CAPUT V.

DE COMPARATIONE QUANTITATUM TRANSCENDENTIUM IN FORMA $\int\frac{P\partial x}{\sqrt{(A+2Bx+Cxx)}}$ CONTENTARUM.

Problema 73.

580.

Proposita inter x et y hac aequatione algebraica

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy=0$$

invenire formulas integrales formae praescriptae, quae inter se comparari queant.

Solutio.

Differentietur aequatio proposita, et ex ejus differentiali

$$2\beta\partial x+2\beta\partial y+2\gamma x\partial x+2\gamma y\partial y+2\delta x\partial y+2\delta y\partial x=0$$

colligetur haec aequatio

$$\partial x(\beta+\gamma x+\delta y)+\partial y(\beta+\gamma y+\delta x)=0.$$

Statuatur $\beta+\gamma x+\delta y=p$ et $\beta+\gamma y+\delta x=q$, atque ex priori erit

$$pp=\beta\beta+2\beta\gamma x+2\beta\delta y+\gamma\gamma xx+2\gamma\delta xy+\delta\delta yy,$$

a qua subtrahatur aequatio proposita per γ multiplicata

$$0=\alpha\gamma+2\beta\gamma x+2\gamma\beta y+\gamma\gamma xx+\gamma\gamma yy+2\gamma\delta xy,$$

fietque

$$pp = \beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)y + (\delta\delta - \gamma\gamma)yy.$$

Similique modo reperietur

$$qq = \beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)x + (\delta\delta - \gamma\gamma)xx,$$

unde erit $p\partial x + q\partial y = 0$. Cum jam sit p functio ipsius y, et similis functio ipsius x, ponatur

$$\beta\beta - \alpha\gamma = A, \beta(\delta - \gamma) = B, \text{ et } \delta\delta - \gamma\gamma = C;$$

unde colligitur

$$\delta - \gamma = \frac{B}{\beta} \text{ et } \delta + \gamma = \frac{C}{\delta - \gamma} = \frac{\beta C}{B},$$

hincque

$$\delta = \frac{BB + \beta\beta C}{2B\beta} \text{ et } \gamma = \frac{\beta\beta C - BB}{2B\beta}:$$

prima vero dat

$$\alpha = \frac{\beta\beta - A}{\gamma} = \frac{2B\beta(\beta\beta - A)}{\beta\beta C - BB}.$$

Quibus valoribus pro α, γ, δ assumtis, aequatio $\frac{\partial x}{q} + \frac{\partial y}{p} = 0$ abi in hanc

$$\frac{\partial x}{\sqrt{(A + 2Bx + Cxx)}} + \frac{\partial y}{\sqrt{(A + 2By + Cyy)}} = 0;$$

cui ergo aequationi differentiali satisfacit aequatio

$$\frac{2B\beta(\beta\beta - A)}{\beta\beta C - BB} + 2\beta(x + y) + \frac{\beta\beta C - BB}{2B\beta}(xx + yy) + \frac{BB + \beta\beta C}{B\beta}xy = 0,$$

quae cum contineat constantem novam β, erit adeo integrale completum aequationis differentialis inventae.

Neque vero opus est, ut formulae illae ipsis litteris A, B, C aequentur, sed sufficit ut ipsis sint proportionales, unde fit

$$\frac{\beta\beta - \alpha\gamma}{\beta(\delta - \gamma)} = \frac{A}{B} \text{ et } \frac{\delta + \gamma}{\beta} = \frac{C}{B}.$$

Ergo

$$\delta = \frac{\beta C}{B} - \gamma \text{ et } \alpha = \frac{\beta\beta}{\gamma} - \frac{\beta A}{\gamma B}(\delta - \gamma), \text{ seu}$$
$$\alpha = \frac{\beta\beta}{\gamma} - \frac{\beta\beta AC}{\gamma BB} + \frac{2\beta A}{B}.$$

Quare aequationis differentialis

$$\frac{\partial x}{\sqrt{(A+2Bx+Cxx)}}+\frac{\partial y}{\sqrt{(A+2By+Cyy)}}=0$$

integrale completum est

$$\beta\beta(CB-AC)+2\beta\gamma AB+2\beta\gamma BB(x+y)+\gamma\gamma BB(xx+yy)$$
$$+2\gamma B(\beta C-\gamma B)xy=0,$$

ubi ratio $\frac{\beta}{\gamma}$ constantem arbitrariam exhibet.

Corollarium 1.

581. Ex aequatione proposita radicem extrahendo fit

$$y=\frac{-\beta-\delta x+\sqrt{(\beta\beta+2\beta\delta x+\delta\delta xx-\alpha\gamma-2\beta\gamma x-\gamma\gamma xx)}}{\gamma},$$

seu loco α et δ substitutis valoribus,

$$y=-\frac{\beta}{\gamma}-\frac{(\beta C-\gamma B)}{\gamma B}x+\sqrt{\left(\frac{\beta\beta C-2\beta\gamma B}{\gamma\gamma BB}\right)(A+2Bx+Cxx)}.$$

Corollarium 2.

582. Si ergo $x=0$, fit

$$y=-\frac{\beta}{\gamma}+\sqrt{\frac{\beta\beta AC-2\beta\gamma AB}{\gamma\gamma BB}},$$

ponatur hic valor $=a$, ut sit

$$\gamma Ba+\beta B=\sqrt{(\beta\beta AC-2\beta\gamma AB)},$$

unde sumtis quadratis oritur

$$\gamma\gamma BBaa+2\beta\gamma BBa+\beta\beta BB=\beta\beta AC-2\beta\gamma AB,$$

hincque

$$\frac{\gamma}{\beta}=\frac{-A-Ba+\sqrt{A(A+2Ba+Caa)}}{Baa}, \text{ seu}$$
$$\frac{\beta}{\gamma}=\frac{B(A+Ba+\sqrt{A(A+2Ba+Caa)})}{AC-BB}.$$

Scholion 1.

