CAPUT IV.

DE INTEGRATIONE FORMULARUM LOGARITHMICARUM ET EXPONENTIALIUM.

Problema 18.

189.

Si X designet functionem algebraicam ipsius x, invenire integrale formulae $X\,\partial x\, l\, x$.

Solutio.

Quaeratur integrale $\int X\,\partial x$, quod sit $= Z$, et cum quantitatis $Z\,l\,x$ differentiale sit $= \partial Z\, l x + \frac{Z\,\partial x}{x}$, erit $Z\,l x = \int \partial Z\, l x + \int \frac{Z\,\partial x}{x}$: ideoque

$$\int \partial Z\, l\, x = \int X\,\partial x\, l\, x = Z\, l\, x - \int \frac{Z\,\partial x}{x}.$$

Sicque integratio formulae propositae reducta est ad integrationem hujus $\frac{Z\,\partial x}{x}$, quae, si Z fuerit functio algebraica ipsius x non amplius logarithmum involvit, ideoque per praecedentes regulas tractari poterit. Sin autem $\int X\,\partial x$ algebraice exhiberi nequeat, hinc nihil subsidii nascitur, expedietque indicatione integralis $\int X\,\partial x\, l x$ acquiescere, ejusque valorem per approximationem investigare.

Nisi forte sit $X = \frac{1}{x}$, quo casu manifesto dat $\int \frac{\partial x}{x}\, l x = \frac{1}{2}(l\, x)^2 + C$.

Corollarium 1.

190. Eodem modo, si denotante V functionem quamcunque ipsius x, proposita sit formula $X\partial x\, l V$, erit existente $\int X\,\partial x = Z$, ejus integrale $= Z\, l\, V - \int \frac{Z\,\partial V}{V}$, sicque ad formulam algebraicam reducitur, si modo Z algebraice detur.

Corollarium 2.

191. Pro casu singulari $\frac{\partial x}{x} lx$ notare licet, si posito $lx = u$, fuerit U functio quaecunque algebraica ipsius u, integrationem hujus formulae $\frac{U \partial x}{x}$ non fore difficilem, quia ob $\frac{\partial x}{x} = \partial u$ abit in $U \partial u$, cujus integratio ad praecedentia capita refertur.

Scholion.

192. Haec reductio innititur isti fundamento, quod cum sit $\partial . xy = y \partial x + x \partial y$, hinc vicissim fiat $xy = \int y \partial x + \int x \partial y$, ideoque $\int y \partial x = xy - \int x \partial y$, ita ut hoc modo in genere integratio formulae $y \partial x$ ad integrationem formulae $x \partial y$ reducatur. Quod si ergo, proposita quacunque formula $V \partial x$, functio V in duos factores, puta $V = PQ$, resolvi queat, ita ut integrale $\int P \partial x = S$ assignari queat, ob $P \partial x = \partial S$, erit $V \partial x = PQ \partial x = Q \partial S$, hincque $\int V \partial x = QS - \int S \partial Q$. Hujusmodi reductio insignem usum affert, cum formula $\int S \partial Q$ simplicior fuerit quam proposita $\int V \partial x$, eaque insuper simili modo ad simpliciorem reduci queat. Interdum etiam commode evenit, ut hac methodo tandem ad formulam propositae similem perveniatur, quo casu integratio pariter obtinetur. Veluti si ulteriori reductione inveniremus $\int S \partial Q = T + n \int V \partial x$, foret utique $\int V \partial x = QS - T - n \int V \partial x$, hincque $\int V \partial x = \frac{QS - T}{n + 1}$. Tum igitur talis reductio insignem praestat usum, cum vel ad formulam simpliciorem, vel ad eandem perducit. Atque ex hoc principio praecipuos casus, quibus formula $X \partial x \, lx$ vel integrationem admittit, vel per seriem commode exhiberi potest, evolvamus.

Exemplum 1.

193. *Formulae differentialis* $x^n \partial x \, l x$ *integrale invenire denotante* n *numerum quemcunque.*

Cum sit $\int x^n \partial x = \frac{1}{n+1} x^{n+1}$, erit

$$\int x^n \partial x \, l x = \frac{1}{n+1} x^{n+1} \, l x - \int \frac{1}{n+1} x^{n+1} \partial . l x$$

$$= \frac{1}{n+1} x^{n+1} \, l x - \frac{1}{n+1} \int x^n \partial x = \frac{1}{n+1} x^{n+1} \, l x - \frac{1}{(n+1)^2} x^{n+1};$$

ideoque

$$\int x^n \partial x \, l x = \frac{1}{n+1} x^{n+1} \left(l x - \frac{1}{n+1} \right).$$

Sicque haec formula absolute est integrabilis.

Corollarium 1.

194. Casus simpliciores, quibus n est numerus integer sive positivus sive negativus, tenuisse juvabit:

$$\int \partial x \, l x = x \, l x - x; \qquad \int \frac{\partial x}{x x} \, l x = -\frac{1}{x} \, l x - \frac{1}{x};$$

$$\int x \, \partial x \, l x = \tfrac{1}{2} x x \, l x - \tfrac{1}{4} x x; \int \frac{\partial x}{x^3} \, l x = -\frac{1}{2xx} \, l x - \frac{1}{4xx};$$

$$\int x^2 \, \partial x \, l x = \tfrac{1}{3} x^3 \, l x - \tfrac{1}{9} x^3; \int \frac{\partial x}{x^4} \, l x = -\frac{1}{3 x^3} \, l x - \frac{1}{9 x^3};$$

$$\int x^3 \, \partial x \, l x = \tfrac{1}{4} x^4 \, l x - \tfrac{1}{16} x^4; \int \frac{\partial x}{x^5} \, l x = -\frac{1}{4 x^4} \, l x - \frac{1}{16 x^4}.$$

Corollarium 2.

195. Casum $\int \frac{\partial x}{x} \, l x = \frac{1}{2} (l x)^2$, qui est omnino singularis, jam supra annotavimus, sequitur vero etiam ex reductione ad eandem formulam. Namque per superiorem reductionem habemus

$$\int \frac{\partial x}{x} \, l x = l x \, . \, l x - \int l x \, . \, \partial \, l x = (l x)^2 - \int \frac{\partial x}{x} \, l x:$$

hincque

$$2 \int \frac{\partial x}{x} \, l x = (l x)^2, \text{ consequenter } \int \frac{\partial x}{x} \, l x = \tfrac{1}{2} (l x)^2.$$

Exemplum 2.

