CAPUT VIII.

DE
VALORIBUS INTEGRALIUM QUOS CERTIS TANTUM CASIBUS RECIPIUNT.

Problema 38.

330.

Integralis $\int \frac{x^m \partial x}{\sqrt{(1-xx)}}$ valorem, quem posito $x = 1$ recipit, assignare, integrali scilicet ita determinato, ut evanescat posito $x = 0$.

Solutio.

Pro casibus simplicissimis, quibus $m = 0$ vel $m = 1$, habemus posito $x = 1$, post integrationem

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2} \text{ et } \int \frac{x \partial x}{\sqrt{(1-xx)}} = 1.$$

Deinde supra §. 119. vidimus esse in genere

$$\int \frac{x^{m+1} \partial x}{\sqrt{(1-xx)}} = \frac{m}{m+1} \int \frac{x^{m-1} \partial x}{\sqrt{(1-xx)}} - \frac{1}{m+1} x^m \sqrt{(1-xx)}:$$

casu ergo $x = 1$ erit

$$\int \frac{x^{m+1} \partial x}{\sqrt{(1-xx)}} = \frac{m}{m+1} \int \frac{x^{m-1} \partial x}{\sqrt{(1-xx)}},$$

unde a simplicissimis ad majores exponentis m valores progrediendo obtinebimus:

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2} \qquad \int \frac{x\,\partial x}{\sqrt{(1-xx)}} = 1$$

$$\int \frac{x^2\,\partial x}{\sqrt{(1-xx)}} = \frac{1}{2}\cdot\frac{\pi}{2} \qquad \int \frac{x^3\,\partial x}{\sqrt{(1-xx)}} = \frac{2}{3}$$

$$\int \frac{x^4\,\partial x}{\sqrt{(1-xx)}} = \frac{1.3}{2.4}\cdot\frac{\pi}{2} \qquad \int \frac{x^5\,\partial x}{\sqrt{(1-xx)}} = \frac{2.4}{3.5}$$

$$\int \frac{x^6\,\partial x}{\sqrt{(1-xx)}} = \frac{1.3.5}{2.4.6}\cdot\frac{\pi}{2} \qquad \int \frac{x^7\,\partial x}{\sqrt{(1-xx)}} = \frac{2.4.6}{3.5.7}$$

$$\int \frac{x^8\,\partial x}{\sqrt{(1-xx)}} = \frac{1.3.5.7}{2.4.6.8}\cdot\frac{\pi}{2} \qquad \int \frac{x^9\,\partial x}{\sqrt{(1-xx)}} = \frac{2.4.6.8}{3.5.7.9}$$

$$\vdots \qquad\qquad \vdots$$

$$\int \frac{x^{2n}\,\partial x}{\sqrt{(1-xx)}} = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots\;2n}\cdot\frac{\pi}{2} \qquad \int \frac{x^{2n+1}\,\partial x}{\sqrt{(1-xx)}} = \frac{2.4.6\ldots\;2n}{3.5.7\ldots(2n+1)}.$$

Corollarium 1.

331. Integrale ergo $\int \frac{x^m\,\partial x}{\sqrt{(1-xx)}}$, posito $x = 1$, algebraice exprimitur casibus, quibus exponens m est numerus integer impar; casibus autem, quibus est par, quadraturam circuli involvit; semper enim π designat peripheriam circuli, cujus diameter $= 1$.

Corollarium 2.

332. Si binas postremas formulas in se multiplicemus prodit:

$$\int \frac{x^{2n}\,\partial x}{\sqrt{(1-xx)}}\cdot\int \frac{x^{2n+1}\,\partial x}{\sqrt{(1-xx)}} = \frac{1}{2n+1}\cdot\frac{\pi}{2}$$

posito scilicet $x = 1$, quam veram esse patet, etiamsi n non sit numerus integer.

Corollarium 3.

333. Haec ergo aequalitas subsistet, si ponamus $x = z^{\nu}$, iisdem conditionibus, quia sumto $x = 0$ vel $x = 1$ fit $z = 0$ vel $z = 1$. Erit ergo

$$\nu\nu \int \frac{z^{2n\nu+\nu-1}\,\partial z}{\sqrt{(1-z^{2\nu})}} \cdot \int \frac{z^{2n\nu+2\nu-1}\,\partial z}{\sqrt{(1-z^{2\nu})}} = \frac{1}{2n+1} \cdot \frac{\pi}{2},$$

et posito $2n\nu + \nu - 1 = \mu$, fiet posito $z = 1$

$$\int \frac{z^{\mu}\,\partial z}{\sqrt{(1-z^{2\nu})}} \cdot \int \frac{z^{\pi+\nu}\,\partial z}{\sqrt{(1-z^{2\nu})}} = \frac{1}{\nu(\mu+1)} \cdot \frac{\pi}{2}.$$

Scholion 1.

334. Quod tale productum binorum integralium exhiberi queat, eo magis est notatu dignum, quod aequalitas haec subsistit, etiamsi neutra formula neque algebraice neque per π exhiberi queat. Veluti si $\nu = 2$ et $\mu = 0$, fit

$$\int \frac{\partial z}{\sqrt{(1-z^4)}} \cdot \int \frac{zz\,\partial z}{\sqrt{(1-z^4)}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4},$$

similique modo:

$\nu = 3,\ \mu = 0$ fit $\int \frac{\partial z}{\sqrt{(1-z^6)}} \cdot \int \frac{z^3\,\partial z}{\sqrt{(1-z^6)}} = \frac{1}{3} \cdot \frac{\pi}{2} = \frac{\pi}{6}$;

$\nu = 3,\ \mu = 1$ fit $\int \frac{z\,\partial z}{\sqrt{(1-z^6)}} \cdot \int \frac{z^4\,\partial z}{\sqrt{(1-z^6)}} = \frac{1}{6} \cdot \frac{\pi}{2} = \frac{\pi}{12}$;

$\nu = 4,\ \mu = 0$ fit $\int \frac{\partial z}{\sqrt{(1-z^8)}} \cdot \int \frac{z^4\,\partial z}{\sqrt{(1-z^8)}} = \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{8}$;

$\nu = 4,\ \mu = 2$ fit $\int \frac{zz\,\partial z}{\sqrt{(1-z^8)}} \cdot \int \frac{z^6\,\partial z}{\sqrt{(1-z^8)}} = \frac{1}{12} \cdot \frac{\pi}{2} = \frac{\pi}{24}$;

$\nu = 5,\ \mu = 0$ fit $\int \frac{\partial z}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^5\,\partial z}{\sqrt{(1-z^{10})}} = \frac{1}{5} \cdot \frac{\pi}{2} = \frac{\pi}{10}$;

$\nu = 5,\ \mu = 1$ fit $\int \frac{z\,\partial z}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^6\,\partial z}{\sqrt{(1-z^{10})}} = \frac{1}{20} \cdot \frac{\pi}{2} = \frac{\pi}{40}$;

$\nu = 5,\ \mu = 2$ fit $\int \frac{zz\partial z}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^7 \partial z}{\sqrt{(1-z^{10})}} = \frac{1}{15} \cdot \frac{\pi}{2} = \frac{\pi}{30}$;

$\nu = 5,\ \mu = 3$ fit $\int \frac{z^3\partial z}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^8 \partial z}{\sqrt{(1-z^{10})}} = \frac{1}{20} \cdot \frac{\pi}{2} = \frac{\pi}{40}$;

quae Theoremata sine dubio omni attentione sunt digna.

