CAPUT III.

DE INTEGRATIONE FORMULARUM DIFFERENTIALIUM PER SERIES INFINITAS.

Problema 12.

126.

Si X fuerit functio rationalis fracta ipsius x, formulae differentialis $\partial y = X \partial x$ integrale per seriem infinitam exhibere.

Solutio.

Cum X sit functio rationalis fracta, ejus valor semper ita evolvi potest, ut fiat

$$X = A x^{m} + B x^{m+n} + C x^{m+2n} + D x^{m+3n} + E x^{m+4n} + \text{etc.}$$

ubi coëfficientes A, B, C, etc. seriem recurrentem constituent, ex denominatore fractionis determinandam. Multiplicentur ergo singuli termini per ∂x, et integrentur, quo facto integrale y per sequentem seriem exprimetur

$$y = \frac{A x^{m+1}}{m+1} + \frac{B x^{m+n+1}}{m+n+1} + \frac{C x^{m+2n+1}}{m+2n+1} + \text{etc.} + \text{Const.}$$

ubi si in serie pro X occurrat hujusmodi terminus $\frac{M}{x}$, inde in integrale ingredietur terminus $M l x$.

Scholion.

127. Cum integrale $\int X \partial x$, nisi sit algebraicum, per logarithmos et angulos exprimatur, hinc valores logarithmorum et angulorum per series infinitas exhiberi possunt. Cujusmodi series cum jam in Introductione plures sint traditae, non solum eaedem, sed etiam infinitae aliae hic per integrationem erui possunt. Hoc exem-

plis declarasse juvabit, ubi potissimum ejusmodi formulas evolvemus, in quibus denominator est binomium; tum vero etiam casus aliquot denominatore trinomio vel multinomio praeditos contemplabimur. Imprimis autem ejusmodi eligemus, quibus fractio in aliam, cujus denominator est binomius, transmutari potest.

Exemplum 1.

128. *Formulam differentialem $\frac{\partial x}{a+x}$ per seriem integrare.*

Sit $y = \int \frac{\partial x}{a+x}$, erit $y = l(a+x) +$ Const., unde integrali ita determinato, ut evanescat posito $x = 0$, erit $y = l(a+x) - la$. Jam cum sit

$$\frac{1}{a+x} = \frac{1}{a} - \frac{x}{a^2} + \frac{xx}{a^3} - \frac{x^3}{a^4} + \frac{x^4}{a^5} - \text{etc.}$$

erit eadem lege integrale definiendo:

$$y = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \text{etc.}$$

unde colligimus, uti quidem jam constat:

$$l(a+x) = la + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \text{etc.}$$

Corollarium 1.

129. Si capiamus x negativum, ut sit $\partial y = \frac{-\partial x}{a-x}$, eodem modo patebit esse:

$$l(a-x) = la - \frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3} - \frac{x^4}{4a^4} - \text{etc.}$$

hisque combinandis:

$$l(aa-xx) = 2la - \frac{xx}{aa} - \frac{x^4}{2a^4} - \frac{x^6}{3a^6} - \frac{x^8}{4a^8} - \text{etc. et}$$

$$l\frac{a+x}{a-x} = \frac{2x}{a} + \frac{2x^3}{3a^3} + \frac{2x^5}{5a^5} + \frac{2x^7}{7a^7} + \text{etc.}$$

Corollarium 2.

134. Cum sit $\int \frac{xx\partial x}{1+x^3} = \frac{1}{3} l (1+x^3)$, erit eodem modo

$$\tfrac{1}{3} l (1+x^3) = \tfrac{1}{3} x^3 - \tfrac{1}{6} x^6 + \tfrac{1}{9} x^9 - \tfrac{1}{12} x^{12} + \text{etc.}$$

qua serie illis adjecta, omnes potestates ipsius x occurrent.

Exemplum 4.

135. *Integrale hoc* $y = \int \frac{(1+xx)\partial x}{1+x^4}$ *per seriem exprimere.*

Cum sit $\frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} + x^{16} -$ etc. erit

$$y = x + \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 + \tfrac{1}{11}x^{11} - \tfrac{1}{13}x^{13} - \tfrac{1}{15}x^{15} + \text{etc.}$$

Verum per §. 82. ubi $m = 1$ et $n = 4$, posito $\frac{\pi}{4} = \omega$, fit integrale idem:

$$y = \sin.\omega \text{ Arc. tang } \frac{x \sin.\omega}{1 - x\cos.\omega}$$
$$+ \sin.3\omega \text{ Arc. tang. } \frac{x \sin.3\omega}{1 - x\cos.3\omega}:$$

At ob $\frac{\pi}{4} = \omega = 45°$, est $\sin.\omega = \frac{1}{\sqrt{2}}$; $\cos.\omega = \frac{1}{\sqrt{2}}$; $\sin.3\omega = \frac{1}{\sqrt{2}}$; $\cos.3\omega = \frac{-1}{\sqrt{2}}$; hinc habebimus:

$$y = \tfrac{1}{\sqrt{2}} \text{ Arc. tang. } \frac{x}{\sqrt{2}-x} + \tfrac{1}{\sqrt{2}} \text{ Arc. tang. } \frac{x}{\sqrt{2}+x}$$
$$= \tfrac{1}{\sqrt{2}} \text{ Arc. tang. } \frac{x\sqrt{2}}{1-xx}.$$

Exemplum 5.

136. *Integrale hoc* $y = \int \frac{(1+x^4)\partial x}{1+x^6}$ *per seriem exprimere.*

Cum fit $\frac{1}{1+x^6} = 1 - x^6 + x^{12} - x^{18} + x^{24} -$ etc. erit

$$y = x + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 - \tfrac{1}{11}x^{11} + \tfrac{1}{13}x^{13} + \tfrac{1}{17}x^{17} - \text{etc.}$$

At per §. 82. ubi $m = 1$, $n = 6$, et $\omega = \frac{\pi}{6} = 30^\circ$, est

$$y = \tfrac{2}{3}\sin.\omega \text{ Arc. tang. } \frac{x \sin.\omega}{1 - x\cos.\omega} + \tfrac{2}{3}\sin.3\omega \text{ Arc. tang. } \frac{x\sin.3\omega}{1 - x\cos.3\omega}$$
$$+ \tfrac{2}{3}\sin.5\omega \text{ Arc. tang. } \frac{x\sin.5\omega}{1 - x\cos.5\omega}:$$

est vero $\sin.\omega = \frac{1}{2}$; $\cos.\omega = \frac{\sqrt{3}}{2}$; $\sin.3\omega = 1$; $\cos.3\omega = 0$; $\sin.5\omega = \frac{1}{2}$; $\cos.5\omega = -\frac{\sqrt{3}}{2}$, ergo

$$y = \tfrac{1}{3}\text{ Arc. tang. } \frac{x}{2 - x\sqrt{3}} + \tfrac{2}{3}\text{ Arc. tang. } x + \tfrac{1}{3}\text{ Arc. tang. } \frac{x}{2 + x\sqrt{3}}:$$

seu

$$y = \tfrac{1}{3}\text{ Arc. tang. } \frac{x}{1 - xx} + \tfrac{2}{3}\text{ Arc. tang. } x = \tfrac{1}{3}\text{ Arc. tang. } \frac{3x(1 - xx)}{3 - 4xx + x^4}$$

Corollarium 1.

137. Sit $z = \int \frac{xx\partial x}{1 + x^6} = \frac{1}{3}x^3 - \frac{1}{9}x^9 + \frac{1}{15}x^{15} - \frac{1}{21}x^{21} + \text{etc.}$ at facto $x^3 = u$, est

$$z = \tfrac{1}{3}\int \frac{\partial u}{1 + uu} = \tfrac{1}{3}\text{ Arc. tang. } u = \tfrac{1}{3}\text{ Arc. tang. } x^3.$$

Hinc series hujusmodi mixta formatur:

$$x + \tfrac{n}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 - \tfrac{n}{9}x^9 - \tfrac{1}{11}x^{11} + \tfrac{1}{13}x^{13} + \tfrac{n}{15}x^{15} + \tfrac{1}{17}x^{17} - \text{etc.}$$

cujus summa est $= \frac{1}{3}\text{ Arc. tang. } \frac{3x(1 - xx)}{1 - 4xx + x^4} + \frac{n}{3}\text{ Arc. tang. } x^3$.

Corollarium 2.

138. Si hic capiatur $n = -1$, binos angulos in unum colligendo, fit

$$\tfrac{1}{3}\text{ Arc. tang. } \frac{3x(1 - xx)}{1 - 4xx + x^4} - \tfrac{1}{3}\text{ Arc. tang. } x^3$$
$$= \tfrac{1}{3}\text{ Arc. tang. } \frac{3x - 4x^3 + 4x^5 - x^7}{1 - 4xx + 4x^4 - 3x^6}:$$

quae fractio per $1 - xx + x^4$ dividendo, reducitur ad $\frac{3x - x^3}{1 - 3xx}$, quae est tangens tripli anguli x pro tangente habentis, ita ut sit $\frac{1}{3}$ Arc. tang. $\frac{3x - x^3}{1 - 3xx}$ = Arc. tang. x, quod idem series inventa manifesto indicat.

Exemplum 6.