196. *Formulae* $\frac{\partial x}{1-x} \, l x$ *integrale per seriem exprimere.*

Reductione ante adhibita parum lucramur, prodit enim:

$$\int \frac{\partial x}{1-x} \, l x = l \frac{1}{1-x} \, . \, l x - \int \frac{\partial x}{x} \, l \frac{1}{1-x}.$$

Cum autem sit

$$l\frac{1}{1-x}=x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3+\tfrac{1}{4}x^4+\text{etc. erit}$$

$$\int\frac{\partial x}{x}\,l\frac{1}{1-x}=x+\tfrac{1}{4}x^2+\tfrac{1}{9}x^3+\tfrac{1}{16}x^4+\tfrac{1}{25}x^5+\text{etc.}$$

ideoque

$$\int\frac{\partial x}{1-x}\,lx=l\frac{1}{1-x}\,.\,lx-x-\tfrac{1}{4}x^2-\tfrac{1}{9}x^3-\tfrac{1}{16}x^4-\tfrac{1}{25}x^5-\text{etc.}$$

quod integrale evanescit casu $x=0$, etsi enim lx tum in infinitum abit, tamen $l\frac{1}{1-x}=x+\frac{1}{2}x^2+\frac{1}{3}x^3+$etc. ita evanescit, ut etiam si per lx multiplicetur, in nihilum abeat, est enim in genere $x^n\,lx=0$ posito $x=0$, dum n numerus positivus.

Corollarium 1.

197. Si ponamus $1-x=u$, fit

$$\frac{\partial x}{1-x}\,lx=-\frac{\partial u}{u}\,l(1-u)=\frac{\partial u}{u}\,l\frac{1}{1-u},$$

ideoque

$$\int\frac{\partial x}{1-x}\,lx=C+u+\tfrac{1}{4}u^2+\tfrac{1}{9}u^3+\tfrac{1}{16}u^4+\tfrac{1}{25}u^5+\text{etc.}$$

quae, ut etiam casu $x=0$ seu $u=1$, evanescat, capi debet

$$C=-1-\tfrac{1}{4}-\tfrac{1}{9}-\tfrac{1}{16}-\tfrac{1}{25}-\text{etc.}=-\tfrac{1}{6}\pi\pi.$$

Corollarium 2.

198. Sumto ergo $1-x=u$ seu $x+u=1$, aequales erunt inter se hae expressiones:

$$-lx\,.\,lu-x-\tfrac{1}{4}x^2-\tfrac{1}{9}x^3-\tfrac{1}{16}x^4-\text{etc.}$$
$$=-\tfrac{1}{6}\pi^2+u+\tfrac{1}{4}u^2+\tfrac{1}{9}u^3+\text{etc.}$$

seu erit

$$\tfrac{1}{6}\pi^2-lx\,.\,lu=x+u+\tfrac{1}{4}(x^2+u^2)+\tfrac{1}{9}(x^3+u^3)$$
$$+\tfrac{1}{16}(x^4+u^4)+\text{etc.}$$

Corollarium 3.

199. Haec series maxime convergit, ponendo $x=u=\frac{1}{2}$: hoc ergo casu habebimus

$$\tfrac{1}{6}\pi - (l2)^2 = 1 + \frac{1}{2.4} + \frac{1}{4.9} + \frac{1}{8.16} + \frac{1}{16.25} + \frac{1}{32.36} + \text{etc.}$$

Hujus ergo seriei

$$x + \tfrac{1}{4}x^2 + \tfrac{1}{9}x^3 + \tfrac{1}{16}x^4 + \tfrac{1}{25}x^5 + \text{etc.}$$

summa habetur non solum casu $x = 1$, quo est $= \frac{\pi\pi}{6}$, sed etiam casu $x = \frac{1}{2}$, quo est $= \frac{1}{12}\pi^2 - \frac{1}{2}(l\,2)^2$.

Corollarium 4.

200. Si ponamus $x = \frac{1}{3}$, et $u = \frac{2}{3}$, erit hujus seriei

$$1 + \frac{5}{3^2.4} + \frac{9}{3^3.9} + \frac{17}{3^4.16} + \frac{33}{3^5.25} + \frac{65}{3^6.36} + \text{etc.}$$

cujus terminus generalis $= \dfrac{1 + 2^n}{3^n\,nn}$, summa $= \frac{1}{6}\pi^2 - l\,3.\,l\frac{3}{2}$: neque vero hinc seriei $x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \frac{1}{16}x^4 + \text{etc.}$ binos casus $x = \frac{1}{3}$ et $x = \frac{2}{3}$ seorsim summare licet.

Exemplum 3.

201. *Formulae* $\frac{\partial x}{(1-x)^2}\,l\,x$ *integrale invenire, idemque in seriem convertere.*

Cum sit $\int \frac{\partial x}{(1-x)^2} = \frac{1}{1-x}$, erit

$\int \frac{\partial x}{(1-x)^2}\,l\,x = \frac{1}{1-x}\,l\,x - \int \frac{\partial x}{x\,(1-x)}$: at ob

$\frac{1}{x\,(1-x)} = \frac{1}{x} + \frac{1}{1-x}$, fit $\int \frac{\partial x}{x\,(1-x)} = l\,x + l\,\frac{1}{1-x}$,

unde colligimus integrale

$$\int \frac{\partial x}{(1-x)^2}\,lx = \frac{l\,x}{1-x} - lx - l\,\frac{1}{1-x} = \frac{x\,l\,x}{1-x} - l\,\frac{1}{1-x};$$

ita sumtum, ut evanescat posito $x = 0$.