283. Ut aequatio assumta

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy=0$$

satisfaciat aequationi differentiali

$$\frac{\partial x}{\sqrt{(A+2Bx+Cxx)}}+\frac{\partial y}{\sqrt{(A+2By+Cyy)}}=0,$$

necesse est ut sit

$$\beta\beta-\alpha\gamma=mA,\ \beta(\delta-\gamma)=mB \text{ et } \delta\delta-\gamma\gamma=mC,$$

unde fit

$$\beta+\gamma y+\delta x=\sqrt{m}(A+2Bx+Cxx) \text{ et}$$
$$\beta+\gamma x+\delta y=\sqrt{m}(A+2By+Cyy).$$

At ex datis A, B, C, litterarum α, β, γ, δ et m tres tantum definiuntur; quare cum binae maneant indeterminatae, aequatio assumta, etiamsi per quemvis coëfficientium dividatur, unam tamen constantem continet novam, ex quo ea pro integrali completo erit habenda. Quare etsi aequationis differentialis neutra pars integrationem algebraice admittit, tamen integrale completum algebraice exhiberi potest. Loco constantis arbitrariae is valor ipsius y introduci potest, quem recipit posito $x=0$: cum autem evenire possit, ut hic valor fiat imaginarius, conveniet istam constantem ita definiri, ut posito $x=a$ fiat $y=b$, quo pacto ad omnes casus applicatio fieri poterit. Hinc erit

$$\frac{\beta+\gamma b+\delta a}{\beta+\gamma a+\delta b}=\sqrt{\frac{A+2Ba+Caa}{A+2Bb+Cbb}},$$

unde colligitur

$$\beta=\frac{(\gamma a+\delta b)\sqrt{(A+2Ba+Caa)}-(\gamma b+\delta a)\sqrt{(A+2Bb+Cbb)}}{-\sqrt{(A+2Ba+Caa)}+\sqrt{(A+2Bb+Cbb)}} \text{ et}$$

$$\sqrt{m}(A+2Ba+Caa)=\frac{(\delta-\gamma)(b-a)\sqrt{(A+2Ba+Caa)}}{\sqrt{(A+2Bb+Cbb)}-\sqrt{(A+2Ba+Caa)}}$$

seu

$$\sqrt{m}=\frac{(\delta-\gamma)(b-a)}{\sqrt{(A+2Bb+Cbb)}-\sqrt{(A+2Ba+Caa)}}.$$

Ponatur brevitatis gratia

$$\sqrt{(A+2Ba+Caa)}=\mathfrak{A} \text{ et } \sqrt{(A+2Bb+Cbb)}=\mathfrak{B},$$

ut sit

$$\sqrt{m}=\frac{(\delta-\gamma)(b-a)}{\mathfrak{B}-\mathfrak{A}} \text{ et}$$

$$\beta=\frac{\mathfrak{A}(\gamma a+\delta b)-\mathfrak{B}(\gamma b+\delta a)}{\mathfrak{B}-\mathfrak{A}},$$

et aequatio $\beta(\delta-\gamma)=m\mathrm{B}$ induet hanc formam

$$\mathfrak{A}(\gamma a+\delta b)-\mathfrak{B}(\gamma b+\delta a)=\frac{\mathrm{B}(\delta-\gamma)(b-a)^2}{\mathfrak{B}-\mathfrak{A}}$$

unde fit

$$\left.\begin{array}{l}+\gamma\mathfrak{A}\mathfrak{B}-\gamma\mathrm{A}-\gamma\mathrm{B}(a+b)-\gamma\mathrm{C}(aa-ab+bb)\\+\delta\mathfrak{A}\mathfrak{B}-\delta\mathrm{A}-\delta\mathrm{B}(a+b)-\delta\mathrm{C}ab\end{array}\right\}=0.$$

Statuatur ergo

$$\gamma=n\mathfrak{A}\mathfrak{B}-n\mathrm{A}-n\mathrm{B}(a+b)-n\mathrm{C}ab$$

$$\delta=n\mathrm{A}+n\mathrm{B}(a+b)+n\mathrm{C}(aa-ab+bb)-n\mathfrak{A}\mathfrak{B}$$

$$\sqrt{m}=\frac{n(b-a)(\mathfrak{A}^2+\mathfrak{B}^2-2\mathfrak{A}\mathfrak{B})}{\mathfrak{B}-\mathfrak{A}}=n(b-a)(\mathfrak{B}-\mathfrak{A})$$

$$\beta=n\mathrm{B}(b-a)^2,\ \text{ergo}\ \delta-\gamma=\frac{m}{n(b-a)^2};$$

unde cum sit $\delta+\gamma=n\mathrm{C}(b-a)^2$, erit utique $\delta\delta-\gamma\gamma=m\mathrm{C}$. Superest ut fiat $\alpha\gamma=\beta\beta-m\mathrm{A}$, hoc est

$$\alpha\gamma=nn\mathrm{BB}(b-a)^4-nn\mathrm{A}(b-a)^2(\mathfrak{B}-\mathfrak{A})^2 \text{ seu}$$

$$\alpha\gamma=nn(b-a)^2[\mathrm{BB}(b-a)^2-\mathrm{A}(\mathfrak{B}-\mathfrak{A})^2].$$

Vel cum posito $x=a$ fiat $y=b$, erit quoque

$$\alpha=-2\beta(a+b)-\gamma(aa+bb)-2\delta ab,$$

hincque

$$\alpha=n(a-b)^2[\mathrm{A}-\mathrm{B}(a+b)-\mathrm{C}ab-\mathfrak{A}\mathfrak{B}];$$

unde aequatio nostra assumta est

$$\begin{array}{l}(b-a)^2[\mathrm{A}-\mathrm{B}(a+b)-\mathrm{C}ab-\mathfrak{A}\mathfrak{B}]+2\mathrm{B}(b-a)^2(x+y)\\ \quad-[\mathrm{A}+\mathrm{B}(a+b)+\mathrm{C}ab-\mathfrak{A}\mathfrak{B}](xx+yy)\\ \quad+2[\mathrm{A}+\mathrm{B}(a+b)+\mathrm{C}(aa-ab+bb)-\mathfrak{A}\mathfrak{B}]xy=0.\end{array}$$

Scholion 2.

584. Si ponatur $\beta=0$, ut aequatio sit

$$\alpha + \gamma(xx + yy) + 2\delta xy = 0, \text{ erit}$$
$$y = \frac{-\delta x + \sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma)xx]}}{\gamma}.$$

