CAPUT VI.

DE EVOLUTIONE INTEGRALIUM PER SERIES, SECUNDUM SINUS COSINUSVE ANGULORUM MULTIPLORUM PROGREDIENTES.

Problema 32.

272.

Integrale formulae $\frac{\partial \phi}{1 + n \cos. \phi}$ per seriem, secundum sinus angulorum multiplorum progredientem, exprimere.

Solutio.

Cum sit more consueto per seriem

$$\frac{1}{1 + n \cos. \phi} = 1 - n \cos. \phi + n^2 \cos. \phi^2 - n^3 \cos. \phi^3 + n^4 \cos. \phi^4 - \text{etc.}$$

potestates cosinus in cosinus angulorum multiplorum convertantur ope formularum in introductione traditarum: ac primo pro potestatibus imparibus:

$$\cos. \phi = \cos. \phi;$$

$$\cos. \phi^3 = \tfrac{3}{4} \cos. \phi + \tfrac{1}{4} \cos. 3\phi;$$

$$\cos. \phi^5 = \tfrac{10}{16} \cos. \phi + \tfrac{5}{16} \cos. 3\phi + \tfrac{1}{16} \cos. 5\phi;$$

$$\cos. \phi^7 = \tfrac{35}{64} \cos. \phi + \tfrac{21}{64} \cos. 3\phi + \tfrac{7}{64} \cos. 5\phi + \tfrac{1}{64} \cos. 7\phi;$$

$$\cos. \phi^9 = \tfrac{126}{256} \cos. \phi + \tfrac{84}{256} \cos. 3\phi + \tfrac{36}{256} \cos. 5\phi + \tfrac{9}{256} \cos. 7\phi + \tfrac{1}{256} \cos. 9\phi;$$

ubi notandum est, si ponatur in genere

$$\cos. \phi^{2\lambda-1} = A \cos. \phi + B \cos. 3\phi + C \cos. 5\phi + D \cos. 7\phi + E \cos. 9\phi + \text{etc.}$$

fore

$$A = 2.\frac{1.3.5\ldots\ldots(2\lambda-1)}{2.4.6\ldots\ldots\quad 2\lambda} = \frac{2}{2^{2\lambda-1}}.\frac{6}{2}.\frac{10}{3}.\frac{14}{4}\ldots\frac{4\lambda-2}{\lambda};$$

$$B = \tfrac{\lambda-1}{\lambda+1}A;\ C = \tfrac{\lambda-2}{\lambda+2}B;\ D = \tfrac{\lambda-3}{\lambda+3}C;\ E = \tfrac{\lambda-4}{\lambda+4}D;\ \text{etc.}$$

Pro paribus vero potestatibus est

$$\cos.\phi^0 = 1$$

$$\cos.\phi^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos.2\phi$$

$$\cos.\phi^4 = \tfrac{3}{8} + \tfrac{4}{8}\cos.2\phi + \tfrac{1}{8}\cos.4\phi$$

$$\cos.\phi^6 = \tfrac{10}{32} + \tfrac{15}{32}\cos.2\phi + \tfrac{6}{32}\cos.4\phi + \tfrac{1}{32}\cos.6\phi$$

$$\cos.\phi^8 = \tfrac{35}{128} + \tfrac{56}{128}\cos.2\phi + \tfrac{28}{128}\cos.4\phi + \tfrac{8}{128}\cos.4\phi + \tfrac{1}{128}\cos.8\phi.$$

In genere autem si ponatur:

$$\cos.\phi^{2\lambda} = \mathfrak{A} + \mathfrak{B}\cos.2\phi + \mathfrak{C}\cos.4\phi + \mathfrak{D}\cos.6\phi + \mathfrak{E}\cos.8\phi + \text{etc. erit}$$

$$\mathfrak{A} = \frac{1.3.5\ldots\ldots(2\lambda-1)}{2.4.6\ldots\ldots\quad 2\lambda} = \frac{1}{2^{2\lambda-1}}.\frac{6}{2}.\frac{10}{3}.\frac{14}{4}\ldots\ldots\frac{4\lambda-2}{\lambda}.$$

$$\mathfrak{B} = \tfrac{2\lambda}{\lambda+1}\mathfrak{A};\ \mathfrak{C} = \tfrac{\lambda-1}{\lambda+2}\mathfrak{B};\ \mathfrak{D} = \tfrac{\lambda-2}{\lambda+3}\mathfrak{C};\ \mathfrak{E} = \tfrac{\lambda-3}{\lambda+4}\mathfrak{D};\ \text{etc.}$$

Quodsi nunc isti valores substituantur, erit $\frac{1}{1+n\cos.\phi} =$

$$\begin{array}{lllllll}
1 & -n\cos.\phi & +\tfrac{1}{2}nn\cos.2\phi & -\tfrac{1}{4}n^3\cos.3\phi & +\tfrac{1}{8}n^4\cos.4\phi & -\tfrac{1}{16}n^5\cos.5\phi & +\tfrac{1}{32}n^6\cos.6\phi \\
+\tfrac{1}{2}nn & -\tfrac{3}{4}n^3 & +\tfrac{4}{8}n^4 & -\tfrac{5}{16}n^5 & +\tfrac{6}{32}n^6 & -\tfrac{7}{64}n^7 & +\tfrac{8}{128}n^8 \\
+\tfrac{3}{8}n^4 & -\tfrac{10}{16}n^5 & +\tfrac{15}{32}n^6 & -\tfrac{21}{64}n^7 & +\tfrac{28}{128}n^8 & -\tfrac{36}{256}n^9 & \\
+\tfrac{10}{32}n^6 & -\tfrac{35}{64}n^7 & +\tfrac{56}{128}n^8 & -\tfrac{84}{256}n^9 & & & \\
+\tfrac{35}{128}n^8 & & & & & &
\end{array}$$

unde patet, si ponatur

$$\frac{1}{1+n\cos.\phi} = A - B\cos.\phi + C\cos.2\phi - D\cos.3\phi + E\cos.4\phi - \text{etc.}$$

est $$A = 1 + \tfrac{1}{2} nn + \tfrac{3}{8} n^4 + \tfrac{10}{32} n^6 + \text{etc. seu}$$
$$A = 1 + \tfrac{1}{2} n^2 + \tfrac{1.3}{2.4} n^4 + \tfrac{1.3.5}{2.4.6} n^6 + \tfrac{1.3.5.7}{2.4.6.8} n^8 + \text{etc.}$$

sicque evidens est esse $A = \frac{1}{\sqrt{(1 - nn)}}$.

Simili modo est

$$B = n + \tfrac{3}{4} n^3 + \tfrac{10}{16} n^5 + \text{etc.} = \tfrac{2}{n} \left(\tfrac{1}{2} n^2 + \tfrac{1.3}{2.4} n^4 + \tfrac{1.3.5}{2.4.6} n^5 + \text{etc.}\right)$$

ideoque $B = \frac{2}{n} \left(\frac{1}{\sqrt{(1 - nn)}} - 1\right)$. Verum et hunc valorem et sequentes facilius hoc modo definire licet. Cum sit

$$\frac{1}{1 + n \cos.\phi} = A - B \cos.\phi + C \cos. 2\phi - D \cos. 3\phi$$
$$+ E \cos. 4\phi - \text{etc.}$$

multiplicetur per $1 + n \cos.\phi$, et quia

$$\cos.\phi \cos.\lambda\phi = \tfrac{1}{2} \cos.(\lambda - 1)\phi + \tfrac{1}{2} \cos.(\lambda + 1)\phi, \text{ fiet}$$

$$\begin{array}{lllll} 1 = A & - B \cos.\phi & + C \cos. 2\phi & - D \cos. 3\phi & + E \cos. 4\phi - \text{etc.} \\ \quad + An & - \tfrac{1}{2} Bn & + \tfrac{1}{2} Cn & - \tfrac{1}{2} Dn & \\ - \tfrac{1}{2} Bn + \tfrac{1}{2} Cn & - \tfrac{1}{2} Dn & + \tfrac{1}{2} En & - \tfrac{1}{2} Fn & \end{array}$$

unde quia A jam definivimus, reliqui coëfficientes ita determinantur:

$$B = \tfrac{2}{n}(A - 1); \qquad E = \frac{2D - Cn}{n}$$
$$C = \frac{2B - An}{n}; \qquad F = \frac{2E - Dn}{n}$$
$$D = \frac{2C - Bn}{n}; \qquad G = \frac{2F - En}{n}$$

etc.