Jam pro serie commodissime invenienda, statuatur $1 - x = u$, et nostra formula fit

$$= \frac{-\partial u}{u\,u}\,l(1-u) = \frac{\partial u}{u\,u}\,l\frac{1}{1-u} = \frac{\partial u}{u\,u}\,(u + \tfrac{1}{2}u^2 + \tfrac{1}{3}u^3 + \tfrac{1}{4}u^4 + \tfrac{1}{5}u^5 + \text{etc.})$$

Quocirca integrando nanciscimur:

$$\int \frac{\partial x}{(1-x)^2}\, l x = C + lu + \frac{u}{1.2} + \frac{uu}{2.3} + \frac{u^3}{3.4} + \frac{u^4}{4.5} + \text{etc.}$$

quae expressio ut etiam evanescat, facto $x = 0$ seu $u = 1$, oportet sit:

$$C = -\frac{1}{1.2} - \frac{1}{2.3} - \frac{1}{3.4} - \frac{1}{4.5} - \text{etc.} = -1.$$

Quare ob $x = 1 - u$, obtinebimus:

$$\frac{u}{1.2} + \frac{u^2}{2.3} + \frac{u^3}{3.4} + \frac{u^4}{4.5} + \text{etc.} = 1 - lu + \frac{(1-u)l(1-u)}{u} + lu$$

$$= 1 + \frac{(1-u)\,l(1-u)}{u}.$$

Corollarium 1.

202. Simili modo si $\partial y = \frac{\partial u}{u\sqrt{u}}\, l\frac{1}{1-u}$, erit

$$y = -\frac{2}{\sqrt{u}}\, l\frac{1}{1-u} + \int \frac{2\partial u}{(1-u)\sqrt{u}};$$

at posito $u = xx$, fit

$$\int \frac{2\partial u}{(1-u)\sqrt{u}} = 4\int \frac{\partial x}{1-xx} = 2\, l\frac{1+x}{1-x}. \quad \text{Ergo}$$

$$y = 2\, l\frac{1+\sqrt{u}}{1-\sqrt{u}} - \frac{2}{\sqrt{u}}\, l\frac{1}{1-u}.$$

At quia per seriem

$$\partial y = \frac{\partial u}{u\sqrt{u}}\left(u + \tfrac{1}{2}uu + \tfrac{1}{3}u^3 + \tfrac{1}{4}u^4 + \text{etc.}\right)$$

erit etiam

$$y = +2\sqrt{u} + \frac{2}{2.3}u\sqrt{u} + \frac{2}{3.5}u^2\sqrt{u} + \frac{2}{4.7}u^3\sqrt{u} + \text{etc.}$$

Corollarium 2.

203. Si ergo multiplicemus per $\frac{\sqrt{u}}{2}$, adipiscimur:

$$u + \frac{uu}{2.3} + \frac{u^3}{3.5} + \frac{u^4}{4.7} + \frac{u^5}{5.9} + \text{etc.} = \sqrt{u}\,.\,l\frac{1+\sqrt{u}}{1-\sqrt{u}} + l(1-u);$$

quae summa est etiam

$$= (1+\sqrt{u})\,l(1+\sqrt{u}) + (1-\sqrt{u})\,l(1-\sqrt{u}).$$

Quare sumto $u = 1$, ob $(1-\sqrt{u})\,l(1-\sqrt{u}) = 0$, erit

$$1 + \frac{1}{2.3} + \frac{1}{3.5} + \frac{1}{4.7} + \frac{1}{5.9} + \frac{1}{6.11} + \text{etc.} = 2\,l\,2.$$

Problema 19.

204. Si P denotet functionem ipsius x, invenire integrale hujus formulae $\partial y = \partial P\,(lx)^n$.

Solutio.

Per reductionem supra monstratam fit

$$y = P\,(lx)^n - \int P\,\partial \cdot (lx)^n = P\,(lx)^n - n\int \frac{P\,\partial x}{x}\,(lx)^{n-1}.$$

Hinc si sit $\int \frac{P\,\partial x}{x} = Q$, erit simili modo

$$\int \frac{P\,\partial x}{x}\,(lx)^{n-1} = Q\,(lx)^{n-1} - (n-1)\int \frac{Q\,\partial x}{x}\,(lx)^{n-2}.$$

Quo modo si ulterius progredimur, haecque integralia capere liceat

$$\int \frac{P\,\partial x}{x} = Q;\ \int \frac{Q\,\partial x}{x} = R;\ \int \frac{R\,\partial x}{x} = S;\ \int \frac{S\,\partial x}{x} = T;\ \text{etc.}$$

obtinebimus integrale quaesitum:

$$\int \partial P\,(lx)^n = P\,(lx)^n - nQ\,(lx)^{n-1} + n\,(n-1)\,R\,(lx)^{n-2} - n\,(n-1)\,(n-2)\,S\,(lx)^{n-3} + \text{etc.}$$

ae si exponens n fuerit numerus integer positivus, integrale forma finita exprimetur.

Exemplum 1.

205. *Formulae $x^m\,\partial x\,(lx)^2$ integrale assignare.*

Hic est $n = 2$, et $P = \frac{x^{m+1}}{m+1}$; hinc $Q = \frac{x^{m+1}}{(m+1)^2}$, et $R = \frac{x^{m+1}}{(m+1)^3}$: unde colligimus

$$\int x^m\,\partial x\,(lx)^2 = x^{m+1}\left(\frac{(lx)^2}{m+1} - \frac{2\,lx}{(m+1)^2} + \frac{2 \cdot 1}{(m+1)^3}\right),$$

quod integrale evanescit posito $x = 0$, dum sit $m + 1 > 0$.

Corollarium 1.

206. Hinc posito $x = 1$, fit $\int x^m\,\partial x\,(lx)^2 = \frac{2 \cdot 1}{(m+1)^3}$. Ex praecedentibus autem patet, si formula $\int x^m\,\partial x\,lx$ ita integretur, ut evanescat posito $x = 0$, tum facto $x = 1$, fieri $\int x^m\,\partial x\,lx = \frac{-1}{(m+1)^2}$.

Corollarium 2.

207. At si sit $m = -1$, ut habeatur $\frac{\partial x}{x}(lx)^2$, erit ejus integrale $\int \frac{\partial x}{x}(lx)^2 = \frac{1}{3}(lx)^3$, qui solus casus ex formula generali est excipiendus.

Exemplum 2.

208. *Formulae $x^{m-1}\partial x\,(lx)^3$ integrale assignare.*

Hic est $n = 3$ et $P = \frac{x^m}{m}$, hinc $Q = \frac{x^m}{m^2}$; $R = \frac{x^m}{m^3}$ et $S = \frac{x^m}{m^4}$: unde integrale quaesitum fit

$$\int x^{m-1}\partial x\,(lx)^3 = x^m\left(\frac{(lx)^3}{m} - \frac{3(lx)^2}{m^2} + \frac{3.2\,lx}{m^3} - \frac{3.2.1}{m^4}\right);$$

quod integrale evanescit, posito $x = 0$, dum sit $m > 0$.