Posito ergo $-\alpha\gamma = m\text{A}$ et $\delta\delta - \gamma\gamma = m\text{C}$, ut sit
$$\gamma y + \delta x = \sqrt{m(\text{A} + \text{C}xx)}, \text{ erit}$$
$$\frac{\partial x}{\sqrt{(\text{A}+\text{C}xx)}} + \frac{\partial y}{\sqrt{(\text{A}+\text{C}yy)}} = 0,$$
cujus aequationis integrale completum erit ipsa aequatio assumta, pro qua habebitur $\frac{\text{C}}{\text{A}} = \frac{\gamma\gamma - \delta\delta}{\alpha\gamma}$, seu $\delta = \sqrt{(\gamma\gamma - \frac{\alpha\gamma\text{C}}{\text{A}})}$. Sin autem posito $x = 0$ fieri debeat $y = b$, ob $\gamma b = \sqrt{m\text{A}}$, erit $\gamma = \frac{\sqrt{m\text{A}}}{b}$; tum $\alpha = -b\sqrt{m\text{A}}$ et $\delta = \sqrt{(\frac{m\text{A}}{bb} + m\text{C})}$. Habebitur ergo haec aequatio
$$\frac{y\sqrt{m\text{A}}}{b} + \frac{x\sqrt{m(\text{A}+\text{C}bb)}}{\text{A}} = \sqrt{m(\text{A}+\text{C}xx)},$$
quae praebet
$$y = -x\sqrt{\frac{\text{A}+\text{C}bb}{\text{A}}} + b\sqrt{\frac{\text{A}+\text{C}xx}{\text{A}}},$$
quae est integrale completum aequationis illius differentialis. Quare si x capiatur negative, hujus aequationis differentialis
$$\frac{\partial x}{\sqrt{(\text{A}+\text{C}xx)}} = \frac{\partial y}{\sqrt{(\text{A}+\text{C}yy)}},$$
integrale completum est
$$y = x\sqrt{\frac{\text{A}+\text{C}bb}{\text{A}}} + b\sqrt{\frac{\text{A}+\text{C}xx}{\text{A}}}.$$

Quodsi simili modo calculus in genere tractetur, aequationis differentialis
$$\frac{\partial x}{\sqrt{(\text{A}+2\text{B}x+\text{C}xx)}} + \frac{\partial y}{\sqrt{(\text{A}+2\text{B}y+\text{C}yy)}} = 0,$$
si brevitatis gratia ponatur $\sqrt{(\text{A}+2\text{B}b+\text{C}bb)} = \mathfrak{B}$, erit integrale completum
$$y\left(\sqrt{\text{A}} + \frac{\text{B}b}{\sqrt{\text{A}}-\mathfrak{B}}\right) + x\left(\mathfrak{B} + \frac{\text{B}b}{\sqrt{\text{A}}-\mathfrak{B}}\right)$$
$$= \frac{\text{B}bb}{\sqrt{\text{A}}-\mathfrak{B}} + b\sqrt{(\text{A}+2\text{B}x+\text{C}xx)};$$
unde casus praecedens manifesto sequitur, si ponatur $\text{B} = 0$.

Verum ope levis substitutionis hae formulae, ubi adest B, ad illum casum ubi $B = 0$ reduci possunt.

Problema 74.

585. Si $\Pi : z$ significet eam functionem ipsius z, quae oritur ex integratione formulae $\int \frac{\partial z}{\sqrt{(A + Czz)}}$, integrale hoc ita sumto, ut evanescat posito $z = 0$, comparationem inter hujusmodi functiones instituere.

Solutio.

Consideretur haec aequatio differentialis

$$\frac{\partial x}{\sqrt{(A + Cxx)}} = \frac{\partial y}{\sqrt{(A + Cyy)}}$$

unde cum sit per hypothesin

$$\int \frac{\partial x}{\sqrt{(A + Cxx)}} = \Pi : x \text{ et } \int \frac{\partial y}{\sqrt{(A + Cyy)}} = \Pi : y,$$

utroque integrali ita sumto, ut evanescat illud posito $x = 0$, hoc vero posito $y = 0$, integrale completum erit

$$\Pi : y = \Pi : x + C.$$

Ante autem vidimus, hoc integrale esse

$$y = x \sqrt{\frac{A + Cbb}{A}} + b \sqrt{\frac{A + Cxx}{A}},$$

ubi posito $x = 0$ fit $y = b$, quare cum $\Pi : 0 = 0$, erit

$$\Pi : y = \Pi : x + \Pi : b;$$

cui ergo aequationi transcendentali satisfacit haec algebraica

$$y = x \sqrt{\frac{A + Cbb}{A}} + b \sqrt{\frac{A + Cxx}{A}}.$$

Simili modo sumto b negative, haec aequatio

$$\Pi : y = \Pi : x - \Pi : b$$

convenit cum hac

$$y = x \sqrt{\frac{A + Cbb}{A}} - b \sqrt{\frac{A + Cxx}{A}},$$

sicque tam summa, quam differentia duarum hujusmodi functionum per similem functionem exprimi potest. Hic jam nullo habito dis crimine inter quantitates variabiles et constantes, dum $\Pi : z$ functic nem determinatam ipsius z significat, scilicet

$$\Pi : z = \int \frac{\partial z}{\sqrt{(A + Czz)}},$$

quae ut assumsimus evanescat posito $z = 0$, ut hoc signandi mod recepto sit

$$\Pi : r = \Pi : p + \Pi : q,$$

debet esse

$$r = p\sqrt{\frac{A + Cqq}{A}} + q\sqrt{\frac{A + Cpp}{A}};$$

ut vero sit

$$\Pi : r = \Pi : p - \Pi : q,$$

debet esse

$$r = p\sqrt{\frac{A + Cqq}{A}} - q\sqrt{\frac{A + Cpp}{A}},$$

utrinque autem sublata irrationalitate prodit inter p, q, r haec ae quatio

$$p^4 + q^4 + r^4 - 2ppqq - 2pprr - 2qqrr = \frac{4Cppqqrr}{A},$$

cujus forma hanc suppeditat proprietatem, ut si p, q, r sint late cujusdam trianguli, eique circumscribatur circulus, cujus diamet vocetur $= T$, semper sit $A + CTT = 0$. Illa autem aequatio c plures quas complectitur radices, satisfacit huic relationi

$$\Pi : p \pm \Pi : q \pm \Pi : r = 0.$$

Corollarium 1.

586. Hinc statim deducitur nota arcuum circularium comp ratio, ponendo $A = 1$ et $C = -1$. Tum enim fit

$$\Pi : z = \int \frac{\partial z}{\sqrt{(1 - zz)}} = \text{Ang. sin. } z,$$

hincque ut sit

$$\text{Ang. sin. } r = \text{Ang. sin. } p + \text{Ang. sin. } q,$$

oportet esse

$$r = p\sqrt{(1-qq)} + q\sqrt{(1-pp)},$$

et ut sit

$$\text{Ang. sin. } r = \text{Ang. sin } p - \text{Ang. sin. } q,$$

debet esse

$$r = p\sqrt{(1-qq)} - q\sqrt{(1-pp)},$$

uti constat.

Corollarium 2.

587. Si sit $A = 1$ et $C = 1$, erit

$$\Pi : z = \int \frac{\partial z}{\sqrt{(1+zz)}} = l\,[z + \sqrt{(1+zz)}],$$

unde ut sit

$$l[r + \sqrt{(1+rr)}] = l[p + \sqrt{(1+pp)}] + l[q + \sqrt{(1+qq)}],$$

erit

$$r = p\sqrt{(1+qq)} + q\sqrt{(1+pp)};$$

ut autem sit

$$l[r + \sqrt{(1+rr)}] = l[p + \sqrt{(1+pp)}] - l\,[q + \sqrt{(1+qq)}],$$

erit

$$r = p\sqrt{(1+qq)} - q\sqrt{(1+pp)},$$

uti ex indole logarithmorum sponte liquet.

