CAPUT III.

DE INVESTIGATIONE AEQUATIONUM DIFFERENTIALIUM QUAE PER MULTIPLICATORES DATAE FORMAE INTEGRABILES REDDANTUR.

Problema 65.

493.

Definire functiones P et Q ipsius x, ut aequatio differentialis $Py\partial x + (y+Q)\partial y = 0$, per multiplicatorem $\frac{1}{y^3 + Myy + Ny}$, ubi M et N sunt functiones ipsius x, fiat integrabilis.

Solutio.

Necesse igitur est, ut factoris ipsius ∂x, qui est $\frac{Py}{y^3 + Myy + Ny}$, differentiale ex variabilitate ipsius y natum, aequale sit differentiali factoris ipsius ∂y, qui est $\frac{y+Q}{y^3 + Myy + Ny}$, dum sola x variabilis sumitur. Horum valorum aequalium, neglecto denominatore communi, aequalitas dat

$$-2Py^3 - PMy^2 = (y^3 + Myy + Ny)\frac{\partial Q}{\partial x} - (y+Q)\frac{(yy\partial M + y\partial N)}{\partial x},$$

quae secundum potestates ipsius y ordinata praebet

$$\begin{array}{llll} 0 = 2Py^3\partial x & + PMy^2\partial x & & \\ \quad + y^3\partial Q & + My^2\partial Q & + Ny\partial Q & \\ \quad - y^3\partial M & - y^2\partial N & & \\ & - Qy^2\partial M & - Qy\partial N & \end{array}$$

$$(n-2)Py^{n+1}+(n-1)PMy^n+nPNy^{n-1}=(yy+My+N)y^{n-1}\frac{\partial Q}{\partial x} - (y^n+Qy^{n-1})\left(\frac{y\partial M}{\partial x}+\frac{\partial N}{\partial x}\right),$$

sive ordinata aequatione

$$\left.\begin{array}{lll} (n-2)Py^{n+1}\partial x + & (n-1)PMy^n\partial x + & nPNy^{n-1}\partial x \\ -y^{n+1}\partial Q & -My^n\partial Q & -Ny^{n-1}\partial Q \\ +y^{n+1}\partial M & +y^n\partial N & +y^{n-1}Q\partial N \\ & +y^nQ\partial M & \end{array}\right\} = 0;$$

unde singulis membris ad nihilum reductis, fit

$$\text{I. } (n-2)P\partial x = \partial Q - \partial M$$
$$\text{II. } (n-1)MP\partial x = M\partial Q - Q\partial M - \partial N$$
$$\text{III. } nNP\partial x = N\partial Q - Q\partial N.$$

Sit $P\partial x = \partial V$, eritque ex prima $Q = A + M + (n-2)V$, quo valore in secunda substituto prodit

$$M\partial V + (n-2)V\partial M + A\partial M + \partial N = 0$$

et tertia fit

$$2N\partial V + (n-2)V\partial N + M\partial N - N\partial M + A\partial N = 0:$$

unde eliminando ∂V reperitur

$$(n-2)V + A = \frac{MM\partial N - MN\partial M - 2N\partial N}{2N\partial M - M\partial N}.$$

Verum si hinc vellemus V elidere, in aequationem differentio-differentialem illaberemur. Casus tamen quo $n=2$ expediri potest.

Exemplum.

498. *Sit in evolutione hujus casus $n=2$, ut per se integrabilis esse debeat haec aequatio*

$$\frac{y[Py\partial x + (y+Q)\partial y]}{yy+My+N} = 0.$$

Ac primo esse oportet $Q = A + M$, tum vero

$$2AN\partial M - AM\partial N = M(M\partial N - N\partial M) - 2N\partial N,$$

quam ergo aequationem integrare debemus, quae cum in nulla jam tractatarum contineatur, videndum est, quomodo tractabilior reddi queat. Ponatur ergo $M = Nu$, ut fiat

$$M\partial N - N\partial M = - NN\partial u, \text{ et}$$
$$2N\partial M - M\partial N = 2NN\partial u + Nu\partial N, \text{ hinc}$$
$$2ANN\partial u + ANu\partial N + N^3 u\partial u + 2N\partial N = 0, \text{ sive}$$
$$\frac{2\partial N}{NN} + \frac{Au\partial N}{NN} + \frac{2A\partial u}{N} + u\partial u = 0:$$

statuatur porro $\frac{1}{N} = v$, seu $N = \frac{1}{v}$, habebitur

$$-2\partial v - Au\partial v + 2Av\partial u + u\partial u = 0, \text{ seu}$$
$$\partial v - \frac{2Av\partial u}{2+Au} = \frac{u\partial u}{2+Au}.$$

Ubi variabilis v unicam habet dimensionem, et hanc ob rem patet, hanc aequationem integrabilem reddi, si dividatur per $(2+Au)^2$; prodibitque

$$\frac{v}{(2+Au)^2} = \int \frac{u\partial u}{(2+Au)^3} = \frac{C}{AA} - \frac{1-Au}{AA(2+Au)^2},$$

ideoque $v = \frac{C(2+Au)^2 - 1 - Au}{AA}$. Sumto ergo pro u functione quacunque ipsius x, erit

$$N = \frac{AA}{C(2+Au)^2 - 1 - Au}, \text{ et } M = \frac{AAu}{C(2+Au)^2 - 1 - Au},$$

atque $Q = \frac{AC(2+Au)^2 - A}{C(2+Au)^2 - 1 - Au}$. Jam ex tertia aequatione adipiscimur $2NP\partial x = N\partial Q - Q\partial N$, seu $2P\partial x = N\partial . \frac{Q}{N}$, at $\frac{Q}{N} = \frac{C(2+Au)^2 - 1}{A}$, unde $\partial . \frac{Q}{N} = 2C\partial u(2+Au)$, ideoque

$$P\partial x = \frac{AAC\partial u(2+Au)}{C(2+Au)^2 - 1 - Au}.$$

Quocirca aequatio nostra per se integrabilis est

$$\frac{AACyy\partial u(2+Au) + y\partial y[C(2+Au)^2 y - (1+Au)y + AC(2+Au)^2 - A)}{C(2+Au)^2 yy - (1+Au)yy + AAuy + AA} = 0,$$

quae posito $Au + 2 = t$, induet hanc formam

$$y . \frac{ACyt\partial t + y\partial y(Ctt - t + 1) + A\partial y(Ctt - 1)}{Cttyy - (t-1)yy + A(t-2)y + AA} = 0.$$

Hinc autem posito $A = \alpha$; $C = \frac{\alpha\gamma}{\beta\beta}$ et $t = -\frac{\beta x}{\alpha}$, invenimus

$$y . \frac{\alpha\gamma xy\partial x + y\partial y(\alpha + \beta x + \gamma xx) - \alpha\partial y(\alpha - \gamma xx)}{(\alpha + \beta x + \gamma xx)yy - \alpha(2\alpha + \beta x)y + \alpha^3} = 0.$$

Corollarium 1.

499. Hoc igitur modo integrari potest haec aequatio

$$\alpha\gamma xy\partial x + y\partial y(\alpha + \beta x + \gamma xx) - \alpha\partial y(\alpha - \gamma xx) = 0,$$

quae quomodo ad separationem reduci debeat, non statim patet. Est autem multiplicator idoneus

$$\frac{y}{(\alpha + \beta x + \gamma xx)yy - \alpha(2\alpha + \beta x)y + \alpha^3}.$$

Corollarium 2.

500. Hic multiplicator etiam hoc modo exprimi potest, ut ejus denominator in factores resolvatur

$$\frac{(\alpha + \beta x + \gamma xx)y}{[(\alpha+\beta x+\gamma xx)y-(\alpha+\tfrac{1}{2}\beta x)+\alpha x\sqrt{(\tfrac{1}{4}\beta\beta-\alpha\gamma)}][(\alpha+\beta x+\gamma xx)y-\alpha(\alpha+\tfrac{1}{2}\beta x)-\alpha x\sqrt{(\tfrac{1}{4}}$$

Corollarium 3.

501. Si ergo ponamus

$$(\alpha + \beta x + \gamma xx)y - \alpha(\alpha + \tfrac{1}{2}\beta x) = \alpha z,$$

erit multiplicator

$$\frac{\alpha + \tfrac{1}{2}\beta x + z}{[z + x\sqrt{(\tfrac{1}{4}\beta\beta - \alpha\gamma)}][z - x\sqrt{(\tfrac{1}{4}\beta\beta - \alpha\gamma)}]}.$$

At ob $y = \dfrac{\alpha\alpha + \tfrac{1}{2}\alpha\beta x + \alpha z}{\alpha + \beta x + \gamma xx}$, aequatio nostra erit

$$\gamma xy\partial x + \partial y(z + \tfrac{1}{2}\beta x + \gamma xx) = 0.$$

At est

$$\partial y = \frac{-\tfrac{1}{2}\alpha(\alpha\beta + 4\alpha\gamma x + \beta\gamma xx)\partial x - \alpha z\partial x(\beta + 2\gamma x) + \alpha\partial z(\alpha + \beta x + \gamma xx)}{(\alpha - \beta x + \gamma xx)^2}$$

hoc autem valore substituto prodit aequatio nimis complicata.

