CAPUT VII.

METHODUS GENERALIS

INTEGRALIA QUAECUNQUE PROXIME INVENIENDI.

Problema 36.

297.

Formulae integralis cujuscunque $y = \int X \partial x$ valorem vero proxime indagare.

Solutio.

Cum omnis formula integralis per se sit indeterminata, ea semper ita determinari solet, ut si variabili x certus quidam valor, puta a, tribuatur, ipsum integrale $y = \int X \partial x$ datum valorem, puta b, obtineat. Integratione igitur hoc modo determinata, quaestio huc redit, si variabili x alius quicunque valor ab a diversus tribuatur, valor, quem tum integrale y sit habiturum, definiatur. Tribuamus ergo ipsi x primo valorem parum ab a discrepantem, puta $x = a + \alpha$, ut α sit quantitas valde parva: et quia functio X parum variatur, sive pro x scribatur a sive $a + \alpha$, eam tanquam constantem spectare licebit. Hinc ergo formulae differentialis $X \partial x$ integrale erit $X x +$ Const. $= y$; sed quia posito $x = a$, fieri debet $y = b$, et valor ipsius X quasi manet immutatus, erit $X a +$ Const. $= b$, ideoque Const. $= b - X a$, unde consequimur $y = b + X(x - a)$. Quare si ipsi x valorem $a + \alpha$ tribuamus, habebimus valorem convenientem ipsius y, qui sit $= b + \beta$; ac jam simili modo ex hoc casu definire poterimus y, si ipsi x tribuatur alius valor parum superans $a + \alpha$: posito igitur $a + \alpha$ loco x, valor ipsius X inde ortus denuo pro constante haberi poterit, indeque fiet $y = b + \beta + X(x - a - \alpha)$. Hanc igitur operationem continuare licet quousque lubuerit, cujus ratio quo melius perspiciatur, rem ita repraesentemus:

si $x = a$ fiat $X = A$ et $y = b$
si $x = a'$ . . $X = A'$ . . $y = b' = b + A(a' - a)$
si $x = a''$ . . $X = A''$ . . $y = b'' = b' + A'(a'' - a')$
si $x = a'''$ . . $X = A'''$ . . $y = b''' = b'' + A''(a''' - a'')$
etc.

ubi valores a, a', a'', a''', etc. secundum differentias valde parvas procedere ponuntur. Erit ergo $b' = b + A(a' - a)$, quippe in quam abit formula inventa $y = b + X(x - a)$: fit enim $X = A$, quia ponitur $x = a$, tum vero tribuitur ipsi x valor $= a'$, cui respondet $y = b'$: simili modo erit $b'' = b' + A'(a'' - a')$; tum $b''' = b'' + A''(a''' - a'')$ etc. uti supra posuimus. Restituendo ergo valores praecedentes habebimus:

$$b' = b + A(a' - a)$$
$$b'' = b + A(a' - a) + A'(a'' - a')$$
$$b''' = b + A(a' - a) + A'(a'' - a') + A''(a''' - a'')$$
$$b'''' = b + A(a' - a) + A'(a'' - a') + A''(a''' - a'') + A'''(a'''' - a''')$$
etc.

unde si x quantumvis excedet a, series a', a'', a''', etc. crescendo continuetur ad x, et ultimum aggregatum dabit valorem ipsius y.

Corollarium 1.

298. Si incrementa, quibus x augetur, aequalia statuantur scilicet $= \alpha$, ut sit $a' = a + \alpha$, $a'' = a + 2\alpha$, $a''' = a + 3\alpha$, etc. quibus valoribus pro x substitutis functio X abeat in A', A'';

A''', etc. atque ultimus illorum, puta $a + n\alpha$, sit $= x$, horum vero X, erit

$$y = b + \alpha (A + A' + A'' + A''' \ldots \ldots + X).$$

Corollarium 2.

299. Valor ergo integralis y per summationem seriei A, A', A'' .... X, cujus termini ex formula X formantur, ponendo loco x successive a, $a + \alpha$, $a + 2\alpha \ldots a + n\alpha$; eruitur. Summa enim illius seriei per differentiam α multiplicata et ad b adjecta, dabit valorem ipsius y, qui ipsi $x = a + n\alpha$ respondet.

Corollarium 3.

300. Quo minores statuuntur differentiae, secundum quas valor ipsius x increscat, eo accuratius hoc modo valor ipsius y definitur. Siquidem termini seriei A, A', A'', etc. inde etiam secundum parvas differentias progrediantur, nisi enim hoc eveniat, illa determinatio nimis erit incerta.

Corollarium 4.

301. Haec ergo approximatio ex doctrina serierum ita explicatur:

Ex indicibus a, a', a'', a''' . . . . x formetur
series A, A', A'', A''' X

cujus ergo terminus generalis X ex formula differentiali $\partial y = X \partial x$ datur. Tum in hac serie sit terminus ultimum praecedens ′X, respondens indici ′x; hincque nova formetur series

$$A(a' - a);\ A'(a'' - a');\ A''(a''' - a'') \ldots \ldots {}'X(x - {}'x),$$

cujus summa si ponatur $= S$, erit integrale $y = \int X \partial x = b + S$, proxime.

Scholion 1.

302. Hoc modo integratio vulgo explicari solet, ut dicatur, esse summatio omnium valorum formulae differentialis $X\partial x$, si variabili x successive omnes valores a dato quodam a usque ad x tribuantur, qui secundum differentiam ∂x procedunt, hanc differentiam autem infinite parvam accipi oportere. Similis igitur haec ratio integrationem repraesentandi est illi, qua in Geometria lineae ut aggregata infinitorum punctorum concipi solent, quae idea, quemadmodum si rite explicetur, admitti potest, ita etiam illi integrationis explicatio tolerari potest, dummodo ad vera principia, uti hic fecimus, revocetur, ut omni cavillationi occurratur. Ex methodo igitur exposita utique patet, integrationem per summationem vero proxime obtineri posse, neque vero exacte expediri, nisi differentiae infinite parvae, hoc est nullae, statuantur. Atque ex hoc fonte tam nomen integrationis, quae etiam summatio vocari solet, quam signum integralis $\int$ est natum, quae, re bene explicata, omnino retineri possunt.

Scholion 2.

303. Si pro singulis intervallis, in quae saltum ab a ad x distinximus, quantitates A, A', A'', A''', etc. revera essent constantes, integrale $\int X\partial x$ accurate impetraremus. Eatenus ergo error inest, quatenus pro singulis illis intervallis istae quantitates non sunt constantes. Ac pro primo quidem intervallo, quo variabilis x a termino a ad a' procedit, A est valor ipsius X termino a conveniens, alteri autem termino a' respondet A'; unde quatenus non est $A' = A$, eatenus error irrepit: cum igitur in istius intervalli initio sit $X = A$, in fine autem $X = A'$, conveniret potius medium quoddam inter A et A' assumi, id quod in correctione hujus methodi mox tradenda observabitur. Interim hic notasse juvabit, pari jure pro quovis intervallo valorem tam finalem quam initialem capi posse, ubi simul hoc perspicitur, si altero modo in excessu pecce-

tur, altero plerumque in defectu errari. Ex quo hinc binas expressiones eruere licet, quarum altera valorem ipsius y nimis magnum, altera nimis parvum sit praebitura, ita ut illae quasi limites veri valoris ipsius y constituant. Quemadmodum ergo rem repraesentavimus §. 301. valor ipsius $y = \int X \partial x$ intra hos duos limites continebitur

$$b + A\,(a'-a) + A'\,(a''-a') + A''\,(a'''-a'') \ldots\ldots + {}'X\,(x-{}'x) \text{ et}$$
$$b + A'(a'-a) + A''(a''-a') + A'''(a'''-a'') \ldots\ldots + X\,(x-{}'x)$$

quibus cognitis, ad veritatem propius accedere licet.