Corollarium 3.

588. Si ponamus in priori formula generali $q = p$, ut sit

$$\Pi : r = 2\,\Pi : p, \text{ erit}$$

$$r = 2\,p\sqrt{\frac{A + Cpp}{A}}.$$

Hinc porro si fiat

$$q = 2\,p\sqrt{\frac{A + Cpp}{A}}, \text{ erit}$$

$$\Pi : r = \Pi : p + 2\,\Pi : p = 3\,\Pi : p,$$

sumto

$$r = p\sqrt{\frac{A+Cqq}{A}} + q\sqrt{\frac{A+Cpp}{A}}.$$

Est vero

$$\sqrt{\frac{A+Cqq}{A}} = \sqrt{\left[1 + \frac{4Cpp}{A}\left(1 + \frac{Cpp}{A}\right)\right]} = 1 + \frac{2Cpp}{A},$$

unde ut sit

$\Pi : r = 3\,\Pi : p$ fit

$$r = p\left(1 + \frac{2Cpp}{A}\right) + 2p\left(1 + \frac{Cpp}{A}\right) = 3p + \frac{4Cp^3}{A}.$$

Scholion.

589. Quo haec multiplicatio facilius continuari queat, praeter relationem aequationi

$$\Pi : r = \Pi : p + \Pi : q$$

respondentem, quae est

$$r = p\sqrt{\frac{A+Cqq}{A}} + q\sqrt{\frac{A+Cpp}{A}},$$

notetur aequatio

$$\Pi : p = \Pi : r - \Pi : q,$$

cui respondet relatio

$p = r\sqrt{\frac{A+Cqq}{A}} - q\sqrt{\frac{A+Crr}{A}}$; unde fit

$$\sqrt{\frac{A+Crr}{A}} = \frac{r}{q}\sqrt{\frac{A+Cqq}{A}} - \frac{p}{q} = \frac{p}{q}\left(\frac{A+Cqq}{A}\right) + \sqrt{\left(\frac{A+Cpp}{A}\right)\left(\frac{A+Cqq}{A}\right)} - \frac{p}{q},$$ seu

$$\sqrt{\frac{A+Crr}{A}} = \frac{Cpq}{A} + \sqrt{\left(\frac{A+Cpp}{A}\right)\left(\frac{A+Cqq}{A}\right)}.$$

Quare ut sit

$$\Pi : r = \Pi : p + \Pi : q,$$

habemus non solum

$$r = p\sqrt{\left(1 + \frac{C}{A}qq\right)} + q\sqrt{\left(1 + \frac{C}{A}pp\right)},$$

sed etiam

$$\sqrt{\left(1 + \frac{C}{A}rr\right)} = \frac{C}{A}pq + \sqrt{\left(1 + \frac{C}{A}pp\right)\left(1 + \frac{C}{A}qq\right)}.$$

Ponamus brevitatis gratia $\sqrt{(1+\frac{C}{A}pp)} = P$, et sumto $q = p$ ut sit

$$\Pi : r = 2\,\Pi : p,$$ erit

$$r = 2\,Pp \text{ et } \sqrt{(1+\tfrac{C}{A}rr)} = \tfrac{C}{A}pp + PP,$$

qui valor ipsius r pro q sumtus dabit

$$\Pi : r = 3\,\Pi : p,$$

existente

$$r = \tfrac{C}{A}p^3 + 3\,PPp, \text{ et}$$

$$\sqrt{(1+\tfrac{C}{A}rr)} = \tfrac{3C}{A}Ppp + P^3.$$

Hic valor ipsius r denuo pro q sumtus, dabit

$$\Pi : r = 4\,\Pi : p,$$

existente

$$r = \tfrac{4C}{A}Pp^3 + 4\,P^3 p, \text{ et}$$

$$\sqrt{(1+\tfrac{C}{A}rr)} = \tfrac{CC}{AA}p^4 + \tfrac{6C}{A}PPpp + P^4.$$

Loco q substituatur hic valor ipsius r, ut prodeat

$$\Pi : r = 5\,\Pi : p,$$

existente

$$r = \tfrac{CC}{AA}p^5 + \tfrac{10C}{A}PPp^3 + 5\,P^4 p, \text{ et}$$

$$\sqrt{(1+\tfrac{C}{A}rr)} = \tfrac{5CC}{AA}Pp^4 + \tfrac{10C}{A}P^3pp + P^5.$$

Atque hinc generatim concludere licet, ut sit

$$\Pi : r = n\,\Pi : p,$$

esse debere

$$r\sqrt{\tfrac{C}{A}} = \tfrac{1}{2}(P + p\sqrt{\tfrac{C}{A}})^n - \tfrac{1}{2}(P - p\sqrt{\tfrac{C}{A}})^n, \text{ et}$$

$$\sqrt{(1+\tfrac{C}{A}rr)} = \tfrac{1}{2}(P + p\sqrt{\tfrac{C}{A}})^n + \tfrac{1}{2}(P - p\sqrt{\tfrac{C}{A}})^n, \text{ seu}$$

$$r = \tfrac{\sqrt{A}}{2\sqrt{C}}(P + p\sqrt{\tfrac{C}{A}})^n - \tfrac{\sqrt{A}}{2\sqrt{C}}(P - p\sqrt{\tfrac{C}{A}})^n.$$

Haec igitur relatio inter p et r satisfaciet huic aequationi differentiali

$$\frac{\partial r}{\sqrt{(A + Crr)}} = \frac{n\,\partial p}{\sqrt{(A + Cpp)}}.$$

dum meminerimus esse $P = \sqrt{(1 + \frac{Cpp}{A})}$.

Problema 75.

590. Si ponatur $\int \frac{\partial z}{\sqrt{(A + Czz)}} = \Pi : z$, integrali ita sum ut evanescat posito $z = f$, unde $\Pi : z$ fit functio determinata ipsius x, comparationem inter hujusmodi functiones instituere.

Solutio.

