae, \; \delta = -MM - bb + ae,$$
$$\Delta = M(M-b)^2 + acM - abc + aae = (M-b)^3 + b(M-b)^2 + ac(M-b) + aae;$$

hincque integrale completum ob constantem M ab arbitrio nostro pendentem, erit

$$\beta + \delta pp + \varepsilon p^4 + qq(\gamma + 2\varepsilon pp + \zeta p^4) = \pm 2p\sqrt{\Delta(a + bp^2 + cp^4 + ep^6)}$$
$$\beta + \delta qq + \varepsilon q^4 + pp(\gamma + 2\varepsilon qq + \zeta q^4) = \pm 2q\sqrt{\Delta(a + bq^2 + cq^4 + cq^6)},$$

quae binae quidem aequationes inter se conveniunt, sed ob ambiguitatem signorum in ipsa aequatione differentiale ambae notari debent, ambiguitate inde sublata. Utrinque autem haec aequatio rationalis resultat

$$0 = \alpha + 2\beta(pp+qq) + \gamma(p^4+q^4) + 2\delta ppqq + 2\varepsilon ppqq(pp+qq) + \zeta p^4 q^4.$$

Corollarium 1.

636. Si constans M ita sumatur, ut fiat $\Delta = 0$, obtinetur integrale particulare hujus formae $qq = \frac{E + Fpp}{G + Hpp}$, quod etiam a posteriori cognoscere licet. Ut enim satisfaciat sumi debet

$$aG^3 + bEGG + cEEG + eE^3 = 0;$$

unde ratio E : G definitur, tum vero invenitur $F = -G$ et denique

$$H = \frac{-cEG - 2eEE}{aG} = \frac{2aGG + 2bEG + cEE}{aE}.$$

Corollarium 2.

637. Constans M ita mutetur, ut sit $M - b = \frac{a}{ff}$, fietque

$$\alpha = -aa, \qquad \beta = \frac{aa}{ff}, \qquad \gamma = -\frac{aa}{f^4},$$
$$\zeta = 4be - cc + \frac{4ae}{ff}, \; \varepsilon = \frac{ac}{ff} + 2ae, \; \delta = \frac{aa}{f^4} + \frac{2ab}{ff} + ac, \text{ et}$$
$$\Delta = \frac{aa}{f^6}(a + bff + cf^4 + ef^6),$$

et aequatio integralis erit

$$aaff + a(a + 2bff + cf^4)pp + aff(c + 2eff)p^4$$
$$-qq[aa - 2aff(c + 2eff)pp + ff(ccff - 4beff - 4ae)p^4]$$
$$= \pm 2afp\sqrt{(a + bff + cf^4 + ef^6)(a + bpp + cp^4 + ep^6)};$$

unde patet posito $p = 0$ fore $qq = ff$.

Corollarium 3.

638. Haec aequatio facile in hanc formam transmutatur

$$aff(a + bpp + cp^4 + ep^6) + app(a + bff + cf^4 + ef^6)$$
$$-qq(a - cffpp)^2 - aeffpp(ff - pp)^2 + 4effppqq(aff + app + bffpp)$$
$$= \pm 2fp\sqrt{a(a + bff + cf^4 + ef^6)a(a + bpp + cp^4 + ep^6)};$$

unde statim patet si sit $e = 0$, fore hanc aequationem, radicem extrahendo

$$f\sqrt{a(a + bpp + cp^4)} \pm p\sqrt{a(a + bff + cf^4)} = q(a - cffpp)$$

quae est integralis completa hujus differentialis

$$\frac{\partial p}{\pm\sqrt{(a + bpp + cp^4)}} + \frac{\partial q}{\pm\sqrt{(a + bqq + cq^4)}} = 0$$

prorsus ut supra jam invenimus.

Corollarium 4.

639. Simili modo patet in genere, quando e non evanescit, integrale completum ita commodius exprimi posse

$$[f\sqrt{a(a + bpp + cp^4 + ep^6)} \pm p\sqrt{a(a + bff + cf^4 + ef^6)}]^2 =$$
$$qq(a - cffpp)^2 + aeffpp(ff - pp)^2 - 4effppqq(aff + app + bffpp),$$

quae ergo cum posito $p = 0$ fiat $q = f$, respondet huic functionum transcendentium relationi

$$\pm \Pi : p \pm \Pi : q = \pm \Pi : 0 \pm \Pi : f.$$

Scholion 1.