139. *Hanc formulam* $\partial y = \frac{(x^{m-1} + x^{n-m-1})\partial x}{1 + x^n}$, *per seriem integrare.*

Ob $\frac{1}{1 + x^n} = 1 - x^n + x^{2n} - x^{3n} + x^{4n} -$ etc. habebitur

$$y = \frac{x^m}{m} + \frac{x^{n-m}}{n-m} - \frac{x^{n+m}}{n+m} - \frac{x^{2n-m}}{2n-m} + \frac{x^{2n+m}}{2n+m} + \frac{x^{3n-m}}{3n-m} - \text{etc.}$$

Haec ergo series per §. 82. aggregatum aliquot arcuum circularium exprimit, quos ibi videre licet.

Corollarium.

140. Eodem modo proposita formula $\partial z = \frac{(x^{m-1} - x^{n-m-1})\partial x}{1 - x^n}$,

ob $\frac{1}{1 - x^n} = 1 + x^n + x^{2n} + x^{3n} +$ etc. invenitur:

$$z = \frac{x^m}{m} - \frac{x^{n-m}}{n-m} + \frac{x^{n+m}}{n+m} - \frac{x^{2n-m}}{2n-m} + \frac{x^{2n+m}}{2n+m} - \frac{x^{3n-m}}{3n-m} + \text{etc.}$$

cujus valor §. 84. est exhibitus.

Exemplum 7.

141. *Hanc formulam* $\partial y = \frac{(1 + 2x)\partial x}{1 + x + xx}$, *per seriem integrare.*

Primo integrale est manifesto $y = l(1 + x + xx)$; ut autem in seriem convertatur, multiplicetur numerator et denominator per $1 - x$, ut fiat $\partial y = \frac{(1 + x - 2xx)\partial x}{1 - x^3}$. Cum nunc sit $\frac{1}{1 - x^3} = 1 + x^3 + x^6 + x^9 + x^{12} + \text{etc.}$ erit integrando:

$$y = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \frac{2x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} - \frac{2x^9}{9} + \text{etc.}$$

Corollarium 1.

142. Eodem modo inveniri potest

$$y = l(1 + x + xx + x^3)$$

per seriem. Cum enim fiat $y + l(1 - x) = l(1 - x^4)$, erit

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} + \frac{x^{10}}{10} + \text{etc.}$$
$$- x^4 \qquad\qquad - \frac{x^8}{2}$$

sive

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3x^8}{8} + \frac{x^9}{9} + \text{etc.}$$

Corollarium 2.

143. At fractio $\frac{1 + 2x}{1 + x + xx}$ per seriem recurrentem evoluta dat

$$1 + x - 2xx + x^3 + x^4 - 2x^5 + x^6 + x^7 - 2x^8 + \text{etc.}$$

unde per integrationem eadem series obtinetur, quae ante.

Exemplum 8.

144. *Hanc formulam* $\partial y = \frac{\partial x}{1 - 2x\cos.\zeta + xx}$ *per seriem integrare.*

Per §. 64. ubi $A = 1$, $B = 0$, $a = 1$, et $b = 1$, est hujus formulae integrale $y = \frac{1}{\sin.\zeta}$ Arc. tang. $\frac{x \sin.\zeta}{1 - x \cos.\zeta}$. At per seriem recurrentem reperimus

$$\frac{1}{1 - 2x\cos.\zeta + xx} = 1 + 2x\cos.\zeta + (4\cos.\zeta^2 - 1)xx$$
$$+ (8\cos.\zeta^3 - 4\cos.\zeta)x^3 + (16\cos.\zeta^4 - 12\cos.\zeta^2 + 1)x^4$$
$$+ (32\cos.\zeta^5 - 32\cos.\zeta^3 + 6\cos.\zeta)x^5 + \text{etc.}$$

qua serie per ∂x multiplicata et integrata, obtinetur quaesitum. Potestatibus autem ipsius $\cos.\zeta$ in cosinus angulorum multiplorum conversis, reperitur:

$$y = x + \tfrac{1}{2}xx(2\cos.\zeta) + \tfrac{1}{3}x^3(2\cos.2\zeta + 1)$$
$$+ \tfrac{1}{4}x^4(2\cos.3\zeta + 2\cos.\zeta) + \tfrac{1}{5}x^5(2\cos.4\zeta + 2\cos.2\zeta + 1)$$
$$+ \tfrac{1}{6}x^6(2\cos.5\zeta + 2\cos.3\zeta + 2\cos.\zeta) + \text{etc.}$$

Corollarium 1.

145. Si ponatur $\partial z = \frac{(1 - x\cos.\zeta)\partial x}{1 - 2x\cos.\zeta + xx}$, erit per §. 63. $A = 1$, $B = -\cos.\zeta$, $a = 1$ et $b = 1$, ideoque

$$z = -\cos.\zeta\, l\sqrt{(1 - 2x\cos.\zeta + xx)} + \sin.\zeta \text{ Arc. tang. } \frac{x\sin.\zeta}{1 - x\cos.\zeta}.$$

At per seriem

ob $\frac{1 - x\cos.\zeta}{1 - 2x\cos.\zeta + xx} = 1 + x\cos.\zeta + x^2\cos.2\zeta$
$+ x^3\cos.3\zeta + x^4\cos.4\zeta +$ etc. fit

$$z = x + \tfrac{1}{2}xx\cos.\zeta + \tfrac{1}{3}x^3\cos.2\zeta + \tfrac{1}{4}x^4\cos.3\zeta$$
$$+ \tfrac{1}{5}x^5\cos.4\zeta + \text{etc.}$$

Corollarium 2.

146. At quia $\partial z = \frac{\partial x(-x\cos.\zeta + \cos.\zeta^2 + \sin.\zeta^2)}{1 - 2x\cos.\zeta + xx}$, erit

$$z = -\cos.\zeta\, l\sqrt{(1 - 2x\cos.\zeta + xx)} + \sin.\zeta^2 \int \frac{\partial x}{1 - 2x\cos.\zeta + xx}.$$

Hinc ergo pro $y = \int \frac{\partial x}{1 - 2x\cos.\zeta + xx}$ alia reperitur series infinita cum logarithmo connexa, scilicet

$$y = \frac{\cos.\zeta}{\sin.\zeta^2}\, l\sqrt{(1 - 2x\cos.\zeta + xx)}$$
$$+ \frac{1}{\sin.\zeta^2}(x + \tfrac{1}{2}xx\cos.\zeta + \tfrac{1}{3}x^3\cos.2\zeta + \tfrac{1}{4}x^4\cos.3\zeta + \text{etc.})$$

Problema 12.

147. Formulam differentialem irrationalem

$\partial y = x^{m-1}\partial x\,(a + bx^n)^{\frac{\mu}{\nu}}$ per seriem infinitam integrare.

Solutio.

Sit $a^{\frac{\mu}{\nu}} = c$, erit $\partial y = cx^{m-1}\partial x\,(1 + \frac{b}{a}x^n)^{\frac{\mu}{\nu}}$, ubi quidem assumimus c non esse quantitatem imaginariam. Cum igitur sit

$$(1+\frac{b}{a}x^n)^{\frac{\mu}{\nu}} = 1 + \frac{\mu b}{1\nu.a}x^n + \frac{\mu(\mu-\nu)bb}{1\nu.2\nu.aa}x^{2n} + \frac{\mu(\mu-\nu)(\mu-2\nu)b^3}{1\nu.2\nu.3\nu.a^3}x^{3n} + \text{etc.}$$

erit integrando:

$$y = c\left(\frac{x^m}{m} + \frac{\mu b}{\nu.a}\cdot\frac{x^{m+n}}{m+n} + \frac{\mu(\mu-\nu)bb}{1\nu.2\nu.aa}\cdot\frac{x^{m+2n}}{m+2n} + \frac{\mu(\mu-\nu)(\mu-2\nu)b^3}{1\nu.2\nu.3\nu.a^3}\cdot\frac{x^{m+3n}}{m+3n} + \text{etc.}\right),$$

quae series in infinitum excurrit, nisi $\frac{\mu}{\nu}$ sit numerus integer positivus.

Sin autem casu, quo ν numerus par, a fuerit quantitas negativa, expressio nostra ita est repraesentanda,

$$\partial y = x^{m-1}\partial x\,(bx^n - a)^{\frac{\mu}{\nu}} = b^{\frac{\mu}{\nu}}x^{m+\frac{\mu n}{\nu}-1}\partial x\,(1 - \frac{a}{b}x^{-n})^{\frac{\mu}{\nu}}.$$

Cum igitur sit

$$(1 - \frac{a}{b}x^{-n})^{\frac{\mu}{\nu}} = 1 - \frac{\mu a}{1\nu.b}x^{-n} + \frac{\mu(\mu-\nu)a^2}{1\nu.2\nu.b^2}x^{-2n} - \frac{\mu(\mu-\nu)(\mu-2\nu)a^3}{1\nu.2\nu.3\nu.b^3}x^{-3n} + \text{etc.}$$

erit integrando

$$y = b^{\frac{\mu}{\nu}} \left(\frac{\nu x^{m + \frac{\mu n}{\nu}}}{m\nu + \mu n} - \frac{\mu a}{1\nu . b} \cdot \frac{\nu x^{m + \frac{(\mu - \nu) n}{\nu}}}{m\nu + (\mu - \nu) n} + \frac{\mu(\mu - \nu) a^2}{1\nu . 2\nu . b^2} \cdot \frac{\nu x^{m + \frac{(\mu - 2\nu) n}{\nu}}}{m\nu + (\mu - 2\nu) n} - \text{etc.} \right)$$

Si a et b sint numeri positivi, utraque evolutione uti licet.