Problema 66.

502. Invenire aequationem differentialem hujus formae

$$yP\partial x + (Qy + R)\partial y = 0$$

in qua P, Q et R sint functiones ipsius x, ut ea integrabilis evadat per hunc multiplicatorem $\frac{y^m}{(1+Sy)^n}$, ubi S est etiam functio ipsius x.

Solutio.

Quia ∂x per $\frac{y^{m+1}P}{(1+Sy)^n}$ et ∂y per $\frac{Qy^{m+1}+Ry^m}{(1+Sy)^n}$ multiplicatur, oportet sit

$$(m+1)Py^m(1+Sy) - nPSy^{m+1}$$
$$= \frac{(1+Sy)(y^{m+1}\partial Q + y^m\partial R) - ny\partial S(Qy^{m+1}+Ry^m)}{\partial x},$$

qua evoluta aequatione erit

$$\left.\begin{array}{lll} (m+1)Py^m\partial x + (m+1-n)PSy^{m+1}\partial x - y^{m+2}S\partial Q \\ -y^m\partial R \quad -y^{m+1}\partial Q & & +ny^{m+2}Q\partial S \\ \quad -y^{m+1}S\partial R & & \\ \quad +ny^{m+1}R\partial S & & \end{array}\right\} = 0$$

hinc fit $P\partial x = \frac{\partial R}{m+1}$ et $S\partial Q = nQ\partial S$, ideoque $Q = AS^n$ et $\partial Q = nAS^{n-1}\partial S$, quibus in membro medio substitutis fit

$$\frac{m+1-n}{m+1}S\partial R - nAS^{n-1}\partial S - S\partial R + nR\partial S = 0, \text{ seu}$$

$$-\frac{S\partial R}{m+1} - AS^{n-1}\partial S + R\partial S = 0, \text{ ideoque}$$

$$\partial R - \frac{(m+1)R\partial S}{S} = -(m+1)AS^{n-2}\partial S,$$

quae per S^{m+1} divisa et integrata praebet

$$\frac{R}{S^{m+1}} = B - \frac{(m+1)\,A\,S^{n-m-2}}{n-m-2}.$$

Ponamus $A = (m+2-n)\,C$, ut sit $Q = (m+2-n)\,C\,S^n$, et $R = B\,S^{m+1} + (m+1)\,C\,S^{n-1}$, ideoque

$$P\partial x = B\,S^m\,\partial S + (n-1)\,C\,S^{n-2}\,\partial S.$$

Quocirca habebimus hanc aequationem

$$y\partial S\,[B\,S^m + (n-1)\,C\,S^{n-2}] + \partial y\,[(m+2-n)\,C\,S^n\,y + B\,S^{m+1} + (m+1)\,C\,S^{n-1}] = 0,$$

quae multiplicata per $\frac{y^m}{(1+S\,y)^n}$ fit integrabilis, ubi pro S functionem quamcunque ipsius x capere licet.

Corollarium 1.

503. Integrari ergo poterit haec aequatio

$$B\,y\,S^m\partial S + B\,S^{m+1}\partial y + (n-1)\,C\,y\,S^{n-2}\partial S + (m+1)\,C\,S^{n-1}\partial y + (m+2-n)\,C\,S^n\,y\,\partial y = 0,$$

quae sponte resolvitur in has duas partes

$$B\,S^m(y\partial S + S\partial y) + C\,S^{n-2}[(n-1)\,y\partial S + (m+1)\,S\partial y + (m+2-n)\,S^2\,y\,\partial y] = 0,$$

quarum utraque seorsim per $\frac{y^m}{(1+S\,y)^n}$ multiplicata fit integrabilis.

Corollarium 2.

504. Prior pars $B\,S^m\,(y\partial S + S\partial y)$ integrabilis redditur per hunc multiplicatorem $\frac{1}{S^m}\,\Phi : S\,y$; est enim haec formula

$B(y\partial S + S\partial y)\,\Phi : Sy$ per se integrabilis. Unde pro hac parte multiplicator erit $S^{\lambda - m}\,y^{\lambda}\,(1 + Sy)^{\mu}$, qui utique continet assumtum $\frac{y^m}{(1 + Sy)^n}$, si quidem capiatur $\lambda = m$ et $\mu = -n$. Est vero

$$\int \frac{y^m}{(1 + Sy)^n} \,.\, BS^m\,(y\partial S + S\partial y) = B\int \frac{v^m\,\partial v}{(1 + v)^n},$$

posito $Sy = v$.

Corollarium 3.

505. Pro altera parte, quae posito $S = \frac{1}{v}$ abit in

$$\frac{C}{v^n}\,[-(n-1)\,y\,\partial v + (m+1)\,v\,\partial y + (m+2-n)\,y\,\partial y],$$

habebimus

$$-\frac{(n-1)\,C\,y}{v^n}\left(\partial v - \frac{(m+1)\,v\,\partial y}{(n-1)\,y} - \frac{(m+2-n)\,\partial y}{(n-1)}\right) =$$

$$-\frac{(n-1)\,C\,y^{\frac{m+n}{n-1}}}{v^n}\left(y^{\frac{-m-1}{n-1}}\,\partial v - \frac{m+1}{n-1}\,y^{\frac{-m-n}{n-1}}\,v\partial y - \frac{m+2-n}{n-1}\,y^{\frac{-m-1}{n-1}}\,\partial y\right)$$

$$= -\frac{(n-1)\,C\,y^{\frac{m+n}{n-1}}}{v^n}\,\partial\,.\left(y^{\frac{-m-1}{n-1}}\,v + y^{\frac{n-m-2}{n-1}}\right).$$

Ideoque haec altera pars ita repraesentabitur

$$-(n-1)\,C\,S^n\,y^{\frac{m+n}{n-1}}\,\partial\,.\,\frac{1 + Sy}{y^{\frac{m+1}{n-1}}\,S}.$$

Multiplicator ergo hanc partem integrabilem reddens erit in genere

$$\frac{1}{S^n\,y^{\frac{m+n}{n-1}}}\,\Phi : \frac{1 + Sy}{S\,y^{\frac{m+1}{n-1}}}.$$

Corollarium 4.

506. Pro altera ergo parte multiplicator erit

$\frac{(1+Sy)^\mu}{S^{n+\mu}\, y^{\frac{m+n+\mu(m+1)}{n-1}}}$, quo haec pars fit:

$$-(n-1)\,C\,.\,\frac{(1+Sy)^\mu}{S^\mu\, y^{\frac{\mu(m+1)}{n-1}}}\,\partial\,.\,\frac{1+Sy}{y^{\frac{m+1}{n-1}}\,S},$$

cujus integrale est

$$-\frac{(n-1)\,C\,z^{\mu+1}}{\mu+1},\ \text{posito}\ z=\frac{1+Sy}{y^{\frac{m+1}{n-1}}\,S}.$$

Corollarium 5.

507. Jam multiplicator pro prima parte
$S^{\lambda-m}\, y^\lambda\,(1+Sy)^\mu$
congruens reddetur cum multiplicatore alterius partis modo exhibito, si sumatur $\lambda=m$ et $\mu=-n$, unde resultat multiplicator communis $\frac{y^m}{(1+Sy)^n}$, hincque posito $Sy=v$ et $\frac{1+Sy}{y^{\frac{m+1}{n-1}}\,S}=z$, nostrae aequationis integrale erit:

$$B\int\frac{v^m\,\partial v}{(1+v)^n}+C\,z^{1-n}=D\ \text{sive}$$
$$B\int\frac{v^m\,\partial v}{(1+v)^n}+\frac{C\,S^{n-1}\,y^{m+1}}{(1+Sy)^{n-1}}=D.$$

Scholion.