640. Genera igitur functionum transcendentium, quas hoc modo perinde atque arcus circulares inter se comparare licet, in his binis formulis integralibus continentur

$$\int \frac{\partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}} \text{ et } \int \frac{\partial z}{\sqrt{(a + bzz + cz^4 + ez^6)}}$$

neque haec methodus ad alias formas magis complexas extendi posse videtur. Neque etiam posterior in denominatore potestates impares ipsius z admittit: nisi forte simplex substitutio reductioni ad illam formam sufficiat. Facile autem patet hujusmodi formam

$$\int \frac{\partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4 + Fz^5 + Gz^6)}},$$

hac methodo tractari certe non posse, si enim coëfficientes ita essent comparati, ut radicis extractio succederet, talis formula $\int \frac{\partial z}{a + bz + czz + ez^3}$ prodiret, cujus integratio, cum tam logarithmos quam arcus circulares involvat, fieri omnino nequit, ut plures hujusmodi functiones algebraice inter se comparentur. Caeterum prior formula latius patet quam posterior, cum haec ex illa nascatur posito $A = 0$, si zz loco z scribatur. De priori autem notari meretur, quod eandem formam servet, etiamsi transformetur hac substitutione $z = \frac{\alpha + \beta y}{\gamma + \delta y}$; prodit enim

$$\int \frac{(\beta\gamma - \alpha\delta)\,\partial y}{\left\{\sqrt{\left[A(\gamma + \delta y)^4 + 2B(\alpha + \beta y)(\gamma + \delta y)^3 + C(\alpha + \beta y)^2(\gamma + \delta y)^2 + 2D(\alpha + \beta y)^3(\gamma + \delta y) + E(\alpha + \beta y)^4\right]}\right\}},$$

ex quo intelligitur quantitates α, β, γ, δ, ita accipi posse, ut potestates impares evanescant. Vel etiam ita definiri poterunt, ut terminus primus et ultimus evanescat, tum enim posito $y = uu$, iterum forma a potestatibus imparibus immunis nascitur.

Scholion 2.

641. Sublatio autem potestatum imparium ita commodissime instituitur. Cum formula

$$A + 2Bz + Czz + 2Dz^3 + Ez^4$$

certe semper habeat duos factores reales, ita exhibeatur formula integralis

$$\int \frac{\partial z}{\sqrt{(a + 2bz + czz)(f + 2gz + hzz)}},$$

quae posito $z = \frac{\alpha + \beta y}{\gamma + \delta y}$, abit in

$$\int \frac{(\beta\gamma - \alpha\delta)\partial y}{\left\{\sqrt{[a(\gamma + \delta y)^2 + 2b(\alpha + \beta y)(\gamma + \delta y) + c(\alpha + \beta y)^2][f(\gamma + \delta y)^2 + 2g(\alpha + \beta y)(\gamma + \delta y) + h(\alpha + \beta y)^2]}\right\}},$$

ubi denominatoris factores evoluti sunt

$$(a\gamma\gamma + 2b\alpha\gamma + c\alpha\alpha) + 2(a\gamma\delta + b\alpha\delta + b\beta\gamma + c\alpha\beta)y + (a\delta\delta + 2b\beta\delta + c\beta\beta)yy$$

$$(f\gamma\gamma + 2g\alpha\gamma + h\alpha\alpha) + 2(f\gamma\delta + g\alpha\delta + g\beta\gamma + h\alpha\beta)y + (f\delta\delta + 2g\beta\delta + h\beta\beta)yy$$

quodsi jam utroque terminus medius evanescens reddatur, fit

$$\frac{\delta}{\beta} = \frac{-b\gamma - c\alpha}{a\gamma + b\alpha} = \frac{-g\gamma - h\alpha}{f\gamma + g\alpha},$$

hincque

$$bf\gamma\gamma + (bg + cf)\alpha\gamma + cg\alpha\alpha = ag\gamma\gamma + (ah + bg)\alpha\gamma + bh\alpha\alpha$$

seu

$$\gamma\gamma = \frac{(ah - cf)\alpha\gamma + (bh - cg)\alpha\alpha}{bf - ag},$$

unde fit

$$\frac{\gamma}{\alpha} = \frac{ah - cf + \sqrt{[(ah - cf)^2 + 4(bf - ag)(bh - cg)]}}{2(bf - ag)}.$$

Hinc sufficere posset eas tantum formulas, in quibus potestates impares desunt, tractasse, id quod initio hujus capitis fecimus, sed si insuper numerator accedat, haec reductio non amplius locum habet.