Scholion 2.

335. Facile hinc etiam colligitur valor integralis $\int \frac{x^m \partial x}{\sqrt{(x-xx)}}$ posito $x = 1$, si enim scribamus $x = zz$, fiet hoc integrale $2\int \frac{z^{2m}\partial z}{\sqrt{(1-zz)}}$; quocirca pro casu $x = 1$ nanciscimur sequentes valores:

$$\int \frac{\partial x}{\sqrt{(x-xx)}} = \pi \qquad \int \frac{x^4 \partial x}{\sqrt{(x-xx)}} = \frac{1.3.5.7}{2.4.6.8}\pi;$$

$$\int \frac{x\partial x}{\sqrt{(x-xx)}} = \tfrac{1}{2}\cdot\pi \qquad \int \frac{x^5 \partial x}{\sqrt{(x-xx)}} = \frac{1.3.5.7.9}{2.4.6.8.10}\pi;$$

$$\int \frac{x^2\partial x}{\sqrt{(x-xx)}} = \frac{1.3}{2.4}\cdot\pi \qquad \vdots$$

$$\int \frac{x^3\partial x}{\sqrt{(x-xx)}} = \frac{1.3.5}{2.4.6}\cdot\pi \qquad \int \frac{x^m \partial x}{\sqrt{(x-xx)}} = \frac{1.3.5\ldots(2m-1)}{2.4.6\ldots\ 2m}\pi.$$

Hinc ergo integralium hujusmodi formulas involventium, quae magis sunt complicata, valores, quos posito $x = 1$ recipiunt, per series succincte exprimi possunt, quem usum aliquot exemplis declaremus.

Exemplum 1.

336. *Valorem integralis* $\int \frac{\partial x}{\sqrt{(1-x^4)}}$, *posito* $x = 1$, *per seriem exhibere.*

Integrali detur haec forma $\int \frac{\partial x}{\sqrt{(1-xx)}} \cdot (1+xx)^{-\frac{1}{2}}$, ut habeamus

$$\int \frac{\partial x}{\sqrt{(1-x^4)}} = \int \frac{\partial x}{\sqrt{(1-xx)}} \left(1 - \tfrac{1}{2} xx + \tfrac{1.3}{2.4} x^4 - \tfrac{1.3.5}{2.4.6} x^6 + \tfrac{1.3.5.7}{2.4.6.8} x^8 - \text{etc.}\right)$$

singulis ergo terminis pro casu $x = 1$ integratis, orietur

$$\int \frac{\partial x}{\sqrt{(1-x^4)}} = \frac{\pi}{2}\left(1 - \tfrac{1}{4} + \tfrac{1.9}{4.16} - \tfrac{1.9.25}{4.16.36} + \tfrac{1.9.25.49}{4.16.36.64} - \text{etc.}\right)$$

Corollarium.

337. Simili modo pro eodem casu $x = 1$ reperitur:

$$\int \frac{x\partial x}{\sqrt{(1-x^4)}} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.} = \frac{\pi}{4}$$

$$\int \frac{xx\partial x}{\sqrt{(1-x^4)}} = \frac{\pi}{2}\left(\tfrac{1}{2} - \tfrac{1^2.3}{2^2.4} + \tfrac{1^2.3^2.5}{2^2.4^2.6} - \tfrac{1^2.3^2.5^2.7}{2^2.4^2.6^2.8} + \text{etc.}\right)$$

$$\int \frac{x^3\partial x}{\sqrt{(1-x^4)}} = \tfrac{2}{3} - \tfrac{4}{3.5} + \tfrac{6}{5.7} - \tfrac{8}{7.9} + \tfrac{10}{9.11} - \text{etc.}$$

est autem $\int \frac{x^3\partial x}{\sqrt{(1-x^4)}} = \frac{1}{2} - \frac{1}{2}\sqrt{(1-x^4)}$, ideoque $= \frac{1}{2}$, posito $x = 1$, unde haec postrema series $= \frac{1}{2}$, quod manifestum est.

Exemplum 2.

338. *Valorem integralis* $\int \partial x \sqrt{\frac{1+axx}{1-xx}}$, *casu* $x = 1$, *per seriem exhibere.*

Cum sit

$$\sqrt{(1+axx)} = 1 + \tfrac{1}{2} axx - \tfrac{1.1}{2.4} a^2 x^4 + \tfrac{1.1.3}{2.4.6} a^3 x^6 - \text{etc.}$$

erit per $\int \frac{\partial x}{\sqrt{(1-xx)}}$ multiplicando et integrando

$$\int \partial x \sqrt{\frac{1+axx}{1-xx}} = \frac{\pi}{2}\left(1 + \tfrac{1.1}{2.2} a - \tfrac{1.1.1.3}{2.2.4.4} a^2 + \tfrac{1.1.1.3.3.5}{2.2.4.4.6.6} a^3 - \text{etc.}\right)$$

unde peripheriam ellipsis cognoscere licet.

Exemplum 3.

339. *Valorem integralis* $\int \frac{\partial x}{\sqrt{x(1-xx)}}$, *casu* $x = 1$; *per seriem exhibere.*

Repraesentetur haec formula ita $\int \frac{\partial x (1+x)^{-\frac{1}{2}}}{\sqrt{(x-xx)}}$, ut sit $=$

$$\int \frac{\partial x}{\sqrt{(x-xx)}} \left(1 - \tfrac{1}{2} x + \tfrac{1.3}{2.4} x^2 - \tfrac{1.3.5}{2.4.6} x^3 + \text{etc.}\right);$$

unde series haec obtinetur:

$$\frac{\partial x}{\sqrt{x(1-xx)}} = \pi\left(1 - \tfrac{1}{4} + \tfrac{1.9}{4.16} - \tfrac{1.9.25}{4.16.36} + \text{etc.}\right)$$

quae ab exemplo primo haud differt: quod non mirum, cum posito $x = zz$, haec formula ad illam reducatur.