Scholion 3.

304. Jam notavimus intervalla illa, per quae x successive increscere assumimus, ideo valde parva statui debere, ut valores respondentes A, A′, A″, etc. parum a se invicem discrepent: atque hinc potissimum judicari oportet, utrum illa intervalla $a' - a$, $a' - a'$, $a''' - a''$, etc. inter se aequalia an inaequalia capi conveniat. Ubi enim valor ipsius X, mutando x, parum mutatur, ibi intervalla, per quae x procedit, tuto majora constitui possunt; ubi autem evenit, ut ipsi x levi mutatione inducta, functio X vehementer varietur, ibi intervalla minima accipi debent. Veluti si sit $X = \frac{1}{\sqrt{(1-xx)}}$, perspicuum est, ubi x proxime ad unitatem accedit, quantumvis parvum intervallum, per quod x augeatur, accipiatur, functionem X maximam mutationem pati posse, quia tandem sumto $x = 1$, ea adeo in infinitum excrescit. His igitur casibus ista approximatione pro eo saltem intervallo, in cujus altero termino X fit infinita, uti non licet; sed huic incommodo facile remedium affertur, dum formula ope idoneae substitutionis in aliam transformatur, vel dum pro hoc saltem intervallo peculiaris integratio instituitur. Veluti si proposita sit formula $\frac{x\partial x}{\sqrt{(1-x^3)}}$, pro intervallo ab $x = 1 - \omega$ ad $x = 1$, illa methodo integrale non reperitur: at posito $x = 1 - z$, quia termini ipsius z sunt 0 et ω, erit z

quantitas minima, unde formula erit $\frac{\partial z(1-z)}{\sqrt{(3z-3z^3+z^3)}} = \frac{\partial z}{\sqrt{3z}}$, cujus integrale $\frac{2\sqrt{z}}{\sqrt{3}}$ pro intervallo illo praebet partem integralis $\frac{2\sqrt{\omega}}{\sqrt{3}}$. Quod artificium in omnibus hujusmodi casibus adhiberi potest; ipsam autem methodum descriptam aliquot exemplis illustrari opus est.

Exemplum 1.

305. *Integrale* $y = \int x^n \partial x$ *ita sumtum, ut evanescat posito* $x = 0$, *proxime exhibere.*

Hic est $a = 0$ et $b = 0$, tum $X = x^n$, jam valores ipsius x a 0 crescant per communem differentiam α, ut sint

indices $0, \alpha, 2\alpha, 3\alpha, 4\alpha \ldots x$
series $0, \alpha^n, 2^n\alpha^n, 3^n\alpha^n, 4^n\alpha^n \ldots x^n$

et terminus ultimum praecedens est $(x-\alpha)^n$, quare integralis $y = \int x^n \partial x = \frac{1}{n+1} x^{n+1}$ limites sunt

$$\alpha\,[0 + \alpha^n + 2^n\alpha^n + 3^n\alpha^n + \ldots + (x-\alpha)^n] \text{ et}$$
$$\alpha\,(\alpha^n + 2^n\alpha^n + 3^n\alpha^n + \ldots\ldots + x^n)$$

qui eo erunt arctiores, quo minus intervallum α accipiatur. Ita si $\alpha = 1$. erunt limites:

$$0 + 1 + 2^n + 3^n + 4^n \ldots\ldots + (x-1)^n \text{ et}$$
$$1 + 2^n + 3^n + 4^n + \ldots\ldots + x^n,$$

si sumatur $\alpha = \frac{1}{2}$, erunt limites

$$\frac{1}{2^{n+1}}\,[0 + 1 + 2^n + 3^n + 4^n + \ldots + (2x-1)^n] \text{ et}$$

$$\frac{1}{2^{n+1}}\,[1 + 2^n + 3^n + 4^n + \ldots\ldots + (2x)^n];$$

ac si in genere sit $\alpha = \frac{1}{m}$, erunt limites:

$$\frac{1}{m^{n+1}}\,[0 + 1 + 2^n + 3^n + 4^n + \ldots + (mx-1)^n] \text{ et}$$

$$\frac{1}{m^{n+1}}\,[1 + 2^n + 3^n + 4^n + \ldots\ldots + (mx)^n];$$

quorum hic illum superat excessu $\frac{x^n}{m}$; unde patet si numerus m sumatur infinitus, utrumque limitem verum integralis $y = \frac{1}{n+1} x^{n+1}$ esse praebiturum valorem.

Corollarium 1.

306. Seriei ergo $1 + 2^n + 3^n + 4^n + \ldots + (mx)^n$ summa eo propius ad $\frac{1}{n+1}(mx)^{n+1}$ accedit, quo major capiatur numerus m; quare posito $mx = z$, hujus progressionis

$$1 + 2^n + 3^n + 4^n + \ldots\ldots + z^n,$$

summa eo propius ad $\frac{1}{n+1} z^{n+1}$ accedit, quo major fuerit numerus z.

Corollarium 2.

307. Ex priore autem limite posito $mx = z$, eadem quantitas $\frac{1}{n+1} z^{n+1}$ proxime exhibet summam hujus seriei

$$0 + 1 + 2^n + 3^n + 4^n + \ldots\ldots + (z - 1)^n,$$

unde medium sumendo erit accuratius:

$$1 + 2^n + 3^n + 4^n \ldots\ldots + (z - 1)^n + \tfrac{1}{2} z^n = \frac{1}{n+1} z^{n+1}$$

seu addendo utrinque $\frac{1}{2} z^n$, habebimus

$$1 + 2^n + 3^n + 4^n \ldots\ldots + z^n = \frac{1}{n+1} z^{n+1} + \tfrac{1}{2} z^n,$$

proxime quod congruit cum iis, quae de vera hujus progressionis summa sunt cognita.

Exemplum 2.

308. *Integrale* $y = \int \frac{\partial x}{x^n}$ *ita sumtum, ut evanescat posito* $x = 1$, *proxime exhibere.*

Erit ergo $a = 1$ et $b = 0$, unde si ab a ad x intervallum progressionis statuatur $= \alpha$, erunt indices

$$a,\ a + \alpha,\ a + 2\alpha,\ a + 3\alpha,\ \ldots\ldots\ x,$$

et termini seriei

$$\frac{1}{a^n},\ \frac{1}{(a+\alpha)^n},\ \frac{1}{(a+2\alpha)^n},\ \frac{1}{(a+3\alpha)^n},\ \ldots\ldots\ \frac{1}{x^n},\ = \mathrm{X},$$

ubi terminus ultimum praecedens est $\frac{1}{(x-\alpha)^n} = \mathrm{X}'$. Cum nunc nostrum integrale sit $y = \frac{1}{n-1} - \frac{1}{(n-1)\,x^{n-1}}$, ejus valor intra hos limites continebitur:

$$\alpha\,[1 + \frac{1}{(1+\alpha)^n} + \frac{1}{(1+2\alpha)^n} + \frac{1}{(1+3\alpha)^n} + \ldots + \frac{1}{(x-\alpha)^n}] \text{ et}$$

$$\alpha\left[\frac{1}{(1+\alpha)^n} + \frac{1}{(1+2\alpha)^n} + \frac{1}{(1+3\alpha)^n} + \ldots\ldots + \frac{1}{x^n}\right].$$

Quare posito $\alpha = \frac{1}{m}$, erunt hi limites:

$$m^{n-1}\left[\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \ldots + \frac{1}{(mx-1)^n}\right] \text{ et}$$

$$m^{n-1}\left[\frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \frac{1}{(m+4)^n} + \ldots + \frac{1}{(mx)^n}\right]$$

qui, quo major accipiatur numerus m, eo propius ad valorem integralis $\frac{1}{n-1} - \frac{1}{(n-1)\,x^{n-1}}$ accedunt. Notandum autem est, casu $n = 1$ integrale fore $= lx$.