508. Aequatio ergo, quam hoc problemate integrare didicimus, per principia jam supra stabilita tractari potest, dum pro binis ejus partibus seorsim multiplicatores quaeruntur, iique inter

se congruentes redduntur, cujus methodi hic insignem usum declaravimus. Possemus etiam multiplicatori hanc formam dare $\frac{y^m}{(1+Sy+Tyy)^n}$, ita ut haec aequatio

$$\frac{y^m\,[yP\partial x+(Qy+R)\,\partial y]}{(1+Sy+Tyy)^n}=0$$

per se debeat esse integrabilis, et calculo ut ante instituto inveniemus

$$y^m\left\{\begin{matrix}+(m+1)P\partial x\\-\partial R\end{matrix}\right\}+y^{m+1}\left\{\begin{matrix}+(m+1-n)PS\partial x\\-\partial Q\\-S\partial R\\+nR\partial S\end{matrix}\right\}+y^{m+2}\left\{\begin{matrix}+(m+1-2n)PT\partial x\\-S\partial Q\\-T\partial R\\+nQ\partial S\\+nR\partial T\end{matrix}\right\}$$

$$+y^{m+3}\left\{\begin{matrix}-T\partial Q\\+nQ\partial T\end{matrix}\right\}=0,$$

unde ex ultimo membro $-T\partial Q+nQ\partial T=0$ concludimus $Q=AT^n$, et ex primo $P\partial x=\frac{\partial R}{m+1}$, qui valores in binis mediis substituti praebent

$$R\partial S-\frac{S\partial R}{m+1}-AT^{n-1}\partial T=0\quad\text{et}$$
$$R\partial T-\frac{2T\partial R}{m+1}+AT^n\partial S-AST^{n-1}\partial T=0,$$

quarum illa fit integrabilis per se si $m=-2$, haec vero integrari potest si $m=2n-1$, fit enim

$$R\partial T-\frac{T\partial R}{n}+AT^{n-1}(T\partial S-S\partial T)=0,\ \text{seu}$$
$$\frac{nR\partial T-T\partial R}{nT^{n+1}}+\frac{A(T\partial S-S\partial T)}{TT}=0,$$

cujus integrale est $\frac{-R}{nT^n}+\frac{AS}{T}=\frac{-B}{n}$; hincque

$$R=BT^n+nAT^{n-1}S.$$

Praeterea vero notari meretur casus $m = -1$, quem cum illis in subjunctis exemplis evolvamus.

Exemplum 1.

509. *Definire hanc aequationem*

$$yP\partial x + (Qy + R)\partial y = 0,$$

ut multiplicata per $\frac{1}{y(1 + Sy + Tyy)^n}$ *fiat per se integrabilis.*

Ob $m = -1$, habemus statim $\partial R = 0$, ideoque $R = C$: tum est ut ante $Q = AT^n$ et $\partial Q = nAT^{n-1}\partial T$, unde binae reliquae determinationes erunt:

$$-PS\partial x + AT^{n-1}\partial T + C\partial S = 0$$
$$-2PT\partial x - AST^{n-1}\partial T + AT^n\partial S + C\partial T = 0,$$

hinc eliminando $P\partial x$ prodit

$$ASST^{n-1}\partial T - 2AT^n\partial T - AT^n S\partial S$$
$$+ 2CT\partial S - CS\partial T = 0.$$

Statuatur hic $SS = Tv$, ut fiat

$$2T\partial S - S\partial T = TS\left(\frac{2\partial S}{S} - \frac{\partial T}{T}\right) = \frac{TS\partial v}{v} = \frac{T\partial v\sqrt{T}}{\sqrt{v}},$$

eritque

$$\tfrac{1}{2}AT^n v\partial T - 2AT^n\partial T - \tfrac{1}{2}AT^{n+1}\partial v + \frac{CT\partial v\sqrt{T}}{\sqrt{v}} = 0,$$

seu hoc modo

$$-\tfrac{1}{2}AT^{n+2}\partial . \frac{v-4}{T} + \frac{CT\partial v\sqrt{T}}{\sqrt{v}} = 0,$$

cujus prior pars integrabilis redditur per multiplicatorem

$$\frac{1}{T^{n+2}}\Phi : \frac{v-4}{T},$$

posterior vero per $\frac{1}{T\sqrt{T}}\Phi : v$, unde communis multiplicator erit $\frac{1}{T(v-4)^{n+\frac{1}{2}}\sqrt{T}}$, hincque aequatio elicitur integralis haec

$$\frac{A T^{n-\frac{1}{2}}}{(2n-1)(v-4)^{n-\frac{1}{2}}} + C\int \frac{\partial v}{(v-4)^{n+\frac{1}{2}}\sqrt{v}} = D,$$

unde T definitur per v; tum vero est $S = \sqrt{Tv}$, $R = C$,

$$Q = A T^n, \text{ et } P\partial x = \frac{C\partial S - A T^{n-1}\partial T}{S}.$$

Corollarium 1.

510. Casu quo est $n = \frac{1}{2}$, ob $\frac{1}{0} z^0 = lz$, habetur

$$\tfrac{1}{2} A l \frac{T}{v-4} + C\int \frac{\partial v}{(v-4)\sqrt{v}} = \tfrac{1}{2} D, \text{ seu}$$
$$\tfrac{1}{2} A l \frac{T}{v-4} - \tfrac{1}{2} C l \frac{\sqrt{v}+2}{\sqrt{v}-2} = \tfrac{1}{2} D:$$

unde posito $v = 4uu$ et $C = \lambda A$, erit

$$l\frac{T}{1-uu} - \lambda l \frac{1+u}{1-u} = \text{Const. seu}$$
$$T = E(1-uu)\left(\frac{1+u}{1-u}\right)^{\lambda}. \text{ Hinc porro}$$

$$S = 2u\sqrt{T} = 2u\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}, \text{ et}$$

$R = C = \lambda A$; tum $Q = A\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}$, atque

$$P\partial x = \frac{\lambda A\partial u}{u} + \frac{\lambda A\partial T}{2T} - \frac{A\partial T}{2Tu}.$$

At est $\frac{\partial T}{T} = \frac{-2u\partial u + 2\lambda\partial u}{1-uu}$. Ergo $P\partial x = \frac{A\partial u(1+\lambda\lambda-2\lambda u)}{1-uu}$

Quocirca pro hac aequatione

$$\frac{Ay\partial u(1+\lambda\lambda-2\lambda u)}{1-uu} + A\partial y\left[\lambda + y\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}\right] = 0$$

multiplicator erit

$$\frac{1}{y\sqrt{\left[1+2uy\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}+Eyy(1-uu)\left(\frac{1+u}{1-u}\right)^{\lambda}\right]}}.$$

Corollarium 2.

511. Casu quo $n = -\frac{1}{2}$ habemus

$$-\frac{A(v-4)}{2T} + 2C\sqrt{v} = -2D, \text{ seu } T = \frac{A(v-4)}{4D+4C\sqrt{v}}$$

Ponamus $v = 4uu$, ut sit $T = \frac{A(uu-1)}{D+2Cu}$, tum fit

$$S = 2u\sqrt{T} = 2u\sqrt{\frac{A(uu-1)}{D+2Cu}},$$

$$R = C,\ Q = \sqrt{\frac{A(D+2Cu)}{uu-1}}, \text{ et}$$

$$P\partial x = \frac{C\partial u}{u} + \frac{C\partial T}{2T} - \frac{A\partial T}{2TTu} = \frac{\partial u(C+Du+Cuu)(Cu^3-3Cu-D)}{u(uu-1)^2(D+2Cu)},$$

unde tam aequatio quam multiplicator definitur.

Exemplum 2.

512. *Definire aequationem*

$$yP\partial x + (Qy+R)\partial y = 0,$$

ut multiplicata per $\frac{1}{y^2(1+Sy+Tyy)^n}$, *fiat per se integrabilis.*

Ob $m = -2$, ex superioribus habemus:

$$RS = \frac{A}{n}T^n + B, \text{ seu } R = \frac{AT^n}{nS} + \frac{B}{S},$$

qui valor in altera aequatione substitutus praebet

$$\frac{(2n+1)AT^n\partial T}{nS} - \frac{2AT^{n+1}\partial S}{nSS} + AT^n\partial S - AST^{n-1}\partial T$$
$$+\frac{B\partial T}{S} - \frac{2BT\partial S}{SS} = 0,$$

quae in has tres partes distinguatur

$$\frac{AS}{nT^n}\left(\frac{(2n+1)T^{2n}\partial T}{S^2} - \frac{2T^{2n+1}\partial S}{S^3}\right) + AT^{n+1}\left(\frac{\partial S}{T} - \frac{S\partial T}{TT}\right)$$
$$+ BS\left(\frac{\partial T}{SS} - \frac{2\ T\ \partial S}{S^3}\right) = 0, \text{ seu}$$

$$\frac{AS}{nT^n}\partial \cdot \frac{T^{2n+1}}{SS} + AT^{n+1}\partial \cdot \frac{S}{T} + BS\partial \cdot \frac{T}{SS} = 0.$$

Statuamus ad abbreviandum

$$\frac{T^{2n+1}}{SS} = p,\ \frac{S}{T} = q \text{ et } \frac{T}{SS} = r,$$

fiet $S = \frac{1}{qr}$, $T = \frac{1}{qqr}$, hinc $p = \frac{1}{q^{4n}\, r^{2n-1}}$; nostraque aequatio ita se habebit

$$\frac{A}{nq\sqrt{pr}}\partial p + \frac{A\sqrt{p}}{qqr\sqrt{r}}\partial q + \frac{B}{qr}\partial r = 0, \text{ seu}$$
$$\frac{A\sqrt{r}}{n\sqrt{p}}\partial p + \frac{A\sqrt{p}}{q\sqrt{r}}\partial q + B\partial r = 0.$$