Problema 39.

340. Valorem integralis $\int x^{m-1}\partial x\,(1-xx)^{n-\frac{1}{2}}$, quod posito $x=0$ evanescat, definire casu $x=1$.

Solutio.

Reductiones supra §. 118. datae praebent pro hoc casu

$$\int x^{m-1}\partial x\,(1-xx)^{\frac{\mu}{2}+1} = \frac{x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2}$$
$$+\frac{\mu+2}{m+\mu+2}\int x^{m-1}\partial x\,(1-xx)^{\frac{\mu}{2}}:$$

sumto ergo $\mu = 2n-1$, erit

$$\int x^{m-1}\partial x\,(1-xx)^{n+\frac{1}{2}} = \frac{2n+1}{m+2n+1}\int x^{m-1}\partial x\,(1-xx)^{n-\frac{1}{2}}$$

posito $x=1$. Cum igitur in praecedente problemate valor $\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}}$ sit assignatus, quem brevitatis gratia ponamus $=M$, hinc ad sequentes progrediamur:

$$\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}} = M;$$

$$\int x^{m-1}\partial x\,(1-xx)^{\frac{1}{2}} = \frac{1}{m+1}M;$$

$$\int x^{m-1}\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1\,.\,3}{(m+1)(m+3)}M;$$

$$\int x^{m-1}\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1\,.\,3\,.\,5}{(m+1)(m+3)(m+5)};$$

et in genere

$$\int x^{m-1}\partial x\,(1-xx)^{n-\frac{1}{2}} = \frac{1.\;3.\;5.\;\ldots\;(2n-1)}{(m+1)(m+3)(m+5)\ldots(m+2n-1)}\,\mathrm{M}.$$

Jam duo casus sunt perpendendi, prout $m-1$ est vel numerus par vel impar: si enim

$m-1$ sit par, erit $\mathrm{M} = \frac{1.3.5\ldots\ldots(m-2)}{2.4.6\ldots\ldots(m-1)}\cdot\frac{\pi}{2}$;

$m-1$ sit impar, erit $\mathrm{M} = \frac{2.4.6\ldots\ldots(m-2)}{3.5.7\ldots\ldots(m-1)}$.

Hinc sequentes deducuntur valores:

$\int\partial x\sqrt{(1-xx)} = \frac{\pi}{4}$	$\int x\,\partial x\sqrt{(1-xx)} = \frac{1}{3}$
$\int x^2\partial x\sqrt{(1-xx)} = \frac{1}{4}\cdot\frac{\pi}{4}$	$\int x^3\partial x\sqrt{(1-xx)} = \frac{1}{3}\cdot\frac{2}{5}$
$\int x^4\partial x\sqrt{(1-xx)} = \frac{1.3}{4.6}\cdot\frac{\pi}{4}$	$\int x^5\partial x\sqrt{(1-xx)} = \frac{1}{3}\cdot\frac{2.4}{5.7}$
$\int x^6\partial x\sqrt{(1-xx)} = \frac{1.3.5}{4.6.8}\cdot\frac{\pi}{4}$	$\int x^7\partial x\sqrt{(1-xx)} = \frac{1}{3}\cdot\frac{2.4.6}{5.7.9}$

$\int\partial x\,(1-xx)^{\frac{3}{2}} = \frac{3\pi}{16}$	$\int x\,\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1}{5}$
$\int x^2\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1}{6}\cdot\frac{3\pi}{16}$	$\int x^3\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1}{5}\cdot\frac{2}{7}$
$\int x^4\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1.3}{6.8}\cdot\frac{3\pi}{16}$	$\int x^5\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1}{5}\cdot\frac{2.4}{7.9}$
$\int x^6\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1.3.5}{6.8.10}\cdot\frac{3\pi}{16}$	$\int x^7\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1}{5}\cdot\frac{2.4.6}{7.9.11}$

$\int\partial x\,(1-xx)^{\frac{5}{2}} = \frac{5\pi}{32}$	$\int x\,\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1}{7}$
$\int x^2\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1}{8}\cdot\frac{5\pi}{32}$	$\int x^3\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1}{7}\cdot\frac{2}{9}$
$\int x^4\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1.3}{8.10}\cdot\frac{5\pi}{32}$	$\int x^5\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1}{7}\cdot\frac{2.4}{9.11}$
$\int x^6\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1.3.5}{8.10.12}\cdot\frac{5\pi}{32}$	$\int x^7\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1}{7}\cdot\frac{2.4.6}{9.11.13}$

etc.

Problema 40.

341. Valores integralium $\int \frac{x^m \partial x}{\sqrt[3]{(1-x^3)}}$ et $\int \frac{x^m \partial x}{\sqrt[3]{(1-x^3)^2}}$, posito $x = 1$, assignare.

Solutio.

Ponamus pro casibus simplicissimis:

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = \text{A}; \int \frac{x \partial x}{\sqrt[3]{(1-x^3)}} = \text{B}; \int \frac{x x \partial x}{\sqrt[3]{(1-x^3)}} = \text{C}$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = \text{B}'; \int \frac{x \partial x}{\sqrt[3]{(1-x^3)^2}} = \text{B}'; \int \frac{x x \partial x}{\sqrt[3]{(1-x^3)^2}} = \text{C}'$$

et ex reductione prima §. 118. posito $a = 1$ et $b = -1$, pro casu $x = 1$ habemus

$$\int x^{m+n-1} \partial x (1-x^n)^{\frac{\mu}{\nu}} = \frac{m\nu}{m\nu + n\mu + n\nu} \int x^{m-1} \partial x (1-x^n)^{\frac{\mu}{\nu}},$$

ergo pro priori ubi $n = 3$, $\nu = 3$ et $\mu = -1$,

$$\int x^{m+2} \partial x (1-x^3)^{-\frac{1}{3}} = \frac{m}{m+2} \int x^{m-1} \partial x (1-x^3)^{-\frac{1}{3}}$$

et pro posteriori, ubi $n = 3$, $\nu = 3$ et $\mu = -2$

$$\int x^{m+2} \partial x (1-x^3)^{-\frac{2}{3}} = \frac{m}{m+1} \int x^{m-1} \partial x (1-x^3)^{-\frac{2}{3}}$$

hinc obtinemus pro forma priori:

$\int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = \text{A}$	$\int \frac{x \partial x}{\sqrt[3]{(1-x^3)}} = \text{B}$	$\int \frac{x x \partial x}{\sqrt[3]{(1-x^3)}} = \text{C}$
$\int \frac{x^3 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{1}{3}\text{A}$	$\int \frac{x^4 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{2}{4}\text{B}$	$\int \frac{x^5 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{3}{5}\text{C}$
$\int \frac{x^6 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{1.4}{3.6}\text{A}$	$\int \frac{x^7 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{2.5}{4.7}\text{B}$	$\int \frac{x^8 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{3.6}{5.8}\text{C}$
$\int \frac{x^9 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{1.4.7}{3.6.9}\text{A}$	$\int \frac{x^{10} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{2.5.8}{4.7.10}\text{B}$	$\int \frac{x^{11} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{3.6.9}{5.8.11}\text{C}$
$\int \frac{x^{12} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{1.4.7.10}{3.6.9.12}\text{A}$	$\int \frac{x^{13} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{2.5.8.11}{4.7.10.13}\text{B}$	$\int \frac{x^{14} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{3.6.9.12}{5.8.11.14}\text{C}$

etc.

at pro forma posteriori

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = A' \qquad \int \frac{x\partial x}{\sqrt[3]{(1-x^3)^2}} = B' \qquad \int \frac{xx\partial x}{\sqrt[3]{(1-x^3)^2}} = C'$$

$$\int \frac{x^3\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{1}{2} A' \qquad \int \frac{x^4\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{2}{3} B' \qquad \int \frac{x^5\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{3}{4} C'$$

$$\int \frac{x^6\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{1.4}{2.5} A' \qquad \int \frac{x^7\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{2.5}{3.6} B' \qquad \int \frac{x^8\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{3.6}{4.7} C'$$

$$\int \frac{x^9\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{1.4.7}{2.5.8} A' \qquad \int \frac{x^{10}\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{2.5.8}{3.6.9} B' \qquad \int \frac{x^{11}\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{3.6.9}{4.7.10} C'$$

$$\int \frac{x^{12}\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{1.4.7.10}{2.5.8.11} A' \qquad \int \frac{x^{13}\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{2.5.8.11}{3.6.9.12} B' \qquad \int \frac{x^{14}\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{3.6.9.12}{4.7.10.13} C'$$

unde concludimus fore generaliter:

$$\int \frac{x^{3n}\partial x}{\sqrt[3]{(1-x^3)}} = \frac{1.4.\;7\ldots(3n-2)}{3.6.\;9\ldots\;3n} A \qquad \int \frac{x^{3n}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1.4.\;7\ldots(3n-2)}{2.5.\;8\ldots(3n-1)} A'$$

$$\int \frac{x^{3n+1}\partial x}{\sqrt[3]{(1-x^3)}} = \frac{2.5.\;8\ldots(3n-1)}{4.7.10\ldots(3n+1)} B \qquad \int \frac{x^{3n+1}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2.5.\;8\ldots(3n-1)}{3.6.\;9\ldots\;3n} B'$$

$$\int \frac{x^{3n+2}\partial x}{\sqrt[3]{(1-x^3)}} = \frac{3.6.\;9\ldots\;3n}{5.8.11\ldots(3n+2)} C \qquad \int \frac{x^{3n+2}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{3.6.\;9\ldots\;3n}{4.7.10\ldots(3n+1)} C'$$

notandum autem est esse $C = \frac{1}{2}$ et $C' = 1$.

Corollarium 1.

342. Hae formulae variis modis combinari possunt, ut egregia Theoremata inde oriantur; erit scilicet;

$$\int \frac{x^{3n}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{3n+2}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{A\,C'}{3n+1} = \frac{1}{3n+1} \int \frac{\partial x}{\sqrt[3]{(1-x^3)}}$$

$$\int \frac{x^{3n+1}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{3n}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{A\,B}{3n+1} = \frac{1}{3n+1} \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x\partial x}{\sqrt[3]{(1-x^3)}}$$

$$\int \frac{x^{3n+2}\,\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{3n+1}\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2B'C}{3n+2} = \frac{1}{3n+2}\int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}}.$$

Corollarium 2.

343. Quia nunc ratio exponentium ad ternarium non amplius in computum ingreditur, erit generaliter:

$$\int \frac{x^{\lambda-1}\,\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\lambda+1}\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda}\int \frac{\partial x}{\sqrt[3]{(1-x^3)}}$$

$$\int \frac{x^{\lambda}\,\partial x}{\sqrt[3]{(\ -x^3)}} \cdot \int \frac{x^{\lambda-1}\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda}\int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}}$$

$$\int \frac{x^{\lambda}\,\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\lambda-1}\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda}\int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}}.$$

quare ex binis postremis consequimur

$$\int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = \int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}}$$

Corollarium 3.

344. Ponatur $x = z^n$ et $\lambda n = m$, et nostra Theoremata sequentes induent formas:

$$\int \frac{z^{m-1}\,\partial z}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{m+2n-1}\,\partial z}{\sqrt[3]{(1-z^{3n})^2}} = \frac{1}{m}\int \frac{z^{n-1}\,\partial z}{\sqrt[3]{(1-z^{3n})}}$$

$$\int \frac{z^{m+n-1}\,\partial z}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{m-1}\,\partial z}{\sqrt[3]{(1-z^{3n})^2}} = \frac{n}{m}\int \frac{z^{2n-1}\,\partial z}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{n-1}\,\partial z}{\sqrt[3]{(1-z^{3n})^2}}$$

$$= \frac{1}{m}\int \frac{z^{2n-1}\,\partial z}{\sqrt[3]{(1-z^{3n})^2}}.$$

Problema 41.

345. Dato integrali $\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$, assignare integrale hujus formulae $\int \frac{x^{m+\lambda n-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$, posito $x=1$.

Solutio.

Ut integrale sit finitum necesse est, ut m et k sint numeri positivi. Cum igitur per reductionem generalem sit

$$\int x^{m+n-1}\partial x\,(1-x^n)^{\frac{\mu}{\nu}} = \frac{m\nu}{m\nu+n(\mu+\nu)}\int x^{m-1}\partial x\,(1-x^n)^{\frac{\mu}{\nu}};$$

ponatur $\nu=n$ et $\mu=k-n$, ut sit $\mu+\nu=k$, erit

$$\int \frac{x^{m+n-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \frac{m}{m+k}\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}.$$

Ponatur ergo hujus formulae valor, quia datur, $=A$, haecque reductio repetita continuo dabit, posito brevitatis gratia P pro $(1-x^n)^{\frac{n-k}{n}}$,

$$\int \frac{x^{m-1}\partial x}{P} = A$$

$$\int \frac{x^{m+n-1}\partial x}{P} = \frac{m}{m+k}A$$

$$\int \frac{x^{m+2n-1}\partial x}{P} = \frac{m(m+n)}{(m+k)(m+n+k)}A$$

$$\int \frac{x^{m+3n-1}\partial x}{P} = \frac{m(m+n)(m+2n)}{(m+k)(m+n+k)(m+2n+k)}A$$

$$\int \frac{x^{m+\alpha n-1}\partial x}{P} = \frac{m(m+n)(m+2n)\;\ldots\;[m+(\alpha-1)n]}{(m+k)(m+n+k)(m+2n+k)\ldots[m+(\alpha-1)n+k]}A.$$

Corollarium 1.