Corollarium 1.

309. Quodsi ponamus $mx = m + z$; ut sit $x = \frac{m+z}{m}$, prodibunt hae progressiones:

$$m^{n-1}\left(\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\ldots\ldots+\frac{1}{(m+z-1)^n}\right) \text{ et}$$

$$m^{n-1}\left(\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\ldots\ldots+\frac{1}{(m+z)^n}\right)$$

quarum summa alterius major est, alterius minor quam

$$\frac{1}{n-1}-\frac{m^{n-1}}{(n-1)(m+z)^{n-1}}=\frac{(m+z)^{n-1}-m^{n-1}}{(n-1)(m+z)^{n-1}}:$$

casu autem $n=1$, haec expressio abit in $l\left(1+\frac{z}{m}\right)$.

Corollarium 2.

310. Cum prior progressio major sit quam posterior, erit

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\ldots\ldots$$
$$\ldots\ldots+\frac{1}{(m+z-1)^n}>\frac{(m+z)^{n-1}-m^{n-1}}{(n-1)\,m^{n-1}(m+z)^{n-1}}$$

$$\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\ldots\ldots$$
$$\ldots\ldots+\frac{1}{(m+z)^n}<\frac{(m+z)^{n-1}-m^{n-1}}{(n-1)\,m^{n-1}(m+z)^{n-1}}:$$

addatur hic utrinque $\frac{1}{m^n}$, ibi vero $\frac{1}{(m+z)^n}$, et sumatur medium arithmeticum, erit exactius

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\ldots\ldots+\frac{1}{(m+z)^n}$$
$$=\frac{(2m+n-1)(m+z)^n-(2z+2m-n+1)\,m^n}{2(n-1)\,m^n(m+z)^n},$$

quae expressio casu $n=1$, abit in $l\left(1+\frac{z}{m}\right)+\frac{1}{2m}+\frac{1}{2(m+z)}$.

Corollarium 3.

311. Ponatur $z = mv$, et habebimus sequentis seriei summam proxime expressam:

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \ldots\ldots + \frac{1}{m^n(1+v)^n}$$
$$= \frac{(2m+n-1)(1+v)^n - 2m(1+v) + n - 1}{2(n-1)m^n(1+v)^n},$$

et casu $n = 1$

$$\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{m+mv} = l(1+v) + \frac{2+v}{2m(1+v)}:$$

unde si $v = 1$, erit proxime

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \ldots + \frac{1}{2^n m^n}$$
$$= \frac{2^n(2m+n-1) - 4m + n - 1}{2^{n+1}(n-1)m^n}, \text{ et}$$

$$\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{2m} = l2 + \frac{3}{4m}.$$

Corollarium 4.

312. Hinc nascitur regula, logarithmos quantumvis magnorum numerorum proxime assignandi, dum series vulgares tantum pro numeris parum ab unitate differentibus, valent. Scribamus enim u pro $1+v$, et habebimus

$$lu = \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{mu} - \frac{1-u}{2mu}.$$

unde lu eo accuratius definitur, quo major sumatur numerus m.

Exemplum 3.

313. *Integrale* $y = \int \frac{c\,\partial x}{cc+xx}$ *ita sumtum, ut evanescat posito* $x = 0$; *proxime exprimere.*

Hoc integrale ut novimus, est $y = \text{Ang. tang.}\,\frac{x}{c}$, ad quem valorem proxime exhibendum, est $a = 0$, et $b = 0$; si ergo valor ipsius x ab 0 per differentiam constantem α crescere statuatur, ob $X = \frac{c}{cc+xx}$, erunt ejus valores

pro indicibus $0 \quad \alpha \quad 2\alpha \quad \ldots\ldots \quad x;$

$$\frac{1}{c};\ \frac{c}{cc+\alpha\alpha};\ \frac{c}{cc+4\alpha\alpha};\ \ldots\ldots\ \frac{c}{cc+xx};$$

cujus terminus ultimum praecedens est $X = \frac{c}{cc+(x-\alpha)^2}$.

Quare integralis nostri $y = \text{Ang. tang.} \frac{x}{c}$ valor proxime est

$$\alpha\left(\frac{1}{c} + \frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \ldots\ldots + \frac{c}{cc+(x-\alpha)^2}\right);$$

alter vero proxime minor, quia hic est nimis magnus, est

$$\alpha\left(\frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \frac{c}{cc+9\alpha\alpha} + \ldots\ldots + \frac{c}{cc+xx}\right).$$

Inter quos si medium capiatur, ibi $\alpha \cdot \frac{1}{c}$, hic vero $\alpha \cdot \frac{c}{cc+xx}$ adjiciendo, propius erit

$$\alpha\left(\frac{c}{cc} + \frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \frac{c}{cc+9\alpha\alpha} + \ldots\ldots + \frac{c}{cc+xx}\right)$$
$$= \text{Ang. tang.} \frac{x}{c} + \frac{\alpha}{2}\left(\frac{1}{c} + \frac{c}{cc+xx}\right)$$
$$= \text{Ang. tang.} \frac{x}{c} + \frac{\alpha(2cc+xx)}{2c(cc+xx)}.$$

Pro hoc ergo angulo valorem proxime verum habemus

$$\text{Ang tang.} \frac{x}{c} = \alpha c\left(\frac{1}{cc} + \frac{1}{cc+\alpha\alpha} + \frac{1}{cc+4\alpha\alpha} + \ldots + \frac{1}{cc+xx}\right)$$
$$- \frac{\alpha(2cc+xx)}{2c(cc+xx)},$$

qui eo minus a veritate discrepabit, quo minor fuerit α numerus ratione ipsius c. Quodsi ergo pro c numerum valde magnum sumamus, pro α unitatem accipere licet; unde posito $x = cv$, erit

$$\text{Ang. tang. } v = c\left(\frac{1}{cc} + \frac{1}{cc+1} + \frac{1}{cc+4} + \frac{1}{cc+9} + \ldots + \frac{1}{cc+ccvv}\right)$$
$$- \frac{(2+vv)}{2c(1+vv)},$$

idque eo exactius, quo major capiatur numerus c.

Corollarium 1

314. Si ponamus $c = 1$, quo casu error insignis esse debet, fiet

$$\text{Ang.tang.}\,v = 1 + \frac{1}{1+1} + \frac{1}{1+4} + \frac{1}{1+9} + \ldots + \frac{1}{1+vv} - \frac{(2+vv)}{2(1+vv)}.$$

Sit $v = 1$, erit Ang. tang. $1 = \frac{\pi}{4} = 1 + \frac{1}{2} - \frac{3}{4} = \frac{3}{4}$, hincque $\pi = 3$, quod non multum abhorret a vero; si ponamus $c = 2$, prodit

$$\text{Ang.tang.}\,v = 2\left(\frac{1}{4} + \frac{1}{4+1} + \frac{1}{4+4} + \frac{1}{4+9} + \ldots + \frac{1}{4+4vv}\right) - \frac{(2+vv)}{4(1+vv)},$$

unde si $v = 1$, colligitur

$$\text{Ang. tang. } 1 = \frac{\pi}{4} = 2\left(\frac{1}{4} + \frac{1}{4+1} + \frac{1}{4+4}\right) - \frac{3}{8} = \frac{23}{20} - \frac{3}{8} = \frac{31}{40},$$

sicque $\pi = \frac{31}{10} = 3,1$, propius accedens.

Corollarium 2.