Exemplum 1.

148. *Formulam* $\partial y = \frac{\partial x}{\sqrt{(1 - xx)}}$, *per seriem integrare.*

Primo ex superioribus patet esse $y = \text{Arc. sin.}\, x$ qui ergo angulus etiam per seriem infinitam exprimetur. Cum enim sit

$$\frac{1}{\sqrt{(1 - xx)}} = 1 + \tfrac{1}{2} x^2 + \tfrac{1.3}{2.4} x^4 + \tfrac{1.3.5}{2.4.6} x^6 + \tfrac{1.3.5.7}{2.4.6.8} x^8 + \text{etc.}$$

erit

$$y = x + \tfrac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \frac{1.3.5.7}{2.4.6.8} \cdot \frac{x^9}{9} + \text{etc.}$$

utroque valore ita definito, ut evanescat posito $x = 0$.

Corollarium 1.

149. Si ergo sit $x = 1$, ob $\text{Arc. sin.}\, 1 = \frac{\pi}{2}$, erit

$$\frac{\pi}{2} = 1 + \tfrac{1}{2.3} + \tfrac{1.3}{2.4.5} + \tfrac{1.3.5}{2.4.6.7} + \tfrac{1.3.5.7}{2.4.6.8.9} + \text{etc.}$$

At si ponatur $x = \frac{1}{2}$, ob $\text{Arc. sin.}\, \frac{1}{2} = 30^\circ = \frac{\pi}{6}$, erit

$$\frac{\pi}{6} = \tfrac{1}{2} + \frac{1.}{2.2^3.3} + \frac{1.3}{2.4.2^5.5} + \frac{1.3.5}{2.4.6.2^7.7} + \frac{1.3.5.7}{2.4.6.8.2^9.9} + \text{etc.}$$

cujus seriei decem termini additi dant 0,52359877, cujus sextuplum 3,14159262 tantum in octava figura a veritate discrepat.

Corollarium 2.

150. Proposita hac formula $\partial y = \frac{\partial x}{\sqrt{(x - xx)}}$ posito $x = uu$, fit

$$\partial y = \frac{2u\partial u}{\sqrt{(uu - u^4)}} = \frac{2\partial u}{\sqrt{(1-uu)}}:$$

ergo $y = 2$ Arc. sin. $u = 2$ Arc. sin. $\sqrt{x}$. Tum vero per seriem erit:

$$y = 2\left(u + \tfrac{1}{2}\cdot\frac{u^3}{3} + \frac{1.3}{2.4}\cdot\frac{u^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{u^7}{7} + \text{etc.}\right) \text{ seu}$$

$$y = 2\left(1 + \tfrac{1}{2}\cdot\frac{x}{3} + \frac{1.3}{2.4}\cdot\frac{xx}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^3}{7} + \text{etc.}\right)\sqrt{x}.$$

Exemplum 2.

151. *Formulam* $\partial y = \partial x\sqrt{(2ax - xx)}$ *per seriem integrare.*

Posito $x = uu$, fit $\partial y = 2uu\partial u\sqrt{(2a - uu)}$: at per reductionem I. (§. 118.) est $n = 2$; $m = 1$; $a = 2a$; $b = -1$; $\mu = 1$; $\nu = 2$; unde

$$\int uu\partial u\sqrt{(2a - uu)} = -\tfrac{1}{2}u(2a-uu)^{\frac{3}{2}} + \tfrac{1}{2}a\int\partial u\sqrt{(2a-uu)}:$$

et per tertiam, sumendo $m = 1$, $a = 2a$, $b = -1$, $n = 2$, $\mu = -1$, $\nu = 2$, fit

$$\int\partial u\sqrt{(2a - uu)} = \tfrac{1}{2}u\sqrt{(2a - uu)} + a\int\frac{\partial u}{\sqrt{(2a-uu)}}:$$

at est

$$\int\frac{\partial u}{\sqrt{(2a-uu)}} = \text{Arc. sin. } \frac{u}{\sqrt{2a}} = \text{Arc. sin. } \frac{\sqrt{x}}{\sqrt{2a}}, \text{ ideoque}$$

$$\int uu\partial u\sqrt{(2a-uu)} = -\tfrac{1}{4}u(2a-uu)^{\frac{3}{2}} + \tfrac{1}{4}au\sqrt{(2a-uu)} + \tfrac{1}{2}aa\,\text{Arc.sin. }\frac{\sqrt{x}}{\sqrt{2a}}$$
$$= \tfrac{1}{4}u(uu-a)\sqrt{(2a-uu)} + \tfrac{1}{2}aa\,\text{Arc. sin. }\frac{\sqrt{x}}{\sqrt{2a}}.$$

Ergo $$y = \tfrac{1}{2}(x-a)\sqrt{(2ax-xx)} + aa\,\text{Arc. sin. }\frac{\sqrt{x}}{\sqrt{2a}}.$$

Pro serie autem invenienda est $\partial y = \partial x\sqrt{2ax}\left(1 - \frac{x}{2a}\right)^{\frac{1}{2}}$

$$= x^{\frac{1}{2}}\partial x\left(1 - \tfrac{1}{2}\cdot\frac{x}{2a} - \frac{1.1}{2.4}\cdot\frac{xx}{4aa} - \frac{1.1.3}{2.4.6}\cdot\frac{x^3}{8a^3} - \text{etc.}\right)\sqrt{2a}:$$

hincque integrando:

$$y = \left(\tfrac{2}{3} x^{\frac{3}{2}} - \tfrac{1}{2} \cdot \frac{2 x^{\frac{5}{2}}}{5 \cdot 2 a} - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{2 x^{\frac{7}{2}}}{7 \cdot 4 aa} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \frac{2 x^{\frac{9}{2}}}{9 \cdot 8 a^3} - \text{etc.}\right) \sqrt{2a},$$

seu

$$y = \left(\tfrac{x}{3} - \tfrac{1}{2} \cdot \frac{x^2}{5 \cdot 2a} - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{x^3}{7 \cdot 4aa} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \frac{x^4}{9 \cdot 8a^3} - \text{etc.}\right) 2\sqrt{2ax}.$$

Corollarium 1.

152. Integrale facilius inveniri potest, ponendo $x = a - v$, unde fit $\partial y = - \partial v \sqrt{(aa - vv)}$, et per reductionem tertiam

$$\int \partial v \sqrt{(aa - vv)} = \tfrac{1}{2} v \sqrt{(aa - vv)} + \tfrac{1}{2} aa \int \frac{\partial v}{\sqrt{(aa - vv)}}, \text{ hinc}$$

$$y = \text{C} - \tfrac{1}{2} v \sqrt{(aa - vv)} - \tfrac{1}{2} aa \text{ Arc. sin. } \tfrac{v}{a}, \text{ seu}$$

$$y = \text{C} - \tfrac{1}{2}(a - x) \sqrt{(2ax - xx)} - \tfrac{1}{2} aa \text{ Arc. sin. } \tfrac{a-x}{a}.$$

Ut igitur fiat $y = 0$, posito $x = 0$, capi debet $\text{C} = \tfrac{1}{2} aa \text{ Arc. sin. } 1$, ita uta sit

$$y = - \tfrac{1}{2}(a - x) \sqrt{(2ax - xx)} + \tfrac{1}{2} aa \text{ Arc. cos. } \tfrac{a-x}{a}.$$

Est vero

$$\text{Arc. sin. } \frac{\sqrt{x}}{\sqrt{2a}} = \tfrac{1}{2} \text{ Arc. cos. } \frac{a-x}{a}.$$

Corollarium 2.

153. Si ponamus $x = \frac{a}{2}$, fit $y = \frac{-aa\sqrt{3}}{8} + \frac{\pi aa}{6}$, series autem dat

$$y = 2aa\left(\frac{1}{2.3} - \frac{1}{2.5.2^3} - \frac{1.1}{2.4.7.2^5} - \frac{1.1.3}{2.4.6.9.2^7} - \text{etc.}\right)$$

unde colligitur

$$\pi = \frac{3\sqrt{3}}{4} + 6\left(\tfrac{1}{3} - \frac{1}{2.5.2^2} - \frac{1.1}{2.4.7.2^4} - \frac{1.1.3}{2.4.6.9.2^6} - \text{etc.}\right)$$

at per superiorem est

$$\pi = 3\left(1 + \frac{1}{2.3.2^2} + \frac{1.3}{2.4.5.2^4} + \frac{1.3.5}{2.4.6.7.2^6} + \text{etc.}\right) \text{ (§. 149.)}$$

ex quarum combinatione plures aliae formari possunt.

Exemplum 3.