Corollarium 1.

209. Quod si integrali ita sumto, ut evanescat posito $x = 0$, tum ponatur $x = 1$, erit:

$$\int x^{m-1}\partial x = \frac{1}{m};\ \int x^{m-1}\partial x\,lx = -\frac{1}{m^2};\ \int x^{m-1}\partial x\,(lx)^2 = +\frac{1.2}{m^3};\ \text{et}$$

$$\int x^{m-1}\partial x\,(lx)^3 = -\frac{1.2.3}{m^4}.$$

Corollarium 2.

210. Casu autem $m = 0$, erit integrale

$$\int \frac{\partial x}{x}(lx)^3 = \frac{1}{4}(lx)^4,$$

quod ita determinari nequit, ut evanescat posito $x = 0$; oporteret enim constantem infinitam adjici. Hoc autem integrale evanescit posito $x = 1$.

Exemplum 3.

211. *Formulae $x^{m-1}\partial x\,(lx)^n$ integrale assignare.*

Cum hic sit $P = \frac{x^m}{m}$; erit $Q = \frac{x^m}{m^2}$; $R = \frac{x^m}{m^3}$; $S = \frac{x^m}{m^4}$; etc. Hinc integrale quaesitum prodit

$$\int x^{m-1}\partial x\,(l\,x)^n = x^m\left(\frac{(lx)^n}{m} - \frac{n\,(lx)^{n-1}}{m^2} + \frac{n\,(n-1)\,(lx)^{n-2}}{m^3} - \frac{n\,(n-1)\,(n-2)\,(lx)^{n-3}}{m^4} + \text{etc.}\right).$$

Casu autem $m = 0$, est $\int\frac{\partial x}{x}(lx)^n = \frac{1}{n+1}(lx)^{n+1}$.

Corollarium 1.

212. Si $m > 0$ integrale assignatum evanescit, posito $x = 0$: deinceps ergo si sumatur $x = 1$, erit integrale

$$\int x^{m-1}\partial x\,(lx)^n = \pm\frac{1.\,2.\,3\,\ldots\ldots\,n}{m^{n+1}},$$

ubi signum $+$ valet, si n sit numerus par, inferius vero $-$ si n impar.

Corollarium 2.

213. Haec ergo ambiguitas tollitur, si loco lx scribatur $-l\frac{1}{x}$; tum enim integratione eodem modo instituta, positoque $x = 1$, fiet

$$\int x^{m-1}\partial x\,(l\tfrac{1}{x})^n = +\frac{1.\,2.\,3\,\ldots\ldots\,n}{m^{n+1}}.$$

Scholion.

214. Si exponens n sit numerus fractus, integrale inventum per seriem infinitam exprimitur, veluti si sit $n = -\frac{1}{2}$, reperitur

$$\int \frac{x^{m-1}\,\partial x}{\sqrt{l x}} = x^m \left(\frac{1}{m\sqrt{l x}} + \frac{1}{2\,m^2\,(l x)^{\frac{3}{2}}} + \frac{1\,.\,3}{4\,m^3\,(l x)^{\frac{5}{2}}} + \frac{1\,.\,3\,.\,5}{8\,m^4\,(l x)^{\frac{7}{2}}} + \text{etc.} \right),$$

quae etiam, quatenus initio x ab 0 ad 1 crescere sumitur, hoc modo repraesentari potest:

$$\int \frac{x^{m-1}\,\partial x}{\sqrt{l\frac{1}{x}}} = \frac{x^m}{\sqrt{l\frac{1}{x}}} \left(\frac{1}{m} + \frac{1}{2\,m^2\,l x} + \frac{1\,.\,3}{4\,m^3\,(l x)^2} + \frac{1\,.\,3\,.\,5}{8\,m^4\,(l x)^3} + \text{etc.} \right).$$

Si exponens n sit negativus, etsi integer, tamen integrale inventum in infinitum progreditur: verum hoc casu alia ratione integrationem instituere licet, qua tandem reducitur ad hujusmodi formulam $\int \frac{T\,\partial x}{l x}$, cujus integratio nullo modo simplicior reddi potest. Hanc ergo reductionem sequenti problemate doceamus.

Problema 20.

215. Integrationem hujus formulae $\partial y = \frac{X\,\partial x}{(l x)^n}$ continuo ad formulas simpliciores reducere.

Solutio.

Formula proposita ita repraesentetur $\partial y = X x\,.\,\frac{\partial x}{x\,(l x)^n}$, et cum sit $\int \frac{\partial x}{x\,(l x)^n} = \frac{-1}{(n-1)\,(l x)^{n-1}}$, erit

$$y = \frac{-X x}{(n-1)\,(l x)^{n-1}} + \frac{1}{n-1} \int \frac{1}{(l x)^{n-1}}\,.\,\partial\,(X x).$$

Quare si ponamus continuo

$$\partial . (Xx) = P \partial x; \ \partial . (Px) = Q \partial x; \ \partial \ (Qx) = R \partial x \text{ etc.}$$

erit hanc reductionem continuando:

$$y = \frac{-Xx}{(n-1)(lx)^{n-1}} - \frac{Px}{(n-1)(n-2)(lx)^{n-2}}$$
$$- \frac{Qx}{(n-1)(n-2)(n-3)(lx)^{n-3}} - \text{etc.}$$

donec tandem perveniatur ad hanc integralem

$$+ \frac{1}{(n-1)(n-2)\ldots 1} \int \frac{V \partial x}{lx},$$

ita ut quoties n fuerit numerus integer positivus, integratio tandem ad hujusmodi formulam perducatur.

Exemplum 1.

216. *Formulae differentialis* $\partial y = \frac{x^{m-1} \partial x}{(lx)^2}$ *integrale investigare.*

Hic est $n = 2$ et $X = x^{m-1}$, unde fit $P = m x^{m-1}$, hincque integrale

$$y = \int \frac{x^{m-1} \partial x}{(lx)^2} = - \frac{x^m}{lx} + \frac{m}{1} \int \frac{x^{m-1} \partial x}{lx}.$$

At formulae $\frac{x^{m-1} \partial x}{lx}$ integrale exhiberi nequit, nisi casu $m = 0$, quo fit $\int \frac{\partial x}{x\, lx} = llx$. Verum si $m = 0$, formulae propositae integratio ne hinc quidem pendet: fit enim absolute $y = \int \frac{\partial x}{x (lx)^2} = - \frac{1}{lx} + C$.