His igitur coëfficientibus inventis, integrale facile assignatur: nam cum sit $\int \partial\phi \cos.\lambda\phi = \frac{1}{\lambda} \sin.\lambda\phi$, habebimus

$$\int \frac{\partial\phi}{1 + n \cos.\phi} = A\phi - B \sin.\phi + \tfrac{1}{2} C \sin. 2\phi$$
$$- \tfrac{1}{3} D \sin. 3\phi + \tfrac{1}{4} E \sin. 4\phi - \text{etc}$$

quae series secundum sinus angulorum ϕ, 2ϕ, 3ϕ, etc. progreditur, uti desiderabatur.

Corollarium 1.

273. Primo patet hanc resolutionem locum habere non posse, nisi n sit numerus unitate minor; si enim $n > 1$, singuli coëfficientes prodeunt imaginarii. Sin autem sit $n = 1$, ob $1 + \cos.\Phi = 2\cos.\frac{1}{2}\Phi^2$, erit integrale

$$\int \frac{\partial\Phi}{1 + \cos.\Phi} = \int \frac{\frac{1}{2}\partial\Phi}{\cos.\frac{1}{2}\Phi^2} = \text{tang.}\,\tfrac{1}{2}\Phi.$$

Corollarium 2.

274. Cum sit $A = \frac{2}{\sqrt{(1 - nn)}}$ et $B = \frac{2}{n}\left(\frac{1}{\sqrt{(1 - nn)}} - 1\right)$, reliqui coëfficientes C, D, E, etc. seriem recurrentem constituunt, ita ut si bini contigui sint P et Q sequens futurus sit $\frac{2}{n}Q - P$. Hinc cum aequationis $zz = \frac{2}{n}z - 1$ radices sint $\frac{1 \pm \sqrt{(1 - nn)}}{n}$, quisque terminus in hac forma continetur

$$\alpha\left(\frac{1 + \sqrt{(1 - nn)}}{n}\right)^\lambda + \beta\left(\frac{1 - \sqrt{(1 - nn)}}{n}\right)^\lambda.$$

Corollarium 3.

275. Quia autem in nostra lege non A sed 2A sumitur: posito $\lambda = 0$, prodire debet 2A ideoque $\alpha + \beta = \frac{2}{\sqrt{(1 - nn)}}$, deinde facto $\lambda = 1$, fieri debet

$$\frac{\alpha + \beta}{n} + \frac{(\alpha - \beta)\sqrt{(1 - nn)}}{n} = \frac{2 - 2(1 - nn)}{n\sqrt{(1 - nn)}},$$

unde $\alpha - \beta = -\frac{2}{\sqrt{(1 - nn)}}$. Ergo $\alpha = 0$ et $\beta = \frac{2}{\sqrt{(1 - nn)}}$; sicque quilibet terminus praeter A erit =

$$\frac{2}{\sqrt{(1 - nn)}}\left(\frac{1 - \sqrt{(1 - nn)}}{n\sqrt{(1 - nn)}}\right)^\lambda.$$

Corollarium 4.

276. Coëfficientes ergo evoluti ita se habebunt:

$$A = \frac{1}{\sqrt{(1-nn)}}$$

$$B = \frac{2-2\sqrt{(1-nn)}}{n\sqrt{(1-nn)}}$$

$$C = \frac{4-2nn-4\sqrt{(1-nn)}}{nn\sqrt{(1-nn)}}$$

$$D = \frac{8-6nn-2(4-nn)\sqrt{(1-nn)}}{n^3\sqrt{(1-nn)}}$$

$$E = \frac{16-16nn+2n^4-2(8-4nn)\sqrt{(1-nn)}}{n^4\sqrt{(1-nn)}}$$

$$F = \frac{32-40nn+10n^4-2(16-12nn+n^4)\sqrt{(1-nn)}}{n^5\sqrt{(1-nn)}}$$

$$G = \frac{64-96nn+36n^4-2n^6-2(32-28nn+6n^4)\sqrt{(1-nn)}}{n^6\sqrt{(1-nn)}}$$

Corollarium 5.

277. Quia $n < 1$, hi coëfficientes plerumque facilius determinantur per series primum inventas, scilicet:

$$A = 1 + \tfrac{1}{2}n^2 + \tfrac{1.3}{2.4}n^4 + \tfrac{1.3.5}{2.4.6}n^6 + \tfrac{1.3.5.7}{2.4.6.8}n^8 + \text{etc.}$$

$$B = n\left(1 + \tfrac{3}{4}n^2 + \tfrac{3.5}{4.6}n^4 + \tfrac{3.5.7}{4.6.8}n^6 + \tfrac{3.5.7.9}{4.6.8.10}n^8 + \text{etc.}\right)$$

$$C = \tfrac{1}{2}n^2\left(1 + \tfrac{3.4}{2.6}n^2 + \tfrac{3.4.5.6}{2.6.4.8}n^4 + \tfrac{3.4.5.6.7.8}{2.6.4.8.6.10}n^6 + \text{etc.}\right)$$

$$D = \tfrac{1}{4}n^3\left(1 + \tfrac{4.5}{2.8}n^2 + \tfrac{4.5.6.7}{2.8.4.10}n^4 + \tfrac{4.5.6.7.8.9}{2.8.4.10.6.12}n^6 + \text{etc.}\right)$$

$$E = \tfrac{1}{8}n^4\left(1 + \tfrac{5.6}{2.10}n^2 + \tfrac{5.6.7.8}{2.10.4.12}n^4 + \tfrac{5.6.7.8.9.10}{2.10.4.12.6.14}n^6 + \text{etc.}\right)$$

$$F = \tfrac{1}{16}n^6\left(1 + \tfrac{6.7}{2.12}n^2 + \tfrac{6.7.8.9}{2.12.4.14}n^4 + \tfrac{6.7.8.9.10.11}{2.12.4.14.6.16}n^6 + \text{etc.}\right)$$

etc.

Scholion.

278. Cum ex his valoribus sit

$$\int \frac{\partial\Phi}{1+n\cos.\Phi} = A\Phi - B\sin.\Phi + \tfrac{1}{2}C\sin.2\Phi - \tfrac{1}{3}D\sin.3\Phi + \tfrac{1}{4}E\sin.4\Phi - \text{etc.}$$

in hac serie terminus primus $A\phi$ imprimis est notandus, quod crescente angulo ϕ continuo crescat, idque in infinitum usque, dum reliqui termini modo crescent modo descrescent: neque tamen certum limitem excedunt; nam sin. $\lambda\phi$ neque supra $+1$ crescere, neque infra -1 decrescere potest. Cum deinde hoc integrale supra inventum sit

$$\frac{1}{\sqrt{(1-nn)}}\text{ Ang. cos. }\frac{n+\cos.\phi}{1+n\cos.\phi}$$

series illa huic angulo aequatur. Quare si hic angulus vocetur ω, ut sit $\partial\omega = \frac{\partial\phi\sqrt{(1-nn)}}{1+n\cos.\phi}$, erit $\cos.\omega = \frac{n+\cos.\phi}{1+n\cos.\phi}$, hincque $n+\cos.\phi - \cos.\omega - n\cos.\phi\cos.\omega = 0$, ex quo est vicissim $\cos.\phi = \frac{\cos.\omega - n}{1-n\cos.\omega}$, quae formula cum ex illa nascatur sumto n negativo, erit

$$\partial\phi = \frac{\partial\omega\sqrt{(1-nn)}}{1-n\cos.\omega},\text{ et}$$

$$\frac{\phi}{\sqrt{(1-nn)}} = A\omega + B\sin.\omega + \tfrac{1}{2}C\sin.2\omega + \tfrac{1}{3}D\sin.3\omega + \tfrac{1}{4}E\sin.4\omega + \text{etc.}$$

Quia vero est

$$\frac{\omega}{\sqrt{(1-nn)}} = A\phi - B\sin.\phi + \tfrac{1}{2}C\sin.2\phi - \tfrac{1}{3}D\sin.3\phi + \tfrac{1}{4}E\sin.4\phi - \text{etc.}$$

ob $\frac{1}{\sqrt{(1-nn)}} = A$, habebimus:

$$0 = B(\sin.\omega - \sin.\phi) + \tfrac{1}{2}C(\sin.2\omega + \sin.2\phi) + \tfrac{1}{3}D(\sin.3\omega - \sin.3\phi) + \text{etc.}$$

cujusmodi relationes notasse juvabit.