346. Si simili modo alia formula sit $\int \frac{x^{p-1}\,\partial x}{(1-x^n)^{\frac{n-q}{n}}} = \mathrm{B}$, posito $x = 1$, at brevitatis gratia scribatur Q pro $(1-x^n)^{\frac{n-q}{n}}$, habebimus

$$\int \frac{x^{p+\alpha n-1}\,\partial x}{\mathrm{Q}} = \frac{p(p+n)(p+2n) \quad \ldots . \quad [p+(\alpha-1)n]}{(p+q)(p+n+q)(p+2n+q)\ldots .[p+(\alpha-1)n+q]}\,\mathrm{B},$$

quae totidem atque illa continet factores.

Corollarium 2.

347. Statuatur nunc $p = m + k$, ut posterior numerator aequalis fiat priori denominatori, et productum harum duarum formularum est

$$\frac{m(m+n)(m+2n) \ldots \ldots \ldots [m+(\alpha-1)n]}{(m+k+q)(m+n+k+q)(m+2n+k+q)\ldots .[m+(\alpha-1)n+k+q]}\,\mathrm{AB};$$

fiat porro $m + k + q = m + n$, seu $q = n - k$, erit hoc productum $= \frac{m}{m+\alpha n}\,\mathrm{AB}$; ideoque

$$\int \frac{x^{m+\alpha n-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}} \cdot \int \frac{x^{m+k+\alpha n-1}\partial x}{(1-x^n)^{\frac{k}{n}}} = \frac{m}{m+\alpha n}\int \frac{x^{m-1}\,\partial x}{(1+x^n)^{\frac{n-k}{n}}} \cdot \int \frac{x^{m+k-1}\partial x}{(1-x^n)^{\frac{k}{n}}},$$

quod est Theorema omni attentione dignum; cum hic non amplius opus sit, ut α sit numerus integer.

Corollarium 3.

348. Quare loco $m + \alpha n$ scribamus μ, erit:

$$\mu \int \frac{x^{\mu-1}\,\partial x}{(1-x^n)^{\frac{n-k}{n}}} \cdot \int \frac{x^{\mu+k-1}\partial x}{(1-x^n)^{\frac{k}{n}}} = m \int \frac{x^{m-1}\,\partial x}{(1-x^n)^{\frac{n-k}{n}}} \cdot \int \frac{x^{m+k-1}\,\partial x}{(1-x^n)^{\frac{k}{n}}}.$$

Hinc si sumamus $m+k=n$, seu $m=n-k$, ob $\int \frac{x^{n-1}\partial x}{(1-x^n)^{\frac{k}{n}}}$ $= \frac{1-(1-x^n)^{\frac{n-k}{n}}}{n-k} = \frac{1}{n-k}$, posito $x=1$, erit

$$\int \frac{x^{\mu-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int\frac{x^{\mu+k-1}\partial x}{(1-x^n)^{\frac{k}{n}}} = \frac{1}{\mu}\int\frac{x^{n-k-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \frac{\pi}{\mu n \sin.\frac{k\pi}{n}}.$$

Ac posito $x=z^\nu$, tum vero $\mu n = p$, $\nu n = q$, et $k=\lambda n$, habebitur:

$$\int\frac{z^{p-1}\partial z}{(1-z^q)^{1-\lambda}}\cdot\int\frac{z^{p+\lambda q-1}\partial z}{(1-z)^\lambda} = \frac{n}{p}\int\frac{z^{(1-\lambda)q-1}\partial z}{(1-z^q)^{1-\lambda}}.$$

Scholion 1.

349. Theoremata particuliaria, quae hinc consequuntur, ita se habebunt:

I. $n=2$; $k=1$; $\int\frac{x^{\mu-1}\partial x}{\sqrt{(1-xx)}}\cdot\int\frac{x^{\mu}\partial x}{\sqrt{(1-xx)}} = \frac{1}{\mu}\int\frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2\mu}$

II. $n=3$; $k=1$; $\int\frac{x^{\mu-1}\partial x}{\sqrt[3]{(1-x^3)^2}}\cdot\int\frac{x^{\mu}\partial x}{\sqrt[3]{(1-x^3)}} = \frac{1}{\mu}\int\frac{x\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2\pi}{3\mu\sqrt{3}}$

$n=3$; $k=2$; $\int\frac{x^{\mu-1}\partial x}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{\mu+1}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\mu}\int\frac{\partial x}{\sqrt[3]{(1-x^3)}} = \frac{2\pi}{3\mu\sqrt{3}}$

III. $n=4$; $k=1$; $\int\frac{x^{\mu-1}\partial x}{\sqrt[4]{(1-x^4)^3}}\cdot\int\frac{x^{\mu}\partial x}{\sqrt[4]{(1-x^4)}} = \frac{1}{\mu}\int\frac{xx\partial x}{\sqrt[4]{(1-x^4)^3}} = \frac{\pi}{2\mu\sqrt{2}}$

$n=4$; $k=2$; $\int\frac{x^{\mu-1}\partial x}{\sqrt{(1-x^4)}}\cdot\int\frac{x^{\mu+1}\partial x}{\sqrt{(1-x^4)}} = \frac{1}{\mu}\int\frac{x\partial x}{\sqrt{(1-x^4)}} = \frac{\pi}{4\mu}$

$n=4$; $k=3$; $\int\frac{x^{\mu-1}\partial x}{\sqrt[4]{(1-x^4)}}\cdot\int\frac{x^{\mu+2}\partial x}{\sqrt[4]{(1-x^4)^3}} = \frac{1}{\mu}\int\frac{\partial x}{\sqrt[4]{(1-x^4)}} = \frac{\pi}{2\mu\sqrt{2}}$

etc.