315. Sit $c = 6$, eritque

$$\text{Ang.tang.}\,v = 6\left(\frac{1}{36} + \frac{1}{36+1} + \frac{1}{36+4} + \ldots + \frac{1}{36+36vv}\right) - \frac{(2+vv)}{12(1+vv)},$$

unde si $v = \frac{1}{2}$ et $v = \frac{1}{3}$, oritur:

$$\text{Ang. tang. } \tfrac{1}{2} = 6\left(\frac{1}{36} + \frac{1}{36+1} + \frac{1}{36+4} + \frac{1}{36+9}\right) - \frac{8}{22},$$

$$\text{Ang. tang. } \tfrac{1}{3} = 6\left(\frac{1}{36} + \frac{1}{36+1} + \frac{1}{36+4}\right) - \frac{19}{120}.$$

At est Ang. tang $\frac{1}{2}$ + Ang. tang. $\frac{1}{3}$ = Ang. tang. $1 = \frac{\pi}{4}$. Ergo

$$\frac{\pi}{4} = 12\left(\frac{1}{36} + \frac{1}{37} + \frac{1}{40}\right) + \frac{2}{15} - \frac{57}{120} = \frac{1063}{1110} - \frac{7}{40} = \frac{695}{888},$$

seu $\pi = \frac{695}{222} = 3,1306$.

Corollarium 3.

316. Sin autem ibi statim ponamus $v = 1$, erit

$$\frac{\pi}{4} = 6\left(\frac{1}{36} + \frac{1}{37} + \frac{1}{40} + \frac{1}{45} + \frac{1}{52} + \frac{1}{61} + \frac{1}{72}\right) - \frac{1}{8},$$

unde fit $\pi = 3,13696$ multo propius veritati; plurium scilicet terminorum additio propius ad veritatem perducit.

Problema 37.

317. Methodum ad integralium valores appropinquandi ante expositam, perfectiorem reddere, ut minus a veritate aberretur.

Solutio.

Sit $y = \int X \partial x$ formula integralis proposita, cujus valorem jam constet esse $y = b$, si ponatur $x = a$, sive is fit datus per ipsam integrationis conditionem, sive jam per aliquot operationes inde derivatus; ac tribuamus jam ipsi x valorem parum superantem illum a, cui respondet $y = b$, tum vero fiat $X = A$, si ponatur $x = a$. In superiori autem methodo assumsimus, dum x parum supra a excrescit, manere X constantem $= A$, ideoque fore $\int X \partial x = A(x - a)$. At quatenus X non est constans, eatenus non est $\int X \partial x = X(x - a)$, sed revera habetur $\int X \partial x = X(x - a) - \int (x - a) \partial X$. Ponamus igitur $\partial X = P \partial x$, eritque $\int (x - a) \partial X = \int P (x - a) \partial x$, et si jam $P = \frac{\partial X}{\partial x}$, quamdiu x non multum a excedit, ut constantem spectemus, habebimus $\int P (x - a) \partial x = \frac{1}{2} P (x - a)^2$ sicque fiet $y = \int X \partial x = b + X (x - a) - \frac{1}{2} P (x - a)^2$; qui valor jam propius ad veritatem accedit, etsi pro X et P ii valores capiantur, quos induunt vel posito $x = a$, vel posito $x = a + \alpha$, majore scilicet valore, ad quem hac operatione x crescere statuimus: ex quo hinc prout vel $x = a$ vel $x = a + \alpha$ ponimus, geminos limites obtinebimus, inter quos veritas subsistit. Simili autem modo ulterius progredi poterimus: cum enim P non sit constans, erit $\int P (x - a) \partial x = \frac{1}{2} P (x - a)^2 - \frac{1}{2} \int (x - a)^2 \partial P$, unde si statuamus $\partial P = Q \partial x$, erit $\int (x - a)^2 \partial P = \int Q (x - a)^2 \partial x = \frac{1}{3} Q (x - a)^3$, si quidem Q ut quantitatem constantem spectemus, ita ut sit

$$y = \int X \partial x = b + X (x - a) - \frac{1}{2} P (x - a)^2 + \frac{1}{2} \cdot \frac{1}{3} Q (x - a)^3.$$

Eadem ergo methodo si ulterius procedamus, ponendo

$$X = \frac{\partial y}{\partial x};\ P = \frac{\partial X}{\partial x};\ Q = \frac{\partial P}{\partial x};\ R = \frac{\partial Q}{\partial x};\ S = \frac{\partial R}{\partial x};\ \text{etc.}$$

inveniemus

$$y = b + X (x - a) - \frac{1}{2} P (x - a)^2 + \frac{1}{2.3} Q (x - a)^3$$
$$- \frac{1}{2.3.4} R (x - a)^4 + \frac{1}{2.3.4.5} S (x - a)^5 - \text{etc.}$$

quae series vehementer convergit, si modo x non multum superet a, atque adeo si in infinitum continuetur, verum valorem ipsius y exhibebit, siquidem in functionibus X, P, Q, R, etc. valor extremus $x = a + \alpha$ substituatur. Nisi autem eam seriem in infinitum extendere velimus, praestabit per intervalla procedere tribuendo ipsi x successive valores a, a', a'', a''', a'''', etc. ac tum pro singulis valores litteris X, P, Q, R, S, etc. convenientes quaeri oportet, qui sint, ut sequuntur:

$$\begin{array}{lllllll}
\text{si fuerit } x = a, & a', & a'', & a''', & a^{\text{IV}}, & a^{\text{V}}, & \text{etc.} \\
\text{fiat } \quad \mathrm{X} = \mathrm{A}, & \mathrm{A}', & \mathrm{A}'', & \mathrm{A}''', & \mathrm{A}^{\text{IV}}, & \mathrm{A}^{\text{V}}, & \text{etc.} \\
\frac{\partial \mathrm{X}}{\partial x} = \mathrm{P} = \mathrm{B}, & \mathrm{B}'', & \mathrm{B}', & \mathrm{B}''', & \mathrm{B}^{\text{IV}}, & \mathrm{B}^{\text{V}}, & \text{etc.} \\
\frac{\partial \mathrm{P}}{\partial x} = \mathrm{Q} = \mathrm{C}, & \mathrm{C}', & \mathrm{C}'', & \mathrm{C}''', & \mathrm{C}^{\text{IV}}, & \mathrm{C}^{\text{V}}, & \text{etc.} \\
\frac{\partial \mathrm{Q}}{\partial x} = \mathrm{R} = \mathrm{D}, & \mathrm{D}', & \mathrm{D}'', & \mathrm{D}''', & \mathrm{D}^{\text{IV}}, & \mathrm{D}^{\text{V}}, & \text{etc.} \\
& & & \text{etc.} & & &
\end{array}$$

tum vero sit

$$y = b,\ b',\ b'',\ b''',\ b^{\text{IV}},\ b^{\text{V}},\ \text{etc.}$$

quibus constitutis erit, ut ex antecedentibus colligere licet:

$$b' = b + \mathrm{A}'(a' - a) - \tfrac{1}{2}\mathrm{B}'(a' - a)^2 + \tfrac{1}{6}\mathrm{C}'(a' - a)^3 - \tfrac{1}{24}\mathrm{D}'\cdot(a' - a)^4 + \text{etc.}$$

$$b' = b' + \mathrm{A}''(a'' - a') - \tfrac{1}{2}\mathrm{B}''(a'' - a')^2 + \tfrac{1}{6}\mathrm{C}''(a'' - a')^3 - \tfrac{1}{24}\mathrm{D}''(a'' - a')^4 + \text{etc.}$$

$$b''' = b'' + \mathrm{A}'''(a''' - a'') - \tfrac{1}{2}\mathrm{B}'''(a''' - a'')^2 + \tfrac{1}{6}\mathrm{C}'''(a''' - a'')^3 - \tfrac{1}{24}\mathrm{D}'''(a''' - a'')^4 + \text{etc.}$$

$$b^{\text{IV}} = b''' + \mathrm{A}^{\text{IV}}(a^{\text{IV}} - a''') - \tfrac{1}{2}\mathrm{B}^{\text{IV}}(a^{\text{IV}} - a''')^2 + \tfrac{1}{6}\mathrm{C}^{\text{IV}}(a^{\text{IV}} - a''')^3 - \tfrac{1}{24}\mathrm{D}^{\text{IV}}(a^{\text{IV}} - a''')^4 + \text{etc.}$$

etc.

quae expressiones eousque continuentur, donec pro valore ipsius x quantumvis ab initiali a discrepante, valor ipsius y obtineatur.