Quas tres partes seorsim consideremus, ac prima fit integrabilis multiplicata per $\frac{\sqrt{p}}{\sqrt{r}}\Phi : p$, secunda vero per $\frac{q\sqrt{r}}{\sqrt{p}}\Phi : q$, tertia tandem per $\Phi : r$. Ut bini primi conveniant, ponatur

$$\frac{\sqrt{p}}{\sqrt{r}} \cdot p^\lambda = \frac{q\sqrt{r}}{\sqrt{p}} \cdot q^\mu \text{ seu } p^{\lambda+1} = q^{\mu+1}\, r, \text{ hinc}$$
$$p = q^{\frac{\mu+1}{\lambda+1}}\, r^{\frac{1}{\lambda+1}} = q^{-4n}\, r^{-2n+1}.$$

Fit ergo

$$\lambda + 1 = -\frac{1}{2n-1} \text{ et } \mu + 1 = -4n(\lambda + 1) = \frac{4n}{2n-1}; \text{ sicque}$$
$$\mu = \frac{2n+1}{2n-1} \text{ et } \lambda = -\frac{2n}{2n-1}.$$

Multiplicetur ergo aequatio per $\dfrac{q^{\frac{4n}{2n-1}}\sqrt{r}}{\sqrt{p}} = q^{2n+\frac{4n}{2n-1}}\, r^n$, ac prodibit

$$\frac{A}{n} p^\lambda \partial p + A q^\mu \partial q + B q^{2n + \frac{4n}{2n-1}} r^n \partial r = 0,$$

seu

$$A \partial . \left(\frac{p^{\lambda+1}}{n(\lambda+1)} + \frac{q^{\mu+1}}{\mu+1} \right) + B q^{\frac{4nn+2n}{2n-1}} r^n \partial r = 0,$$

vel

$$\frac{(2n-1)A}{4n} \partial . q^{\frac{4n}{2n-1}} (1-4r) + B q^{\frac{4nn+2n}{2n-1}} r^n \partial r = 0.$$

Multiplicetur per $q^{\frac{4\nu n}{2n-1}} (1-4r)^\nu$, ut prodeat

$$\frac{(2n-1)A}{4n} . q^{\frac{4\nu n}{2n-1}} (1-4r)^\nu \partial . q^{\frac{4n}{2n-1}} (1-4r)$$
$$+ B q^{\frac{4nn+2n+4\nu n}{2n-1}} r^n \partial r (1-4r)^\nu = 0.$$

Fiat ergo $4\nu + 4n + 2 = 0$ seu $\nu = -n - \frac{1}{2}$, et ambo membra integrari poterunt, eritque

$$\frac{(2n-1)A}{4n(\nu+1)} q^{\frac{4n(\nu+1)}{2n-1}} (1-4r)^{\nu+1} + B \int r^n \partial r (1-4r)^\nu = \text{Const.}$$

at est $\nu + 1 = -n + \frac{1}{2} = \frac{-2n+1}{2}$, sicque habebitur

$$-\frac{A}{2n} q^{-2n} (1-4r)^{\frac{-2n+1}{2}} + B \int \frac{r^n \partial r}{(1-4r)^{\frac{2n+1}{2}}} = \text{Const.}$$

Dabitur ergo q per r, eritque $S = \frac{1}{qr}$, $T = \frac{S}{q}$, tum $R = \frac{AT^n}{nS} + \frac{B}{S}$, $Q = AT^n$ et $P \partial x = -\partial R$.

Corollarium 1.

513. Si sit $n = -\frac{1}{2}$, erit $Aq + \frac{2Br\sqrt{r}}{3} = \frac{C}{3}$, seu $q = \frac{C - 2Br\sqrt{r}}{3A}$; hincque

$$S = \frac{3A}{Cr - 2Br^2\sqrt{r}}, \quad T = \frac{9AA}{r(C-2Br\sqrt{r})^2}, \quad Q = \frac{C\sqrt{r} - 2Brr}{3} \text{ et}$$

$$R = \frac{Q + nB}{nS} = \frac{B - 2Q}{S} = \frac{r(C - 2Br\sqrt{r})(3B - 2C\sqrt{r} + 4Brr)}{9A} \quad \text{seu}$$

$$R = \frac{3BCr - 2CCr\sqrt{r} - 6BBrr\sqrt{r} + 8BCr^3 - 8BBr^4\sqrt{r}}{9A},$$

Corollarium 2.

514. Ponamus eodem casu $r = uu$, erit

$$S = \frac{3A}{Cuu - 2Bu^5}, \quad T = \frac{9AA}{uu(C - 2Bu^3)^2}, \quad Q = \frac{u(C - 2Bu^3)}{3}, \text{ et}$$

$$R = \frac{3BCu^2 - 2CCu^3 - 6BBu^5 + 8BCu^6 - 8BBu^9}{9A}, \text{ hincque}$$

$$P\partial x = \frac{-6BCu + 6CCuu + 30BBu^4 - 48BCu^5 + 72BBu^8}{9A}\partial u,$$

eritque aequatio $yP\partial x + (Qy + R)\partial y = 0$ integrabilis, si multiplicetur per

$$\frac{\sqrt{(1 + Sy + Tyy)}}{yy} = \frac{1}{yy}\sqrt{\left(1 + \frac{3Ay}{uu(C - 2Bu^3)} + \frac{9AAyy}{uu(C - 2Bu^3)^2}\right)}.$$

Exemplum 3.

515. *Definire aequationem*

$$yP\partial x + (Qy + R)\partial y = 0,$$

quae multiplicata per $\dfrac{y^{2n-1}}{(1 + Sy + Tyy)^n}$ *fiat per se integrabilis.*

Hic est $m = 2n - 1$, $Q = AT^n$, et $P\partial x = \frac{\partial R}{2n}$; tum vero ex superioribus $R = nAT^{n-1}S + BT^n$, ac superest aequatio

$$R\partial S - \frac{S\partial R}{2n} - AT^{n-1}\partial T = 0,$$

quae loco R substituto valore invento, abit in

$$(2n-1)AT^{n-1}S\partial S - (n-1)AT^{n-2}SS\partial T - 2AT^{n-1}\partial T$$
$$+ 2BT^n\partial S - BT^{n-1}S\partial T = 0, \text{ seu}$$
$$(2n-1)ATS\partial S - (n-1)ASS\partial T - 2AT\partial T$$
$$+ 2BTT\partial S - BTS\partial T = 0.$$

Prius membrum posito $SS = u$ abit in

$$\left(n - \tfrac{1}{2}\right) AT\partial u - (n-1) Au\partial T - 2AT\partial T, \text{ seu}$$

$$\left(n - \tfrac{1}{2}\right) AT\left(\partial u - \frac{(n-1)u\partial T}{\left(n-\tfrac{1}{2}\right)T} - \frac{2\partial T}{n-\tfrac{1}{2}}\right), \text{ sive}$$

$$\tfrac{1}{2}(2n-1) AT^{\frac{4n-3}{2n-1}}\left(\frac{\partial u}{T^{\frac{2n-2}{2n-1}}} - \frac{2(n-1)u\partial T}{(2n-1)T^{\frac{4n-3}{2n-1}}} - \frac{4\partial T}{(2n-1)T^{\frac{2n-2}{2n-1}}}\right)$$

$$= \tfrac{1}{2}(2n-1) AT^{\frac{4n-3}{2n-1}}\partial.\left(\frac{u}{T^{\frac{2n-2}{2n-1}}} - 4T^{\frac{1}{2n-1}}\right), \text{ vel}$$

$$\tfrac{1}{2}(2n-1) AT^{\frac{4n-3}{2n-1}}\partial.T^{\frac{1}{2n-1}}\left(\frac{SS}{T} - 4\right) + \frac{BT^3}{S}\partial.\frac{SS}{T} = 0, \text{ seu}$$

$$(2n-1) AT^{\frac{-1}{2n-1}}\partial.T^{\frac{1}{2n-1}}\left(\frac{SS}{T} - 4\right) + \frac{2BT}{S}\partial.\frac{SS}{T} = 0.$$

Ponatur $\frac{SS}{T} = p$ et

$$T^{\frac{1}{2n-1}}\left(\frac{SS}{T} - 4\right) = q = T^{\frac{1}{2n-1}}(p-4),$$

ut sit $T^{\frac{1}{2n-1}} = \frac{q}{p-4}$, unde

$$T = \frac{q^{2n-1}}{(p-4)^{2n-1}} \text{ et } S = \sqrt{\frac{pq^{2n-1}}{(p-4)^{2n-1}}}.$$

Ergo

$$\frac{(2n-1)A(p-4)\partial q}{q} + \frac{2B\sqrt{q^{2n-1}}}{\sqrt{p(p-4)^{2n-1}}}\partial p = 0$$

sive

$$\frac{(2n-1)A\partial q}{q^{n+\frac{1}{2}}} + \frac{2B\partial p : \sqrt{p}}{(p-4)^{n+\frac{1}{2}}} = 0,$$

quae integrata praebet

$$\frac{-2A}{q^{n-\frac{1}{2}}} + 2B\int\frac{\partial p : \sqrt{p}}{(p-4)^{n+\frac{1}{2}}} = 2C,$$

et facto $\frac{p}{p-4} = vv$, seu $p = \frac{4vv}{vv-1}$, fiet

$$\frac{+A}{g^{n-\frac{1}{2}}} - \frac{B}{4^{n-1}} \int \partial v\,(vv-1)^{n-1} = C.$$

Scholion.