Problema 83.

642. Denotante n numerum integrum quemcunque, invenire integrale completum algebraice expressum hujus aequationis differentialis

$$\frac{\partial y}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}} = \frac{n\,\partial x}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}.$$

Solutio.

Per functiones transcendentes integrale completum est

$$\Pi : y = n\,\Pi : x + \text{Const.}$$

At ut idem algebraice expressum eruamus, posito $M - C = L$, sit per formulas supra (627.) inventas

$\alpha = 4(AC - BB + AL)$, $\beta = 4AD + 2BL$, $\gamma = 4AE - LL$,

$\zeta = 4(CE - DD + EL)$, $\varepsilon = 4BE + 2DL$, $\delta = 4AE + 4BD + 2CL + LL$,

et

$$\Delta = L^3 + CL^2 + 4(BD - AE) + 4(ADD + BBE - ACE).$$

Quibus positis si fuerit

$$\beta + \delta p + \varepsilon pp + q(\gamma + 2\varepsilon p + \zeta pp) = 2\sqrt{\Delta}\,(A + 2Bp + Cp^2 + 2Dp^3 + Ep^4)$$

$$\beta + \delta q + \varepsilon qq + p(\gamma + 2\varepsilon q + \zeta qq) = -2\sqrt{\Delta}\,(A + 2Bq + Cq^2 + 2Dq^3 + Eq^4)$$

erit $\Pi : q = \Pi : p + \text{Const.}$

Cum autem hae duae aequationes inter se conveniant, et in hac rationali contineantur

$$\alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq = 0$$

si sumamus, posito $p = a$ fieri $q = b$, constans illa L ita definiri debet, ut sit

$$\alpha + 2\beta(a+b) + \gamma(aa+bb) + 2\delta ab + 2\varepsilon ab(a+b) + \zeta aabb = 0,$$

eritque

$$\Pi : q = \Pi : p + \Pi : b - \Pi : a;$$

ubi jam nullum inest discrimen inter constantes et variabiles. Ponamus ergo $p = b$, ut sit

$$\Pi : q = 2\,\Pi : p - \Pi : a$$

atque huic aequationi superiores aequationes algebraicae conveniunt, si modo quantitas L ita definiatur, ut sit

$$\alpha + 2\beta(a+p) + \gamma(aa+pp) + 2\delta ap + 2\varepsilon ap(a+p) + \zeta aapp = 0,$$

unde deducitur

$$\tfrac{1}{2}\mathrm{L}(a-p)^2 = \mathrm{A} + \mathrm{B}(a+p) + \mathrm{C}ap + \mathrm{D}ap(a+p) + \mathrm{E}aapp \pm \sqrt{(\mathrm{A}+2\mathrm{B}a+\mathrm{C}aa+2\mathrm{D}a^3+\mathrm{E}a^4)(\mathrm{A}+2\mathrm{B}p+\mathrm{C}pp+2\mathrm{D}p^3+\mathrm{E}p^4)}.$$

Hoc ergo valore pro L constituto, indeque litteris α, β, γ, δ, ε, ζ per superiores formulas rite definitis, si jam p et q ut variabiles, a vero ut constantem spectemus, erit haec aequatio

$$\alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq = 0,$$

integrale completum hujus aequationis differentialis

$$\frac{\partial q}{\sqrt{(\mathrm{A}+2\mathrm{B}q+\mathrm{C}qq+2\mathrm{D}q^3+\mathrm{E}q^4)}} = \frac{2\,\partial p}{\sqrt{(\mathrm{A}+2\mathrm{B}p+\mathrm{C}pp+2\mathrm{D}p^4+\mathrm{E}p^4)}}.$$

Postquam hoc modo q per p definivimus, determinetur r per hanc aequationem

$$\alpha + 2\beta(q+r) + \gamma(qq+rr) + 2\delta qr + 2\varepsilon qr(q+r) + \zeta qqrr = 0,$$

erit

$$\Pi : r - \Pi : q = \Pi : p - \Pi : a,$$

quoniam, posito $q = a$ et $r = p$, littera L, quae in valores α, β, γ, δ, ε, ζ ingreditur, perinde definitur ut ante. Quare cum sit

$$\Pi : q = 2\,\Pi : p - \Pi : a, \text{ erit } \Pi : r = 3\,\Pi : p - 2\,\Pi : a;$$

unde sumto a constante, illa aequatio algebraica inter q et r, dum q per praecedentem aequationem ex p definitur, erit integrale completum hujus aequationis differentialis

$$\frac{\partial r}{\sqrt{(A+2Br+Crr+2Dr^3+Er^4)}}=\frac{3\,\partial p}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