Problema 36.

279. Integrale formulae $\partial\phi(1+n\cos.\phi)^\nu$ per seriem, secundum sinus angulorum multiplorum ipsius ϕ progredientem, exprimere.

Solutio.

Cum sit

$$(1+n\cos.\phi)^\nu = 1 + \frac{\nu}{1}n\cos.\phi + \frac{\nu(\nu-1)}{1.2}n^2\cos.\phi^2 + \frac{\nu(\nu-1)(\nu-2)}{1.2.3}n^3\cos.\phi^3 + \text{etc.}$$

si ponamus

$$(1+n\cos.\Phi)^{\nu} = A + B\cos.\Phi + C\cos.2\Phi$$
$$+ D\cos 3\Phi + E\cos.4\Phi + \text{etc.}$$

erit per formulas supra indicatas:

$$A = 1 + \frac{\nu(\nu-1)}{1.2}\cdot\tfrac{1}{2}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1.2.3.4}\cdot\frac{1.3}{2.4}n^4$$
$$+ \frac{\nu(\nu-1)\ldots\ldots(\nu-5)}{1.2\ldots\ldots 6}\cdot\frac{1.3.5}{2.4.6}n^6 + \text{etc.}$$
$$B = 2n\left[\frac{\nu}{2} + \frac{\nu(\nu-1)(\nu-2)}{1.\ 2.\ 3}\cdot\frac{1.3}{2.4}n^2\right.$$
$$\left. + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{1.\ 2.\ 3.\ 4.\ 5}\cdot\frac{1.3.5}{2.4.6}n^4 + \text{etc.}\right]$$

quae series ita clarius exhibentur:

$$A = 1 + \frac{\nu(\nu-1)}{2.\ 2.}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2.\ 2.\ 4.\ 4}n^4$$
$$+ \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2.\ 2.\ 4.\ 4.\ 6.\ 6}n^6 + \text{etc.}$$
$$\tfrac{1}{2}B = \frac{\nu}{2} + \frac{\nu(\nu-1)(\nu-2)}{2.\ 2.\ 4}n^3 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{2.\ 2.\ 4.\ 4.\ 6.}n^5 + \text{etc.}$$

Inventis autem his binis coëfficientibus A et B, reliqui ex his sequenti modo commodius determinari poterunt. Cum sit

$$\nu l(1+n\cos.\Phi) = l[A + B\cos.\Phi + C\cos.2\Phi$$
$$+ D\cos.3\Phi + E\cos.4\Phi + \text{etc.}]$$

sumantur differentialia, ac per $-\partial\Phi$ dividendo prodit

$$\frac{\nu n\sin.\Phi}{1+n\cos.\Phi} = \frac{B\sin.\Phi + 2C\sin.2\Phi + 3D\sin.3\Phi + 4E\sin.4\Phi + \text{etc.}}{A + B\cos.\Phi + C\cos.2\Phi + D\cos.3\Phi + E\cos.4\Phi + \text{etc.}}.$$

Jam per crucem multiplicando,

$$\text{ob } \sin.\lambda\Phi\cos.\Phi = \tfrac{1}{2}\sin.(\lambda+1)\Phi + \tfrac{1}{2}\sin.(\lambda-1)\Phi \text{ et}$$
$$\sin.\Phi\cos.\lambda\Phi = \tfrac{1}{2}\sin.(\lambda+1)\Phi - \tfrac{1}{2}\sin.(\lambda-1)\Phi,$$

pervenietur ad hanc aequationem:

$$\begin{array}{llllll}
0 = B\sin.\Phi & + 2C\sin.2\Phi & + 3D\sin.3\Phi & + 4E\sin.4\Phi & + 5F\sin.5\Phi & + \text{etc.} \\
 & + \tfrac{1}{2}Bn & + \tfrac{2}{2}Cn & + \tfrac{3}{2}Dn & + \tfrac{4}{2}En & \\
+ \tfrac{2}{2}Cn & + \tfrac{3}{2}Dn & + \tfrac{4}{2}En & + \tfrac{5}{2}Fn & + \tfrac{6}{2}Gn & \\
- \nu An & - \tfrac{\nu}{2}Bn & - \tfrac{\nu}{2}Cn & - \tfrac{\nu}{2}Dn & - \tfrac{\nu}{2}En & \\
+ \tfrac{\nu}{2}Cn & + \tfrac{\nu}{2}Dn & + \tfrac{\nu}{2}En & + \tfrac{\nu}{2}Fn & + \tfrac{\nu}{2}Gn &
\end{array}$$

unde hae sequuntur determinationes:

$$\begin{array}{l|l}
(\nu + 2)\,Cn + 2B - 2\nu An = 0 & C = \frac{2\nu An - 2B}{(\nu+2)n} \\
(\nu + 3)\,Dn + 4C - (\nu - 1)\,Bn = 0 & D = \frac{(\nu-1)Bn - 4C}{(\nu+3)n} \\
(\nu + 4)\,En + 6D - (\nu - 2)\,Cn = 0 & E = \frac{(\nu-2)Cn - 6D}{(\nu+4)n} \\
(\nu + 5)\,Fn + 8E - (\nu - 3)\,Dn = 0 & F = \frac{(\nu-3)Dn - 8E}{(\nu+5)n} \\
(\nu + 6)\,Gn + 10F - (\nu - 4)\,En = 0 & G = \frac{(\nu-4)En - 10F}{(\nu+6)n}
\end{array}$$

ubi si superiores valores pro A et B substituantur, reperitur:

$$C = 4nn\left[\frac{1\nu(\nu-1)}{2.2.4.} + \frac{2\nu(\nu-1)(\nu-2)(\nu-3)}{2.2.4.4.6}n^2 + \frac{3\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2.2.4.4.6.6.8}n^4 + \text{etc.}\right]$$

$$D = 8n^3\left[\frac{1.2\nu(\nu-1)(\nu-2)}{2.2.4.4.6} + \frac{2.3\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{2.2.4.4.6.6.8}n^2 + \text{etc.}\right]$$

$$E = 16n^4\left[\frac{1.2.3\nu(\nu-1)(\nu-2)(\nu-3)}{2.2.4.4.6.6.8} + \frac{2.3.4\nu(\nu-1)\ldots\ldots(\nu-5)}{2.2.4.4.6.6.8.8.10}n^2 + \text{etc.}\right]$$

etc.

unde forma sequentium serierum colligitur.

His autem inventis coëfficientibus, erit integrale quaesitum

$$\int\partial\phi\,(1 + n\cos.\phi)^\nu = A\phi + B\sin.\phi + \tfrac{1}{2}C\sin.2\phi$$
$$+ \tfrac{1}{3}D\sin 3\phi + \tfrac{1}{4}E\sin.4\phi + \text{etc.}$$

Corollarium 1.

280. Ad similitudinem harum serierum pro C, D, E etc. datarum, etiam valor ipsius B ita exprimi potest:

$$B = 2n\left[\frac{\nu}{2} + \frac{\nu(\nu-1)(\nu-2)}{2.2.4}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{2.2.4.4.6}n^4 + \text{etc.}\right]$$

Series autem pro A inventa formam habet singularem in hac lege non comprehensam.