Exemplum 2.

217. *Formulae differentialis* $\partial y = \frac{x^{m-1} \partial x}{(lx)^n}$ *integrale investigare, casibus, quibus* n *est numerus integer positivus.*

Cum sit $X = x^{m-1}$, erit $P = \frac{\partial . (X x)}{\partial x} = m x^{m-1}$, tum vero $Q = \frac{\partial . (P x)}{\partial x} = m^2 x^{m-1}$; $R = m^3 x^{m-1}$; $S = m^4 x^{m-1}$; etc. Quare integrale hinc ita formabitur, ut sit

$$y = \int \frac{x^{m-1} \partial x}{(l x)^n} = \frac{-x^m}{(n-1)(lx)^{n-1}} - \frac{m x^m}{(n-1)(n-2)(lx)^{n-2}}$$
$$- \frac{m^2 x^m}{(n-1)(n-2)(n-3)(lx)^{n-3}} - \text{etc.}$$
$$\ldots + \frac{m^{n-1}}{(n-1)(n-2)\ldots\ldots 1} \int \frac{x^{m-1} \partial x}{l x}.$$

Corollarium.

218. Pro n ergo successive numeros 1, 2, 3, 4, etc. substituendo, habebimus istas reductiones:

$$\int \frac{x^{m-1} \partial x}{(lx)^2} = \frac{-x^m}{lx} + \frac{m}{1} \int \frac{x^{m-1} \partial x}{lx}$$
$$\int \frac{x^{m-1} \partial x}{(lx)^3} = \frac{-x^m}{2(lx)^2} - \frac{m x^m}{2.1\, lx} + \frac{m^2}{2.1} \int \frac{x^{m-1} \partial x}{lx}$$
$$\int \frac{x^{m-1} \partial x}{(lx)^4} = \frac{-x^m}{3(lx)^3} - \frac{m x^m}{3.2(lx)^2} - \frac{m^2 x^m}{3.2.1 lx} + \frac{m^3}{3.2.1} \int \frac{x^{m-1} \partial x}{lx}.$$

Scholion.

219. Hae ergo integrationes pendent a formula $\int \frac{x^{m-1} \partial x}{lx}$, quae posito $x^m = z$, ob $x^{m-1} \partial x = \frac{1}{m} \partial z$ et $lx = \frac{1}{m} lz$, reducitur ad hanc simplicissimam formam $\int \frac{\partial z}{l z}$, cujus integrale si assignari posset, amplissimum usum in Analysi esset allaturum, verum nullis adhuc artificiis, neque per logarithmos, neque angulos, exhiberi potuit: quomodo autem per seriem exprimi possit, infra ostendemus (§. 227). Videtur ergo haec formula $\int \frac{\partial z}{l z}$ singularem speciem func-

tionum transcendentium suppeditare, quae utique accuratiorem evolutionem meretur. Eadem autem quantitas transcendens in integrationibus formularum exponentialium frequenter occurrit, quas in hoc capite tractare instituimus, propterea quod cum logarithmicis tam arcte cohaerent, ut alterum genus facile in alterum converti possit: veluti ipsa formula modo considerata $\frac{\partial z}{lz}$, posito $lz = x$, ut sit $z = e^x$, et $\partial z = e^x \partial x$, transformatur in hanc exponentialem $e^x . \frac{\partial x}{x}$, cujus ergo integratio aeque est abscondita. Formulas igitur tractabiles evolvamus et ejusmodi quidem, quae non obvia substitutione ad formam algebraicam reduci possunt. Veluti si V fuerit functio quaecunque ipsius v, sitque $v = a^x$, formula $V \partial x$, ob $x = \frac{lv}{la}$ et $\partial x = \frac{\partial v}{v la}$, abit in $\frac{V \partial v}{v la}$, qua ratione variabilis v est algebraica. Hujusmodi ergo formulas $\frac{a^x \partial x}{\sqrt{(1 + a^{nx})}}$, quippe quae posito $a^x = v$, nihil habent difficultatis, hinc excludimus.

Problema 21.

220. Formulae differentialis $a^x X \partial x$, denotante X functionem quamcunque ipsius x, integrale investigare.

Solutio 1.

Cum sit $\partial . a^x = a^x \partial x la$, erit vicissim $\int a^x \partial x = \frac{1}{la} a^x$: quare si formula proposita in hos factores resolvatur, $X . a^x \partial x$, habebitur per reductionem:

$$\int a^x X \partial x = \frac{1}{la} a^x X - \frac{1}{la} \int a^x \partial X.$$

Quodsi ulterius ponamus $\partial X = P \partial x$, ut sit

$$\int a^x P \partial x = \frac{1}{la} a^x P - \frac{1}{la} \int a^x \partial P,$$

prodibit haec reductio

$$\int a^x X \partial x = \frac{1}{la} a^x X - \frac{1}{(la)^2} a^x P + \frac{1}{(la)^2} \int a^x \partial P.$$

Si porro ponamus $\partial P = Q \partial x$, habebitur haec reductio

$$\int a^x X \partial x = \frac{1}{l\,a} a^x X - \frac{1}{(la)^2} a^x P + \frac{1}{(la)^3} a^x Q - \frac{1}{(la)^3} \int a^x \partial Q;$$

sicque ulterius ponendo $\partial Q = R \partial x$, $\partial R = S \partial x$, etc. progredi licet, donec ad formulam vel integrabilem, vel in suo genere simplicissimam perveniatur.

Solutio. 2.

Alio modo resolutio formulae in factores institui potest; ponatur $\int X \partial x = P$ seu $X \partial x = \partial P$, et formula ita relata $a^x . \partial P$, habebitur

$$\int a^x X \partial x = a^x P - la \int a^x P \partial x:$$

simili modo si ponamus $\int P \partial x = Q$, obtinebimus

$$\int a^x X \partial x = a^x P - la . a^x Q + (la)^2 \int a^x Q \partial x.$$

Ponamus porro $\int Q \partial x = R$, et consequimur

$$\int a^x X \partial x = a^x P - la . a^x Q + (la)^2 . a^x R - (la)^3 \int a^x R \partial x,$$

hocque modo quousque lubuerit progredi licet, donec ad formulam vel integrabilem vel in suo genere simplicissimam perveniamus.