154. *Formulam* $\partial y = \frac{\partial x}{\sqrt{(1+xx)}}$, *per seriem integrare.*

Integrale est $y = l[x + \sqrt{(1+xx)}]$, ita sumtum ut evanescat posito $x = 0$. At ob

$$\frac{1}{\sqrt{(1+xx)}} = 1 - \tfrac{1}{2}x^2 + \tfrac{1.3}{2.4}x^4 - \tfrac{1.3.5}{2.4.6}x^6 + \text{etc.}$$

erit idem integrale per seriem expressum:

$$y = x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \text{etc.}$$

Exemplum 4.

155. *Formulam* $\partial y = \frac{\partial x}{\sqrt{(xx-1)}}$ *per seriem integrare.*

Integratio dat $y = l[x + \sqrt{(xx-1)}]$ quod evanescit posito $x = 1$. Jam ob

$$\frac{1}{\sqrt{(xx-1)}} = \frac{1}{x} + \frac{1}{2x^3} + \frac{1.3}{2.4.x^5} + \frac{1.3.5}{2.4.6x^7} + \text{etc.}$$

erit idem integrale:

$$y = C + lx - \frac{1}{2.2x^2} - \frac{1.3}{2.4.4x^4} - \frac{1.3.5}{2.4.6.6x^6} - \text{etc.}$$

quod ut evanescat posito $x = 1$, constans ita definitur, ut fiat:

$$y = lx + \frac{1}{2.2}\left(1 - \frac{1}{xx}\right) + \frac{1.3}{2.4.4}\left(1 - \frac{1}{x^4}\right) + \frac{1.3.5}{2.4.6.6}\left(1 - \frac{1}{x^6}\right) + \text{etc.}$$

Corollarium.

156. Posito $x = 1 + u$ fit

$$\partial y = \frac{\partial u}{\sqrt{(2u+uu)}} = \frac{\partial u}{\sqrt{2u}}\left(1 + \frac{u}{2}\right)^{-\frac{1}{2}} =$$

$$\frac{\partial u}{\sqrt{2u}}\left(1 - \frac{1}{2}\cdot\frac{u}{2} + \frac{1.3}{2.4}\cdot\frac{uu}{4} - \frac{1.3.5}{2.4.6}\cdot\frac{u^3}{8} + \text{etc}\right)$$

unde integrando habebitur:

$$y = \frac{1}{\sqrt{2}}\left(2\sqrt{u} - \frac{1}{2}\cdot\frac{2u^{\frac{3}{2}}}{2.3} + \frac{1.3}{2.4}\cdot\frac{2u^{\frac{5}{2}}}{5.4} - \frac{1.3.5}{2.4.6}\cdot\frac{2u^{\frac{7}{2}}}{7.8} + \text{etc.}\right) \text{ seu}$$

$$y = \left(1 - \frac{1u}{2.3.2} + \frac{1.3.uu}{2.4.5.4} - \frac{1.3.5.u^3}{2.4.6.7.8} + \text{etc.}\right)\sqrt{2u}.$$

Exemplum 5.

157. *Formulam* $\partial y = \frac{\partial x}{(1-x)^n}$ *per seriem integrare*

Per integrationem fit

$$y = \frac{1}{(n-1)(1-x)^{n-1}} - \frac{1}{n-1},$$

facto $y = 0$ si $x = 0$, seu

$$y = \frac{(1-x)^{-n+1} - 1}{n-1}.$$

Jam vero per seriem est

$$\partial y = \partial x\left(1 + nx + \tfrac{n(n+1)}{1.2}x^2 + \tfrac{n(n+1)(n+2)}{1.2.3}x^3 + \text{etc.}\right)$$

unde idem integrale ita exprimetur:

$$y = x + \frac{nx^2}{2} + \frac{n(n+1)x^3}{1.2.3} + \frac{n(n+1)(n+2)x^4}{1.2.3.4} + \text{etc.}$$

Hinc autem quoque manifesto fit

$$(n-1)y + 1 = \frac{1}{(1-x)^{n-1}}.$$

Scholion.

158. Haec autem cum sint nimis obvia, quam ut iis fusius inhaerere sit opus, aliam methodum series eliciendi exponam magis absconditam, quae saepe in Analysi eximium usum afferre potest.

Problema 13.

159. Proposita formula differentiali

$$\partial y = x^{m-1}\,\partial x\,(a + b x^n)^{\frac{\mu}{\nu} - 1},$$

ejus integrale altera methodo in seriem convertere.

Solutio.

Ponatur $y = (a + b x^n)^{\frac{\mu}{\nu}} z$, erit

$$\partial y = (a + b x^n)^{\frac{\mu}{\nu} - 1}\left[\partial z (a + b x^n) + \tfrac{n\mu}{\nu} b x^{n-1} z \partial x\right]:$$

unde fit

$$x^{m-1}\partial x = \partial z\,(a + b x^n) + \tfrac{n\mu}{\nu} b x^{n-1} z \partial x, \text{ seu}$$

$$\nu x^{m-1}\partial x = \nu \partial z\,(a + b x^n) + n\mu b x^{n-1} z \partial x.$$

Jam antequam seriem, qua valor ipsius z definiatur, investigemus, notandum est casu, quo x evanescit, fieri

$$\partial y = a^{\frac{\mu}{\nu} - 1} x^{m-1}\partial x = a^{\frac{\mu}{\nu}}\partial z,$$

ut sit $\partial z = \frac{1}{a} x^{m-1}\partial x$. Statuamus ergo:

$$z = A x^m + B x^{m+n} + C x^{m+2n} + D x^{m+3n} + \text{etc.}$$

eritque

$$\frac{\partial z}{\partial x} = m A x^{m-1} + (m+n) B x^{m+n-1} + (m+2n) C x^{m+2n-1} + \text{etc.}$$

Substituantur hae series loco z et $\frac{\partial z}{\partial x}$ in aequatione

$$\frac{\nu \partial z}{\partial x}(a + b x^n) + n\mu b x^{n-1} z - \nu x^{m-1} = 0,$$

singulisque terminis secundum potestates ipsius x dispositis, orietur ista aequatio:

$$\left.\begin{array}{lll} m\nu a A x^{m-1} & + (m+n)\nu a B x^{m+n-1} & + (m+2n)\nu a C x^{m+2n-1} + \text{etc.} \\ -\nu & + m\nu b A & + (m+n)\nu b B \\ & + n\mu b A & + n\mu b B \end{array}\right\} = 0:$$

unde singulis terminis nihilo aequalibus positis, coëfficientes ficti per sequentes formulas definientur:

$$m\nu a A - \nu = 0; \qquad \text{hinc } A = \frac{1}{ma};$$
$$(m+n)\nu a B + (m\nu + n\mu) b A = 0; \quad B = -\frac{(m\nu+n\mu)b}{(m+n)\nu a} A;$$
$$(m+2n)\nu a C + [(m+n)\nu + n\mu] b B = 0; \quad C = -\frac{[(m+n)\nu+n\mu]b}{(m+2n)\nu a} B;$$
$$(m+3n)\nu a D + [(m+2n)\nu + n\mu] b C = 0; \quad D = -\frac{[(m+2n)\nu+n\mu]b}{(m+3n)\nu a} C;$$

sicque quilibet coëfficiens facile ex praecedente reperitur. Tum vero erit:

$$y = (a+bx^n)^{\frac{\mu}{\nu}} (A x^m + B x^{m+n} + C x^{m+2n} + D x^{m+3n} + \text{etc.})$$

Solutio 2.

Quemadmodum hic seriem secundum potestates ipsius x ascendentem assumsimus, ita etiam descendentem constituere licet:

$$z = A x^{m-n} + B x^{m-2n} + C x^{m-3n} + D x^{m-4n} + \text{etc.}$$

ut sit

$$\frac{\partial z}{\partial x} = (m-n) A x^{m-n-1} + (m-2n) B x^{m-2n-1} + (m-3n) C x^{m-3n-1} + \text{etc.}$$

quibus seriebus substitutis prodit:

$$\left.\begin{array}{llll} (m-n)\nu b A x^{m-1} & + (m-n)\nu a A x^{m-n-1} & + (m-2n)\nu a B x^{m-2n-1} & + (m-3n)\nu a C x^{m-3n-1} \\ + n\mu b A & + (m-2n)\nu b B & + (m-3n)\nu b C & + (m-4n)\nu b D \\ - \nu & + n\mu b B & + n\mu b C & + n\mu b D \end{array}\right\} = 0$$

Hinc ergo sequenti modo litterae A, B, C, etc. determinantur:

$$(m-n)\nu b A + n\mu b A - \nu = 0 \quad \text{ergo} \quad A = \frac{\nu}{(m-n)\nu + n\mu} \cdot \frac{1}{b};$$
$$(m-n)\nu a A + (m-2n)\nu b B + n\mu b B = 0, \quad B = \frac{-(m-n)\nu}{(m-2n)\nu + n\mu} \cdot \frac{a}{b} A;$$
$$(m-2n)\nu a B + (m-3n)\nu b C + n\mu b C = 0, \quad C = \frac{-(m-2n)\nu}{(m-3n)\nu + n\mu} \cdot \frac{a}{b} B;$$
$$(m-3n)\nu a C + (m-4n)\nu b D + n\mu b D = 0, \quad D = \frac{-(m-3n)\nu}{(m-4n)\nu + n\mu} \cdot \frac{a}{b} C;$$

ubi iterum lex progressionis harum litterarum est manifesta.