Corollarium 2.

281. Si series A et B inter se comparemus, varias relationes inter eas observare licet, quarum haec primo se offert:

$$An + \tfrac{1}{2}B = \frac{(\nu+2)}{2} n \left[1 + \frac{\nu(\nu-1)}{2.\ 4} n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2.\ 4.\ 4.\ 6} n^4 + \frac{\nu(\nu-1)\ldots\ldots(\nu-5)}{2.4.\ldots\ldots\ldots 8} n^6 + \text{etc.}\right]$$

quae a serie A tantum secundum denominatores differt.

Corollarium 3.

282. Ponamus $\frac{2Ann+Bn}{\nu+2} = N$, ut sit

$$N = n^2 + \frac{\nu(\nu-1)}{2.\ 4} n^4 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\ 4.\ 4.\ 6} n^6 + \text{etc.}$$

$$A = 1 + \frac{\nu(\nu-1)}{2.\ 2} n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2.\ 2.\ 4.\ 4} n^4 + \text{etc.}$$

Quodsi jam n ut variabilis tractetur, differentiatio praebet:

$$\frac{\partial N}{n\,\partial n} = 2 + \frac{\nu(\nu-1)}{2} n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2.\ 4.\ 4} n^4 + \text{etc.} = 2A.$$

Cum igitur sit

$$\partial N = \frac{4An\partial n + B\partial n + 2nn\partial A + n\partial B}{\nu+2} = 2An\partial n,$$

erit $2\nu An\partial n = 2nn\partial A + B\partial n + n\partial B$.

Corollarium 4.

283. Ex dato ergo coëfficiente A, coëfficiens B ita per integrationem inveniri potest, ut sit

$$Bn = 2\int(\nu An\partial n - nn\partial A):$$

vel erit etiam ex illa forma

$$B = \frac{2(\nu+2)}{n}\int An\partial n - 2An.$$

Ubi notandum est, posito $n = 0$ integrale $\int An\partial n$ evanescere debere, quia hoc casu B evanescit.

Scholion.

284. Series pro litteris B, C, D, etc. inventas etiam sequenti modo per continuos factores exprimere licet:

$$B = \nu n\left(1 + \frac{(\nu-1)(\nu-2)}{2.\ 4} n^2 + \frac{(\nu-3)(\nu-4)}{4.\ 6} Pn^2 + \frac{(\nu-5)(\nu-6)}{6.\ 8} Pn^2 + \text{etc.}\right)$$

$$C = \frac{\nu(\nu-1)}{1.\ 2} \cdot \frac{n^2}{2}\left(1 + \frac{(\nu-2)(\nu-3)}{2.\ 6} n^2 + \frac{(\nu-4)(\nu-5)}{4.\ 8} P n^2 \right.$$
$$\left. + \frac{(\nu-6)(\nu-7)}{6.\ 10} P n^2 + \text{etc.}\right)$$

$$D = \frac{\nu(\nu-1)(\nu-2)}{1.\ 2.\ 3} \cdot \frac{n^3}{4}\left(1 + \frac{(\nu-3)(\nu-4)}{2.\ 8} n^2 + \frac{(\nu-5)(\nu-6)}{4.\ 10} P n^2 \right.$$
$$\left. + \frac{(\nu-7)(\nu-8)}{6.\ 12} P n^2 + \text{etc.}\right)$$

$$E = \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1.\ 2.\ 3.\ 4} \cdot \frac{n^4}{8}\left(1 + \frac{(\nu-4)(\nu-5)}{2.\ 10} n^2 + \frac{(\nu-6)(\nu-7)}{4.\ 12} P n^2 \right.$$
$$\left. + \frac{(\nu-8)(\nu-9)}{6.\ 14} P n^2 + \text{etc.}\right)$$

$$F = \frac{\nu \ldots\ldots (\nu-4)}{1 \ldots\ldots 5} \cdot \frac{n^5}{16}\left(1 + \frac{(\nu-5)(\nu-6)}{2.\ 12} n^2 + \frac{(\nu-7)(\nu-8)}{4.\ 14} P n^2 \right.$$
$$\left. + \frac{(\nu-9)(\nu-10)}{6.\ 16} P n^2 + \text{etc.}\right)$$

etc.

ubi in qualibet serie littera P terminum praecedentem integrum denotat. Atque ope serierum istarum coëfficientes plerumque facilius inveniuntur, quam ex lege ante tradita, qua quisque ex binis praecedentibus determinatur. Quin haec lex defectu laborat, quod si ν fuerit numerus integer negativus praeter -1; quidam coëfficientes plane non definiantur, quos ergo ex his seriebus desumi oportet. Ita si fuerit

$$\nu = -2, \text{ erit } B = \nu A n = -2 A n, \text{ et}$$
$$C = \tfrac{3}{1} \cdot \frac{n^2}{2}\left(1 + \frac{4.\ 5}{2.\ 6} n^2 + \frac{4.\ 5.\ 6.\ 7}{2.\ 6.\ 4.\ 8} n^4 + \frac{4.\ 5.\ 6.\ 7.\ 8.\ 9}{2.\ 6.\ 4.\ 8.\ 6.\ 10} n^6 + \text{etc.}\right)$$

si sit $\nu = -3$, erit $C = -Bn$, et

$$D = -\frac{4.\ 5}{1.\ 2} \cdot \frac{n^3}{4}\left(1 + \frac{6.\ 7}{2.\ 3} n^2 + \frac{6.\ 7.\ 8.\ 9}{2.\ 8.\ 4.\ 10} n^4 + \frac{6.\ 7.\ 8.\ 9.\ 10.\ 11}{2.\ 8.\ 4.\ 10.\ 6.\ 12} n^6 + \text{etc.}\right)$$

si sit $\nu = -4$, erit $D = -Cn$, et

$$E = \frac{5.\ 6.\ 7}{1.\ 2.\ 3} \cdot \frac{n^4}{8}\left(1 + \frac{8.\ 9}{2.\ 10} n^2 + \frac{8.\ 9.\ 10.\ 11}{2.\ 10.\ 4.\ 12} n^4 + \frac{8.\ 9.\ 10.\ 11.\ 12.\ 13}{2.\ 10.\ 4.\ 12.\ 6.\ 14} n^6 + \text{etc.}\right)$$

si sit $\nu = -5$, erit $E = -Dn$, et

$$F = -\frac{6.\ 7.\ 8.\ 9}{1.\ 2.\ 3.\ 4} \cdot \frac{n^5}{16}\left(1 + \frac{10.\ 11}{2.\ 12} n^2 + \frac{10.\ 11.\ 12.\ 13}{2.\ 12.\ 4.\ 14} n^4 \right.$$
$$\left. + \frac{10.\ 11.\ 12.\ 13.\ 14.\ 15}{2.\ 12.\ 4.\ 14.\ 6.\ 16} n^6 + \text{etc.}\right)$$

et ita de reliquis.

Exemplum 1.

285. *Formulae $\partial\Phi\,(1+n\cos.\Phi)^\nu$ integrale evolvere, si ν sit numerus integer positivus.*

Posito $(1+n\cos.\Phi)^\nu = A + B\cos.\Phi + C\cos.2\Phi + D\cos.3\Phi + E\cos.4\Phi + \text{etc.}$

pro singulis valoribus exponentis ν habebimus:

1.) si $\nu=1$; $A=1$; $B=n$; $C=0$; etc.

2.) si $\nu=2$; $A=1+\frac{1}{2}n^2$; $B=2n$; $C=\frac{1}{2}nn$; $D=0$; etc.

3.) si $\nu=3$; $A=1+\frac{3}{2}n^2$; $B=3n(1+\frac{1}{4}n^2)$; $C=\frac{3}{2}n^2$; $D=\frac{1}{4}n^3$; $E=0$; etc.

4.) si $\nu=4$; $A=1+\frac{6}{2}n^2+\frac{3}{8}n^4$; $B=4n(1+\frac{3}{4}n^2)$; $C=3n^2(1+\frac{1}{6}n^2)$; $D=n^3$; $E=\frac{1}{8}n^4$; $F=0$; etc.