Consideretur haec aequatio differentialis

$$\frac{\partial x}{\sqrt{(A + Cxx)}} + \frac{\partial y}{\sqrt{(A + Cyy)}} = 0,$$

unde integrando fit

$$\Pi : x + \Pi : y = \text{Const.}$$

Integrale autem sit quoque

$$\alpha + \gamma (xx + yy) + 2\,\delta xy = 0,$$

quod ut locum habeat necesse est, sit

$$-\alpha\gamma = A\,m, \text{ et } \delta\delta - \gamma\gamma = C\,m:$$

tum vero erit

$$\gamma x + \delta y = \sqrt{m (A + Cyy)}, \text{ et } \gamma y + \delta x = \sqrt{m (A + Cxx)}.$$

Ponamus constantem integratione ingressam ita definiri, ut posito $x = a$ fiat $y = b$, et integrale erit

$$\Pi : x + \Pi : y = \Pi : a + \Pi : b.$$

Pro forma autem algebraica invenienda, sit brevitatis gratia

$$\sqrt{(A + C\,aa)} = \mathfrak{A} \text{ et } \sqrt{(A + C\,bb)} = \mathfrak{B},$$

eritque

$$\gamma a + \delta b = \mathfrak{B}\sqrt{m} \text{ et } \gamma b + \delta a = \mathfrak{A}\sqrt{m};$$

unde colligitur

$$\gamma = \frac{\mathfrak{A}b - \mathfrak{B}a}{bb - aa} \sqrt{m} \text{ et } \delta = \frac{\mathfrak{B}b - \mathfrak{A}a}{bb - aa} \sqrt{m}.$$

Quocirca aequatio integralis algebraica erit

$$(\mathfrak{A}b - \mathfrak{B}a)x + (\mathfrak{B}b - \mathfrak{A}a)y = (bb - aa)\sqrt{(A + Cyy)}$$

seu

$$(\mathfrak{A}b - \mathfrak{B}a)y + (\mathfrak{B}b - \mathfrak{A}a)x = (bb - aa)\sqrt{(A + Cxx)}.$$

Hinc y per x ita definitur, ut sit

$$y = \frac{(\mathfrak{A}a - \mathfrak{B}b)x + (bb - aa)\sqrt{(A + Cxx)}}{\mathfrak{A}b - \mathfrak{B}a},$$

quae fractio supra et infra per $\mathfrak{A}b + \mathfrak{B}a$ multiplicando, ob

$$\mathfrak{A}\mathfrak{A}bb - \mathfrak{B}\mathfrak{B}aa = A(bb - aa) \text{ et}$$

$$(\mathfrak{A}a - \mathfrak{B}b)(\mathfrak{A}b + \mathfrak{B}a) = (\mathfrak{A}\mathfrak{A} - \mathfrak{B}\mathfrak{B})ab - \mathfrak{A}\mathfrak{B}(bb - aa) = - (bb - aa)(Cab + \mathfrak{A}\mathfrak{B});$$

abit in

$$y = -\frac{(Cab + \mathfrak{A}\mathfrak{B})x}{A} + \frac{(\mathfrak{A}b + \mathfrak{B}a)\sqrt{(A + Cxx)}}{A}.$$

Hinc porro colligitur

$$(bb - aa)\sqrt{(A + Cyy)} = (\mathfrak{A}b - \mathfrak{B}a)x - \frac{(\mathfrak{B}b - \mathfrak{A}a)^2 x}{\mathfrak{A}b - \mathfrak{B}a} + \frac{(\mathfrak{B}b - \mathfrak{A}a)(bb - aa)}{\mathfrak{A}b - \mathfrak{B}a}\sqrt{(A + Cxx)},$$

seu

$$\sqrt{(A + Cyy)} = -\frac{C(bb - aa)}{\mathfrak{A}b - \mathfrak{B}a}x + \frac{\mathfrak{B}b - \mathfrak{A}a}{\mathfrak{A}b - \mathfrak{B}a}\sqrt{(A + Cxx)};$$

ubi iterum supra et infra multiplicando per $\mathfrak{A}b + \mathfrak{B}a$, fit

$$\sqrt{(A + Cyy)} = -\frac{C(\mathfrak{A}b + \mathfrak{B}a)}{A}x + \frac{(Cab + \mathfrak{A}\mathfrak{B})}{A}\sqrt{(A + Cxx)}.$$

Necesse autem est valorem formulae $\sqrt{(A + Cyy)}$ hoc modo potius definiri quam extractione radicis, qua ambiguitas implicaretur. Quocirca haec aequatio transcendens

$$\Pi : r + \Pi : s = \Pi : p + \Pi : q$$

praebet sequentem determinationem algebraicam, si quidem brevitatis gratia ponamus $\sqrt{(A + Cpp)} = P$, $\sqrt{(A + Cqq)} = Q$ et $\sqrt{(A + Crr)} = R$, scilicet ut sit

$\Pi : s = \Pi : p + \Pi : q - \Pi : r$, erit

$s = \frac{-PQr - Cpqr + PRq + QRp}{A}$ et

$\surd(A + Css) = \frac{-CPqr - CQpr + CRpq + PQR}{A}$, seu

$\surd(A + Css) = \frac{PQR + C(Rpq - Pqr - Qpr)}{A}$.

Corollarium 1.

591. Quoniam est per hypothesin $\Pi : f = 0$, si ponamu brevitatis gratia $\surd(A + Cff) = F$, et $r = f$, ut sit $R = F$, hae aequatio

$\Pi : s = \Pi : p + \Pi : q$

praebet

$s = \frac{F(Pq + Qp) - PQf - Cfpq}{A}$, et

$\surd(A + Css) = \frac{FPQ + CFpq - Cf(Pq + Qp)}{A}$.

Corollarium 2.

592. Si ponamus $q = f$ et $Q = F$, ut sit $\Pi : q = 0$, hae aequatio

$\Pi : s = \Pi : p - \Pi : r$

praebet

$s = \frac{F(Rp - Pr) + fPR - Cfpr}{A}$ et

$\surd(A + Css) = \frac{FPR - CFpr + Cf(Rp - Pr)}{A}$.

Corollarium 3.

593. Si sit $C = 0$ et $A = 1$, erit

$\Pi : z = \int \partial z = z - f$,

quia integrale ita capi debet, ut evanescat posito $x = f$. Tur ergo erit $P = 1$, $Q = 1$ et $R = 1$; unde ut sit

$$\Pi : s = \Pi : p + \Pi : q - \Pi : r,$$

seu $s = p + q - r$, oportet esse

$$s = -r + q + p \text{ et } \sqrt{(1 + 0ss)} = 1,$$

uti per se constat.

Corollarium 4.

594. Si sumatur $A = 1$ et $C = -1$, fiatque $\Pi : z = \text{Ang. cos. } z$, ut sit $f = 1$, erit

$$\text{Arc. cos. } s = \text{Arc. cos. } p + \text{Arc. cos. } q - \text{Arc. cos. } r,$$

si fuerit

$$s = pqr - PQr + PRq + QRp \text{ et}$$

$$\sqrt{(1 - ss)} = PQR + Pqr + Qpr - Rpq,$$

unde sumto $r = 1$, ut sit $R = 0$, et $\text{Arc. cos. } r = 0$, erit $s = pq - PQ$ et $\sqrt{(1 - ss)} = Pq + Qp$.