516. Haec fusius non prosequor, quia ista exempla eum in finem potissimum attuli, ut methodus supra tradita aequationes differentiales tractandi exerceretur; in his enim exemplis casus non parum difficiles se obtulerunt, quos ita per partes resolvere licuit, ut pro singulis multiplicatores idonei quaererentur, ex iisque multiplicator communis definiretur; nunc igitur alia aequationum genera, quae per multiplicatores integrabiles reddi queant, investigemus.

Problema 67.

517. Ipsius x functiones P, Q, R, S definire, ut haec aequatio $(Py + Q)\,\partial x + y\,\partial y = 0$, per hunc multiplicatorem $(yy + Ry + S)^n$ integrabilis reddatur.

Solutio.

Necesse igitur est, sit

$$\left(\frac{\partial.(Py+Q)(yy+Ry+S)^n}{\partial y}\right) = \left(\frac{\partial.y(yy+Ry+S)^n}{\partial x}\right)$$

unde colligitur per $(yy + Ry + S)^{n-1}$ dividendo

$$P(yy+Ry+S) + n(Py+Q)(2y+R) = \frac{ny(y\,\partial R + \partial S)}{\partial x}$$

seu

$$\left.\begin{array}{lll}(2n+1)Pyy\,\partial x & +(n+1)PRy\,\partial x & +PS\,\partial x\\ -nyy\,\partial R & +2nQy\,\partial x & +nQR\,\partial x\\ & -ny\,\partial S & \end{array}\right\} = 0.$$

**

Hinc ergo concluditur $P\partial x = \frac{n\partial R}{2n+1}$, et

$$\frac{(n+1)R\partial R}{2n+1} + 2Q\partial x - \partial S = 0,$$

$$\frac{S\partial R}{2n+1} + QR\partial x = 0, \text{ porroque}$$

$$Q\partial x = \frac{-S\partial R}{(2n+1)R} = \frac{-(n+1)R\partial R}{2(2n+1)} + \frac{\partial S}{2}: \text{ ergo}$$

$$\partial S + \frac{2S\partial R}{(2n+1)R} = \frac{(n+1)R\partial R}{2n+1},$$

quae per $R^{\frac{2}{2n+1}}$ multiplicata et integrata, dat

$$R^{\frac{2}{2n+1}}S = C + \tfrac{1}{4}R^{\frac{4n+4}{2n+1}}, \text{ hincque}$$

$$S = \tfrac{1}{4}RR + CR^{\frac{-2}{2n+1}}, \text{ atque}$$

$$Q\partial x = \frac{-R\partial R}{4(2n+1)} - \frac{C}{2n+1}R^{\frac{-2n-3}{2n+1}}\partial R, \text{ et } P\partial x = \frac{n\partial R}{2n+1}:$$

unde aequationem obtinemus

$$(ny - \tfrac{1}{4}R - CR^{\frac{-2n-3}{2n+1}})\partial R + (2n+1)y\partial y = 0,$$

quae integrabilis redditur per hunc multiplicatorem

$$(yy + Ry + \tfrac{1}{4}RR + CR^{\frac{-2}{2n+1}})^n.$$

Corollarium 1.

518. Casu quo $n = -\frac{1}{2}$, fit $\partial R = 0$ et $R = A$, et reliquae aequationes sunt

$$(n+1)AP\partial x + 2nQ\partial x - n\partial S = 0 \text{ et}$$

$$PS\partial x + nAQ\partial x = 0.$$

Ergo $P\partial x = \frac{AQ\partial x}{2S} = \frac{2Q\partial x - \partial S}{A}$, ideoque

$$(AA - 4S)Q\partial x = -2S\partial S, \text{ seu}$$

$$Q\partial x = \frac{-2S\partial S}{AA-4S} \text{ et } P\partial x = \frac{-A\partial S}{AA-4S}$$

sicque haec aequatio $\frac{(Ay+2S)\partial S}{4S-AA}+y\partial y=0$ integrabilis redditur per hunc multiplicatorem $\frac{1}{\sqrt{(yy+Ay+S)}}$.

Corollarium 2.

519. Si hic ponamus $A=2a$ et $S=x$, haec aequatio $\frac{(ay+x)\partial x+2y\partial y(x-aa)}{(x-aa)\sqrt{(yy+2ay+x)}}=0$ per se est integrabilis, unde integrale inveniri potest hujus aequationis

$$x\partial x+ay\partial x+2xy\partial y-2aay\partial y=0,$$

quae divisa per $(x-aa)\sqrt{(yy+2ay+x)}$ fit integrabilis.

Corollarium 3.

520. Ad integrale inveniendum, sumatur primo x constans, et partis $\frac{2y\partial y}{\sqrt{(yy+2ay+x)}}$ integrale est

$$2\sqrt{(yy+2ay+x)}+2al[a+y-\sqrt{(yy+2ay+x)}]+X,$$

cujus differentiale sumto y constante

$$\frac{\partial x}{\sqrt{(yy+2ay+x)}}-\frac{a\partial x:\sqrt{(yy+2ay+x)}}{a+y-\sqrt{(yy+2ay+x)}}+\partial X,$$

si alteri aequationis parti $\frac{(ay+x)\partial x}{(x-aa)\sqrt{(yy+2ay+x)}}$ aequetur, reperitur $\partial X=\frac{a\partial x}{aa-x}$ et $X=-al(aa-x)$. Ex quo integrale completum erit

$$\sqrt{(yy+2ay+x)}+al\frac{a+y-\sqrt{(yy+2ay+x)}}{\sqrt{(aa-x)}}=C.$$

Corollarium 4.

521. Memoratu dignus est etiam casus $n=-1$, qui scripto a loco $C+\frac{1}{4}$ praebet hanc aequationem

$$(y+aR)\partial R+y\partial y=0,$$

quae divisa per $yy+Ry+aRR$ fit integrabilis, haec autem aequatio est homogenea.

Scholion.

522. Potest etiam aequationis

$$(Py + Q)\,\partial x + y\,\partial y = 0$$

multiplicator statui $(y + R)^m (y + S)^n$, fierique debet

$$\left(\frac{\partial . (Py + Q)(y + R)^m (y + S)^n}{\partial y}\right) = \left(\frac{\partial . y (y + R)^m (y + S)^n}{\partial x}\right);$$

unde reperitur

$$P\,\partial x (y + R)(y + S) + m\,\partial x (Py + Q)(y + S)$$
$$+ n\,\partial x (Py + Q)(y + R) = my (y + S)\,\partial R + ny (y + R)\,\partial S,$$

quae evolvitur in

$$\left.\begin{array}{ll} (m + n + 1) Pyy\,\partial x & + (n + 1) PRy\,\partial x + PRS\,\partial x \\ - myy\,\partial R & + (m + 1) PSy\,\partial x + mQS\,\partial x \\ - nyy\,\partial S & + (m + n) Qy\,\partial x + nQR\,\partial x \\ & - mSy\,\partial R \\ & - nRy\,\partial S \end{array}\right\} = 0$$

unde colligitur

$$P\,\partial x = \frac{m\,\partial R + n\,\partial S}{m + n + 1} \text{ et } Q\,\partial x = \frac{-PRS\,\partial x}{mS + nR} = \frac{-RS (m\,\partial R + n\,\partial S)}{(m + n + 1)(mS + nR)},$$

hincque

$$\frac{(m\partial R + n\partial S)[(n + \;)R + (m + 1)S]}{m + n + 1} - \frac{(m + n)RS(m\partial R + n\partial S)}{(m + n + 1)(mS + nR)} - mS\,\partial R - nR\,\partial S = 0,$$

seu

$$\begin{array}{l} + m (n + 1) R\,\partial R - mnR\,\partial S \\ + n (m + 1) S\,\partial S - mnS\,\partial R \end{array} - \frac{m (m + n) RS\,\partial R - n (m + n) RS\,\partial S}{mS + nR} = 0,$$