Hoc valore ipsius r per p invento, quaeratur s per hanc aequationem

$$\alpha+2\beta(r+s)+\gamma(rr+ss)+2\delta rs+2\varepsilon rs(r+s)+\zeta rrss=0,$$

retinente L semper valorem primo assignatum, eritque

$$\Pi:s-\Pi:r=\Pi:p-\Pi:a,\ \text{seu}\ \Pi:s=4\,\Pi:p-3\,\Pi:a$$

unde ista aequatio algebraica erit integrale completum hujus aequationis differentialis

$$\frac{\partial s}{\sqrt{(A+2Bs+Css+2Ds^3+Es^4)}}=\frac{4\,\partial p}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

Cum hoc modo quousque libuerit progredi liceat, perspicuum est, ad integrale completum hujus aequationis differentialis inveniendum

$$\frac{\partial z}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}=\frac{n\,\partial p}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

sequentes operationes institui oportere.

1.) Quaeratur quantitas L, ut sit

$$\tfrac{1}{2}L(p-a)^2=A+B(a+p)+Cap+Dap(a+p)+Eaapp$$
$$\pm\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bp+Cpp+2Dp^3+Ep^4)}$$

2.) Hinc determinentur litterae α, β, γ, δ, ε, ζ, per has formulas

$$\alpha=4(AC-BB+AL),\ \beta=4AD+2BL,\ \gamma=4AE-LL,$$
$$\zeta=4(CE-DD+EL),\ \varepsilon=4BE+2DL,\ \delta=4AE+4BD+2CL+LL.$$

3.) Formetur series quantitatum $p, q, r, s, t, \ldots\ldots z$, quarum prima sit p, secunda q, tertia r etc. ultima vero ordine n sit z, quae successive per has aequationes determinentur

$$\alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq = 0$$
$$\alpha + 2\beta(q+r) + \gamma(qq+rr) + 2\delta qr + 2\varepsilon qr(q+r) + \zeta qqrr = 0$$
$$\alpha + 2\beta(r+s) + \gamma(rr+ss) + 2\delta rs + 2\varepsilon rs(r+s) + \zeta rrss = 0$$
etc.

donec ad ultimam z perveniatur.

4.) Relatio quae hinc concluditur inter p et z erit integrale completum aequationis differentialis propositae, et littera a vicem gerit constantis arbitrariae per integrationem ingressae.

Corollarium.

643. Hinc etiam integrale completum inveniri potest hujus aequationis differentialis

$$\frac{m\,\partial y}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}} = \frac{n\,\partial x}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}},$$

designantibus m et n numeros integros. Statuatur enim utrumque membrum $= \frac{\partial u}{\sqrt{(A+2Bu+Cuu+2Du^3+Eu^4)}}$, et quaeratur relatio tam inter x et u, quam inter y et u; unde elisa u orietur aequatio algebraica inter x et y.

Scholion.

644. Ne hic extractio radicis in singulis aequationibus repetenda ambiguitatem creet, loco uniuscujusque uti conveniet binis per extractionem jam erutis. Scilicet ut ex prima valor q rite per p definiatur, primo quidem habemus

$$q = \frac{-\beta - \delta p - \varepsilon pp + 2\sqrt{\Delta(A+2Bp+Cpp+2Dp^3+Ep^4)}}{\gamma + 2\varepsilon p + \zeta pp},$$

tum vero capi debet

$$2\sqrt{\Delta}(A+2Bq+Cqq+2Dq^3+Eq^4) = -\beta - \delta q - \varepsilon qq - p(\gamma + 2\varepsilon q + \zeta qq):$$

simili que modo in relatione inter binas sequentes quantitates investiganda erit procedendum. Caeterum adhuc notari convenit numeros integros m et n positivos esse debere, neque hanc investigationem ad negativos extendi, propterea quod formula differentialis $\frac{\partial z}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}$, posito z negativo, naturam suam mutat. Interim tamen cum hanc aequalitatem

$$\Pi : x + \Pi : y = \text{Const.}$$

supra algebraice expresserimus, ejus ope quoque ii casus resolvi possunt, ubi est m vel n numerus negativus: si enim fuerit

$$\Pi : z = n\,\Pi : p + \text{Const.}$$

quaeratur y, ut sit

$$\Pi : y + \Pi : z = \text{Const.}$$

eritque

$$\Pi : y = -n\,\Pi : p + \text{Const.}$$

Problema 84.