Corollarium 1.

318. Haec igitur approximandi methodus eo utitur Theoremate, cujus veritas jam in calculo differentiali est demonstrata, quod si y ejusmodi fuerit functio ipsius x, quae posito $x = a$, fiat $= b$, ac statuatur

$$\frac{\partial y}{\partial x} = X, \quad \frac{\partial X}{\partial x} = P, \quad \frac{\partial P}{\partial x} = Q, \quad \frac{\partial Q}{\partial x} = R, \text{ etc.}$$

fore generaliter:

$$y = b + (x - a) - \tfrac{1}{2} P (x - a)^2 + \tfrac{1}{6} Q (x - a)^3 - \tfrac{1}{24} R (x - a)^4 + \tfrac{1}{120} S (x - a)^5 - \text{etc.}$$

Corollarium 2.

319. Si hanc seriem in infinitum continuare vellemus, non opus esset, valorem ipsius x parum tantum ab a diversum assumere. Verum quo ista series magis convergens reddatur, expedit saltum ab a ad x in intervalla dispesci, et pro singulis operationem hic descriptam institui.

Corollarium 3.

320. Si valores ipsius x ab a per differentias constantes $= \alpha$ crescere faciamus, sitque ultimus $a + n\alpha = x$, ita ut

si fuerit $x =$	a,	$a + \alpha$,	$a + 2\alpha$,	$a + 3\alpha$,	 x
fiat $X =$	A,	A',	A'',	A''',	 X
$\frac{\partial X}{\partial x} = P =$	B,	B',	B'',	B''',	. . . . P
$\frac{\partial P}{\partial x} = Q =$	C,	C',	C'',	C''',	. . . . Q
$\frac{\partial Q}{\partial x} = R =$	D,	D',	D'',	D''',	. . . . R

etc.

indeque $y = b, \quad b', \quad b'', \quad b''', \quad \ldots\ldots y,$

erit pro valore $x = x$ omnes series colligendo:

$$\begin{aligned} y = b + & \alpha \,(A + A'' + A''' + \ldots\ldots + X) \\ & - \tfrac{1}{2}\alpha^2 (B' + B'' + B'' + \ldots\ldots + P) \\ & + \tfrac{1}{6}\alpha^3 (C' + C' + C'' + \ldots\ldots + Q) \\ & - \tfrac{1}{24}\alpha^4 (D' + D' + D'' + \ldots\ldots + R) \\ & \text{etc.} \end{aligned}$$

Scholion 1.

324. Demonstratio theorematis Corollario 1. memorati, cui haec methodus approximandi innititur, ex natura differentialium ita instruitur: Sit y functio ipsius x, quae posito $x = a$, fiat $y = b$; et quaeramus valorem ipsius y, si x utcunque excedat a: incipiamus a valore ipsius maximo, qui est x, et per differentialia descendamus; atque ex differentialibus patet:

si fuerit x	fore y
$x - \partial x$	$y - \partial y + \partial\partial y - \partial^3 y + \partial^4 y -$ etc.
$x - 2\partial x$	$y - 2\partial y + 3\partial\partial y - 4\partial^3 y + 5\partial^4 y -$ etc.
$x - 3\partial x$	$y - 3\partial y + 6\partial\partial y - 10\partial^3 y + 15\partial^4 y -$ etc.
.	.
.	.
.	.
.	.
$x - n\partial x$	$y - n\partial y + \frac{n(n+1)}{1.2}\partial\partial y - \frac{n(n+1)(n+2)}{1.2.3}\partial^3 y + \frac{n(n+1)(n+2)(n+3)}{1.2.3.4}\partial^4 y -$ etc.

Ponamus nunc $x - n\partial x = a$, erit $n = \frac{x-a}{\partial x}$, ideoque numerus infinitus; tum vero valor pro y resultans per hypothesin esse debet $= b$, quamobrem habebimus

$$b = y - \frac{(x-a)\partial y}{\partial x} + \frac{(x-a)^2\partial\partial y}{1.2\partial x^2} - \frac{(x-a)^3\partial^3 y}{1.2.3\partial x^3} + \frac{(x-a)^4\partial^4 y}{1.2.3.4\partial x^4} - \text{etc.}$$

Quod si jam statuamus

$$\frac{\partial y}{\partial x} = X, \ \frac{\partial X}{\partial x} = P, \ \frac{\partial P}{\partial x} = Q, \ \frac{\partial Q}{\partial x} = R, \text{ etc.}$$

reperimus ut ante:

$$y = b + X(x-a) - \tfrac{1}{2}P(x-a)^2 + \tfrac{1}{6}Q(x-a)^3 - \tfrac{1}{24}R(x-a)^4 + \text{etc.}$$

Unde patet, si x quam minime superet a, sufficere statui $y = b + X(x-a)$, quod est fundamentum approximationis primum propositae, illius scilicet limitis, quo X ex valore majore ipsius x definitur.

Scholion 2:

322. Quemadmodum hoc ratiocinium nobis alterum tantum limitem supra assignatum patefecit, ita ad alterum limitem hoc ratiocinium nos manuducet. Scilicet, uti ante ab x ad a descendimus, ita nunc ab a ad x ascendamus.

si abeat a	tum b abibit in
in $a + \partial a$	$b + \partial b$
$a + 2\partial a$	$b + 2\partial b + \partial\partial b$
$a + 3\partial a$	$b + 3\partial b + 3\partial\partial b + \partial^3 b$
.	..
.	..
.	..
$a + n\partial a$	$b + n\partial b + \frac{n(n-1)}{1.\,2}\partial\partial b + \frac{n(n-1)(n-2)}{1.\,2.\,3}\partial^3 b + \text{etc.}$

Sit jam $a + n\partial a = x$, seu $n = \frac{x-a}{\partial a}$, et valor ipsius b fiet $= y$. Sint autem A, B, C, D, etc. valores superiorum functionum X, P, Q, R, etc. si loco x scribatur a, eritque pro praesenti casu $A = \frac{\partial b}{\partial a}$; $B = \frac{\partial\partial b}{\partial a^2}$; $C = \frac{\partial^3 b}{\partial a^3}$; etc. Quocirca habebimus

$$y = b + A(x-a) + \tfrac{1}{2}B(x-a)^2 + \tfrac{1}{6}C(x-a)^3 + \tfrac{1}{24}D(x-a)^4 + \text{etc.}$$

quae series superiori praeter signa omnino est similis; ac si x parum excedat a, ut $b + A(x - a)$ satis exacte valorem ipsius y indicet, hinc alter limes supra assignatus nascitur. Quodsi autem progressum ab a ad x, ut supra §. 320. in intervalla aequalia secundum differentiam α dispescamus, et termini in singulis seriebus ultimos praecedentes notentur per $'X$, $'P$, $'Q$, $'R$, etc. habebimus pro y quasi alterum limitem