Corollarium 1.

160. Prior series ideo est memorabilis, quod casibus, quibus $(m+in)\nu + n\mu = 0$, seu $-\frac{m}{n} - \frac{\mu}{\nu} = i$, abrumpitur, atque

ipsum integrale algebraicum exhibet. Posterior vero abrumpitur, quoties $m - in = 0$ seu $\frac{m}{n} = i$, denotante i numerum integrum positivum.

Corollarium 2.

161. Utraque vero series etiam incommodo quodam laborat, quod non semper in usum vocari potest. Quando enim vel $m = 0$, vel $m + in = 0$, priori uti non licet: quando vero $(m + in)\nu + n\mu = 0$, seu $\frac{m}{n} + \frac{\mu}{\nu} = i$, usus posterioris tollitur, quia termini fierent infiniti.

Corollarium 3.

162. Hoc vero commode usu venit, ut quoties altera applicari nequit, altera certo in usum vocari possit, iis tantum casibus exceptis, quibus et $-\frac{m}{n}$ et $\frac{\mu}{\nu} + \frac{m}{n}$ sunt numeri integri positivi. Quia autem tum est $\nu = 1$, hi casus sunt rationales integri, nihilque difficultatis habent.

Corollarium 4.

163. Possunt etiam ambae series simul pro z conjungi hoc modo: Sit prior series $= P$, posterior vero $= Q$, ut capi possit tam $z = P$, quam $z = Q$. Binis autem conjungendis, erit $z = \alpha P + \beta Q$, dummodo sit $\alpha + \beta = 1$.

Scholion.

164. Inde autem, quod duas series pro z exhibemus, minime sequitur, has duas series inter se esse aequales, neque enim necesse est, ut valores ipsius y inde orti fiant aequales, dummodo quantitate constante a se invicem differant. Ita si prior series inventa per P, posterior per Q indicetur, quia ex illa fit $y = (a + bx^n)^{\frac{\mu}{\nu}} P$, ex hac vero $y = (a + bx^n)^{\frac{\mu}{\nu}} Q$, certo erit $(a + bx^n)^{\frac{\mu}{\nu}} (P - Q)$ quantitas constans, ideoque $P - Q = C(a + bx^n)^{-\frac{\mu}{\nu}}$. Utraque

scilicet series tantum integrale particulare praebet, quoniam nullam constantem involvit, quae non jam in formula differentiali contineatur. Interim tamen eadem methodo etiam valor completus pro z erui potest: praeter seriem enim assumtam P vel Q statui potest

$$z = P + \alpha + \beta x^n + \gamma x^{2n} + \delta x^{3n} + \varepsilon x^{4n} + \text{etc.}$$

ac substitutione facta, series P ut ante definitur, pro altera vero nova serie efficiendum est, ut sit

$$\left.\begin{array}{llll} n\nu a\beta x^{n-1} & + 2n\nu a\gamma x^{2n-1} & + 3n\nu a\delta x^{3n-1} & + 4n\nu a\varepsilon x^{4n-1} \\ + n\mu b\alpha & + n\nu b\beta & + 2n\nu b\gamma & + 3n\nu b\delta \\ & + n\mu b\beta & + n\mu b\gamma & + n\mu b\delta \end{array}\right\} = 0,$$

unde ducuntur hae determinationes:

$$\beta = \frac{-\mu b}{\nu a}.\alpha;\ \gamma = \frac{-(\mu+\nu)b}{2\nu a}.\beta;\ \delta = \frac{-(\mu+2\nu)b}{3\nu a}.\gamma;$$
$$\varepsilon = \frac{-(\mu+3\nu)b}{4\nu a}.\delta \text{ etc.}$$

ita ut prodeat

$$z = P + \alpha\left(1 - \frac{\mu}{\nu}.\frac{b}{a}x^n + \frac{\mu(\mu+\nu)}{\nu.2\nu}.\frac{b^2}{a^2}x^{2n} - \frac{\mu(\mu+\nu)(\mu+2\nu)}{\nu.2\nu.3\nu}.\frac{b^3}{a^3}x^{3n} + \text{etc.}\right)$$

seu $z = P + \alpha\left(1 + \frac{b}{a}x^n\right)^{-\frac{\mu}{\nu}}$, hincque

$$y = P(a + bx^n)^{\frac{\mu}{\nu}} + \alpha a^{\frac{\mu}{\nu}};$$

quod est integrale completum quia constans α mansit arbitraria

Exemplum 1.

165. *Formulam* $\partial y = \frac{\partial x}{\sqrt{(1-xx)}}$ *hoc modo per seriem integrare.*

Comparatione cum forma generali instituta, sit $a = 1$, $b = -1$, $m = 1$, $n = 2$, $\mu = 1$, $\nu = 2$: unde posito $y = z\sqrt{(1-xx)}$ prima solutio

$z = Ax + Bx^3 + Cx^5 + Dx^7 + \text{etc.}$ praebet

$A = 1$, $B = \frac{2}{3}A$; $C = \frac{4}{5}B$; $D = \frac{6}{7}C$; $E = \frac{8}{9}D$; etc.

unde colligimus:

$$y = \left(x + \tfrac{2}{3}x^3 + \tfrac{2.4}{3.5}x^5 + \tfrac{2.4.6}{3.5.7}x^7 + \text{etc.}\right)\sqrt{(1 - xx)},$$

quod integrale evanescit posito $x = 0$, est ergo $y =$ Arc. sin. x. Altera methodus hic frustra tentatur, ob $\frac{m}{n} + \frac{\mu}{\nu} = 1$.

Corollarium 1.

166. Posito $x = 1$, videtur hinc fieri $y = 0$, ob $\sqrt{(1 - xx)} = 0$: at perpendendum est, fieri hoc casu seriei infinitae summam infinitam, ita ut nihil obstet, quo minus fit $y = \frac{\pi}{2}$. Si ponamus $x = \frac{1}{2}$, fit $y = 30^\circ = \frac{\pi}{6}$, ideoque

$$\frac{\pi}{6} = \left(1 + \frac{2.}{3.4} + \frac{2.4}{3.5.4^2} + \frac{2.4.6}{3.5.7.4^3} + \text{etc.}\right)\frac{\sqrt{3}}{4}.$$

Corollarium 2.

167. Simili modo proposita formula $\partial y = \frac{\partial x}{\sqrt{(1 + xx)}}$ reperitur:

$$y = \left(x - \tfrac{2}{3}x^3 + \tfrac{2.4}{3.5}x^5 - \tfrac{2.4.6}{3.5.7}x^7 + \text{etc.}\right)\sqrt{(1 + xx)}$$

estque $y = l[x + \sqrt{(1 + xx)}]$.

Exemplum 2.

168. *Formulam $\partial y = \frac{\partial x}{x\sqrt{(1 - xx)}}$ hoc modo per seriem integrare.*

Est ergo $m = 0$, $n = 2$, $\mu = 1$, $\nu = 2$, $a = 1$, et $b = -1$, utendum igitur est altera serie sumendo

$$z = \frac{y}{\sqrt{(1 - xx)}} = \mathrm{A}x^{-2} + \mathrm{B}x^{-4} + \mathrm{C}x^{-6} + \mathrm{D}x^{-8} + \text{etc.}$$

sitque

$$\mathrm{A} = 1;\ \mathrm{B} = \tfrac{2}{3}\mathrm{A};\ \mathrm{C} = \tfrac{4}{5}\mathrm{B};\ \mathrm{D} = \tfrac{6}{7}\mathrm{C};\ \text{etc.}$$

Hinc ergo colligimus:

$$y = \left(\frac{1}{xx} + \frac{2}{3x^4} + \frac{2.4}{3.5\,x^6} + \frac{2.4.6}{3.5.7.x^8} + \text{etc.}\right)\sqrt{(1 - xx)}.$$

At integratio praebet $y = l\,\frac{1-\sqrt{(1-xx)}}{x}$, qui valores conveniunt, quia uterque evanescit posito $x = 1$.

Corollarium 1.

169. Cum autem haec series non convergat nisi capiatur $x > 1$, hoc autem casu formula $\sqrt{(1-xx)}$ fiat imaginaria, haec series nullius est usus.

Corollarium 2.

170. Si proponatur $\partial y = \frac{\partial x}{x\sqrt{(xx-1)}}$, eadem pro y series emergit per $\sqrt{-1}$ multiplicata, eritque

$$y = -\left(\frac{1}{xx} + \frac{2}{3x^4} + \frac{2.4}{3.5x^6} + \frac{2.4.6}{3.5.7x^8} + \text{etc.}\right)\sqrt{(xx-1)}.$$

Posito autem $x = \frac{1}{u}$, erit $\partial y = \frac{-\partial u}{\sqrt{(1-uu)}}$, et $y = C - \text{Arc. sin. } u$, seu $y = C - \text{Arc. sin. } \frac{1}{x}$: ubi sumi oportet $C = 0$, quia series illa evanescit posito $x = \infty$: ita ut sit $y = -\text{Arc. sin. } \frac{1}{x}$, quae cum superiori convenit statuendo $\frac{1}{x} = v$.

Exemplum 3.