Scholion.

595. Hinc notae regulae pro cosinibus deducuntur, quas fusius non prosequor. Verum casus facillimus, quo $A = 0$ et $C = 1$, hincque fit $\Pi : z = \int \frac{\partial z}{z} = lz$, existente $f = 1$, insigni difficultate premi videtur, ob expressiones pro s et $\sqrt{(A + Czz)} = z$ in infinitum abeuntes. Cui incommodo ut occurratur, primo quidem numerus A ut infinite parvus spectetur, eritque

$$P = \sqrt{(pp + A)} = p + \frac{A}{2p}, \quad Q = q + \frac{A}{2q}, \quad R = r + \frac{A}{2r}.$$

Quare ut fiat $ls = lp + lq - lr$, reperitur

$$As = -r\left(p + \frac{A}{2p}\right)\left(q + \frac{A}{2q}\right) - pqr$$
$$+ q\left(p + \frac{A}{2p}\right)\left(r + \frac{A}{2r}\right) + p\left(q + \frac{A}{2q}\right)\left(r + \frac{A}{2r}\right);$$

ac singulis membris evolutis

$$As = -\frac{Aqr}{2p} - \frac{Apr}{2q} + \frac{Aqr}{2p} + \frac{Apq}{2r} + \frac{Apr}{2q} + \frac{Apq}{2r}$$

seu $s = \frac{pq}{r}$, uti natura logarithmorum exigit. Caeterum ex formulis inventis haud difficulter multiplicatio hujusmodi functionum transcendentium colligitur, veluti ut sit $\Pi:y = n\Pi:x$, relatio inter x et y algebraice assignari poterit.

Problema 76.

596. Si ponatur $\Pi:z = \int \frac{\partial z\,(L + Mzz)}{\sqrt{(A + Czz)}}$, sumto hoc integrali ita ut evanescat posito $z = 0$, comparationem inter hujusmodi functiones transcendentes investigare.

Solutio.

Statuatur inter binas variabiles x et y ista relatio

$$\alpha + \gamma\,(xx + yy) + 2\,\delta xy = 0,$$

unde fit

$$y = \frac{-\delta x + \sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma)\,xx]}}{\gamma}.$$

Ponatur $-\alpha\gamma = Am$ et $\delta\delta - \gamma\gamma = Cm$, ut sit

$$\gamma y + \delta x = \sqrt{m}\,(A + Cxx) \text{ et}$$

$$\gamma x + \delta y = \sqrt{m}\,(A + Cyy).$$

At illam aequationem differentiando fit

$$\partial x\,(\gamma x + \delta y) + \partial y\,(\gamma y + \delta x) = 0, \text{ seu}$$

$$\frac{\partial x}{\sqrt{(A + Cxx)}} + \frac{\partial y}{\sqrt{(A + Cyy)}} = 0.$$

Jam statuatur

$$\frac{\partial x\,(L + Mxx)}{\sqrt{(A + Cxx)}} + \frac{\partial y\,(L + Myy)}{\sqrt{(A + Cyy)}} = \partial V \sqrt{m},$$

ut sit integrando

$$\Pi:x + \Pi:y = \text{Const.} + V\sqrt{m}.$$

Cum igitur sit

$$\frac{\partial y}{\sqrt{(A + Cyy)}} = \frac{-\partial x}{\sqrt{(A + Cxx)}}, \text{ erit}$$

$$\partial V \sqrt{m} = \frac{M \partial x (xx - yy)}{\sqrt{(A + Cxx)}},$$

hincque ob

$$y = \frac{\sqrt{m(A + Cxx)} - \delta x}{\gamma}, \text{ erit}$$

$$xx - yy = \frac{1}{\gamma\gamma}(\gamma\gamma xx - mA - mCxx - \delta\delta xx + 2\delta x \sqrt{m(A + Cxx)}).$$

At $\gamma\gamma - \delta\delta = -mC$, ergo

$$\partial V \sqrt{m} = \frac{M \partial x (2\delta x \sqrt{m(A + Cxx)} - mA - 2mCxx)}{\gamma\gamma\sqrt{(A + Cxx)}},$$

cujus integrale commode capi potest, dum fit

$$V \sqrt{m} = \frac{\delta Mxx\sqrt{m}}{\gamma\gamma} - \frac{Mmx}{\gamma\gamma}\sqrt{(A + Cxx)},$$

quae formula ob

$$\sqrt{m(A + Cxx)} = \gamma y + \delta x, \text{ abit in}$$

$$V \sqrt{m} = \frac{\delta Mxx - \gamma Mxy - \delta Mxx}{\gamma\gamma} \sqrt{m} = -\frac{Mxy}{\gamma}\sqrt{m}.$$

Quocirca habebimus

$$\Pi : x + \Pi : y = \text{Const.} - \frac{Mxy}{\gamma}\sqrt{m},$$

existente

$$\gamma y + \delta x = \sqrt{m(A + Cxx)} \text{ et } \gamma x + \delta y = \sqrt{m(A + Cyy)},$$

ac praeterea

$$-\alpha\gamma = Am \text{ et } \delta\delta - \gamma\gamma = Cm.$$

Ad constantem definiendam sumamus, posito $x = 0$ fieri $y = b$, ut sit

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mxy}{\gamma}\sqrt{m}.$$

Tum vero est

$$\gamma b = \sqrt{mA} \text{ et } \delta b = \sqrt{(mA + mCbb)},$$

ergo

$$\gamma = \frac{\sqrt{mA}}{b} \text{ et } \delta = \frac{\sqrt{(mA + mCbb)}}{b}$$

Hinc ergo concludimus, si fuerit

$$y\sqrt{A} + x\sqrt{(A + Cbb)} = b\sqrt{(A + Cxx)},$$

et quod eodem redit

$$x\sqrt{A} + y\sqrt{(A + Cbb)} = b\sqrt{(A + Cyy)}, \text{ fore}$$

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}};$$

denotante Π ejusmodi functionem quantitatis suffixae, ut sit

$$\Pi : z = \int \frac{\partial z (L + Mzz)}{\sqrt{(A + Czz)}},$$

integrali hoc ita sumto, ut evanescat posito $z = 0$. Natura harum functionum stabilita, ac sublato discrimine inter quantitates constantes ac variabiles, erit

$$\Pi : r = \Pi : p + \Pi : q + \frac{Mpqr}{\sqrt{A}},$$

si fuerit

$$q\sqrt{A} + p\sqrt{(A + Crr)} = r\sqrt{(A + Cpp)} \text{ et}$$
$$p\sqrt{A} + q\sqrt{(A + Crr)} = r\sqrt{(A + Cqq)}$$

unde fit

$$r = \frac{p\sqrt{(A + Cqq)} + q\sqrt{(A + Cpp)}}{\sqrt{A}} \text{ et}$$
$$\sqrt{(A + Crr)} = \frac{Cpq + \sqrt{(A + Cpp)(A + Cqq)}}{\sqrt{A}}.$$

Corollarium 1.