Ubi notandum est, formulam $\int \frac{x^{n-k-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$ ad rationalitatem reduci posse. Ponatur enim $\frac{x^n}{1-x^n} = z^n$, seu $x^n = \frac{z^n}{1+z^n}$, unde $\frac{\partial x}{x} = \frac{\partial z}{z(1+z^n)}$. Quare cum formula nostra sit $= \int \left(\frac{x^n}{1-x^n}\right)^{\frac{n-k}{n}} . \frac{\partial x}{x}$, evadet ea $= \int \frac{z^{n-k-1}\partial z}{1+z^n}$, cujus integrale ita determinari debet, ut evanescat posito $x = 0$ ideoque $z = 0$; tum vero posito $x = 1$, hoc est $z = \infty$ dabit valorem, quo hic utimur. Mox autem ostendemus valorem hujus integralis $\int \frac{z^{n-k-1}\partial z}{1+z^n}$, posito $z = \infty$, ideoque et hujus $\int \frac{x^{n-k-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$ per angulos exprimi posse, quorum valores hic statim apposui. Deinde etiam notari meretur formulae $\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$ haec transformatio oriunda, posito $1 - x^n = z^n$, quae praebet $-\int \frac{z^{k-1}\partial z}{(1-z^n)^{\frac{n-m}{n}}}$ ita integranda, ut evanescat posito $x = 0$ seu $z = 1$, tum vero statui debet $x = 1$ seu $z = 0$. Quod eodem redit, ac si mutato signo haec formula $\int \frac{z^{k-1}\partial z}{(1-z^n)^{\frac{n-m}{n}}}$ ita integretur, ut evanescat, posito $z = 0$, tum vero ponatur $z = 1$. Cum jam nihil impediat quo minus loco z scribamus x, habebimus hoc insigne Theorema:

$$\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}\partial x}{(1-x^n)^{\frac{n-m}{n}}}.$$

ita ut in hujusmodi formula exponentes m et k inter se commutare liceat, pro casu scilicet $x = 1$. Ita pro praecedente formula ad rationalitatem reducibili, ubi $m = n - k$, erit

$$\int \frac{x^{n-k-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}\partial x}{(1-x^n)^{\frac{k}{n}}},$$

unde sequitur etiam fore, posito $z = \infty$,

$$\int \frac{z^{n-k-1}\partial z}{1+z^n} = \int \frac{z^{k-1}\partial z}{1+z^n}.$$

Scholion 2.

350. Hinc etiam formularum magis compositarum integralia pro casu $x = 1$, per series concinnas exprimi possunt. Cum enim in reductione superiori, posito $m + k = \mu$ seu $k = \mu - m$, sit

$$\int \frac{x^{m+n+1}\partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}} = \frac{m}{\mu}\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}},$$

si habeatur hujusmodi formula differentialis

$$\partial y = \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}}(\mathrm{A} + \mathrm{B}x^n + \mathrm{C}x^{2n} + \mathrm{D}x^{3n} + \text{etc.})$$

quam ita integrari oporteat, ut y evanescat posito $x = 0$, ac requiratur valor ipsius y casu $x = 1$, erit si hoc casu fieri ponamus $\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}} = \mathrm{O}$, iste valor $=$

$$\mathrm{O}\left(\mathrm{A} + \frac{m}{\mu}\mathrm{B} + \frac{m(m+n)}{\mu(\mu+n)}\mathrm{C} + \frac{m(m+n)(m+2n)}{\mu(\mu+n)(\mu+2n)}\mathrm{D} + \text{etc.}\right)$$

Vicissim ergo proposita hac serie

$$\mathrm{A} + \frac{m}{\mu}\mathrm{B} + \frac{m(m+n)}{\mu(\mu+n)}\mathrm{C} + \frac{m(m+n)(m+2n)}{\mu(\mu+n)(\mu+2n)}\mathrm{D} + \text{etc.}$$

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ejus summa aequabitur huic formulae integrali

$$\frac{1}{O}\int \frac{x^{m-1}\,\partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}}(A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.})$$

si post integrationem ponatur $x = 1$. Quod si ergo eveniat, ut hujus seriei $A + Bx^n + Cx^{2n} + \text{etc.}$ summa assignari, indeque integratio absolvi queat, obtinebitur summa illius seriei.

Problema 42.

351. Integralis hujus formulae $\frac{x^{m-1}\,\partial x}{1 + x^n}$ ita determinatum ut posito $x = 0$ evanescat, valorem casu $x = \infty$ assignare.

Solutio.

Hujus formulae integrale jam supra §. 77. exhibuimus, et quidem ita determinatum, ut posito $x = 0$ evanescat, quod posito brevitatis gratia $\frac{\pi}{n} = \omega$, ita se habet:

$$-\frac{2}{n}\cos.m\omega\, l\sqrt{(1 - 2x\cos.\omega + xx)} + \frac{2}{n}\sin.m\omega.\text{Arc.tang.}\frac{x\sin.\omega}{1 - x\cos.\omega}$$
$$-\frac{2}{n}\cos.3m\omega\, l\sqrt{(1 - 2x\cos.3\omega + xx)} + \frac{2}{n}\sin.3m\omega.\text{Arc.tang.}\frac{x\sin.3\omega}{1 - x\cos.3\omega}$$
$$-\frac{2}{n}\cos.5m\omega\, l\sqrt{(1 - 2x\cos.5\omega + xx)} + \frac{2}{n}\sin.5m\omega.\text{Arc.tang.}\frac{x\sin.5\omega}{1 - x\cos.5\omega}$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$-\frac{2}{n}\cos.\lambda m\omega\, l\sqrt{(1 - 2x\cos.\lambda\omega + xx)} + \frac{2}{n}\sin.\lambda m\omega.\text{Arc.tang.}\frac{x\sin.\lambda\omega}{1 - x\cos.\lambda\omega}$$

ubi λ denotat maximum numerum imparem exponente n minorem ac si n fuerit ipse numerus impar, insuper accedit pars $\pm\frac{1}{n}l(1+x)$ prout m fuerit vel numerus impar, vel par; illo scilicet casu signum $+$, hoc vero signum $-$ valet. Hic igitur quaeritur istius inte gralis valor, qui prodit posito $x = \infty$. Primo ergo partes logarithmos implicantes expendamus, et quia ob $x = \infty$ est