171. *Formulam* $\partial y = \frac{\partial x}{\sqrt{(a+bx^4)}}$ *hoc modo per seriem integrare.*

Est hic $m=1$, $n=4$, $\mu=1$, $\nu=2$, ideoque posito $y=z\sqrt{(a+bx^4)}$, prior resolutio dat

$$z = Ax + Bx^5 + Cx^9 + Dx^{13} + \text{etc.}$$

existente

$$A = \frac{1}{a};\ B = \frac{-3b}{5a}A;\ C = \frac{-7b}{9a}B;\ D = \frac{-11b}{13a}C;\ \text{etc.}$$

ita ut sit

$$y = \left(\frac{x}{a} - \frac{3bx^5}{5aa} + \frac{3.7b^2x^9}{5.9a^3} - \frac{3.7.11b^3x^{13}}{5.9.13a^4} + \text{etc.}\right)\sqrt{(a+bx^4)}.$$

Hic autem quoque altera resolutio locum habet, ponendo

$$z = Ax^{-3} + Bx^{-7} + Cx^{-11} + Dx^{-15} + \text{etc.}$$

existente

$$A = \frac{-1}{b};\ B = \frac{-5a}{5b}A;\ C = \frac{-7a}{9b}B;\ D = \frac{-11a}{13b}C;\ \text{etc.}$$

unde colligitur:

$$y = -\left(\frac{1}{bx^3} - \frac{3a}{5b^2x^7} + \frac{3.7aa}{5.9b^3x^{11}} - \frac{3.7.11a^3}{5.9.13b^4x^{15}} + \text{etc.}\right)\sqrt{(a+bx^4)}$$

quarum serierum illa evanescit posito $x = 0$, haec vero posito $x = \infty$.

Corollarium 1.

172. Differentia ergo harum duarum serierum est constans, scilicet:

$$\left\{\begin{array}{l} +\dfrac{x}{a} - \dfrac{3bx^5}{5aa} + \dfrac{3.7b^2x^9}{5.9a^3} - \dfrac{3.7.11b^3x^{13}}{5.9.13a^4} + \text{etc.} \\ +\dfrac{1}{bx^3} - \dfrac{3a}{5bbx^7} + \dfrac{3.7a^2}{5.9b^3x^{11}} - \dfrac{3.7.11a^3}{5.9.13b^4x^{15}} + \text{etc.} \end{array}\right\}\sqrt{(a+bx^4)} = \text{Const.}$$

Corollarium 2.

173. Has ergo binas series colligendo habebimus

$$\frac{a+bx^4}{abx^3} - \frac{3}{5}\cdot\frac{a^3+b^3x^{12}}{a^2b^2x^7} + \frac{3.7}{5.9}\cdot\frac{a^5+b^5x^{20}}{a^3b^3x^{11}} - \text{etc.} = \frac{C}{\sqrt{(a+bx^4)}}$$

ubi quicunque valor ipsi x tribuatur, pro C semper eadem quantitas obtinetur.

Corollarium 3.

174. Ita si $a = 1$ et $b = 1$, erit haec series in $\sqrt{(1+x^4)}$ ducta semper constans, scilicet

$$\left(\frac{1+x^4}{x^3} - \tfrac{3}{5}\cdot\frac{1+x^{12}}{x^7} + \frac{3.7}{5.9}\cdot\frac{1+x^{20}}{x^{11}} - \text{etc.}\right)\sqrt{(1+x^4)} = C.$$

Cum igitur posito $x = 1$, fiat

$$C = \left(1 - \tfrac{3}{5} + \tfrac{3.7}{5.9} - \tfrac{3.7.11}{5.9.13} + \text{etc.}\right) 2\sqrt{2},$$

huicque valori etiam illa series, quicunque valor ipsi x tribuatur, est aequalis.

Corollarium 4.

175. Haec postrema series signis alternantibus procedens, per differentias facile in aliam iisdem signis praeditam transformatur, unde eadem constans concluditur

$$C = \left(1 + \tfrac{1}{5} + \tfrac{1.3}{5.9} + \tfrac{1.3.5}{5.9.13} + \tfrac{1.3.5.7}{5.9.13.17} + \text{etc.}\right)\sqrt{2},$$

quae series satis cito convergit, eritque proxime $C = \frac{13}{7}$.

Scholion.

176. Ista methodus in hoc consistit, ut series quaedam indefinita fingatur, ejusque determinatio ex natura rei derivetur. Ejus usus autem potissimum cernitur in aequationibus differentialibus resolvendis; verum etiam in praesenti instituto saepe utiliter adhibetur. Ejusdem quoque methodi ope quantitates transcendentes reciprocae, veluti exponentiales et sinus cosinusve angulorum, per series exprimuntur, quae etsi jam aliunde sint cognitae, tamen earum investigationem per integrationem exposuisse juvabit, cum simili modo alia praeclara erui queant.

Problema 14.

177. Quantitatem exponentialem $y = a^x$ in seriem convertere.

Solutio.

Sumtis logarithmis, habemus $ly = xla$, et differentiando $\frac{\partial y}{y} = \partial x la$, seu $\frac{\partial y}{\partial x} = yla$: unde valorem ipsius y per seriem quaeri oportet. Cum autem integrale completum latius pateat, notetur nostro casu posito $x = 0$, fieri debere $y = 1$: quare fingatur haec pro y series:

$$y = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{etc.}$$

unde fit

$$\frac{\partial y}{\partial x} = A + 2Bx + 3Cx^2 + 4Dx^3 + \text{etc.}$$

quibus substitutis in aequatione $\frac{\partial y}{\partial x} - yla = 0$, erit

$$\left.\begin{array}{l} A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4 + \text{etc.} \\ -la - Ala - Bla - Cla - Dla - \text{etc.} \end{array}\right\} = 0,$$

hincque coëfficientes ita determinantur:

$$A = la;\ B = \tfrac{1}{2} Ala;\ C = \tfrac{1}{3} Bla;\ D = \tfrac{1}{4} Cla \text{ etc.}$$

sicque consequimur:

$$y = a^x = 1 + \frac{xla}{1} + \frac{x^2(la)^2}{1.2} + \frac{x^3(la)^3}{1.2.3} + \frac{x^4(la)^4}{1.2.3.4} + \text{etc.}$$

quae est ipsa series notissima in Introductione data.

Scholion.

178. Pro sinibus et cosinibus angulorum ad differentialia secundi gradus est descendendum, ex quibus deinceps series integrale referens elici debet. Cum autem gemina integratio duplicem determinationem requirat, series ita est fingenda, ut duabus conditionibus ex natura rei petitis satisfaciat. Verum haec methodus etiam ad alias investigationes extenditur, quae adeo in quantitatibus algebraicis versantur, a cujusmodi exemplo hic inchoëmus.

Problema 15.

179. Hanc expressionem $y = [x + \sqrt{(1 + xx)}]^n$ in seriem, secundum potestates ipsius x progredientem, convertere.

Solutio.

Quia est $ly = nl[x + \sqrt{(1 + xx)}]$ erit $\frac{\partial y}{y} = \frac{n\partial x}{\sqrt{(1 + xx)}}$; jam ad signum radicale tollendum sumantur quadrata, erit

$(1+xx)\partial y^2 = nnyy\partial x^2$. Aequatio, sumto ∂x constante, denuo differentietur, ut per $2\partial y$ diviso prodeat

$$\partial\partial y(1+xx) + x\partial x\partial y - nny\partial x^2 = 0:$$

unde y per seriem elici debet. Primo autem patet, si sit $x = 0$, fore $y = 1$, ac si x infinite parvum, $y = (1+x)^n = 1 + nx$, Fingatur ergo talis series:

$$y = 1 + nx + Ax^2 + Bx^3 + Cx^4 + Dx^5 + Ex^6 + \text{etc.}$$

ex qua colligitur:

$$\frac{\partial y}{\partial x} = n + 2Ax + 3Bxx + 4Cx^3 + 5Dx^4 + 6Ex^5 + \text{etc. et}$$
$$\frac{\partial\partial y}{\partial x^2} = 2A + 6Bx + 12Cxx + 20Dx^3 + 30Ex^4 + \text{etc.}$$

Facta ergo substitutione adipiscimur:

$$\left.\begin{array}{l} 2A + 6Bx + 12Cxx + 20Dx^3 + 30Ex^4 + 42Fx^5 + \text{etc.} \\ \qquad\quad + 2A \quad + 6B \quad + 12C \quad + 20D \quad + \text{etc.} \\ \quad + nx + 2A \quad + 3B \quad + 4C \quad + 5D \quad + \text{etc.} \\ -nn - n^3 - An^2 \quad - Bn^2 \quad - Cn^2 \quad - Dn^2 \quad + \text{etc.} \end{array}\right\} = 0;$$

hincque derivantur sequentes determinationes

$$A = \frac{nn}{2};\ B = \frac{n(nn-1)}{2.3};\ C = \frac{A(nn-4)}{3.4};\ D = \frac{B(nn-9)}{4.5};\ \text{etc.}$$

ita ut sit

$$y = 1 + nx + \frac{nn}{1.2}x^2 + \frac{n(nn-1)}{1.2.3}x^3 + \frac{nn(nn-4)}{1.2.3.4}x^4 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}x^5$$
$$+ \frac{nn(nn-4)(nn-16)}{1.2.3.4.5.6}x^6 + \frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}x^7 + \text{etc.}$$

Corollarium 1.