597. Sumto z negativo est

$$\Pi : -z = -\Pi : z,$$

unde capiendo quantitates p et q negative, fiet

$$\Pi : p + \Pi : q + \Pi : r = \frac{Mpqr}{\sqrt{A}},$$

si fuerit

$$p\sqrt{A} + q\sqrt{(A + Crr)} + r\sqrt{(A + Cqq)} = 0 \text{ seu}$$
$$q\sqrt{A} + p\sqrt{(A + Crr)} + r\sqrt{(A + Cpp)} = 0 \text{ seu}$$
$$r\sqrt{A} + p\sqrt{(A + Cqq)} + q\sqrt{(A + Cpp)} = 0 \text{ vel}$$
$$Cpq - \sqrt{A(A + Crr)} + \sqrt{(A + Cpp)(A + Cqq)} = 0$$

ex qua formatur haec relatio

$$Cpqr + p\sqrt{(A + Cqq)(A + Crr)} + q\sqrt{(A + Cpp)(A + Crr)} + r\sqrt{(A + Cpp)(A + Cqq)} = 0.$$

Corollarium 2.

598. Hac ergo methodo tres hujusmodi functiones $\Pi : z$ exhiberi possunt, quarum summam algebraice exprimere licet; quod autem de summa ostendimus, valet quoque de summa binarum demta tertia.

Corollarium 3.

599. Si ponamus $L = A$ et $M = C$, functio proposita $\Pi : z = \int \partial z \sqrt{(A + Czz)}$, exprimit aream curvae, cujus abscissae z convenit applicata $\sqrt{(A + Czz)}$; et summa trium hujusmodi arearum ita algebraice dabitur:

$$\Pi : p + \Pi : q + \Pi : r = \frac{Cpqr}{\sqrt{A}}$$

si inter p, q, r superior relatio statuatur.

Scholion.

600. Haec proprietas inde est nata, quod differentiale ∂V integrationem admisit. Cum nempe esset

$$\partial V \sqrt{m} = \frac{M \partial x (xx - yy)}{\sqrt{(A + Cxx)}}, \text{ ob}$$

$$\sqrt{m(A + Cxx)} = \gamma y + \delta x, \text{ erit}$$

$$\partial V = \frac{M \partial x (xx - yy)}{\gamma y + \delta x},$$

cujus integrale commode ex aequatione assumta

$$\alpha + \gamma (xx + yy) + 2 \delta xy = 0$$

definiri potest. Ponatur enim

$$xx + yy = tt \text{ et } xy = u, \text{ erit}$$

$$\alpha + \gamma tt + 2 \delta u = 0$$

et differentialibus sumendis

$$x \partial x + y \partial y = t \partial t; \; x \partial y + y \partial x = \partial u \text{ et } \gamma t \partial t + \delta \partial u = 0;$$

ex binis prioribus colligitur

$$(xx - yy)\partial x = xt\partial t - y\partial u, \text{ et ob } t\partial t = -\frac{\delta\partial u}{\gamma}, \text{ erit}$$

$$(xx - yy)\partial x = -\frac{\partial u}{\gamma}(\delta x + \gamma y),$$

ita ut sit

$$\frac{\partial x(xx - yy)}{\gamma y + \delta x} = -\frac{\partial u}{\gamma}, \text{ hincque } \partial V = -\frac{M\partial u}{\gamma},$$

unde manifesto sequitur

$$V = -\frac{Mu}{\gamma} = -\frac{Mxy}{\gamma},$$

uti in solutione operosius eruimus. Verum hac operatione commode uti licebit in sequente problemate, ubi formulas magis complexas sumus contemplaturi.

Problema 77.

604. Si ponatur

$$\Pi : z = \int \frac{\partial z(L + Mz^2 + Nz^4 + Oz^6 + etc.)}{\sqrt{(A + Czz)}},$$

integrali hoc ita sumto ut evanescat posito $z = 0$, comparationem inter hujusmodi functiones transcendentes investigare.

Solutio.

Posita ut ante inter variabiles x et y hac relatione,

$$\alpha + \gamma(xx + yy) + 2\delta xy = 0,$$

sit

$$-\alpha\gamma = Am \text{ et } \delta\delta - \gamma\gamma = Cm,$$

fietque

$$\gamma y + \delta x = \sqrt{m(A + Cxx)} \text{ et } \gamma x + \delta y = \sqrt{m(A + Cyy)},$$

sumtisque differentialibus

$$\frac{\partial x}{\sqrt{(A + Cxx)}} + \frac{\partial y}{\sqrt{(A + Cyy)}} = 0.$$

Jam statuatur

$$\frac{\partial x(L + Mx^2 + Nx^4 + Ox^6)}{\sqrt{(A + Cxx)}} + \frac{\partial y(L + My^2 + Ny^4 + Oy^6)}{\sqrt{(A + Cyy)}} = \partial V\sqrt{m}$$

ut sit

$$\Pi : x + \Pi : y = \text{Const.} + V\sqrt{m}.$$

At ob $\frac{\partial y}{\sqrt{(A+Cyy)}} = -\frac{\partial x}{\sqrt{(A+Cxx)}}$, ista aequatio abit in

$$\frac{\partial x[M(xx-yy)+N(x^4-y^4)+O(x^6-y^6)]}{\sqrt{(A+Cxx)}} = \partial V\sqrt{m},$$

et ob $\sqrt{m}(A+Cxx) = \gamma y + \delta x$, in hanc

$$\frac{\partial x(xx-yy)[M+N(xx+yy)+O(x^4+xxyy+y^4)]}{\gamma y+\delta x} = \partial V.$$