$l\sqrt{(1 - 2x\cos.\lambda\omega + xx)} = l(x - \cos.\lambda\omega) = lx + l(1 - \frac{\cos.\lambda\omega}{x}) = lx$,

ob $\frac{\cos.\lambda\omega}{x} = 0$; unde partes logarithmicae praebent:

$$-\frac{2lx}{n}(\cos.m\omega + \cos.3m\omega + \cos.5m\omega + \ldots + \cos.\lambda m\omega)$$

$$(\pm\frac{lx}{n}, \text{ si } n \text{ impar}).$$

Ponamus hanc seriem cosinuum

$$\cos.m\omega + \cos.3m\omega + \cos.5m\omega + \ldots + \cos.\lambda m\omega = s,$$

eritque per $2\sin.m\omega$ multiplicando

$$2s\sin.m\omega = \sin.2m\omega + \sin.4m\omega + \sin.6m\omega \ldots + \sin.(\lambda+1)m\omega$$
$$- \sin.2m\omega - \sin.4m\omega - \sin.6m\omega,$$

unde fit $s = \frac{\sin.(\lambda+1)m\omega}{2\sin.m\omega}$. Quare si n sit numerus par, erit $\lambda = n - 1$, sicque partes logarithmicae fiunt

$$-\frac{lx}{n}\cdot\frac{\sin.nm\omega}{\sin.m\omega} = -\frac{lx}{n}\cdot\frac{\sin.m\pi}{\sin.m\omega}, \text{ ob } n\omega = \pi.$$

At propter m numerum integrum, est $\sin.m\pi = 0$, unde hae partes evanescunt. Sin autem sit n numerus impar, est $\lambda = n - 2$, et summa partium logarithmicarum fit

$$-\frac{lx}{n}\cdot\frac{\sin.(n-1)m\omega}{\sin.m\omega} \pm \frac{lx}{n};$$

at $\sin.(n-1)m\omega = \sin.(m\pi - m\omega) = \pm\sin.m\omega$, ubi signum superius valet, si m sit numerus impar, contra vero inferius, quod idem de altera ambiguitate est tenendum, ita ut habeamus $\mp\frac{lx}{n}\cdot\frac{\sin.m\omega}{\sin.m\omega} \pm \frac{lx}{n} = 0$. Perpetuo ergo partes logarithmicae se mutuo tollunt; quod etiam inde est perspicuum, quod alioquin integrale foret infinitum, cum tamen manifesto debeat esse finitum.

Relinquuntur ergo soli anguli, quos in unam summam colligamus; consideretur ergo Arc. tang. $\frac{x\sin.\lambda\omega}{1 - x\cos.\lambda\omega}$, qui arcus casu $x = 0$ evanescit, tum vero casu $x = \frac{1}{\cos.\lambda\Phi}$ fit quadrans, ulterius ergo aucta x quadrantem superabit, donec facto $x = \infty$, ejus tangens fiat $= -\frac{\sin. \lambda\omega}{\cos. \lambda\omega} = -\text{tang.}\, \lambda\omega = \text{tang.}\, (\pi - \lambda\omega)$, ideoque ipse arcus $= \pi - \lambda\omega$, ex quo hi arcus junctim sumti dabunt:

$$\tfrac{2}{n}\, [(\pi - \omega) \sin. m\omega + (\pi - 3\omega) \sin. 3m\omega + (\pi - 5\omega) \sin. 5m\omega + \ldots$$
$$\ldots + (\pi - \lambda\omega) \sin. \lambda m\omega]:$$

unde duas series adipiscimur

$$\tfrac{2\pi}{n} (\sin. m\omega + \sin. 3m\omega + \sin. 5m\omega + \ldots\ldots$$
$$\ldots + \sin. \lambda m\omega) = \tfrac{2\pi}{n}\, p;$$
$$\tfrac{-2\omega}{n} (\sin. m\omega + 3 \sin. 3m\omega + 5 \sin. 5m\omega + \ldots\ldots$$
$$\ldots + \lambda \sin. \lambda m\omega) = \tfrac{-2\omega}{n}\, q;$$

quas seorsim investigemus, ac pro posteriori quidem cum ante habuissemus

$$\cos. m\omega + \cos. 3m\omega + \cos. 5m\omega + \ldots\ldots$$
$$\ldots + \cos. \lambda m\omega = s = \frac{\sin. (\lambda+1) m\omega}{2 \sin. m\omega},$$

si angulum ω ut variabilem spectemus, differentiatio praebet

$$-m\partial\omega (\sin. m\omega + 3 \sin. 3m\omega + 5 \sin. 5m\omega + \ldots + \lambda \sin. \lambda m\omega)$$
$$= \frac{(\lambda+1) m\partial\omega \cos. (\lambda+1) m\omega}{2 \sin. m\omega} - \frac{m\partial\omega \sin. (\lambda+1) m\omega \cos. m\omega}{2 \sin. m\omega^2},$$

ergo

$$-q = \frac{(\lambda+1) \cos. (\lambda+1) m\omega}{2 \sin. m\omega} - \frac{\sin. (\lambda+1) m\omega \cos. m\omega}{2 \sin. m\omega^2}, \text{ seu}$$
$$-q = \frac{\lambda \cos. (\lambda+1) m\omega}{2 \sin. m\omega} - \frac{\sin. \lambda m\omega}{2 \sin. m\omega^2}.$$

Pro altera serie

$$p = \sin. m\omega + \sin. 3m\omega + \sin. 5m\omega + \ldots + \sin. \lambda m\omega,$$

multiplicemus utrinque per $2 \sin. m\omega$, fietque

$$2p \sin. m\omega = 1 - \cos. 2m\omega - \cos. 4m\omega - \cos. 6m\omega - \ldots - \cos. (\lambda+1) m\omega$$
$$+ \cos. 2m\omega + \cos. 4m\omega + \cos. 6m\omega$$

sicque erit $p = \frac{1 - \cos. (\lambda+1) m\omega}{2 \sin. m\omega}$.

Quodsi jam fuerit n numerus par, erit $\lambda = n - 1$, indeque

$$\cos.(\lambda + 1) m\omega = \cos. nm\omega = \cos. m\pi, \text{ et}$$
$$\sin.(\lambda + 1) m\omega = \sin. m\pi = 0, \text{ ergo}$$
$$p = \frac{1 - \cos. m\pi}{2 \sin. m\omega} \text{ et } -q = \frac{n \cos. m\pi}{2 \sin. m\omega};$$

hincque omnes arcus junctim sumti

$$\frac{2\pi}{n} \cdot \frac{(1 - \cos. m\pi)}{2 \sin. m\omega} + \frac{2\omega}{n} \cdot \frac{n \cos. m\pi}{2 \sin. m\omega} = \frac{\pi}{n \sin. m\omega}, \text{ ob } n\omega = \pi.$$

Sit nunc n numerus impar, erit $\lambda = n - 2$, indeque

$$\cos.(\lambda + 1) m\omega = \cos.(m\pi - m\omega), \text{ et}$$
$$\sin.(\lambda + 1) m\omega = \sin.(m\pi - m\omega), \text{ seu}$$
$$\cos.(\lambda + 1) m\omega = \cos. m\pi \cos. m\omega, \text{ et}$$
$$\sin.(\lambda + 1) m\omega = -\cos. m\pi \sin. m\omega, \text{ ergo}$$

$$p = \frac{1 - \cos. m\pi \cos. m\omega}{2 \sin. m\omega} \text{ et } -q = \frac{(n-1) \cos. m\pi \cos. m\omega}{2 \sin. m\omega} + \frac{\cos. m\pi \cos. m\omega}{2 \sin. m\omega}:$$

unde summa omnium angulorum

$$\frac{\pi(1 - \cos. m\pi \cos. m\omega)}{n \sin. m\omega} + \frac{\omega(n-1) \cos. m\pi \cos. m\omega}{n \sin. m\omega} + \frac{\omega \cos. m\pi \cos. m\omega}{n \sin. m\omega},$$

quae ob $n\omega = \pi$ reducitur ad $\frac{\pi}{n \sin. m\omega}$.