Corollarium 1.

221. Priori solutione semper uti licet, quia functiones P, Q, R, etc. per differentiationem functionis X eliciuntur, dum est

$$P = \frac{\partial X}{\partial x}; \; Q = \frac{\partial P}{\partial x}; \; R = \frac{\partial Q}{\partial x}; \text{ etc.}$$

Quare si X fuerit functio rationalis integra, tandem ad formulam pervenietur $\int a^x \partial x = \frac{1}{la} . a^x$; ideoque his casibus integrale absolute exhiberi potest.

Corollarium 2.

222. Altera solutio locum non invenit, nisi formulae $X \partial x$ integrale P assignari queat; neque etiam eam continuare licet, nisi

quatenus sequentes integrationes $\int P\partial x = Q$, $\int Q\partial x = R$, etc. succedunt.

Exemplum 1.

223. *Formulae $a^x x^n \partial x$ integrale definire, denotante n numerum integrum positivum.*

Cum sit $X = x^n$, solutione prima utentes habebimus

$$\int a^x x^n \partial x = \frac{1}{la} \cdot a^x x^n - \frac{n}{la} \int a^x x^{n-1} \partial x;$$

hinc ponendo pro n successive numeros 0, 1, 2, 3, etc., quia primo casu integratio constat, sequentia integralia eruemus:

$$\int a^x \partial x = \frac{1}{la} \cdot a^x$$

$$\int a^x x \partial x = \frac{1}{la} \cdot a^x x - \frac{1}{(la)^2} a^x$$

$$\int a^x x^2 \partial x = \frac{1}{la} \cdot a^x x^2 - \frac{2}{(la)^2} a^x x + \frac{2.1}{(la)^3} a^x$$

$$\int a^x x^3 \partial x = \frac{1}{la} \cdot a^x x^3 - \frac{3}{(la)^2} a^x x^2 + \frac{3.2}{(la)^3} a^x x - \frac{3.2.1}{(la)^4} a^x$$

etc.

unde in genere pro quovis exponente n concludimus

$$\int a^x x^n \partial x = a^x \left(\frac{x^n}{la} - \frac{n x^{n-1}}{(la)^2} + \frac{n(n-1) x^{n-2}}{(la)^3} - \frac{n(n-1)(n-2) x^{n-3}}{(la)^4} + \text{etc.} \right).$$

ad quam expressionem insuper constantem arbitrariam adjici oportet, ut integrale completum obtineatur.

Corollarium.

224. Si integrale ita determinari debeat, ut evanescat posito $x = 0$, erit

$$\int a^x \partial x = \frac{1}{la} \cdot a^x - \frac{1}{la}$$

$$\int a^x x \partial x = a^x \left(\frac{x}{la} - \frac{1}{(la)^2}\right) + \frac{1}{(la)^2}$$

$$\int a^x x^2 \partial x = a^x \left(\frac{x^2}{la} - \frac{2x}{(la)^2} + \frac{2.1}{(la)^3}\right) - \frac{2.1}{(la)^3}$$

$$\int a^x x^3 \partial x = a^x \left(\frac{x^3}{la} - \frac{3x^2}{(la)^2} + \frac{3.2x}{(la)^3} - \frac{3.2.1}{(la)^4}\right) + \frac{3.2.1}{(la)^4}$$

etc.

Exemplum 2.

225. *Formulae* $\frac{a^x \partial x}{x^n}$ *integrale investigare, si quidem* n *denotet numerum integrum positivum.*

Hic commode altera solutione utemur, ubi cum sit $X = \frac{1}{x^n}$, erit $P = \frac{-1}{(n-1)x^{n-1}}$; hincque resultat ista reductio

$$\int \frac{a^x \partial x}{x^n} = \frac{-a^x}{(n-1)x^{n-1}} + \frac{la}{n-1} \int \frac{a^x \partial x}{x^{n-1}}.$$

Perspicuum igitur est, posito $n = 1$ hinc nihil concludi posse; qui est ipse casus supra memoratus $\int \frac{a^x \partial x}{x}$, singularem speciem transcendentium functionum complectens, qua admissa integralia sequentium casuum exhibere poterimus:

$$\int \frac{a^x \partial x}{x^2} = C - \frac{a^x}{1x} + \frac{la}{1} \int \frac{a^x \partial x}{x}$$

$$\int \frac{a^x \partial x}{x^3} = C - \frac{a^x}{2x^2} - \frac{a^x la}{2.1x} + \frac{(la)^2}{2.1} \int \frac{a^x \partial x}{x}$$

$$\int \frac{a^x \partial x}{x^4} = C - \frac{a^x}{3x^3} - \frac{a^x la}{3.2x^2} - \frac{a^x (la)^2}{3.2.1x} + \frac{(la)^3}{3.2.1} \int \frac{a^x \partial x}{x}$$

unde in genere colligimus

$$\int \frac{a^x \partial x}{x^n} = C - \frac{a^x}{(n-1)x^{n-1}} - \frac{a^x l a}{(n-1)(n-2)x^{n-2}}$$
$$- \frac{a^x (l a)^2}{(n-1)(n-2)(n-3)x^{n-3}} - \ldots - \frac{a^x (l a)^{n-2}}{(n-1)(n-2)\ldots 1 . x}$$
$$+ \frac{(l a)^{n-1}}{(n-1)(n-2)\ldots 1} \int \frac{a^x \partial x}{x}.$$

Corollarium 1.

226. Admissa ergo quantitate transcendente $\int \frac{a^x \partial x}{x}$, hanc formulam $a^x x^m \partial x$ integrare poterimus, sive exponens m fuerit numerus integer positivus, sive negativus. Illis quidem casibus integratio ab ista nova quantitate transcendente non pendet.

Corollarium 2.