quae reducitur ad hanc formam

$$\left.\begin{array}{l} + (n + 1) RR\,\partial R + (m - n - 1) RS\,\partial R - mSS\,\partial R \\ + (m + 1) SS\,\partial S + (n - m - 1) RS\,\partial S - nRR\,\partial S \end{array}\right\} = 0,$$

quae cum sit homogenea, dividatur per

$$(n + 1) R^3 + (m - 2n - 1) R^2 S + (n - 2m - 1) RS^2 + (m + 1) S^3,$$

seu per

$$(R - S)^2 [(n + 1) R + (m + 1) S]$$

ut fiat integrabilis. At ipsa illa aequatio per R — S divisa, erit

$$(n + 1) R \partial R + m S \partial R - (m + 1) S \partial S = 0.$$

Dividatur per

$$(R - S) [(n + 1) R + (m + 1) S]$$

et resolvatur in fractiones partiales, erit

$$\frac{\partial R}{m + n + 2} \left(\frac{m + n + 1}{R - S} + \frac{n + 1}{(n + 1) R + (m + 1) S} \right)$$
$$+ \frac{\partial S}{m + n + 2} \left(\frac{m + n + 1}{S - R} + \frac{m + 1}{(n + 1) R + (m + 1) S} \right) = 0$$

seu

$$\frac{(m + n + 1)(\partial R - \partial S)}{R - S} + \frac{(n + 1) \partial R + (m + 1) \partial S}{(n + 1) R + (m + 1) S} = 0;$$

unde integrando obtinemus,

$$(R - S)^{m + n + 1} [(n + 1) R + (m + 1) S] = C.$$

Sit $R - S = u$, erit

$$(n + 1) R + (m + 1) S = \frac{C}{u^{m + n + 1}},$$

hincque

$$R = \frac{(m + 1) u}{m + n + 2} + \frac{a}{u^{m + n + 1}}, \text{ et}$$

$$S = \frac{-(n + 1) u}{m + n + 2} + \frac{a}{u^{m + n + 1}},$$

tum vero

$$P \partial x = \frac{(m - n) \partial u}{m + n + 2} - \frac{(m + n) a \partial u}{u^{m + n + 2}}, \text{ et}$$

$$Q \partial x = \frac{\partial u}{u} \left(\frac{a}{u^{m + n + 1}} + \frac{(m + 1) u}{m + n + 2} \right) \left(\frac{a}{u^{m + n + 1}} - \frac{(n + 1) u}{m + n + 2} \right).$$

Corollarium 1.

523. Hinc ergo integrari potest ista aequatio

$$y\partial y+y\partial u\left(\frac{m-n}{m+n+2}-\frac{(m+n)a}{u^{m+n+2}}\right)$$
$$+\frac{\partial u}{u}\left(\frac{aa}{u^{2m+2n+2}}+\frac{(m-n)a}{(m+n+2)\,u^{m+n}}-\frac{(m+1)(n+1)uu}{(m+n+2)^2}\right)=0,$$

quippe quae per se fit integrabilis, si multiplicetur per

$$\left(y+\frac{a}{u^{m+n+1}}+\frac{(m+1)u}{m+n+2}\right)^m\left(y+\frac{a}{u^{m+n+1}}-\frac{(n+1)u}{m+n+2}\right)^n.$$

Corollarium 2.

524. Sit $m=n$, et aequatio nostra erit

$$y\partial y-\frac{2nay\partial u}{u^{2n+2}}+\frac{aa\partial u}{u^{4n+3}}-\tfrac{1}{4}u\partial u=0,$$

cujus multiplicator est $[(y+\frac{a}{u^{2n+1}})^2-\frac{1}{4}uu]^n$. Quare si ponamus $y=z-\frac{a}{u^{2n+1}}$; aequatio prodit

$$z\partial z-\frac{a\partial z}{u^{2n+1}}+\frac{az\partial u}{u^{2n+2}}-\tfrac{1}{4}u\partial u=0,$$

quae integrabilis fit multiplicata per $(zz-\frac{1}{4}uu)^n$. Vel ponatur $z=\frac{1}{2}y$ et $a=\frac{1}{2}b$, erit aequatio

$$y\partial y-u\partial u-\frac{b\partial y}{u^{2n+1}}+\frac{by\partial u}{u^{2n+2}}=0,$$

et multiplicator $(yy-uu)^n$.

Corollarium 3.

525. Si $m=-n$, prodit haec aequatio

$$y\partial y-ny\partial u+\frac{aa\partial u}{u^3}+\tfrac{1}{4}(nn-1)u\partial u-\frac{na\partial u}{u}=0,$$

quae integrabilis redditur multiplicata per

$$[y+\tfrac{a}{u}-\tfrac{1}{2}(n+1)u)]^n[y+\tfrac{a}{u}-\tfrac{1}{2}(n-1)u]^{-n}.$$

Posito autem $y+\frac{a}{u}=z$, prodit haec aequatio

$$z\partial z-nz\partial u+\tfrac{1}{4}(nn-1)u\partial u-\frac{a\partial z}{u}+\frac{az\partial u}{uu}=0,$$

quam integrabilem reddit hic multiplicator

$$[z-\tfrac{1}{2}(n+1)u]^n[z-\tfrac{1}{2}(n-1)u]^{-n}.$$

Corollarium 4.

526. Ponamus hic $z=uv$, et habebitur ista aequatio

$$uuv\partial v+u\partial u[vv-nv+\tfrac{1}{4}(nn-1)]=a\partial v,$$

quae si multiplicetur per $\left(\frac{v-\frac{1}{2}(n+1)}{v-\frac{1}{2}(n-1)}\right)^n$, utrumque membrum fiet integrabile. Posito enim $\frac{v-\frac{1}{2}(n+1)}{v-\frac{1}{2}(n-1)}=s$, seu

$$v=\frac{n+1-(n-1)s}{2(1-s)},$$

oritur

$$\frac{s^{n+1}u\partial u}{(1-s)^2}+\frac{n+1-(n-1)s}{2(1-s)^3}uus^n\partial s=\frac{as^n\partial s}{(1-s)^2},$$

cujus integrale est

$$\frac{s^{n+1}uu}{2(1-s)^2}=a\int\frac{s^n\partial s}{(1-s)^2}.$$

Scholion.

527. Quo nostram aequationem in genere concinniorem reddamus, ponamus $m=-\lambda-1+\mu$ et $n=-\lambda-1-\mu$, ut sit $m+n+2=-2\lambda$, fietque aequatio

$$y\partial y-y\partial u[\tfrac{\mu}{\lambda}-2(\lambda+1)au^{2\lambda}]$$

$$+u\partial u\left(\frac{\mu\mu-\lambda\lambda}{4\lambda\lambda}-\frac{\mu}{\lambda}au^{2\lambda}+aau^{4\lambda}\right)=0,$$

quae per hunc multiplicatorem integrabilis redditur

$$\left(y+au^{2\lambda+1}-\frac{(\mu-\lambda)u}{2\lambda}\right)^{\mu-\lambda-1}\left(y+au^{2\lambda+1}-\frac{(\mu+\lambda)u}{2\lambda}\right)^{-\mu-\lambda-1}.$$

Ponatur $y+au^{2\lambda+1}=uz$, et orietur haec aequatio

$$uz\partial z-au^{2\lambda+1}\partial z+\partial u\left(zz-\frac{\mu}{\lambda}z+\frac{\mu\mu-\lambda\lambda}{4\lambda\lambda}\right)=0,$$

cui respondet multiplicator

$$u^{-2\lambda-1}\left(z+\frac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda-1}\left(z-\frac{\lambda-\mu}{2\mu}\right)^{-\mu-\lambda-1}.$$

Reperitur autem integrale

$$C=a\int\partial z\left(z+\frac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda-1}\left(z-\frac{\lambda-\mu}{2\lambda}\right)^{-\mu-\lambda-1}$$
$$+\frac{1}{2\lambda u^{2\lambda}}\left(z+\frac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda}\left(z-\frac{\lambda-\mu}{2\lambda}\right)^{-\mu-\lambda},$$

quod ergo convenit huic aequationi differentiali

$$z\partial z+\frac{\partial u}{u}\left(z+\frac{\lambda-\mu}{2\lambda}\right)\left(z-\frac{\lambda-\mu}{2\lambda}\right)=au^{2\lambda}\partial z.$$

Problema 68.

528. Ipsius x functiones P, Q, R et X definire, ut haec aequatio $\partial y+yy\partial x+X\partial x=0$ integrabilis reddatur per hunc multiplicatorem $\frac{1}{Pyy+Qy+R}$.

Solutio.