Sive ergo exponens n sit positivus sive negativus, posito $x = \infty$ habemus

$$\int \frac{x^{m-1} \partial x}{1 + x^n} = \frac{\pi}{n \sin. m\omega} = \frac{\pi}{n \sin. \frac{m\pi}{n}}.$$

Corollarium 1.

252. Hinc ergo erit formula supra memorata (349)

$$\int \frac{z^{n-k-1} \partial z}{1 + z^n} = \int \frac{z^{k-1} \partial z}{1 + z^n} = \frac{\pi}{n \sin. \frac{(n-k)\pi}{n}} = \frac{\pi}{n \sin. \frac{k\pi}{n}}, \text{ posito } z = \infty.$$

Unde sequitur fore etiam formulam, cui hanc aequari ostendimus:

$$\int \frac{x^{n-k-1}\,\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}\,\partial x}{(1-x^n)^{\frac{k}{n}}} = \frac{\pi}{n \sin. \frac{k\pi}{n}}, \text{ posito } x = 1.$$

Corollarium 2.

353. Percurramus casus simpliciores, pro utroque formularum genere, posito $z = \infty$ et $x = 1$;

$$\int \frac{\partial z}{1+zz} = \int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2 \sin. \frac{1}{2}\pi} = \frac{\pi}{2};$$

$$\int \frac{\partial z}{1+z^3} = \int \frac{z\,\partial z}{1+z^3} = \int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = \int \frac{x\,\partial x}{\sqrt[3]{(1+x^3)^2}}$$

$$= \frac{\pi}{3 \sin. \frac{1}{3}\pi} = \frac{2\pi}{3\sqrt{3}};$$

$$\int \frac{\partial z}{1+z^4} = \int \frac{zz\,\partial z}{1+z^4} = \int \frac{\partial x}{\sqrt[4]{(1-x^4)}} = \int \frac{xx\,\partial x}{\sqrt[4]{(1-x^4)^3}}$$

$$= \frac{\pi}{4 \sin. \frac{1}{4}\pi} = \frac{\pi}{2\sqrt{2}};$$

$$\int \frac{\partial z}{1+z^6} = \int \frac{z^4\,\partial z}{1+z^6} = \int \frac{\partial x}{\sqrt[6]{(1-x^6)}} = \int \frac{x^4\,\partial x}{\sqrt[6]{(1-x^6)^5}}$$

$$= \frac{\pi}{6 \sin. \frac{1}{6}\pi} = \frac{\pi}{3}.$$

Corollarium 3.

354. Cum sit

$$\frac{1}{(1-x^n)^{\frac{k}{n}}} = 1 + \frac{k}{n} x^n + \frac{k(k+n)}{n.2n} x^{2n} + \frac{k(k+n)(k+2n)}{n.2n\ .\ 3n} x^{3n} + \text{etc.}$$

erit per $x^{k-1}\partial x$ multiplicando, tum integrando, ac $x=1$ ponendo

$$\frac{\pi}{n\,\sin.\frac{k\pi}{n}}=\frac{1}{k}+\frac{k}{n(k+n)}+\frac{k(k+n)}{n.2n(k+2n)}+\frac{k(k+n)(k+2n)}{n.2n.3n.(k+3n)}+\text{etc.}$$

et loco k scribendo $n-k$ erit quoque

$$\frac{\pi}{n\,\sin.\frac{k\pi}{n}}=\frac{1}{n-k}+\frac{n-k}{n(2n-k)}+\frac{(n-k)(2n-k)}{n.2n.(3n-k)}+\frac{(n-k)(2n-k)(3n-k)}{n.2n.3n(4n-k)}\text{etc.}$$

Scholion.

355. Pro formulis quantitates transcendentes continentibus supra jam praecipuos valores, quos integralia dum variabili certus quidam valor tribuitur, recipiunt, evolvimus; ita ut non opus sit hujusmodi formulas hic denuo examinare. Hinc autem intelligitur, eos valores integralis $\int X\partial x$ prae reliquis esse notatu dignos, ac plerumque multo succinctius exprimi posse, qui ejusmodi valoribus variabilis x respondent, quibus functio X vel fit infinita vel in nihilum abit. Ita integralia formularum $\int\frac{x^{m-1}\partial x}{(1-x^n)^{\frac{\mu}{\nu}}}$ et $\int\frac{z^{m-1}\partial z}{1+z^n}$, valores prae reliquis memorabiles recipiunt, si fiat $x=1$ et $z=\infty$, ubi illius denominator evanescit, hujus vero fit infinitus. Caeterum omni attentione dignum est, quod hic ostendimus, formulae integralis $\int\frac{z^{m-1}\partial z}{1+z^n}$ valorem casu $z=\infty$ tam concinne exprimi, ut sit $\frac{\pi}{n\,\sin.\frac{m}{n}\pi}$, cujus demonstratio cum per tot ambages sit adstructa, merito suspicionem excitat, eam via multo faciliori confici posse, etiamsi modus nondum perspiciatur. Id quidem manifestum est, hanc demonstrationem ex ratione sinuum angulorum multiplorum peti oportere; et quoniam in Introductione $\sin.\frac{m}{n}\pi$ per productum infinitorum factorum expressi, mox videbimus, inde eandem veritatem

multo facilius deduci posse, etiamsi ne hanc quidem viam pro maxime naturali haberi velim. Sequens autem caput hujusmodi investigationi destinavi, quo valores integralium, quos uti in hoc capite certo quodam casu recipiunt, per producta infinita seu ex innumeris factoribus constantia exprimere docebo; quandoquidem hinc insignia subsidia in Analysin redundant, pluraque alia incrementa inde expectari possunt.