180. Uti est $y = [x + \sqrt{(1+xx)}]^n$, si statuamus $z = [-x + \sqrt{(1+xx)}]^n$, pro z similis series prodit, in qua x tantum negative capitur, hinc ergo concluditur:

$$\frac{y+z}{2} = 1 + \frac{nn}{1.2}x^2 + \frac{nn(nn-4)}{1.2.3.4}x^4 + \frac{nn(nn-4)(nn-16)}{1.2.3.4.5.6}x^6 + \text{etc. et}$$
$$\frac{y-z}{2} = nx + \frac{n(nn-1)}{1.2.3}x^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}x^5$$
$$+ \frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}x^7 + \text{etc.}$$

Corollarium 2.

181. Si ponatur $x = \sqrt{-1} \cdot \sin.\Phi$, erit $\sqrt{(1+xx)} = \cos.\Phi$; hincque

$$y = (\cos.\Phi + \sqrt{-1}.\sin.\Phi)^n = \cos.n\Phi + \sqrt{-1}.\sin.n\Phi, \text{ et}$$
$$z = (\cos.\Phi - \sqrt{-1}.\sin.\Phi)^n = \cos.n\Phi - \sqrt{-1}.\sin.n\Phi:$$

unde deducimus:

$$\cos.n\Phi = 1 - \frac{nn}{1.2}\sin.\Phi^2 + \frac{nn(nn-4)}{1.2.3.4}\sin.\Phi^4 - \frac{nn(nn-4)(nn-16)}{1.2.3.4.5.6}\sin.\Phi^6 + \text{etc.}$$
$$\sin.n\Phi = n\sin.\Phi - \frac{n(nn-1)}{1.2.3}\sin.\Phi^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}\sin.\Phi^5 - \frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}\sin.\Phi^7 + \text{etc.}$$

Corollarium 3.

182. Hae series ad multiplicationem angulorum pertinent, atque hoc habent singulare, quod prior tantum casibus, quibus n est numerus par, posterior vero, quibus est numerus impar, abrumpatur.

Problema 16.

183. Proposito angulo Φ, tam ejus sinum quam cosinum per seriem infinitam exprimere.

Solutio.

Sit $y = \sin.\Phi$ et $z = \cos.\Phi$, erit $\partial y = \partial\Phi\sqrt{(1-yy)}$ et $\partial z = -\partial\Phi\sqrt{(1-zz)}$. Sumantur quadrata

$$\partial y^2 = \partial\Phi^2(1-yy) \text{ et } \partial z^2 = \partial\Phi^2(1-zz):$$

differentietur sumto $\partial\Phi$ constante, fietque

$$\partial\partial y = -y\partial\Phi^2 \text{ et } \partial\partial z = -z\partial\Phi^2,$$

sicque y et z ex eadem aequatione definiri oportet. Sed pro $y = \sin.\Phi$ observandum est, si Φ evanescat, fieri $y = \Phi$; pro $z = \cos.\Phi$ verum, si Φ evanescat, fieri $z = 1 - \frac{1}{2}\Phi\Phi$, seu $z = 1 + 0\Phi$. Fingatur ergo

$$y = \phi + A\phi^3 + B\phi^5 + C\phi^7 + \text{etc.}$$
$$z = 1 + \alpha\phi^2 + \beta\phi^4 + \gamma\phi^6 + \delta\phi^8 + \text{etc.}$$

fietque substitutione facta:

$$\left.\begin{array}{l} 2.3\,A\phi + 4.5\,B\phi^3 + 6.7\,C\phi^5 + \text{etc.} \\ +\;1 \quad + \quad A \quad + \quad B \end{array}\right\} = 0 \text{ et}$$

$$\left.\begin{array}{l} 1.2\,\alpha + 3.4\,\beta\phi^2 + 5.6\,\gamma\phi^4 + \text{etc.} \\ +\;1 \quad + \quad \alpha \quad + \quad \beta \end{array}\right\} = 0:$$

unde colligimus:

$$A = \frac{-1}{2.3};\; B = \frac{-A}{4.5};\; C = \frac{-B}{6.7};\; D = \frac{-C}{8.9};\; \text{etc.}$$
$$\alpha = \frac{-1}{1.2};\; \beta = \frac{-\alpha}{3.4};\; \gamma = \frac{-\beta}{5.6};\; \delta = \frac{-\gamma}{7.8};\; \text{etc.}$$

unde series jam notissimae obtinentur:

$$\sin.\phi = \frac{\phi}{1} - \frac{\phi^3}{1.2.3} + \frac{\phi^5}{1.2.3.4.5} - \frac{\phi^7}{1.2\ldots\ldots 7} + \text{etc.}$$

$$\cos.\phi = 1 - \frac{\phi^2}{1.2} + \frac{\phi^4}{1.2.3.4} - \frac{\phi^6}{1.2\ldots\ldots 7} + \text{etc}$$

Scholion.

184. Non opus erat ad differentialia secundi gradus descendere: sed ex formularum $y = \sin.\phi$ et $z = \cos.\phi$ differentialibus, quae sunt $\partial y = z\partial\phi$ et $\partial z = -y\partial\phi$, eaedem series facile reperiuntur. Fictis enim seriebus ut ante $y = \phi + A\phi^3 + B\phi^5 + C\phi^7 + \text{etc.}$ et $z = 1 + \alpha\phi^2 + \beta\phi^4 + \gamma\phi^6 + \text{etc.}$ substitutione facta, obtinebitur:

ex priore

$$\left.\begin{array}{l} 1 + 3\,A\phi^2 + 5\,B\phi^4 + 7\,C\phi^6 + \text{etc.} \\ -1 - \quad \alpha \quad - \quad \beta \quad - \quad \gamma \end{array}\right\} = 0$$

ex posteriore

$$\left.\begin{array}{l} 2\,\alpha\phi + 4\,\beta\phi^3 + 6\,\gamma\phi^5 + \text{etc.} \\ +\;1 \quad + \quad A \quad + \quad B \end{array}\right\} = 0:$$

unde colliguntur hae determinationes:

$$\alpha = -\tfrac{1}{2};\ A = \tfrac{\alpha}{3};\ \beta = \tfrac{-A}{4};\ B = \tfrac{\beta}{5};\ \gamma = \tfrac{-B}{6};\ C = \tfrac{\gamma}{7};$$

ideoque

$$\alpha = -\tfrac{1}{2};\ \beta = +\tfrac{1}{2.3.4};\ \gamma = -\tfrac{1}{2.3.4.5.6};\ \text{etc.}$$

$$A = -\tfrac{1}{2.3};\ B = +\tfrac{1}{2.3.4.5};\ C = -\tfrac{1}{2.3.4.5.6.7};\ \text{etc.}$$

qui valores cum praecedentibus conveniunt. Hinc intelligitur, quomodo saepe duae aequationes simul facilius per series evolvuntur, quam si alteram seorsim tractare velimus.

Problema 17.

185. Per seriem exprimere valorem quantitatis y, qui satisfaciat huic aequationi $\frac{m\,\partial y}{\sqrt{(a+byy)}} = \frac{n\,\partial x}{\sqrt{(f+gxx)}}$.

Solutio.

Integratio hujus aequationis suppeditat:

$$\frac{m}{\sqrt{b}}\, l\,[\sqrt{(a+byy)} + y\sqrt{b}] = \frac{n}{\sqrt{g}}\, l\,[\sqrt{(f+gxx)} + x\sqrt{g}] + C,$$

unde deducimus:

$$y = \frac{1}{2\sqrt{b}}\left(\frac{\sqrt{(f+gxx)}+x\sqrt{g}}{h}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - \frac{a}{2\sqrt{b}}\left(\frac{\sqrt{(f+axx)}-x\sqrt{g}}{k}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}.$$

constantes h et k ita capiendo, ut sit $hk = f$. Hinc discimus, si x sumatur evanescens, fore

$$y = \frac{1}{2\sqrt{b}}\left(\frac{\sqrt{f}+x\sqrt{g}}{h}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - \frac{a}{2\sqrt{b}}\left(\frac{\sqrt{f}-x\sqrt{g}}{k}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}, \text{ seu}$$

$$y = \frac{1}{2\sqrt{b}}\left[\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right]$$

$$+ \frac{nx}{2m\sqrt{f}}\left[\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} + a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right],$$

vel posito $y = \mathrm{A} + \mathrm{B}x$, erit $\mathrm{B} = \frac{n\sqrt{(\mathrm{AA}b + a)}}{m\sqrt{f}}$, ita ut constans B definiatur ex constante

$$\mathrm{A} = \frac{1}{2\sqrt{b}}\left[\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right]:$$

et vicissim

$$\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} = \mathrm{A}\sqrt{b} + \sqrt{(a + b\mathrm{AA})}, \text{ atque}$$

$$a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} = -\mathrm{A}\sqrt{b} + \sqrt{(a + b\mathrm{AA})}.$$