Hi autem casus nihil habent difficultatis. Ad usum sequentem tantum juvabit primum terminum absolutum A notasse:

si $\nu=1$; $A=1$;

si $\nu=2$; $A=1+\frac{2.1}{2.2}n^2$;

si $\nu=3$; $A=1+\frac{3.2}{2.2}n^2$;

si $\nu=4$; $A=1+\frac{4.3}{2.2}n^2+\frac{4.3.2.1}{2.2.4.4}n^4$;

si $\nu=5$; $A=1+\frac{5.4}{2.2}n^2+\frac{5.4.3.2}{2.2.4.4}n^4$;

si $\nu=6$; $A=1+\frac{6.5}{2.2}n^2+\frac{6.5.4.3}{2.2.4.4}n^4+\frac{6.5.4.3.2.1}{2.2.4.4.6.6}n^6$;

si $\nu=7$; $A=1+\frac{7.6}{2.2}n^2+\frac{7.6.5.4}{2.2.4.4}n^4+\frac{7.6.5.4.3.2}{2.2.4.4.6.6}n^6$;

etc.

Exemplum 2.

286. *Formulae $\dfrac{\partial\Phi}{(1+n\cos.\Phi)^\mu}$ integrale per seriem evolvere.*

Posito $\frac{1}{(1+n\cos.\Phi)^{\mu}} = A + B\cos.\Phi + C\cos.2\Phi$
$+ D\cos.3\Phi + E\cos.4\Phi + \text{etc.}$

ex praecedentibus formulis ponendo $\nu = -\mu$ erit

$$A = 1 + \frac{\mu(\mu+1)}{2\cdot 2}n^2 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{2\cdot 2\cdot 4\cdot 4}n^4 + \frac{\mu(\mu+\ )(\mu+2)(\mu+3)(\mu+4)(\mu+5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6 + \text{etc.}$$

$$B = -\mu n\left(1 + \frac{(\mu+1)(\mu+2)}{2\cdot 4}n^2 + \frac{(\mu+3)(\mu+4)}{4\cdot 6}Pn^2 + \frac{(\mu+5)(\mu+6)}{6\cdot 8}Pn^2 + \text{etc.}\right);$$

$$C = \frac{\mu(\mu+1)}{1\cdot 2}\cdot\frac{n^2}{2}\left(1 + \frac{(\mu+2)(\mu+3)}{2\cdot 6}n^2 + \frac{(\mu+4)(\mu+5)}{4\cdot 8}Pn^2 + \frac{(\mu+6)(\mu+7)}{6\cdot 10}Pn^2 + \text{etc.}\right)$$

$$D = -\frac{\mu(\mu+1)(\mu+2)}{1\cdot 2\cdot 3}\cdot\frac{n^3}{4}\left(1 + \frac{(\mu+3)(\mu+4)}{2\cdot 8}n^2 + \frac{(\mu+5)(\mu+6)}{4\cdot 10}Pn^2 + \frac{(\mu+7)(\mu+8)}{6\cdot 12}Pn^2 + \text{etc.}\right);$$

etc.

ubi ut ante in quaque serie P terminum praecedentem denotat. Hi autem coëfficientes ita a se invicem pendent, ut sit

$$B = \frac{-2(\mu-2)}{n}\int An\partial n - 2An \quad \text{et}$$

$$C = \frac{2B + 2\mu An}{(\mu-2)n}; \qquad D = \frac{4C + (\mu+1)Bn}{(\mu-3)n};$$

$$E = \frac{6D + (\mu+2)Cn}{(\mu-4)n}; \qquad F = \frac{8E + (\mu+3)Dn}{(\mu-5)n};$$

$$G = \frac{10F + (\mu+4)En}{(\mu-6)n}; \qquad H = \frac{12G + (\mu+5)Fn}{(\mu-7)n};$$

etc.

Ubi incommodo, quando μ est numerus integer, supra jam remedium est allatum. Hic igitur praecipue investigamus quomodo coëfficientes cujusque casus ex casu praecedente determinari queant, quod ita fieri poterit. Cum sit

$$\frac{1}{(1+n\cos.\Phi)^{\mu}} = A + B\cos.\Phi + C\cos.2\Phi + D\cos.3\Phi + \text{etc.}$$

ponatur

$$\frac{f}{(1+n\cos.\phi)^{\mu+1}} = A' + B'\cos.\phi + C'\cos.2\phi + D'\cos.3\phi + \text{etc.}$$

haec igitur series per $1 + n\cos.\phi$ multiplicata in illam abire debet, est autem productum

$$\begin{array}{llll} A' & + B'\cos.\phi + & C'\cos.2\phi + & D'\cos.3\phi + \text{etc.} \\ & + A'n & + \tfrac{1}{2}B'n & + \tfrac{1}{2}C'n \\ + \tfrac{1}{2}B'n + \tfrac{1}{2}C'n & + \tfrac{1}{2}D'n & + \tfrac{1}{2}E'n & \end{array}$$

unde colligimus

$$B' = \frac{2(A-A')}{n}; \qquad C' = \frac{2(B-B')-2A'n}{n};$$
$$D' = \frac{2(C-C')-B'n}{n}; \qquad E' = \frac{2(D-D')-C'n}{n};$$

dummodo ergo coëfficiens A' constaret, sequentes B', C', D' etc. haberemus. Videamus igitur quomodo A' ex A determinari possit: quia est

$$A = 1 + \frac{\mu(\mu+1)}{2\cdot 2}n^2 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{2\cdot 2\cdot 4\cdot 4}n^4 + \text{etc.}$$
$$A' = 1 + \frac{(\mu+1)(\mu+2)}{2\cdot 2}n^2 + \frac{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2\cdot 2\cdot 4\cdot 4}n^4 + \text{etc.}$$

tractetur n ut variabilis, ac prior series per n^μ multiplicata differentietur, ut prodeat

$$\frac{\partial A n^\mu}{\partial n} = \mu n^{\mu-1} + \frac{\mu(\mu+1)(\mu+2)n^{\mu+1}}{2.\ 2} + \frac{\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2.\ 2.\ 4.\ 4}n^{\mu+3} + \text{etc.}$$

quae series manifesto est $= \mu n^{\mu-1}A'$; quocirca A' ita per A determinatur, ut sit

$$A' = \frac{\partial . A n^\mu}{\partial . n^\mu} = A + \frac{n\partial A}{\mu\partial n}.$$

Cum igitur pro casu $\mu = 1$ invenerimus

$$A = \frac{1}{\sqrt{(1 - nn)}}; \text{ ob } \frac{\partial A}{\partial n} = \frac{n}{(1 - nn)^{\frac{3}{2}}}; \text{ erit}$$

$$A' = \frac{1}{\sqrt{(1 - nn)}} + \frac{nn}{(1 - nn)^{\frac{3}{2}}} = \frac{1}{(1 - nn)^{\frac{3}{2}}}.$$

Hic jam est valor ipsius A pro $\mu = 2$, unde ob

$$\frac{\partial A}{\partial n} = \frac{3n}{(1 - nn)^{\frac{5}{2}}}, \text{ fiet pro } \mu = 3,$$

$$A = \frac{1}{(1 - nn)^{\frac{3}{2}}} + \frac{3nn}{2(1 - nn)^{\frac{5}{2}}} = \frac{1 + \frac{1}{2}nn}{(1 - nn)^{\frac{5}{2}}}.$$

Hoc modo si ulterius progrediamur, reperiemus:

$$\text{si } \mu = 1; \; A = \frac{1}{\sqrt{(1 - nn)}};$$

$$\text{si } \mu = 2; \; A = \frac{1}{(1 - nn)\sqrt{(1 - nn)}};$$

$$\text{si } \mu = 3; \; A = \frac{1 + \frac{1}{2}nn}{(1 - nn)^2\sqrt{(1 - nn)}};$$

$$\text{si } \mu = 4; \; A = \frac{1 + \frac{3}{2}nn}{(1 - nn)^3\sqrt{(1 - nn)}};$$

$$\text{si } \mu = 5; \; A = \frac{1 + 3nn + \frac{3}{8}n^4}{(1 - nn)^4\sqrt{(1 - nn)}}.$$

Corollarium 1.