645. Si $\Pi : z$ ejusmodi functionem transcendentem ipsius z denotet, ut sit

$$\Pi : z = \int \frac{\partial z\,(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}z^3+\mathfrak{E}z^4)}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}},$$

comparationem inter hujusmodi functiones investigare.

Solutio.

Ex coëfficientibus A, B, C, D, E, una cum constante arbitraria L determinentur sequentes valores

$\alpha = 4(AC-BB+AL)$, $\beta = 4AD+2BL$, $\gamma = 4AE-LL$,
$\zeta = 4(CE-DD+EL)$, $\varepsilon = 4BE+2DL$, $\delta = 4AE+4BD+2CL+LL$,

et inter binas variabiles x et y haec constituatur relatio

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy = 0,$$

eritque

$$\frac{\partial x}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}} + \frac{\partial y}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}} = 0,$$

pro qua sine ambiguitate habetur

$$\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx) = 2\sqrt{\Delta}(A+2Bx+Cxx+2Dx^3+Ex^4)$$
$$\beta+\delta y+\varepsilon yy+x(\gamma+2\varepsilon y+\zeta yy) = 2\sqrt{\Delta}(A+2By+Cyy+2Dy^3+Ey^4)$$

existente

$$\Delta = L^3 + CL^2 + 4(BD-AE)L + 4(ADD+BBE-ACE).$$

Quare si ponamus

$$\frac{\partial x(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\mathfrak{E}x^4)}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}+\frac{\partial y(\mathfrak{A}+\mathfrak{B}y+\mathfrak{C}y^2+\mathfrak{D}y^3+\mathfrak{E}y^4)}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=2\partial V\sqrt{\Delta},$$

ut sit

$$\Pi : x+\Pi : y = \text{Const.} + 2V\sqrt{\Delta}, \text{ erit}$$

$$\frac{\partial x[\mathfrak{B}(x-y)+\mathfrak{C}(x^2-y^2)+\mathfrak{D}(x^3-y^3)+\mathfrak{E}(x^4-y^4)]}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}=2\partial V\sqrt{\Delta}, \text{ seu}$$

$$\partial V=\frac{\partial x[\mathfrak{B}(x-y)+\mathfrak{C}(x^2-y^2)+\mathfrak{D}(x^3-y^3)+\mathfrak{E}(x^4-y^4)]}{\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)}.$$

Ponatur nunc $x+y=t$ et $xy=u$, et quia $\partial x+\partial y=\partial t$ et $x\partial y+y\partial x=\partial u$, erit $\partial x=\frac{x\partial t-\partial u}{x-y}$, seu $(x-y)\partial x=x\partial t-\partial u$, tum vero est $x=\frac{1}{2}t+\sqrt{(\frac{1}{4}tt-u)}$. At his positionibus aequatio assumta induit hanc formam

$$\alpha+2\beta t+\gamma tt+2(\delta-\gamma)u+2\varepsilon tu+\zeta uu=0,$$

unde fit differentiando

$$\partial t(\beta+\gamma t+\varepsilon u)+\partial u(\delta-\gamma+\varepsilon t+\zeta u)=0, \text{ ergo}$$

$$\partial t=\frac{-\partial u(\delta-\gamma+\varepsilon t+\zeta u)}{\beta+\gamma t+\varepsilon u}, \text{ et}$$

$$x\partial t-\partial u=\frac{-\partial u[\beta+\gamma t+\varepsilon u+(\delta-\gamma)x+\varepsilon tx+\zeta ux]}{\beta+\gamma t+\varepsilon u} \text{ sive}$$

$$x\partial t-\partial u=\frac{-\partial u[\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)]}{\beta+\gamma t+\varepsilon u}$$

sicque habebimus

$$\frac{\partial x(x-y)}{\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)}=\frac{-\partial u}{\beta+\gamma t+\varepsilon u}; \text{ ergo}$$

$$\partial V=\frac{-\partial u[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{\beta+\gamma t+\varepsilon u} \text{ seu}$$

$$\partial V=\frac{+\partial t[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{\delta-\gamma+\varepsilon t+\zeta u}.$$