227. At si m fuerit fractus numerus, neutra solutio negotium conficit, sed utraque seriem infinitam pro integrali exhibet. Veluti si sit $m = -\frac{1}{2}$, habebimus ex priore

$$\int \frac{a^x \partial x}{\sqrt{x}} = a^x \left(\frac{1}{la} + \frac{1}{2x(la)^2} + \frac{1.3}{4x^2(la)^3} + \frac{1.3.5}{8x^3(la)^4} + \text{etc.} \right) : \sqrt{x} + C,$$

ex posteriore autem;

$$\int \frac{a^x \partial x}{\sqrt{x}} = C + \frac{a^x}{\sqrt{x}} \left(\frac{2x}{1} - \frac{4x^2 la}{1.3} + \frac{8x^3 (la)^2}{1.3.5} - \frac{16x^4 (la)^3}{1.3.5.7} + \text{etc.} \right).$$

Scholion 1.

228. Hinc quantitas transcendens $\int \frac{a^x \partial x}{x}$ per seriem exprimi potest secundum potestates ipsius x progredientem. Cum enim sit

$$a^x = 1 + x\,la + \frac{x^2\,(la)^2}{1.\,2} + \frac{x^3\,(la)^3}{1.\,2.\,3} + \text{etc. erit}$$

$$\int \frac{a^x\,\partial x}{x} = \mathrm{C} + lx + \frac{x\,la}{1} + \frac{x^2\,(la)^2}{1.\,2.\,2} + \frac{x^3\,(la)^3}{1.\,2.\,3.\,3} + \frac{x^4\,(la)^4}{1.\,2.\,3.\,4.\,4} + \text{etc.}$$

Ac si pro a sumamus numerum, cujus logarithmus hyperbolicus est unitas, quem numerum littera e indicemus, habebimus

$$\int \frac{e^x\,\partial x}{x} = \mathrm{C} + lx + \frac{x}{1} + \frac{1}{2}\cdot\frac{x^2}{1.2} + \frac{1}{3}\cdot\frac{x^3}{1.2.3} + \frac{1}{4}\cdot\frac{x^4}{1.2.3.4} + \text{etc.}$$

Atque hinc etiam ponendo $e^x = z$, ut sit $x = lz$, formulam supra memoratam $\frac{\partial z}{lz}$ per seriem integrare poterimus, eritque

$$\int \frac{\partial z}{lz} = \mathrm{C} + llz + \frac{lz}{1} + \tfrac{1}{2}\cdot\frac{(lz)^2}{1.2} + \tfrac{1}{3}\cdot\frac{(lz)^3}{1.2.3} + \tfrac{1}{4}\cdot\frac{(lz)^4}{1.2.3.4} + \text{etc.}$$

quod integrale si debeat evanescere, sumto $z = 0$, constans C fit infinita, unde pro reliquis casibus nihil concludi potest. Idem incommodum locum habet, si evanescens reddamus casu $z = 1$, quia $llz = l0$ fit infinitum. Caeterum patet, si integrale sit reale, pro valoribus ipsius z unitate minoribus, ubi lz est negativus, tum pro valoribus unitate majoribus fieri imaginarium, et vicissim. Hinc ergo natura hujus functionis transcendentis parum cognoscitur.

Scholion 2.

229. Quando vel integratio non succedit, vel series ante inventae minus idoneae videntur, hinc quantitatem a^x in seriem resolvendo, statim sine aliis subsidiis formulae $a^x\,\mathrm{X}\,\partial x$ integrale per seriem exhiberi potest, erit enim

$$\int a^x\,\mathrm{X}\,\partial x = \int \mathrm{X}\,\partial x + \frac{la}{1}\int \mathrm{X}\,x\,\partial x + \frac{(la)^2}{1.\,2}\int \mathrm{X}\,x^2\,\partial x + \frac{(la)^3}{1.2.3}\int \mathrm{X}\,x^3\,\partial x + \text{etc.}$$

Ita si sit $X = x^n$, habebitur

$$\int a^x x^n \partial x = C + \frac{x^{n+1}}{n+1} + \frac{x^{n+2}\, la}{1\,(n+2)} + \frac{x^{n+3}\,(la)^2}{1.2\,(n+3)} + \frac{x^{n+4}\,(la)^3}{1.2.3\,(n+4)} + \text{etc.}$$

ubi notandum, si n fuerit numerus integer negativus, puta $n = -i$, loco $\frac{x^{n+i}}{n+i}$ scribi debere lx.

Exemplum 3.

230. *Formulae* $\frac{a^x \partial x}{1-x}$ *integrale per seriem infinitam exprimere.*

Per priorem solutionem obtinemus, ob

$$X = \frac{1}{1-x};\quad P = \frac{\partial X}{\partial x} = \frac{1}{(1-x)^2};\quad Q = \frac{\partial P}{\partial x} = \frac{1.2}{(1-x)^3};\quad R = \frac{\partial Q}{\partial x} = \frac{1.2.3}{(1-x)^4}\ \text{etc.}$$

hincque sequentem seriem:

$$\int \frac{a^x \partial x}{1-x} = a^x\left(\frac{1}{(1-x)\,la} - \frac{1}{(1-x)^2(la)^2} + \frac{1.2}{(1-x)^3(la)^3} - \frac{1.2.3}{(1-x)^4(la)^4} + \text{etc.}\right)$$

Aliae series reperiuntur, si vel a^x, vel fractio $\frac{1}{1-x}$ in seriem evolvatur. Commodissima autem videtur, quae seriem fingendo eruitur: brevitatis gratia pro a sumamus numerum e, ut $le = 1$, ac statuatur $\partial y = \frac{e^x \partial x}{1-x}$ seu

$$\frac{\partial y}{\partial x}(1-x) - 1 - x - \frac{x^2}{1.2} - \frac{x^3}{1.2.3} - \frac{x^4}{1.2.3.4} - \text{etc.} = 0.$$

Jam pro y fingatur haec series

$$y = \int \frac{e^x \partial x}{1-x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

eritque facta substitutione

$$\left.\begin{array}{l} B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \text{etc.} \\ \quad - B \quad - 2C \quad - 3D \quad - 4E \\ -1 - 1 - \tfrac{1}{2} \quad - \tfrac{1}{6} \quad - \tfrac{1}{24} \end{array}\right\} = 0:$$

unde eliciuntur istae determinationes:

$$\begin{array}{l|l} B = 1 & E = \tfrac{1}{4}(1 + 1 + \tfrac{1}{2} + \tfrac{1}{6}) \\ C = \tfrac{1}{2}(1 + 1) & F = \tfrac{1}{5}(1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24}) \\ D = \tfrac{1}{3}(1 + 1 + \tfrac{1}{2}) & \quad \text{etc.} \end{array}$$

Problema 22.