Sit nunc $xx+yy = tt$ et $xy = u$, ut habeatur

$$\alpha + \gamma tt + 2\delta u = 0 \text{ et } \gamma t\partial t + \delta\partial u = 0,$$

seu $t\partial t = -\frac{\delta\partial u}{\gamma}$,

atque ob

$$x\partial x + y\partial y = t\partial t \text{ et } x\partial y + y\partial x = \partial u$$

hinc colligimus

$$(xx-yy)\partial x = xt\partial t - y\partial u = -\frac{\partial u}{\gamma}(\gamma y+\delta x),$$

ideoque

$$\frac{\partial x(xx-yy)}{\gamma y+\delta x} = -\frac{\partial u}{\gamma},$$

unde habebimus

$$\partial V = -\frac{\partial u}{\gamma}[M+N(xx+yy)+O(x^4+xxyy+y^4)].$$

At est

$$xx+yy = tt = \frac{-\alpha-\delta u}{\gamma} \text{ et}$$

$$x^4+xxyy+y^4 = t^4 - uu.$$

Notetur autem esse $\frac{\partial u}{\gamma} = -\frac{t\partial t}{\delta}$, unde concludimus

$$\partial V = -\frac{M\partial u}{\gamma} + \frac{Nt^3\partial t}{\delta} + \frac{Ot^5\partial t}{\delta} + \frac{Ouu\partial u}{\gamma},$$

sicque prodit integrando

$$V = -\frac{Mu}{\gamma} + \frac{Nt^4}{4\delta} + \frac{Ot^6}{6\delta} + \frac{Ou^3}{3\gamma}.$$

Quodsi jam ponamus fieri $y=b$ si $x=0$, erit $\gamma = \frac{\sqrt{mA}}{b}$, $\delta = \frac{\sqrt{m(A+Cbb)}}{b}$ et $a = -b\sqrt{mA}$, tum vero

$$y\surd A + x\surd(A + Cbb) = b\surd(A + Cxx)$$
$$x\surd A + y\surd(A + Cbb) = b\surd(A + Cyy) \text{ et}$$
$$b\surd A = x\surd(A + Cyy) + y\surd(A + Cxx).$$

Hinc cum sit

$$V = -\frac{Mbxy}{\surd mA} + \frac{Nb(xx+yy)^2}{4\surd m(A+Cbb)} + \frac{Ob(xx+yy)^3}{6\surd m(A+Cbb)} + \frac{Obx^3y^3}{3\surd mA},$$

nostra relatio, cui satisfaciunt praecedentes determinationes, inter functiones transcendentes, erit

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\surd A} + \frac{Nb(xx+yy)^2}{4\surd(A+Cbb)} + \frac{Ob(xx+yy)^3}{6\surd(A+Cbb)}$$
$$+ \frac{Obx^3y^3}{3\surd A} - \frac{Nb^5}{4\surd(A+Cbb)} - \frac{Ob^7}{6\surd(A+Cbb)}:$$

ubi notandum est esse in rationalibus

$$-b\surd A + \frac{(xx+yy)\surd A}{b} + \frac{2xy\surd(A+Cbb)}{b} = 0, \text{ seu}$$
$$xx + yy = bb - \frac{2xy\surd(A+Cbb)}{\surd A}.$$

Hinc colligitur

$$(xx+yy)^2 - b^4 = -\frac{4bbxy\surd(A+Cbb)}{\surd A} + \frac{4xxyy(A+Cbb)}{A} \text{ et}$$
$$(xx+yy)^3 - b^6 = -\frac{6b^4xy\surd(A+Cbb)}{\surd A} + \frac{12bbxxyy(A+Cbb)}{A}$$
$$- \frac{8x^3y^3(A+Cbb)^{\frac{3}{2}}}{A\surd A},$$

ita ut nostra aequatio sit

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\surd A} - \frac{Nb^3xy}{\surd A} + \frac{Nbxxyy}{A}\surd(A+Cbb) - \frac{Ob^5xy}{\surd A}$$
$$+ \frac{2Ob^3xxyy}{A}\surd(A+Cbb) - \frac{Obx^3y^3}{3A\surd A}(3A + 4Cbb).$$

Corollarium 1.

602. Si ponamus $b = r$, $x = -p$, $y = -q$, erit nostra aequatio

$$\Pi : p + \Pi : q + \Pi : r = \frac{pqr}{\surd A}(M + Nrr + Or^4)$$
$$- \frac{ppqq\surd(A+Crr)}{A}(Nr + 2Or^3) + \frac{Op^3q^3r}{3A\surd A}(3A + 4Crr),$$

existente $pp + qq = rr - \frac{2pq}{\sqrt{A}} \sqrt{(A + Crr)}$, unde fit
$\frac{\sqrt{(A + Crr)}}{\sqrt{A}} = \frac{rr - pp - qq}{2pq}$.

Corollarium 2.

603. Substituto hoc valore pro $\frac{\sqrt{(A + Crr)}}{\sqrt{A}}$, sequens obtinebitur aequatio, in quam ternae quantitates p, q, r aequaliter ingrediuntur

$$\Pi : p + \Pi : q + \Pi : r = \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2\sqrt{A}}(pp + qq + rr)$$
$$+ \frac{Opqr}{3\sqrt{A}}(p^4 + q^4 + r^4 + ppqq + pprr + qqrr)$$

cui satisfaciunt formulae supra datae, vel haec rationalis

$$\frac{4Cppqqrr}{A} = p^4 + q^4 + r^4 - 2ppqq - 2pprr - 2qqrr.$$

Corollarium 3.

604. Si numeratori formulae integralis adhuc adjecissemus terminum Pz^8, ut esset

$$\Pi : z = \int \frac{\partial z (L + Mz^2 + Nz^4 + Oz^6 + Pz^8)}{\sqrt{(A + Czz)}},$$

ad aequationem modo inventam adhuc accessisset terminus

$$\frac{Ppqr}{4\sqrt{A}}(p^6 + q^6 + r^6 + ppq^4 + ppr^4 + p^4qq + p^4rr + q^4rr + qqr^4 + \tfrac{4}{3}ppqqrr).$$

Scholion.

605. Istae relationes quoque ex superioribus reductionibus derivari possunt, cum enim inde sit $\Pi : z = E\int \frac{\partial z}{\sqrt{(A + Czz)}}$ + *quantitate algebraica*, si hic pro z successive quantitates p, q, r substituamus, ita a se invicem pendentes, ut ante declaravimus, erit

$$\int \frac{\partial p}{\sqrt{(A + Cpp)}} + \int \frac{\partial q}{\sqrt{(A + Cqq)}} + \int \frac{\partial r}{\sqrt{(A + Crr)}} = 0:$$

unde concludimus

$$\Pi : p + \Pi : q + \Pi : r = f : p + f : q + f : r,$$

denotante f functionem quandam algebraicam quantitatis suffixae:

atque summa harum trium functionum rediret ad expressionem ante inventam, si modo relationis inter p, q, r datae ratio habeatur: scilicet inde littera C eliminari deberet. Haec autem reductio ingentem laborem requireret. Hic vero imprimis methodum, qua hic sum usus, spectari convenit, quae cum sit prorsus singularis, ad magis arduam deducere videtur. Certe comparatio functionum transcendentium, quam in capite sequente sum traditurus, vix alia methodo investigari posse videtur, unde hujus methodi utilitas in sequenti capite potissimum cernetur.