$$\begin{aligned} y = b &+ \alpha\,(A + A' + A'' + \ldots\ldots + {}'X) \\ &+ \tfrac{1}{2}\alpha^2\,(B + B' + B'' + \ldots\ldots + {}'P) \\ &+ \tfrac{1}{6}\alpha^3\,(C + C' + C'' + \ldots\ldots + {}'Q) \\ &+ \tfrac{1}{24}\alpha^4\,(D + D' + D'' + \ldots\ldots + {}'R) \\ &\text{etc.} \end{aligned}$$

ita ut etiam in hac methodo emendata binos habebimus limites, inter quos verus valor ipsius y contineatur. Propius ergo valorem assequemur, si inter hos limites medium arithmeticum capiamus; unde prodibit

$$\begin{aligned} y = b &+ \alpha\,(A + A' + A'' + \ldots\ldots + X) - \tfrac{1}{2}\alpha\,(A + X) + \tfrac{1}{4}\alpha^2\,(B - P) \\ &+ \tfrac{1}{6}\alpha^3(C + C' + C'' + \ldots\ldots + Q) - \tfrac{1}{12}\alpha^3(C + Q) + \tfrac{1}{48}\alpha^4\,(D - R) \\ &+ \tfrac{1}{120}\alpha^5(E + E' + E'' + \ldots\ldots + S) - \tfrac{1}{240}\alpha^5(E + S) + \tfrac{1}{1440}\alpha^6\,(F - T) \\ &\text{etc.} \end{aligned}$$

Atque hinc superiores approximationes tantum addendo membrum $\frac{1}{4}\alpha^2\,(B - P)$, non mediocriter corrigentur.

Exemplum 1.

323. *Logarithmum cujusvis numeri* x *proxime exprimere.*

Hic igitur est $y = \int \frac{\partial x}{x}$, quod integrale ita capitur, ut evanescat posito $x = 1$: erit ergo $a = 1$, $b = 0$ et $X = \frac{1}{x}$, Sumamus jam, ab unitate ad x per intervalla $= \alpha$ ascendi, et cum sit $P = \frac{\partial X}{\partial x} = -\frac{1}{xx}$; $Q = \frac{\partial P}{\partial x} = \frac{2}{x^3}$; $R = \frac{\partial Q}{\partial x} = -\frac{6}{x^4}$; pro indicibus

**

$$\begin{array}{llllll}
x = & 1; & 1+\alpha; & 1+2\alpha; & 1+3\alpha; \;\ldots\ldots & x, \text{ erit} \\
X = & 1; & \frac{1}{1+\alpha}; & \frac{1}{1+2\alpha}; & \frac{1}{1+3\alpha}; \;\ldots\ldots & \frac{1}{x} \\
P = & -1; & \frac{1}{(1+\alpha)^2}; & \frac{1}{(1+2\alpha)^2}; & \frac{1}{(1+3\alpha)^2}; \;\ldots\ldots & -\frac{1}{xx} \\
Q = & 2; & \frac{2}{(1+\alpha)^3}; & \frac{2}{(1+2\alpha)^3}; & \frac{2}{(1+3\alpha)^3}; \;\ldots\ldots & +\frac{2}{x^3} \\
R = & -6; & \frac{6}{(1+\alpha)^4}; & \frac{6}{(1+2\alpha)^4}; & \frac{6}{(1+3\alpha)^4}; \;\ldots\ldots & -\frac{6}{x^4}
\end{array}$$

etc.

unde adipiscimur

$$lx = \alpha\left[1 + \frac{1}{1+\alpha} + \frac{1}{1+2\alpha} + \frac{1}{1+3\alpha} + \ldots\ldots + \frac{1}{x}\right]$$

$$- \tfrac{1}{2}\alpha\left(1+\frac{1}{x}\right) - \tfrac{1}{4}\alpha\alpha\left(1-\frac{1}{xx}\right)$$

$$+ \tfrac{1}{3}\alpha^3\left[1 + \frac{1}{(1+\alpha)^3} + \frac{1}{(1+2\alpha)^3} + \frac{1}{(1+3\alpha)^3} + \ldots\ldots + \frac{1}{x^3}\right]$$

$$- \tfrac{1}{6}\alpha^3\left(1+\frac{1}{x^3}\right) - \tfrac{1}{8}\alpha^4\left(1-\frac{1}{x^4}\right)$$

$$+ \tfrac{1}{5}\alpha^5\left[1 + \frac{1}{(1+\alpha)^5} + \frac{1}{(1+2\alpha)^5} + \frac{1}{(1+3\alpha)^5} + \ldots\ldots + \frac{1}{x^5}\right]$$

$$- \tfrac{1}{10}\alpha^5\left(1+\frac{1}{x^5}\right) - \tfrac{1}{12}\alpha^6\left(1-\frac{1}{x^6}\right)$$

etc.

Quare si sumamus $\alpha = \frac{1}{m}$, erit

$$lx = \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{mx}$$

$$- \frac{(x+1)}{2mx} - \frac{(xx-1)}{4mmxx}$$

$$+\tfrac{1}{3}\Big[\frac{1}{m^3}+\frac{1}{(m+1)^3}+\frac{1}{(m+2)^3}+\ldots\ldots+\frac{1}{(mx)^3}\Big]$$
$$-\frac{(x^3+1)}{6m^3x^3}-\frac{(x^4-1)}{8m^4x^4}$$
$$+\tfrac{1}{5}\Big[\frac{1}{m^5}+\frac{1}{(m+1)^5}+\frac{1}{(m+2)^5}+\ldots\ldots+\frac{1}{(mx)^5}\Big]$$
$$-\frac{(x^5+1)}{10m^5x^5}-\frac{(x^6-1)}{12m^6x^6}$$
etc.

Corollarium.

324. Si hae progressiones in infinitum continuentur, erit postremarum partium summa:

$$=-\tfrac{1}{2}\,l\,\frac{m}{m-1}-\tfrac{1}{2}\,l\,\frac{mx+1}{mx}=-\tfrac{1}{2}\,l\,\frac{mx+1}{(m-1)x},$$

primarum vero $=\frac{1}{2}\,l\,\frac{m+1}{m-1}$: unde cum sit

$$lx+\tfrac{1}{2}\,l\,\frac{mx+1}{(m-1)x}+\tfrac{1}{2}\,l\,\frac{m-1}{m+1}=\tfrac{1}{2}\,l\,\frac{x(mx+1)}{m+1},$$

erit

$$l\,\frac{x(mx+1)}{m+1}=2\,\Big(\frac{1}{m+1}+\frac{1}{m+2}+\frac{1}{m+3}+\ldots\ldots+\frac{1}{mx}\Big)$$
$$+\tfrac{2}{3}\,\Big(\frac{1}{(m+1)^3}+\frac{1}{(m+2)^3}+\frac{1}{(m+3)^3}+\ldots\ldots+\frac{1}{m^3x^3}\Big)$$
$$+\tfrac{2}{5}\,\Big(\frac{1}{(m+1)^5}+\frac{1}{(m+2)^5}+\frac{1}{(m+3)^5}+\ldots\ldots+\frac{1}{m^5x^5}\Big)$$
etc.

quae expressio adeo, si in infinitum continuetur, verum valorem log. $\frac{x(mx+1)}{m+1}$ praebet

Exemplum 2.