231. Formulae differentialis $\partial y = x^{nx} \partial x$ integrale investigare, ac per seriem infinitam exprimere.

Solutio.

Commodius hoc praestari nequit, quam ut formula exponentialis x^{nx} in seriem infinitam convertatur, quae est

$$x^{nx} = 1 + nx\,lx + \frac{n^2x^2(lx)^2}{1.2} + \frac{n^3x^3(lx)^3}{1.2.3} + \frac{n^4x^4(lx)^4}{1.2.3.4} + \text{etc.}$$

qua per ∂x multiplicata, et singulis terminis integratis, erit:

$$\int \partial x = x;$$

$$\int x \partial x\, lx = x^2 \left(\frac{lx}{2} - \frac{1}{2^2}\right);$$

$$\int x^2 \partial x (lx)^2 = x^3 \left(\frac{(lx)^2}{3} - \frac{2\,lx}{3^2} + \frac{2.1}{3^3}\right);$$

$$\int x^3 \partial x (lx)^3 = x^4 \left(\frac{(lx)^3}{4} - \frac{3(lx)^2}{4^2} + \frac{3.2\,lx}{4^3} - \frac{3.2.1}{4^4}\right);$$

$$\int x^4 \partial x (lx)^4 = x^5 \left(\frac{(lx)^4}{5} - \frac{4(lx)^3}{5^2} + \frac{4.3(lx)^2}{5^3} - \frac{4.3.2\,lx}{5^4} + \frac{4.3.2.1}{5^5}\right).$$

etc.

Quare si hae series substituantur, et secundum potestates ipsius lx disponantur, integrale quaesitum exprimetur per has innumerabiles series infinitas:

$$y = \int x^{nx} \partial x = + x\left(1 - \frac{nx}{2^2} + \frac{n^2x^2}{3^3} - \frac{n^3x^3}{4^4} + \frac{n^4x^4}{5^5} - \text{etc.}\right)$$
$$+ \frac{nx^2lx}{1}\left(\frac{1}{2^1} - \frac{nx}{3^2} + \frac{n^2x^2}{4^3} - \frac{n^3x^3}{5^4} + \frac{n^4x^4}{6^5} - \text{etc.}\right)$$
$$+ \frac{n^2x^3(lx)^2}{1.2}\left(\frac{1}{3^1} - \frac{nx}{4^2} + \frac{n^2x^2}{5^3} - \frac{n^3x^3}{6^4} + \frac{n^4x^4}{7^5} - \text{etc.}\right)$$
$$+ \frac{n^3x^4(lx)^3}{1.2.3}\left(\frac{1}{4^1} - \frac{nx}{5^2} + \frac{n^2x^2}{6^3} - \frac{n^3x^3}{7^4} + \frac{n^4x^4}{8^5} - \text{etc.}\right)$$
etc.

quod integrale ita est sumtum, ut evanescat, posito $x = 0$.

Corollarium.

232. Hac ergo lege instituta integratione, si ponatur $x = 1$, valor integralis $\int x^{nx} \partial x$ huic seriei aequatur

$$1 - \frac{n}{2^2} + \frac{n^2}{3^3} - \frac{n^3}{4^4} + \frac{n^4}{5^5} - \frac{n^5}{6^6} + \text{etc.}$$

quae ob concinnitatem terminorum omnino est notatu digna.

Scholion.

233. Eodem modo reperitur integrale hujus formulae:

$$y = \int x^{nx} x^m \partial x = \int x^m \partial x\left(1 + nxlx + \frac{n^2x^2(lx)^2}{1.2} + \frac{n^3x^3(lx)^3}{1.2.3} + \text{etc.}\right)$$

erit enim singulis terminis integrandis:

$$\int x^m \partial x = \frac{x^{m+1}}{m+1};$$

$$\int x^{m+1} \partial x\,(lx) = x^{m+2}\left(\frac{lx}{m+2} - \frac{1}{(m+2)^2}\right);$$

$$\int x^{m+2} \partial x\,(lx)^2 = x^{m+3}\left(\frac{(lx)^2}{m+3} - \frac{2\,lx}{(m+3)^2} + \frac{2.1}{(m+3)^3}\right);$$

$$\int x^{m+3} \partial x\,(lx)^3 = x^{m+4}\left(\frac{(lx)^3}{m+4} - \frac{3\,(lx)^2}{(m+4)^2} + \frac{3.2\,lx}{(m+4)^3} - \frac{3.2.1}{(m+4)^4}\right);$$

etc.

Quod si ergo integrale ita determinetur, ut evanescat posito $x=0$, tum vero statuatur $x=1$, pro hoc casu valor formulae integralis $\int x^{nx}\, x^m\, \partial x$ exprimetur hac serie satis memorabili:

$$\frac{1}{m+1} - \frac{n}{(m+2)^2} + \frac{nn}{(m+3)^3} - \frac{n^3}{(m+4)^4} + \frac{n^4}{(m+5)^5} - \text{etc.}$$

quae uti manifestum est, locum habere nequit, quoties m est numerus integer negativus.

Alia exempla formularum exponentialium non adjungo, quia plerumque integralia nimis inconcinne exprimuntur, methodus autem eas tractandi hic sufficienter est exposita. Interim tamen singularem attentionem merentur formulae integrationem absolute admittentes, quae in hac forma continentur $e^x(\partial P + P\partial x)$ cujus integrale manifesto est $e^x P$. Hujusmodi autem casibus difficile est regulas tradere integrale inveniendi, et conjecturae plerumque plurimum est tribuendum. Veluti si proponeretur haec formula $\frac{e^x x\partial x}{(1+x)^2}$, facile est suspicari integrale, si datur, talem formam esse habiturum $\frac{e^x z}{1+x}$. Hujus ergo differentiale $\frac{e^x[\partial z(1+x) + xz\partial x]}{(1+x)^2}$ cum illo comparatum dat $\partial z(1+x) + xz\partial x = x\partial x$, ubi statim patet esse $z=1$, quod nisi per se pateret, ex regulis difficulter cognosceretur. Quare transeo ad alterum genus formularum transcendentium jam in Analysin receptarum, quae vel angulos vel sinus, tangentesve angulorum complectuntur.