287. Eodem modo etiam reliqui coëfficientes B′, C′ etc. ex analogis B, C etc. definientur, eruntque omnes istae relationes inter se similes, scilicet uti est

$$A' = \frac{\partial. A n^\mu}{\partial. n^\mu} = A + \frac{n \partial A}{\mu \partial n}, \text{ ita erit}$$
$$B' = \frac{\partial. B n^\mu}{\partial. n^\mu} = B + \frac{n \partial B}{\mu \partial n};$$
$$C' = \frac{\partial. C n^\mu}{\partial. n^\mu} = C + \frac{n \partial C}{\mu \partial n};$$
etc.

Corollarium 2.

288. At ante invenimus $B' = \frac{2(A - A')}{n}$, unde fiet
$$B' = -\frac{2 \partial A}{\mu \partial n} = B + \frac{n \partial B}{\mu \partial n}, \text{ hincque}$$
$$\mu B \partial n + n \partial B + 2 \partial A = 0:$$
multiplicetur per $n^{\mu - 1}$ ut sit
$$\partial. B n^\mu + 2 n^{\mu-1} \partial A = 0,$$
unde integrando
$$B n^\mu = -2 \int n^{\mu-1} \partial A = -2 n^{\mu-1} A + 2 (\mu - 1) \int A n^{\mu-2} \partial n:$$
ideoque
$$B = -\frac{2 A}{n} + \frac{2(\mu - 1)}{n^\mu} \int A n^{\mu-2} \partial n.$$
At ante habueramus
$$B = -2 A n - \frac{2(\mu - 2)}{n} \int A n \partial n.$$

Corollarium 3.

289. His valoribus aequatis, obtinetur aequatio inter A et n, qua quantitas A per n determinatur, erit enim
$$n^{-\mu} \int n^{\mu-1} \partial A = A n + \frac{(\mu - 2)}{n} \int A n \partial n:$$
unde per duplicem differentiationem prodit
$$(1 - nn) \partial\partial A + \frac{\partial n \partial A}{n} - 2(\mu + 1) n \partial n \partial A - \mu(\mu + 1) A \partial n^2 = 0.$$

Scholion 1.

290. Si hos valores ipsius A cum superioribus, ubi μ erat numerus integer negativus inter se compararemus, eximiam convenientiam deprehendemus.

Pro superioribus.	Pro his formulis.
si $\nu = 0$; $A = 1$	si $\mu = 1$; $A = \frac{1}{\sqrt{(1-nn)}}$
$\nu = 1$; $A = 1$	$\mu = 2$; $A = \frac{1}{(1-nn)\sqrt{(1-nn)}}$
$\nu = 2$; $A = 1 + \frac{1}{2}n^2$	$\mu = 3$; $A = \frac{1 + \frac{1}{2}nn}{(1-nn)^2\sqrt{(1-nn)}}$
$\nu = 3$; $A = 1 + \frac{3}{2}n^2$	$\mu = 4$; $A = \frac{1 + \frac{3}{2}nn}{(1-nn)^3\sqrt{(1-nn)}}$
$\nu = 4$; $A = 1 + 3n^2 + \frac{3}{8}n^4$	$\mu = 5$; $A = \frac{1 + 3nn + \frac{3}{8}n^4}{(1-nn)^4\sqrt{(1-nn)}}$
	etc.

unde concludimus, si fuerit

$$(1 + n\cos.\Phi)^{\nu} = A + B\cos.\Phi + C\cos.2\Phi + \text{etc.}$$
$$(1 + n\cos.\Phi)^{-\nu-1} = \mathfrak{A} + \mathfrak{B}\cos.\Phi + \mathfrak{C}\cos.2\Phi + \text{etc.}$$

fore $\mathfrak{A} = \frac{A}{(1-nn)^{\nu}\sqrt{(1-nn)}}$.

Quare cum pro casibus, quibus ν est numerus integer positivus, valor ipsius A facile definiatur, etiam pro casibus, quibus est negativus, inde expedite assignabitur.

Scholion 2.

291. Cum pro casu $\mu = 1$, supra valores singularum litterarum A, B, C, D etc. sint inventi, scilicet posito brevitatis gratia $\frac{1-\sqrt{(1-nn)}}{n} = m$,

$A = \frac{1}{\sqrt{(1-nn)}}$; $B = \frac{2m}{\sqrt{(1-nn)}}$; $C = \frac{2mm}{\sqrt{(1-nn)}}$; $D = \frac{2m^3}{\sqrt{(1-nn)}}$;

et in genere pro termino quocunque $N = \frac{2m^\lambda}{\sqrt{(1-nn)}}$: si pro simili termino casu $\mu = 2$, scribamus N', erit $N' = \frac{\partial. Nn}{\partial n}$. Nunc autem est $\frac{\partial. Nn}{\partial n} = \frac{2m^\lambda}{(1-nn)^{\frac{3}{2}}} + \frac{2\lambda n m^{\lambda-1}\partial m}{\partial n \sqrt{(1-nn)}}$: tum vero $\frac{\partial m}{\partial n} = \frac{m}{n\sqrt{(1-nn)}}$, unde colligimus

$$N' = \frac{2m^\lambda}{(1-nn)^{\frac{3}{2}}} + \frac{2\lambda m^\lambda}{1-nn} = \frac{2m^\lambda[1+\lambda\sqrt{(1-nn)}]}{(1-nn)\sqrt{(1-nn)}}.$$

Quare si statuamus:

$$\frac{1}{(1+n\cos.\phi)^2} = A + B\cos.\phi + C\cos.2\phi + D\cos.3\phi + E\cos.4\phi + \text{etc.}$$

erit

$$A = \frac{1}{(1-nn)^{\frac{3}{2}}};\ B = \frac{2m[1+\sqrt{(1-nn)}]}{(1-nn)^{\frac{3}{2}}};\ C = \frac{2m^2[1+2\sqrt{(1-nn)}]}{(1-nn)^{\frac{3}{2}}};$$
$$D = \frac{2m^3[1+3\sqrt{(1-nn)}]}{(1-nn)^{\frac{3}{2}}};\ \text{etc.}$$

Verum si exponens μ fuerit numerus fractus, coëfficientes A, B, C, D, E, etc. haud aliter, ac per series supra datas definiri posse videntur. Primus autem A modo peculiari vero proxime assignari potest, quemadmodum in problemate sequente docemus.

Problema 34.

292. Pro evolutione formulae $(1+n\cos.\phi)^\nu$ in hujusmodi seriem $A + B\cos.\phi + C\cos.2\phi + D\cos.3\phi + E\cos.4\phi +$ etc. terminum absolutum A vero proxime definire.

Solutio.