Debet ergo esse

$$\frac{1}{\partial y}\partial\cdot\frac{yy+X}{Pyy+Qy+R}=\frac{1}{\partial x}\partial\cdot\frac{1}{Pyy+Qy+R},$$

hincque

$$2y(Pyy+Qy+R)-(yy+X)(2Py+Q)$$
$$=-\frac{yy\partial P-y\partial Q-\partial R}{\partial x};$$

ergo fieri debet

$$\left.\begin{array}{l} +Qyy\partial x + 2Ry\partial x - QX\partial x \\ +yy\partial P - 2PXy\partial x + \partial R \\ +y\partial Q \end{array}\right\} = 0.$$

Quare habetur $Q = -\frac{\partial P}{\partial x} = \frac{\partial R}{X\partial x}$, et $X = -\frac{\partial R}{\partial P}$. Sumto ergo ∂x constante est $\partial Q = -\frac{\partial\partial P}{\partial x}$, unde fieri oportet

$$2R\partial x + \frac{2P\partial R\partial x}{\partial P} - \frac{\partial\partial P}{\partial x} = 0, \text{ seu}$$

$$R\partial P + P\partial R = \frac{\partial P\partial\partial P}{2\partial x^2},$$

cujus integratio praebet $PR = \frac{\partial P^2}{4\partial x^2} + C$, hinc $R = \frac{\partial P^2}{4P\partial x^2} + \frac{C}{P}$, tum

$$Q = -\frac{\partial P}{\partial x}, \text{ et } X = \frac{C}{PP} + \frac{\partial P^2}{4PP\partial x^2} - \frac{\partial\partial P}{2P\partial x^2}.$$

Ponamus $P = SS$, ut S sit functio quaecunque ipsius x, obtinebimusque

$$P = SS,\ Q = -\frac{2S\partial S}{\partial x},\ R = \frac{C}{SS} + \frac{\partial S^2}{\partial x^2}, \text{ et } X = \frac{C}{S^4} - \frac{\partial\partial S}{S\partial x^2},$$

quibus sumtis valoribus, per se integrabilis erit haec aequatio

$$\frac{\partial y + yy\partial x + X\partial x}{Pyy + Qy + R} = 0.$$

Scholion.

529. Haec solutio commodius institui poterit, si multiplicatori tribuatur haec forma $\frac{P}{yy + 2Qy + R}$, ut fieri debeat

$$\frac{1}{\partial y}\partial \cdot \frac{P(yy + X)}{yy + 2Qy + R} = \frac{1}{\partial x}\partial \cdot \frac{P}{yy + 2Qy + R}.$$

unde oritur

$$\left.\begin{array}{l} 2PQyy\partial x + 2PRy\partial x - 2PQX\partial x \\ -yy\partial P - 2PXy\partial x - R\partial P \\ -2Qy\partial P + P\partial R \\ +2Py\partial Q \end{array}\right\} = 0,$$

ubi ex singulis commode definitur $\frac{\partial P}{P}$: scilicet

$$\frac{\partial P}{P} = 2Q\partial x = \frac{R\partial x - X\partial x + \partial Q}{Q} = \frac{\partial R - 2QX\partial x}{R}.$$

Hinc colligitur $2Q(R+X)\partial x = \partial R$, unde nunc ipsum elementum ∂x definiamus, $\partial x = \frac{\partial R}{2Q(R+X)}$, quo valore substituto adipiscimur

$$\frac{Q\partial R}{R+X} = \frac{(R-X)\partial R}{2Q(R+X)} + \partial Q \text{ seu}$$

$$2QQ\partial R = R\partial R - X\partial R + 2QR\partial Q + 2QX\partial Q:$$

unde colligimus

$$X = \frac{2QQ\partial R - 2QR\partial Q - R\partial R}{2Q\partial Q - \partial R}, \text{ et } R+X = \frac{2(QQ-R)\partial R}{2Q\partial Q - \partial R},$$

hinc $\partial x = \frac{2Q\partial Q - \partial R}{4Q(QQ-R)}$, atque $\frac{\partial P}{P} = \frac{2Q\partial Q - \partial R}{2(QQ-R)}$; ideoque
$P = A\sqrt{(QQ-R)}$.

Fiat $QQ - R = S$, ac reperietur

$$\partial x = \frac{\partial S}{4QS}, \quad X = \frac{4QS\partial Q}{\partial S} - QQ - S, \quad R = QQ - S,$$

atque $P = A\sqrt{S}$. Quocirca habebimus hanc aequationem

$$\partial y + \frac{yy\partial S}{4QS} + \partial Q - \frac{(QQ+S)\partial S}{4QS} = 0,$$

quae integrabilis redditur per hunc multiplicatorem

$$\frac{\sqrt{S}}{yy+2Qy+QQ-S} = \frac{\sqrt{S}}{(y+Q)^2-S}.$$

Ad ejus integrale inveniendum, sumantur Q et S constantes, prodibitque

$$\int \frac{\partial y\sqrt{S}}{(y+Q)^2-S} = \tfrac{1}{2}l\frac{y+Q-\sqrt{S}}{y+Q+\sqrt{S}} + V,$$

existente V certa functione ipsius S vel Q. Jam differentietur haec forma sumta y constante, proditque

$$\frac{\partial Q\sqrt{S} - \frac{(Q+y)\partial S}{2\sqrt{S}}}{(y+Q)^2 - S} + \partial V = \frac{yy\partial S + 4QS\partial Q - QQ\partial S - S\partial S}{4Q[(y+Q)^2-S]\sqrt{S}},$$

ideoque

$$\partial V = \frac{yy\partial S + 2Qy\partial S + QQ\partial S - S\partial S}{4Q[(y+Q)^2-S)\sqrt{S}} = \frac{\partial S}{4Q\sqrt{S}}.$$

Ex quo aequationis nostrae integrale est

$$\tfrac{1}{2}l\frac{y+Q-\sqrt{S}}{y+Q+\sqrt{S}} + \tfrac{1}{4}\int\frac{\partial S}{Q\sqrt{S}} = C.$$

Corollarium 1.

530. Singularis est casus, quo $R = QQ$, fit enim

$$\frac{\partial P}{P} = 2Q\partial x = \frac{QQ\partial x - X\partial x + \partial Q}{Q} = \frac{2\partial Q - 2X\partial x}{Q},$$

unde has duas aequationes elicimus

$$QQ\partial x + X\partial x - \partial Q = 0 \text{ et } QQ\partial x + X\partial x - \partial Q = 0,$$

quae cum inter se conveniant, erit

$$X\partial x = \partial Q - QQ\partial x, \text{ et } lP = 2\int Q\partial x.$$

Corollarium 2.

531. Sumto ergo Q negativo, ut habeamus hanc aequationem

$$\partial y + yy\partial x - \partial Q - QQ\partial x = 0,$$

haec integrabilis redditur per hunc multiplicatorem

$$\frac{e^{-2\int Q\partial x}}{(y-Q)^2}.$$ Et integrale erit

$$\frac{-1}{y-Q}e^{-2\int Q\partial x} + V = \text{Const.}$$

ubi V est functio ipsius x, ad quam definiendam, differentietur sumta y constante

$$\frac{-\partial Q}{(y-Q)^2}e^{-2\int Q\partial x} + \frac{2Q\partial x}{y-Q}e^{-2\int Q\partial x} + \partial V = \frac{yy\partial x - \partial Q - QQ\partial x}{(y-Q)^2}e^{-2\int Q\partial x},$$

unde fit $V = \int e^{-2\int Q\partial x}\partial x$, ita ut integrale fit

$$\int e^{-2\int Q\partial x}\partial x - \frac{e^{-2\int Q\partial x}}{y-Q} = C.$$

Corollarium 3.

532. Proposita ergo aequatione

$$\partial y + yy\partial x + X\partial x = 0,$$

si ejus integrale particulare quoddam constet $y = Q$, ut sit

$$\partial Q + QQ\partial x + X\partial x = 0,$$

ideoque

$$\partial y + yy\partial x - \partial Q - QQ\partial x = 0,$$

multiplicator pro ea erit $\frac{1}{(y-Q)^2} e^{-2\int Q\partial x}$, et integrale completum

$$Ce^{2\int Q\partial x} + \frac{1}{(y-Q)} = e^{2\int Q\partial x} \int e^{-2\int Q\partial x}\partial x.$$

Scholion.

533. Aequatio autem in praecedente scholio inventa

$$\partial y + \frac{yy\partial S}{4QS} + \partial Q - \frac{(QQ+S)\partial S}{4QS} = 0,$$

non multum habet in recessu, posito enim $y + Q = z$ prodit

$$\partial z - \frac{z\partial S}{2S} + \frac{\partial S(zz-S)}{4QS} = 0,$$

in qua, ut bini priores termini in unum contrahantur, ponatur $z = v\sqrt{S}$, reperieturque

$$\partial v\sqrt{S} + \frac{vv\partial S}{4Q} - \frac{\partial S}{4Q} = 0, \text{ seu } \frac{\partial v}{vv-1} + \frac{\partial S}{4Q\sqrt{S}} = 0,$$

quae cum sit separata integrale erit $\frac{1}{2} l \frac{1+v}{1-v} = \frac{1}{4}\int \frac{\partial S}{Q\sqrt{S}}$, ubi est $v = \frac{y+Q}{\sqrt{S}}$.