325. *Arcum circuli cujus tangens est* $=\frac{x}{c}$ *hac methodo proxime exprimere.*

Quaestio igitur est de integrali $y = \int \frac{c\,\partial x}{cc + xx}$, quod posito $x = 0$ evanescit; eritque $a = 0$, et $b = 0$, tum vero

$$X = \frac{c}{cc + xx}; \quad P = \frac{\partial X}{\partial x} = \frac{-2cx}{(cc + xx)^2}; \quad Q = \frac{\partial P}{\partial x} = \frac{-2c(cc - 3xx)}{(cc + xx)^3};$$

$$R = \frac{\partial Q}{\partial x} = \frac{6cx(3cc - 4xx)}{(cc + xx)^4}; \quad S = \frac{\partial R}{\partial x} = \frac{6c(3c^4 - 33ccxx + 20x^4)}{(cc + xx)^5}; \text{ etc.}$$

quae formae in infinitum continuatae dant

$$y = \frac{cx}{cc + xx} + \frac{cx^3}{(cc + xx)^2} - \frac{cx^3(cc - 3xx)}{3(cc + xx)^3} - \frac{cx^5(3cc - 4xx)}{4(cc + xx)^4}$$
$$+ \frac{cx^5(3c^4 - 33ccxx + 20x^4)}{20(cc + xx)^5} + \text{etc.}$$

Verum si x per intervalla $= 1$, ut sit $\alpha = 1$, crescere ponamus, erit

$$A = \frac{c}{cc}; \quad B = 0; \quad C = \frac{-2c^3}{c^6}; \quad D = 0;$$
$$A' = \frac{c}{cc + 1}; \quad B' = \frac{-2c}{(cc + 1)^2}; \quad C' = \frac{-2c(cc - 3)}{(cc + 1)^3}; \quad D' = \frac{6c(3cc - 4)}{(cc + 1)^4};$$
$$A'' = \frac{c}{cc + 4}; \quad B'' = \frac{-4c}{(cc + 4)^2}; \quad C'' = \frac{-2c(cc - 12)}{(cc + 4)^3}; \quad D'' = \frac{12c(3cc - 16)}{(cc + 4)^4};$$
$$A''' = \frac{c}{cc + 9}; \quad B''' = \frac{-6c}{(cc + 9)^2}; \quad C''' = \frac{-2c(cc - 27)}{(cc + 9)^3}; \quad D''' = \frac{18c(3cc - 36)}{(cc + 9)^4};$$

.
.
.
.

$$X = \frac{c}{cc + xx}; \quad P = \frac{-2cx}{(cc + xx)^2}; \quad Q = \frac{-2c(cc - 3xx)}{(cc + xx)^3}; \quad R = \frac{6cx(3cc - 4xx)}{(cc + xx)^4};$$

hincque

$$y = c\left(\frac{1}{cc} + \frac{1}{cc + 1} + \frac{1}{cc + 4} + \frac{1}{cc + 9} + \ldots + \frac{1}{cc + xx}\right)$$
$$- \frac{1}{2c} - \frac{c}{2(cc + xx)} + \frac{cx}{2(cc + xx)^2}$$
$$- \frac{c}{3}\left(\frac{1}{c^4} + \frac{cc - 3}{(cc + 1)^3} + \frac{cc - 12}{(cc + 4)^3} + \frac{cc - 27}{(cc + 9)^3} + \ldots + \frac{cc - 3xx}{(cc + xx)^3}\right)$$
$$+ \frac{1}{6c^3} + \frac{c(cc - 3xx)}{6(cc + xx)^3} - \frac{cx(3cc - 4xx)}{8(cc + xx)^4}$$
$$\text{etc.}$$

Corollarium.

326. Posito ergo $c = x = 1$, ut fiat $y = \text{Ang. tang. } 1 = \frac{\pi}{4}$, erit

$$\tfrac{\pi}{4} = \tfrac{1}{4} + \tfrac{4}{17} + \tfrac{4}{20} + \tfrac{4}{25} + \tfrac{1}{8} - \tfrac{1}{8} - \tfrac{1}{16} + \tfrac{1}{128}$$
$$- \tfrac{4}{3}\left(\tfrac{1}{256} + \tfrac{13}{17^3} + \tfrac{4}{20^3} - \tfrac{11}{25^3} - \tfrac{32}{32^3}\right) + \tfrac{1}{384} - \tfrac{1}{1536} + \tfrac{1}{196 \cdot 256}$$

cujus valor non multum a veritate discedit; sed haec exempla tantum illustrationis causa afferro, non ut approximatio facilior, quam aliae methodi suppeditant, inde expectetur.

Exemplum 3.

327. *Integrale* $y = \int \frac{e^{-\frac{1}{x}} \partial x}{x}$ *ita sumtum, ut evanescet posito* $x = 0$, *vero proxime assignare.*

Per reductiones supra expositas est

$$\int \frac{e^{-\frac{1}{x}} \partial x}{x} = e^{-\frac{1}{x}} x - \int e^{-\frac{1}{x}} \partial x,$$

et pars $e^{-\frac{1}{x}} x$ evanescit, posito $x = 0$. Quaeramus ergo integrale $z = \int e^{-\frac{1}{x}} \partial x$, quia eo invento habetur $y = e^{-\frac{1}{x}} x - z$; ac supra jam observavimus, alias methodos approximandi in hoc exemplo frustra tentari. Cum igitur, posito $x = 0$, evanescat z, erit $a = 0$ et $b = 0$, tum vero $X = e^{-\frac{1}{x}}$, hincque $P = \frac{\partial X}{\partial x} = e^{-\frac{1}{x}} \frac{1}{xx}$; $Q = \frac{\partial P}{\partial x} = e^{-\frac{1}{x}} \left(\frac{1}{x^4} - \frac{2}{x^3}\right)$; $R = \frac{\partial Q}{\partial x} = e^{-\frac{1}{x}} \left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}\right)$; $S = \frac{\partial R}{\partial x} = e^{-\frac{1}{x}} \left(\frac{1}{x^8} - \frac{12}{x^7} + \frac{36}{x^6} - \frac{24}{x^5}\right)$ etc., quibus valoribus in infinitum continuatis, erit

$$z = e^{-\frac{1}{x}} \left[x - \tfrac{1}{2} + \tfrac{1}{6} x^3 \left(\tfrac{1}{x^4} - \tfrac{2}{x^3}\right) - \tfrac{1}{24} x^4 \left(\tfrac{1}{x^6} - \tfrac{6}{x^5} + \tfrac{6}{x^4}\right) + \tfrac{1}{120} x^5 \left(\tfrac{1}{x^8} - \tfrac{12}{x^7} + \tfrac{36}{x^6} - \tfrac{24}{x^5}\right) - \text{etc.} \right] \text{ seu}$$

$$z = e^{-\frac{1}{x}} \left[x - \tfrac{1}{2} + \tfrac{1}{6}\left(\tfrac{1}{x} - 2\right) - \tfrac{1}{24}\left(\tfrac{1}{xx} - \tfrac{6}{x} + 6\right) + \tfrac{1}{120}\left(\tfrac{1}{x^3} - \tfrac{12}{x^2} + \tfrac{36}{x} - 24\right) - \tfrac{1}{720}\left(\tfrac{1}{x^4} - \tfrac{20}{x^3} + \tfrac{120}{x^2} - \tfrac{240}{x} + 120\right) + \text{etc.} \right]$$

quae series parum convergit, quicunque valor ipsi x tribuatur. Per intervalla igitur a 0 usque ad x ascendamus, ponendo pro x successive 0, α, 2α, 3α, etc. ubi notandum fore $A = 0$, $B = 0$, $C = 0$, $D = 0$, etc. ac regula nostra praebet:

$$z = \alpha\left(e^{-\frac{1}{\alpha}} + e^{-\frac{1}{2\alpha}} + e^{-\frac{1}{3\alpha}} + \ldots + e^{-\frac{1}{x}}\right) - \tfrac{1}{2}\alpha e^{-\frac{1}{x}} - \tfrac{1}{4}\alpha^2 e^{-\frac{1}{x}} \frac{1}{xx}$$
$$+ \tfrac{1}{6}\alpha^3 \left[e^{-\frac{1}{\alpha}}\left(\frac{1}{\alpha^4} - \frac{2}{\alpha^3}\right) + e^{-\frac{1}{2\alpha}}\left(\frac{1}{16\alpha^4} - \frac{2}{8\alpha^3}\right) + e^{-\frac{1}{3\alpha}}\left(\frac{1}{81\alpha^4} - \frac{2}{27\alpha^3}\right) \ldots\right.$$
$$\left. + e^{-\frac{1}{x}}\left(\frac{1}{x^4} - \frac{2}{x^3}\right)\right] - \tfrac{1}{12}\alpha^3 e^{-\frac{1}{x}}\left(\frac{1}{x^4} - \frac{2}{x^3}\right) - \tfrac{1}{48}\alpha^4 e^{-\frac{1}{x}}\left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}\right).$$