Cum necessario sit $n < 1$, series quidem supra inventa pro A convergit, verum si n parum ab unitate deficiat, permultos terminos actu evolvi oportet, antequam valor ipsius A satis exacte prodeat, praecipue si ν fuerit numerus mediocriter magnus tam positivus quam negativus. Quoniam tamen posita evolutione hujus formulae $(1 + n\cos.\phi)^{-\nu-1} = \mathfrak{A} + \mathfrak{B}\cos.\phi + \mathfrak{C}\cos.2\phi +$ etc. a termino $\mathfrak{A}$ ille A ita pendet, ut sit $A = (1 - nn)^{\nu+\frac{1}{2}}\mathfrak{A}$, pro hoc termino A inveniendo duplicem habemus seriem

$$A = 1 + \frac{\nu(\nu-1)}{2.2}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2.\ 2.\ 4.\ 4}n^4 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2.\ 2.\ 4.\ 4.\ 6.\ 6}n^6 + \text{etc.}$$

$$A = (1 - nn)^{\nu+\frac{1}{2}}\Big(1 + \frac{(\nu+1)(\nu+2)}{2.\ 2}n^2 + \frac{(\nu+1)(\nu+2)(\nu+3)(\nu+4)}{2.\ 2.\ 4.\ 4}n^4 + \frac{(\nu+1)(\nu+2)(\nu+3)(\nu+4)(\nu+5)(\nu+6)}{2.\ 2.\ 4.\ 4.\ 6.\ 6}n^6 + \text{etc.}\Big)$$

quovis casu ea usurpari potest, quae magis convergit. Verum tamen quia reliqui coëfficientes B, C, D, E, etc. tandem convergere debent, hinc alia via ad valorem ipsius A appropinquandi patet. Quoniam enim hi coëfficientes alternatim per pares et impares potestates ipsius n definiuntur, sumto angulo quocunque α erit

$$(1 + n\cos.\alpha)^\nu = A + B\cos.\alpha + C\cos.2\alpha + D\cos.3\alpha + E\cos.4\alpha + \text{etc. et}$$

$$(1 - n\cos.\alpha)^\nu = A - B\cos.\alpha + C\cos.2\alpha - D\cos.3\alpha + E\cos.4\alpha - \text{etc.}$$

His igitur additis prodit

$$\tfrac{1}{2}(1 + n\cos.\alpha)^\nu + \tfrac{1}{2}(1 - n\cos.\alpha)^\nu = A + C\cos.2\alpha + E\cos.4\alpha + G\cos.6\alpha + \text{etc.}$$

ubi si pro α scribamus $90^\circ - \alpha'$) erit

$$\tfrac{1}{2}(1 + n\sin.\alpha)^\nu + \tfrac{1}{2}(1 - n\sin.\alpha)^\nu = A - C\cos.2\alpha + E\cos.4\alpha - G\cos.6\alpha + \text{etc.}$$

unde his additis, semissis terminorum denuo tollitur. Formemus plures hujusmodi expressiones, ac ponamus brevitatis gratia:

$$\tfrac{1}{4}(1+n\cos.\alpha)^\nu+\tfrac{1}{4}(1-n\cos.\alpha)^\nu+\tfrac{1}{4}(1+n\sin.\alpha)^\nu+\tfrac{1}{4}(1-n\sin.\alpha)^\nu=\mathfrak{A}$$
$$\tfrac{1}{4}(1+n\cos.\beta)^\nu+\tfrac{1}{4}(1-n\cos.\beta)^\nu+\tfrac{1}{4}(1+n\sin.\beta)^\nu+\tfrac{1}{4}(1-n\sin.\beta)^\nu=\mathfrak{B}$$
$$\tfrac{1}{4}(1+n\cos.\gamma)^\nu+\tfrac{1}{4}(1-n\cos.\gamma)^\nu+\tfrac{1}{4}(1+n\sin.\gamma)^\nu+\tfrac{1}{4}(1-n\sin.\gamma)^\nu=\mathfrak{C}$$
etc.

et pro coëfficientibus B, C, D, E, etc. scribamus respective (1), (2), (3), (4), etc. quo facilius terminos ab initio quantumvis remotos repraesentare possimus. Habebimus ergo

$$\mathfrak{A}=A+(4)\cos.4\alpha+(8)\cos.8\alpha+(12)\cos.12\alpha+\text{etc.}$$
$$\mathfrak{B}=A+(4)\cos.4\beta+(8)\cos.8\beta+(12)\cos.12\beta+\text{etc.}$$
$$\mathfrak{C}=A+(4)\cos.4\gamma+(8)\cos.8\gamma+(12)\cos.12\gamma+\text{etc.}$$
etc.

Atque hinc sequentes approximationes adipiscimur.

I. Si capiamus $4\alpha=\frac{\pi}{2}$ seu $\alpha=\frac{\pi}{8}$, prodit

$$\mathfrak{A}=A-(8)+(16)-(24)+\text{etc.}\quad \text{Ergo}$$
$$A=\mathfrak{A}+(8)-(16)+(24)-\text{etc.}$$

Quare si termini (8) et sequentes ob parvitatem contemni queant, erit satis exacte $A=\mathfrak{A}$.

II. Sumamus duas series ac statuamus $4\alpha=\frac{\pi}{4}$ et $4\beta=\frac{5\pi}{4}$ ut sit $\alpha=\frac{\pi}{16}$ et $\beta=\frac{5\pi}{16}$ erit $\cos.4\alpha+\cos.4\beta=0$, $\cos.8\alpha+\cos.8\beta=0$, $\cos.12\alpha+\cos.12\beta=0$ et $\cos.16\alpha+\cos.16\beta=-2$, unde sequitur:

$$\mathfrak{A}+\mathfrak{B}=2A-2(16)+2(32)-2(48)+\text{etc.}$$

ideoque

$$A=\tfrac{1}{2}(\mathfrak{A}+\mathfrak{B})+(16)-(32)+\text{etc.}$$

ubi numeri (16), (32) plerumque tam erunt parvi, ut negligi queant.

III. Addamus tres series, ac statuamus $4\alpha=\frac{\pi}{6}$; $4\beta=\frac{3\pi}{6}$; $4\gamma=\frac{5\pi}{6}$; ut sit $\alpha=\frac{\pi}{24}$; $\beta=\frac{\pi}{8}$; $\gamma=\frac{5\pi}{24}$; eritque

$\cos. 4\alpha + \cos. 4\beta + \cos. 4\gamma = 0$ | $\cos. 16\alpha + \cos. 16\beta + \cos. 16\gamma = 0$
$\cos. 8\alpha + \cos. 8\beta + \cos. 8\gamma = 0$ | $\cos. 20\alpha + \cos. 20\beta + \cos. 20\gamma = 0$
$\cos. 12\alpha + \cos. 12\beta + \cos. 12\gamma = 0$ | $\cos. 24\alpha + \cos. 24\beta + \cos. 24\gamma = -3$;

unde colligitur

$$A = \tfrac{1}{3}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C}) + (24) - (48) + \text{etc.}$$

IV. Si haec determinatio non satis exacta videatur, addantur quatuor ejusmodi expressiones $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, sitque

$$4\alpha = \tfrac{\pi}{8};\ 4\beta = \tfrac{3\pi}{8};\ 4\gamma = \tfrac{5\pi}{8};\ 4\delta = \tfrac{7\pi}{8};$$

ac reperietur

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D} = 4A - 4(32) + 4(64) - \text{etc.}$$

ergo multo propius

$$A = \tfrac{1}{4}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D}).$$

Corollarium 1.

293. Ex invento autem valore A sequens B satis expedite reperitur, cum sit

$$B = \frac{2(\nu+2)}{n}\int A n \partial n - 2An.$$

Quatenus ergo in A ingreditur membrum $(1 \pm n \cos. \alpha)^\nu$, vel $(1 + nf)^\nu$, dum f omnes illos sinus et cosinus complectitur, inde pro B oritur

$$\frac{2(\nu+2)}{n}\int n\partial n(1+nf)^\nu - 2n(1+nf)^\nu = \frac{2 - 2(1-\nu nf)(1+nf)^\nu}{(\nu+1)nff}$$

Corollarium 2.

294. Cognitis autem coëfficientibus A et B, quemadmodum sequentes omnes ex illis derivari possint, supra ostendimus. Iis vero inventis integratio formulae $\partial\Phi(1 + n\cos.\Phi)^\nu$ per se est manifesta.

Problema 35.

295. Integrale formulae $\partial \phi\, l(1 + n \cos.\phi)$ per seriem secundum sinus angulorum ϕ, 2ϕ, 3ϕ, etc. progredientem evolvere.

Solutio.