Est vero aequatione illa resoluta

$$t=\frac{-\beta-\varepsilon u+\sqrt{[\beta\beta-\alpha\gamma+2(\gamma\gamma+\beta\varepsilon-\gamma\delta)u+(\varepsilon\varepsilon-\gamma\zeta)uu]}}{\gamma} \text{ seu}$$

$$t=\frac{-\beta-\varepsilon u+2\sqrt{\Delta(A+Lu+Euu)}}{\gamma},$$

unde conficitur

$$\partial V=\frac{-\partial u[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{2\sqrt{\Delta(A+Lu+Euu)}},$$

ideoque

$$\Pi : x+\Pi : y=\text{Const.}-\int\frac{\partial u[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{\sqrt{(A+Lu+Euu)}}.$$

**

Vel cum reperiatur

$$u = \frac{-(\delta-\gamma)-\varepsilon t+\sqrt{[(\delta-\gamma)^2-\alpha\zeta+2[(\delta-\gamma)\varepsilon-\beta\zeta]t+(\varepsilon\varepsilon-\gamma\zeta)tt]}}{\gamma}$$

quae expressio abit in hanc

$$u = \frac{-(\delta-\gamma)-\varepsilon t+2\sqrt{\Delta(\mathrm{L}+2\mathrm{D}t+\mathrm{E}tt)}}{\zeta}$$

unde fit

$$\partial \mathrm{V} = \frac{\partial t[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{2\sqrt{\Delta(\mathrm{L}+\mathrm{C}+2\mathrm{D}t+\mathrm{E}tt)}}$$

sicque habebimus per t

$$\Pi:x+\Pi:y = \text{Const.} + \int\frac{\partial t[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{\sqrt{(\mathrm{L}+\mathrm{C}+2\mathrm{D}t+\mathrm{E}tt)}},$$

quae expressio, nisi sit algebraica, certe vel per logarithmos, vel arcus circulares exhiberi potest. Tum vero post integrationem tantum opus est, ut loco t restituatur ejus valor $x+y$.

Corollarium 1.

646. Si velimus, ut posito $x=a$ fiat $y=b$; constans L ita debet definiri, ut sit

$$\tfrac{1}{2}\mathrm{L}(b-a)^2 = \mathrm{A}+\mathrm{B}(a+b)+\mathrm{C}ab+\mathrm{D}ab(a+b)+\mathrm{E}aabb$$
$$\pm\sqrt{(\mathrm{A}+2\mathrm{B}a+\mathrm{C}aa+2\mathrm{D}a^3+\mathrm{E}a^4)(\mathrm{A}+2\mathrm{B}b+\mathrm{C}bb+2\mathrm{D}b^3+\mathrm{E}b^4)};$$

tum igitur constans nostra erit $=\Pi:a+\Pi:b$, integrali postremo ita sumto, ut evanescat posito $t=a+b$.

Corollarium 2.

647. Eodem modo etiam differentia functionum $\Pi:x-\Pi:y$ exprimi potest, mutando alterutrius formulae radicalis signum, quo pacto formularum differentialium signum alterius convertetur.

Corollarium 3.

648. Qvantitas V comparationi harum functionum inserviens, erit algebraica, si haec formula differentialis

$$\frac{\partial t[\mathfrak{B}+\mathfrak{C}\zeta t+\mathfrak{D}(\delta-\gamma+\varepsilon t+\zeta tt)+\mathfrak{E}[2(\delta-\gamma)+2\varepsilon t+\zeta tt]t]}{\zeta\sqrt{(\mathrm{L}+\mathrm{C}+2\mathrm{D}t+\mathrm{E}tt)}}$$

integrationem admittat; quia altera pars $\frac{-2\partial t\sqrt{\Delta}}{\zeta}(\mathfrak{D}+2\mathfrak{E}t)$ per se est integrabilis.

Scholion.

649. Hoc ergo argumentum plane novum de comparatione hujusmodi functionum transcendentium tam copiose pertractavimus, quam praesens institutum postulare videbatur. Quando autem ejus applicatio ad comparationem arcuum curvarum, quorum longitudo hujusmodi functionibus exprimitur, erit facienda, uberiori evolutione erit opus, ubi contemplatio singularium proprietatum, quae hoc modo eruuntur, eximium usum afferre poterit. Commode autem hoc argumentum ad doctrinam de resolutione aequationum differentialium referri videtur, siquidem inde ejusmodi aequationum integralia completa et quidem algebraice exhiberi possunt, quae aliis methodis frustra indagantur. Hunc igitur huic sectionis finem faciet methodus generalis omnium aequationum differentialium integralia proxime determinandi.