Si hinc valorem ipsius z pro casu $x = 1$ determinare velimus, et pro α fractionem parvam $\frac{1}{n}$ assumamus, habebimus:

$$z = \tfrac{1}{n}\left(e^{-\frac{n}{1}} + e^{-\frac{n}{2}} + e^{-\frac{n}{3}} + e^{-\frac{n}{4}} + \ldots + e^{-\frac{n}{n}}\right) - \frac{1}{2ne} - \frac{1}{4nne}$$
$$+ \tfrac{1}{6}\left[e^{-\frac{n}{1}}\left(\frac{n-2}{1}\right) + e^{-\frac{n}{2}}\left(\frac{n-4}{16}\right) + e^{-\frac{n}{3}}\left(\frac{n-6}{81}\right) + \ldots + e^{-\frac{n}{n}}\left(\frac{n-2n}{n^4}\right)\right]$$
$$+ \frac{1}{12n^3e} - \frac{1}{48n^4e}.$$

Si hic pro n sumatur numerus mediocriter magnus vel uti 10, valor ipsius z ad partem millionesimam unitatis exactus reperitur, ac vicies exactior prodiret, si pro n sumeremus 20.

Scholion 1.

328. Hoc exemplum sufficiat eximium usum hujus methodi approximandi ostendisse. Interim tamen occurrunt casus, quibus ne hac quidem methodo uti licet, etiamsi totum spatium, per quod variabilis x crescit, in minima intervalla dividamus. Evenit hoc, quando functio X pro quopiam intervallo, dum variabili x certus quidam valor tribuitur, in infinitum excrescit, cum tamen ipsa quantitas integralis $y = \int X\,\partial x$ hoc casu non fiat infinita: veluti si fuerit $y = \int \frac{\partial x}{\sqrt{(a-x)}}$, ubi $X = \frac{1}{\sqrt{(a-x)}}$, quae posito $x = a$ fit infinita, integrale vero $y = C - 2\sqrt{(a-x)}$ hoc casu est finitum.

Hoc autem semper usu venit, quoties hujusmodi factor $a - x$ in denominatore habet exponentem unitate minorem, tum enim idem factor in integrali in numeratorem transit; sin autem ejusdem factoris exponens in denominatore est unitas, vel adeo unitate major, tum etiam ipsum integrale casu $x = a$ fit infinitum, quo casu quia approximatio cessat, hic tantum de iis sermo est, ubi exponens unitate est minor; quoniam tum approximatio revera turbatur. Verum huic incommodo facile medela afferri potest, cum enim differentiale ejusmodi formam sit habiturum $\frac{X\partial x}{(a-x)^{\lambda:\mu}}$, existente $\lambda < \mu$, ponatur $a - x = z^{\mu}$, ut sit $x = a - z^{\mu}$ et $\partial x = -\mu z^{\mu-1}\partial z$, et differentiale nostrum erit $= -\mu X z^{\mu-\lambda-1}\partial z$, quod casu $x = a$ seu $z = 0$, non amplius fit infinitum. Vel quod eodem redit, pro iis intervallis, quibus functio X fit infinita, integratio seorsim revera instituatur, ponendo $x = a \pm \omega$, tum enim formula $X\partial x$ satis fiet simplex ob ω valde parvum, ut integratio nihil habeat difficultatis. Veluti si valorem ipsius $y = \int \frac{xx\partial x}{\sqrt{(a^4 - x^4)}}$ per intervalla ab $x = 0$ usque ad $x = a - \alpha$, jam simus consecuti, pro hoc ultimo intervallo ponamus $x = a - \omega$, et integrari oportebit $\frac{(a-\omega)^2\partial\omega}{\sqrt{(4a^3\omega - 6aa\omega\omega + 4a\omega^3 - \omega^4)}}$, quod ob ω valde parvum abit in

$$\frac{\partial\omega\sqrt{a}}{2\sqrt{\omega}}\left(1 - \frac{\omega}{2a} + \frac{7\omega\omega}{8aa}\right),$$

cujus integrale, sumto $\omega = \alpha$, est

$$\sqrt{a\alpha} - \frac{\alpha\sqrt{\alpha}}{6\sqrt{a}} + \frac{7\alpha\alpha\sqrt{\alpha}}{4a\sqrt{a}},$$

quod si ad plures terminos continuetur, non solum pro ultimo intervallo sed pro duobus pluribusve postremis, ponendo $\omega = 2\alpha$ vel $\omega = 3\alpha$ adhiberi potest. Pro quibus enim intervallis denominator jam fit satis parvus, praestat hac methodo uti, quam ea quae ante est exposita.

Scholion 2.

329. Interdum etiam illud incommodum occurrit, ut denominator duobus casibus evanescat, veluti si fuerit $y = \int \frac{X\partial x}{\sqrt{(a-x)(x-b)}}$,

ubi variabilis x semper inter limites b et a contineri debet, ita ut cum a b ad a creverit, deinceps iterum ab a ad b decrescat; interea autem integrale y continuo crescere pergat, cujus igitur valor per intervalla commode determinari non potest. Hoc ergo casu in subsidium vocetur haec substitutio $x = \frac{1}{2}(a + b) - \frac{1}{2}(a - b)\cos.\Phi$, qua fit $\partial x = + \frac{1}{2}(a - b)\partial\Phi \sin.\Phi$, et $(a - x)(x - b) = [\frac{1}{2}(a - b) + \frac{1}{2}(a - b)\cos.\Phi][\frac{1}{2}(a - b) - \frac{1}{2}(a - b)\cos.\Phi]$, seu $(a - x)(x - b) = \frac{1}{4}(a - b)^2 \sin.\Phi^2$: unde oritur $y = \int X\partial\Phi$, quae nullo amplius incommodo laborat, cum angulum Φ continuo ulterius aequabiliter augere licet. Hoc etiam ad casus patet, ubi bini factores in denominatore non eundem habent exponentem, veluti si fuerit $y = \int \frac{X\partial x}{\sqrt[2\lambda]{(a - x)^\mu (x - b)^\nu}}$, ita ut μ et ν sint minores quam 2λ, quem exponentem parem suppono. Si jam μ et ν non sint aequales sed $\nu < \mu$, ad aequalitatem reducantur hoc modo, $y = \int \frac{X\partial x \sqrt[2\lambda]{(x - b)^{\mu-\nu}}}{\sqrt[2\lambda]{(a - x)^\mu (x - b)^\mu}}$. Quodsi jam ut ante ponatur $x = \frac{1}{2}(a + b) - \frac{1}{2}(a - b)\cos.\Phi$, obtinebitur

$$y = \left(\frac{a-b}{2}\right)^{\frac{2\lambda-\mu-\nu}{2\lambda}} \int X\partial\Phi \sin.\Phi^{\frac{\lambda-\mu}{\lambda}} (1 - \cos.\Phi)^{\frac{\mu-\nu}{2\lambda}},$$

ubi angulum Φ quousque libuerit continuare et methodo per intervalla procedente uti licet. Quibus observatis vix quicquam amplius hanc methodum approximandi remorabitur.