Nunc ad seriem inveniendam, aequatio proposita, sumtis quadratis

$$mm(f + gxx)\,\partial y^2 = nn(a + byy)\,\partial x^2,$$

denuo differentietur, sumto ∂x constante, ut facta divisione per $2\,\partial y$ prodeat:

$$mm\,\partial\partial y\,(f + gxx) + mmgx\,\partial x\,\partial y - nnby\,\partial x^2 = 0.$$

Jam pro y fingatur series:

$$y = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3 + \mathrm{E}x^4 + \mathrm{F}x^5 + \text{etc.}$$

qua substituta habebitur

$$\left.\begin{array}{rrrrr} 2mmf\mathrm{C} & + 6mmf\mathrm{D}x & + 12mmf\mathrm{E}x^2 & + 20mmf\mathrm{F}x^3 & + \text{etc.} \\ & & + 2mmg\mathrm{C} & + 6mmg\mathrm{D} & + \text{etc.} \\ & + mmgb\mathrm{B} & + 2mmg\mathrm{C} & + 3mmg\mathrm{D} & + \text{etc.} \\ - nnb\mathrm{A} & - nnb\mathrm{B} & - nnb\mathrm{C} & - nnb\mathrm{D} & - \text{etc.} \end{array}\right\} = 0.$$

Cum ergo A et B dentur, reliquae litterae ita determinantur:

$$\mathrm{C} = \frac{nnb}{2mmf}\mathrm{A};$$

$$\mathrm{D} = \frac{nnb - mmg}{2.3mmf}\mathrm{B};\quad \mathrm{E} = \frac{nnb - 4mmg}{3.4mmf}\mathrm{C};$$

$$\mathrm{E} = \frac{nnb - 9mmg}{4.5mmf}\mathrm{D};\quad \mathrm{G} = \frac{nnb - 16mmg}{5.6mmf}\mathrm{E};$$

$$\mathrm{H} = \frac{nnb - 25mmg}{6.7mmf}\mathrm{F};\quad \mathrm{J} = \frac{nnb - 36mmg}{7.8mmf}\mathrm{G};$$

sicque series pro y erit cognita.

Exemplum 1.

186. *Functionem transcendentem* $c^{\text{Arc. sin.}\, x}$ *per seriem secundum potestates ipsius* x *progredientem exprimere.*

Ponatur $y = c^{\text{Arc. sin.}\, x}$, erit $ly = lc\, .\, \text{Arc. sin.}\, x$, et $\frac{\partial y}{y} = \frac{\partial x\, lc}{\sqrt{(1-xx)}}$: hinc $\partial y^2 (1 - xx) = yy \partial x^2 (lc)^2$ et differentiando $\partial\partial y (1 - xx) - x \partial x \partial y - y \partial x^2 (lc)^2 = 0$. Observetur jam, posito x evanescente, fore $y = c^x = 1 + xlc$; hinc fingatur series $y = 1 + xlc + Ax^2 + Bx^3 + Cx^4 + Dx^5 +$ etc. qua substituta habebitur:

$$\left.\begin{array}{llll} 1.2\,A + 2.3\,B\,x + & 3.4\,C\,x^2 + & 4.5\,D\,x^3 + & 5.6\,E\,x^4 \\ & -1.2\,A & -2.3\,B & -3.4\,C \\ -lc & -\;\;2\,A & -\;\;3\,B & -\;\;4\,C \\ -(lc)^2 - (lc)^3 & -A\,(lc)^2 & -B\,(lc)^2 & -C\,(lc)^2 \end{array}\right\} \text{etc.} = 0.$$

Unde reliqui coëfficientes ita definiuntur:

$$A = \frac{(l\,)^2}{1.2}; \qquad B = \frac{[1 + (lc)^2]\,lc}{2.3};$$

$$C = \frac{4 + (lc)^2}{3.4} A; \qquad D = \frac{9 + (lc)^2}{4.5} B;$$

$$E = \frac{16 + (lc)^2}{5.6} C; \qquad F = \frac{25 + (lc)^2}{6.7} D; \text{ etc.}$$

Sit brevitatis gratia $lc = \gamma$, eritque

$$c^{\text{Arc. sin.}\, x} = 1 + \gamma x + \frac{\gamma\gamma}{1.2} x^2 + \frac{\gamma(1+\gamma\gamma)}{1.2.3} x^3 + \frac{\gamma\gamma(4+\gamma\gamma)}{1.2.3.4} x^4$$

$$+ \frac{\gamma(1+\gamma\gamma)(9+\gamma\gamma)}{1.2.3.4.5} x^5 + \frac{\gamma\gamma(4+\gamma\gamma)(16+\gamma\gamma)}{1.2.3.4.5.6} x^6 + \text{etc.}$$

Exemplum 2.

187. *Posito* $x = \sin. \Phi$, *invenire seriem secundum potestates ipsius* x *progredientem, quae sinum anguli* $n\Phi$ *exprimat.*

Ponatur $y = \sin. n\Phi$, ac notetur evanescente Φ, fieri $x = \Phi$ et $y = n\Phi = nx$, hoc est $y = 0 + nx$, quod est seriei quaesitae initium. Nunc autem est

$\partial \phi = \frac{\partial x}{\sqrt{(1-xx)}}$, et $n \partial \phi = \frac{\partial y}{\sqrt{(1-yy)}}$. Ergo

$\frac{\partial y}{\sqrt{(1-yy)}} = \frac{n \partial x}{\sqrt{(1-xx)}}$,

et sumtis quadratis

$$(1-xx)\partial y^2 = nn \partial x^2 (1-yy) \text{: hinc}$$

$$\partial\partial y (1-xx) - x\partial x \partial y + nny\partial x^2 = 0.$$

Quare fingatur haec series

$$y = nx + Ax^3 + Bx^5 + Cx^7 + Dx^9 + \text{etc.}$$

qua substituta habebitur:

$$\left.\begin{array}{llll} 2.3\,A x & + 4.5\,B x^3 & + 6.7\,C x^5 & + 8.9\,D x^7 \\ & - 2.3\,A & - 4.5\,B & - 6.7\,C \\ -n & - 3\,A & - 5\,B & - 7\,C \\ +n^3 & + nnA & + nnB & + nnC \end{array}\right\} \text{etc.} = 0.$$

Unde hae determinationes colliguntur:

$$A = \frac{-n(nn-1)}{2.3}; \quad B = \frac{-(nn-9)A}{4.5}; \quad C = \frac{-(nn-25)B}{6.7}; \text{ etc.}$$

ita ut sit:

$$y = nx - \frac{n(nn-1)}{1.2.3}x^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}x^5 - \frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}x^7 + \text{etc.}$$

sive

$$\sin. n\phi = n\sin.\phi - \frac{n(nn-1)}{1.2.3}\sin.\phi^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}\sin.\phi^5 - \text{etc.}$$

Scholion.

188. Quia haec series tantum casibus, quibus n est numerus impar, abrumpitur, pro paribus notandum est, seriem commode exprimi posse per productum ex sin. ϕ in aliam seriem, secundum cosinus ipsius ϕ potestates progredientem. Ad quam inveniendam ponamus cos. $\phi = u$, fitque sin. $n\phi = z$ sin. $\phi = z\sqrt{(1-uu)}$; unde ob $\partial\phi = -\frac{\partial u}{\sqrt{(1-uu)}}$, erit differentiando

$$-\frac{n\partial u \cos. n\phi}{\sqrt{(1-uu)}} = \partial z\sqrt{(1-uu)} - \frac{zu\partial u}{\sqrt{(1-uu)}}, \text{ seu}$$

$$-n\partial u \cos. n\phi = \partial z(1-uu) - zu\partial u,$$

quae, sumto ∂u constante, denuo differentiata dat: $-\frac{nn\partial u^2 \sin. n\Phi}{\sqrt{(1-uu)}}$ $= \partial\partial z(1-uu) - 3u\partial u\partial z - z\partial u^2 = -nnz\partial u^2$, ob $\frac{\sin. n\Phi}{\sqrt{(1-uu)}} = z$. Quocirca series quaesita pro $z = \frac{\sin. n\Phi}{\sin. \Phi}$ ex hac aequatione erui debet

$$\partial\partial z(1-uu) - 3u\partial u\partial z - z\partial u^2 + nnz\partial u^2 = 0,$$

ubi notandum est, quia $u = \cos. \Phi$ evanescente u, quo casu sit $\Phi = 90°$, fore vel $z = 0$, si n numerus par, vel $z = 1$, si $n = 4\alpha + 1$; vel $z = -1$, si $n = 4\alpha - 1$. Qui singuli casus seorsim sunt evolvendi: et quo principium cujusque seriei pateat, sit $\Phi = 90° - \omega$, et evanescente ω, fit $u = \cos. \Phi = \omega$; $\sin. \Phi = 1$; $\sin. n\Phi = \sin. (90.n - n\omega) = z$.

Nunc pro casibus singulis:

I. si $n = 4\alpha$; fit $z = -\sin. n\omega = -nu$
II si $n = 4\alpha + 1$; fit $z = \cos. n\omega = 1$
III. si $n = 4\alpha + 2$; fit $z = \sin. n\omega = +nu$
IV. si $n = 4\alpha + 3$; fit $z = -\cos. n\omega = -1$

unde series jam satis notae deducuntur.