Aequatio autem in ipsa solutione inventa

$$\partial y + yy\partial x + \frac{C\partial x}{S^4} - \frac{\partial\partial S}{S\partial x} = 0,$$

ubi S est functio quaecunque ipsius x, et $\frac{\partial\partial S}{\partial x} = \partial \cdot \frac{\partial S}{\partial x}$, magis ardua videtur, dum per se fit integrabilis, si dividatur per

$$SSyy - \frac{2Sy\partial S}{\partial x} + \frac{\partial S^2}{\partial x^2} + \frac{C}{SS} = \left(Sy - \frac{\partial S}{\partial x}\right)^2 + \frac{C}{SS}.$$

At sumto x constante integrale reperitur

$$\frac{1}{\sqrt{C}} \text{ Arc. tang. } \frac{SSy\partial x - S\partial S}{\partial x\sqrt{C}} + V = \text{Const.}$$

nunc ergo ad functionem V inveniendam, sumatur differentiale posita y constante, quod est

$$\frac{2Sy\partial S - \frac{S\partial\partial S}{\partial x} - \frac{\partial S^2}{\partial x}}{SS(Sy - \frac{\partial S}{\partial x})^2 + C} + \partial V,$$

et aequari debet alteri parti

$$\frac{\frac{C\partial x}{S^4} - \frac{\partial\partial S}{S\partial x} + yy\partial x}{(Sy - \frac{\partial S}{\partial x})^2 + \frac{C}{SS}} = \frac{\frac{C\partial x}{SS} - \frac{S\partial\partial S}{\partial x} + SSyy\partial x}{SS(Sy - \frac{\partial S}{\partial x})^2 + C}.$$

Ergo

$$\partial V = \frac{SSyy\partial x - 2Sy\partial S + \frac{\partial S^2}{\partial x} + \frac{C\partial x}{SS}}{SS(Sy - \frac{\partial S}{\partial x})^2 + C} = \frac{\partial x}{SS}.$$

Quocirca integrale completum est

$$\frac{1}{\sqrt{C}} \text{Arc. tang.} \frac{SSy\partial x - S\partial S}{\partial x\sqrt{C}} + \int \frac{\partial x}{SS} = D.$$

Quod si sumamus $S = x$, hujus aequationis

$$\partial y + yy\partial x + \frac{C\partial x}{x^4} = 0,$$

integrale completum est

$$\frac{1}{\sqrt{C}} \text{Arc. tang.} \frac{xxy - x}{\sqrt{C}} - \frac{1}{x} = D.$$

Sin autem fit $S = x^n$,

ob $\frac{\partial S}{\partial x} = nx^{n-1}$ et $\partial . \frac{\partial S}{\partial x} = n(n-1)x^{n-2}\partial x$,

integrari poterit haec aequatio

$$\partial y + yy\partial x + \frac{C\partial x}{x^{4n}} - \frac{n(n-1)\partial x}{xx} = 0,$$

integrale enim erit

$$\frac{1}{\sqrt{C}} \text{Arc. tang.} \frac{x^{2n}y - nx^{2n-1}}{\sqrt{C}} - \frac{1}{(2n-1)x^{2n-1}} = D.$$

Supra autem invenimus hanc aequationem

$\partial y + yy \partial x + C x^m \partial x = 0$, ad separationem reduci posse, quoties fuerit $m = \frac{-4i}{2i+1}$, iisdem ergo casibus functionem S assignare licebit, ut fiat $\frac{C}{S^4} - \frac{\partial \partial S}{S \partial x^2} = C x^m$; quod cum ad aequationes differentiales secundi gradus pertineat, hic non attingemus.

Problema 69.

534. Definire functiones P et Q ambarum variabilium x et y, ut aequatio differentialis $P \partial x + Q \partial y = 0$, divisa per $P x + Q y$ fiat per se integrabilis.

Solutio.

Cum formula $\frac{P \partial x + Q \partial y}{P x + Q y}$ debeat esse integrabilis, statuamus $Q = P R$, ut habeamus $\frac{\partial x + R \partial y}{x + R y}$, sitque $\partial R = M \partial x + N \partial y$. Quare fieri oportet

$$\frac{1}{\partial y} \partial \cdot \frac{1}{x + R y} = \frac{1}{\partial x} \partial \cdot \frac{R}{x + R y},$$

unde nanciscimur $\frac{-R - N y}{(x + R y)^2} = \frac{M x - R}{(x + R y)^2}$ seu $N = -\frac{M x}{y}$; hinc fit $\partial R = M \partial x - \frac{M x \partial y}{y} = M y . \frac{y \partial x - x \partial y}{y y}$, quae formula cum debeat esse integrabilis, necesse est sit $M y$ functio ipsius $\frac{x}{y}$, quia $\frac{y \partial x - x \partial y}{y y} = \partial . \frac{x}{y}$: atque ex hac integratione prodit $R = \phi : \frac{x}{y}$, seu quod eodnm redit, R erit functio nullius dimensionis ipsarum x et y. Quocirca cum $\frac{Q}{P} = R$, manifestum est huic conditioni satisfieri, si P et Q fuerint functiones homogeneae ejusdem dimensionum numeri ipsarum x et y; hoc ergo modo eandem integrationem aequationum homogenearum sumus assecuti, quam in capite superiori docuimus.

Corollarium 1.

535. Cum igitur $\frac{\partial t + R \partial u}{t + R u}$ sit integrabile, si fuerit $R = \phi : \frac{t}{u}$, seu $R = \frac{t}{u} \phi : \frac{t}{u}$, erit etiam haec formula

$\frac{\frac{\partial t}{t}+\frac{\partial u}{u}\phi:\frac{t}{u}}{1+\phi:\frac{t}{u}}$ integrabilis, quae ita repraesentari potest

$$\frac{\frac{\partial t}{t}+\frac{\partial u}{u}\phi:\left(\int\frac{\partial t}{t}-\int\frac{\partial u}{u}\right)}{1+\phi:\left(\int\frac{\partial t}{t}-\int\frac{\partial u}{u}\right)},$$

ubi littera ϕ denotat functionem quamcunque quantitatis suffixae.

Corollarium 2.

536. Ponatur $\frac{\partial t}{t}=\frac{\partial x}{X}$ et $\frac{\partial u}{u}=\frac{\partial y}{Y}$, atque haec formula

$$\frac{\frac{\partial x}{X}+\frac{\partial y}{Y}\phi:\left(\int\frac{\partial x}{X}-\int\frac{\partial y}{Y}\right)}{1+\phi:\left(\int\frac{\partial x}{X}-\int\frac{\partial y}{X}\right)}=\frac{\partial x+\frac{X\partial y}{Y}\phi:\left(\int\frac{\partial x}{X}-\int\frac{\partial y}{Y}\right)}{X+X\phi:\left(\int\frac{\partial x}{X}-\int\frac{\partial y}{Y}\right)}$$

erit per se integrabilis. Quare posito $R=\frac{X}{Y}\phi:\left(\int\frac{\partial x}{X}-\int\frac{\partial y}{Y}\right)$, haec formula $\frac{\partial x+R\partial y}{X+RY}$ erit per se integrabilis, quaecunque functio sit X ipsius x, et Y ipsius y.

Corollarium 3.

537. Quare si quaerantur functiones P et Q, ut haec aequatio $P\partial x+Q\partial y=0$ fiat integrabilis, si dividatur per $PX+QY$, existente X functione quacunque ipsius x, et Y ipsius y, decet esse $\frac{Q}{P}=\frac{X}{Y}\phi:\left(\int\frac{\partial x}{X}-\int\frac{\partial y}{Y}\right)$.

Corollarium 4.

538. Quare si signa ϕ et ψ functiones quascunque indicent, fueritque

$$P=\frac{V}{X}\phi:\left(\int\frac{\partial x}{X}-\int\frac{\partial y}{Y}\right) \text{ et } Q=\frac{V}{Y}\psi:\left(\int\frac{\partial x}{X}-\int\frac{\partial y}{Y}\right),$$

haec aequatio $P\partial x+Q\partial y=0$ integrabilis reddetur, si dividatur per $PX+QY$.

Scholion.

539. Hinc ergo innumerabiles aequationes proferri possunt, quas integrare licebit, etiamsi alioquin difficillime pateat, quomodo

eae ad separationem variabilium reduci queant. Verum haec investigatio proprie ad librum secundum Calculi Integralis est referenda, cujus jam egregia specimina hic habentur; definivimus enim functionem R binarum variabilium x et y ex certa conditione inter M et N proposita scilicet $Mx + Ny = 0$ seu $x\left(\frac{\partial R}{\partial x}\right) + y\left(\frac{\partial R}{\partial y}\right) = 0$, hoc est ex certa differentialium conditione.