Cum sit

$$l(1 + n \cos.\phi) = n \cos.\phi - \tfrac{1}{2} n^2 \cos.\phi^2 + \tfrac{1}{3} n^3 \cos.\phi^3 - \tfrac{1}{4} n^4 \cos.\phi^4 + \text{etc.}$$

erit his potestatibus ad simplices cosinus reductis.

$$l(1+n\cos.\phi) = \begin{array}{lllll} & +n\cos.\phi & -\tfrac{1}{2}\cdot\tfrac{1}{2}n^2\cos.2\phi & +\tfrac{1}{3}\cdot\tfrac{1}{4}n^3\cos.3\phi & -\tfrac{1}{4}\cdot\tfrac{1}{8}n^4\cos.4\phi \\ -\tfrac{1}{2}\cdot\tfrac{1}{2}n^2 & +\tfrac{1}{3}\cdot\tfrac{3}{4}n^3 & -\tfrac{1}{4}\cdot\tfrac{4}{8}n^4 & +\tfrac{1}{5}\cdot\tfrac{5}{16}n^5 & \\ -\tfrac{1}{4}\cdot\tfrac{3}{8}n^4 & +\tfrac{1}{5}\cdot\tfrac{10}{16}n^5 & -\tfrac{1}{6}\cdot\tfrac{15}{32}n^6 & & \\ -\tfrac{1}{6}\cdot\tfrac{10}{32}n^6 & +\tfrac{1}{7}\cdot\tfrac{35}{64}n^7 & & & \\ -\tfrac{1}{8}\cdot\tfrac{35}{128}n^8. & & & & \end{array}$$

Quare si ponamus

$$l(1 + n\cos.\phi) = -A + B\cos.\phi - C\cos.2\phi + D\cos.3\phi - \text{etc.}$$

erit

$$A = +\frac{1}{2}\cdot\frac{n^2}{2} + \frac{1.3}{2.4}\cdot\frac{n^4}{4} + \frac{1.3.5}{2.4.6}\cdot\frac{n^6}{6} + \frac{1.3.5.7}{2.4.6.8}\cdot\frac{n^8}{8} + \text{etc.}$$

considerato ergo numero n ut variabili, erit

$$\frac{n\,\partial A}{\partial n} = \tfrac{1}{2}nn + \frac{1.3}{2.4}n^4 + \frac{1.3.5}{2.4.6}n^6 + \text{etc.} = \frac{1}{\sqrt{(1-nn)}} - 1.$$

Hinc $\partial A = \frac{\partial n}{n\sqrt{(1-nn)}} - \frac{\partial n}{n}$, unde integratio praebet;

$$A = l\,\frac{1-\sqrt{(1-nn)}}{n} - ln + C = l\,\frac{2-2\sqrt{(1-nn)}}{nn}:$$

hoc enim modo evanescente n, fit $A = l\,1 = 0$. Tum vero erit

$$\tfrac{1}{2}B = \tfrac{1}{2}\cdot n + \frac{1.3}{2.4}\cdot\frac{n^3}{3} + \frac{1.3.5}{2.4.6}\cdot\frac{n^5}{5} + \text{etc.}$$

unde differentiatio praebet

$$\frac{nn\partial B}{2\partial n} = \tfrac{1}{2}nn + \frac{1.3}{2.4}n^4 + \frac{1.3.5}{2.4.6}n^6 + \text{etc.} = \frac{1}{\sqrt{(1-nn)}} - 1:$$

ergo $\frac{1}{2}\partial B = \frac{\partial n}{nn\sqrt{(1-nn)}} - \frac{\partial n}{nn}$, et integrando

$$\tfrac{1}{2}B = \frac{-\sqrt{(1-nn)}}{n} + \frac{1}{n} + C = \frac{1-\sqrt{(1-nn)}}{n}$$

integrali ita determinato, ut evanescat posito $n = 0$.

Quocirca pro binis primis terminis habemus:

$$A = l\frac{2-2\sqrt{(1-nn)}}{nn} \text{ et } B = \frac{2-2\sqrt{(1-nn)}}{n};$$

ut sit $A = l\frac{B}{n}$. At pro reliquis differentiemus aequationem assumtam

$$\frac{-n\partial\Phi\sin.\Phi}{1+n\cos.\Phi} = -B\partial\Phi\sin.\Phi + 2C\partial\Phi\sin.2\Phi - 3D\partial\Phi\sin.3\Phi + 4E\partial\Phi\sin.4\Phi - \text{etc.}$$

seu

$$0 = \frac{n\sin.\Phi}{1+n\cos.\Phi} - B\sin.\Phi + 2C\sin.2\Phi - 3D\sin.3\Phi + 4E\sin.4\Phi - \text{etc.}$$

Quare per $2 + 2n\cos.\Phi$ multiplicando prodit:

$$\begin{array}{rlllll} 0 = 2n\sin.\Phi & -2B\sin.\Phi & +4C\sin.2\Phi & -6D\sin.3\Phi & +8E\sin.4\Phi & -\text{etc.} \\ & & -Bn & +2Cn & -3Dn & \\ & +2Cn & -3Dn & +4En & -5Fn & \end{array}$$

unde colligimus:

$$C = \frac{B-n}{n};\ D = \frac{4C-Bn}{3n};\ E = \frac{6D-2Cn}{4n};\ F = \frac{8E-3Dn}{5n};$$

Cum igitur sit $B = \frac{2-2\sqrt{(1-nn)}}{n}$, erit $C = \frac{2-nn-2\sqrt{(1-nn)}}{nn}$, seu $C = \left(\frac{1-\sqrt{(1-nn)}}{n}\right)^2$: tum vero

$$D = \tfrac{2}{3}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^3;\ E = \tfrac{2}{4}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^4;\ F = \tfrac{2}{5}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^5;\ \text{etc.}$$

Hinc si brevitatis gratia ponamus $\frac{1-\sqrt{(1-nn)}}{n} = m$, erit

$$l(1+n\cos.\Phi) = -l\frac{2m}{n} + \tfrac{2}{1}m\cos.\Phi - \tfrac{2}{2}m^2\cos.2\Phi + \tfrac{2}{3}m^3\cos.3\Phi - \tfrac{2}{4}m^4\cos.4\Phi + \text{etc.}$$

ideoque integrale quaesitum:

$$\int \partial \Phi\, l(1+n\cos.\Phi) = \text{Const.} - \Phi\, l\frac{2m}{n} + \tfrac{2}{1} m \sin.\Phi - \tfrac{2}{4} m^2 \sin. 2\Phi$$
$$+ \tfrac{2}{9} m^3 \sin. 3\Phi - \tfrac{2}{16} m^4 \sin. 4\Phi + \tfrac{2}{25} m^5 \sin. 5\Phi - \text{etc.}$$

Corollarium 1.

296. Quodsi ergo ponamus $n = 1$, erit $m = 1$ et

$$l(1+\cos.\Phi) = -l2 + \tfrac{2}{1}\cos.\Phi - \tfrac{2}{2}\cos. 2\Phi + \tfrac{2}{3}\cos. 3\Phi - \tfrac{2}{4}\cos. 4\Phi + \text{etc.}$$

et

$$l(1-\cos.\Phi) = -l2 - \tfrac{2}{1}\cos.\Phi - \tfrac{2}{2}\cos. 2\Phi - \tfrac{2}{3}\cos. 3\Phi - \tfrac{2}{4}\cos. 4\Phi - \text{etc.}$$

Cum jam sit

$1 + \cos.\Phi = 2\cos.\tfrac{1}{2}\Phi^2$ et $1 - \cos.\Phi = 2\sin.\tfrac{1}{2}\Phi^2$, erit

$l\cos.\tfrac{1}{2}\Phi = -l2 + \cos.\Phi - \tfrac{1}{2}\cos. 2\Phi + \tfrac{1}{3}\cos. 3\Phi - \tfrac{1}{4}\cos. 4\Phi + \text{etc.}$ et

$l\sin.\tfrac{1}{2}\Phi = -l2 - \cos.\Phi - \tfrac{1}{2}\cos. 2\Phi - \tfrac{1}{3}\cos. 3\Phi - \tfrac{1}{4}\cos. 4\Phi - \text{etc.}$

hinc

$$l\,\text{tang.}\tfrac{1}{2}\Phi = -2\cos.\Phi - \tfrac{2}{3}\cos. 3\Phi - \tfrac{2}{5}\cos. 5\Phi - \tfrac{2}{7}\cos. 7\Phi - \text{etc.}$$