CAPUT II.

DE

INTEGRATIONE FORMULARUM DIFFERENTIALIUM IRRATIONALIUM.

Problema 6.

88.

Proposita formula differentiali $\partial y = \frac{\partial x}{\sqrt{(\alpha + \beta x + \gamma xx)}}$, ejus integrale invenire.

Solutio.

Quantitas $\alpha + \beta x + \gamma xx$, vel habet duos factores reales vel secus.

I. Priori casu formula proposita erit hujusmodi $\partial y = \frac{\partial x}{\sqrt{(a + bx)(f + gx)}}$. Statuatur ad irrationalitatem tollendam

$$(a + bx)(f + gx) = (a + bx)^2 zz,$$

erit $x = \frac{f - azz}{bzz - g}$, ideoque

$$\partial x = \frac{2(ag - bf)z\partial z}{(bzz - g)^2} \text{ et } \sqrt{(a + bx)(f + gx)} = -\frac{(ag - bf)z}{bzz - g}:$$

unde fit $\partial y = \frac{-2\partial z}{bzz - g} = \frac{2\partial z}{g - bzz}$, atque $z = \sqrt{\frac{f + gx}{a - bx}}$. Quare si litterae b et g paribus signis sunt affectae, integrale per logarithmos, sin autem signis disparibus, per angulos exprimetur.

II. Posteriori casu habebimus $\partial y = \frac{\partial x}{\sqrt{(aa - 2abx\cos.\zeta + bbxx)}}$. Statuatur

$$bbxx - 2abx\cos.\zeta + aa = (bx - az)^2, \text{ erit}$$

$$-2bx\cos.\zeta + a = -2bxz + azz \text{ et } x = \frac{a(1 - zz)}{2b(\cos.\zeta - z)};$$

hinc $\partial x = \frac{a\partial z\,(1-2z\cos.\zeta+zz)}{2b\,(\cos.\zeta-z)^2}$, et

$\sqrt{}(aa-2abx\cos.\zeta+bbxx) = \frac{a\,(1-2z\cos.\zeta+zz)}{2\,(\cos.\zeta-z)}$: ergo

$$\partial y = \frac{\partial z}{b\,(\cos.\zeta-z)}, \text{ et } y = -\frac{1}{b}\,l\,(\cos.\zeta-z).$$

At est

$$z = \frac{bx-\sqrt{}(aa-2abx\cos.\zeta+bbxx)}{a}, \text{ ideoque}$$

$$y = -\frac{1}{b}\,l\,\frac{a\cos.\zeta-bx+\sqrt{}(aa-2abx\cos.\zeta+bbxx)}{a}, \text{ vel}$$

$$y = \frac{1}{b}\,l\,[-a\cos.\zeta+bx+\sqrt{}(aa-2abx\cos.\zeta+bbxx)]+C.$$

Corollarium 1.

89. Casus ultimus latius patet, et ad formulam $\partial y = \frac{\partial x}{\sqrt{}(\alpha+\beta x+\gamma xx)}$, accomodari potest, dummodo fuerit γ quantitas positiva: namque ob $b=\sqrt{}\gamma$ et $a\cos.\zeta = \frac{-\beta}{2\sqrt{}\gamma}$, oritur,

$$y = \frac{1}{\sqrt{}\gamma}\,l\,[\tfrac{\beta}{2\sqrt{}\gamma}+x\sqrt{}\gamma+\sqrt{}(\alpha+\beta x+\gamma xx)]+C \text{ seu}$$

$$y = \frac{1}{\sqrt{}\gamma}\,l\,[\tfrac{1}{2}\beta+\gamma x+\sqrt{}\gamma\,(\alpha+\beta x+\gamma xx)]+C.$$

Corollarium 2.

90. Pro casu priori cum sit

$$\int\frac{2\partial z}{g-bzz} = \frac{1}{\sqrt{}bg}\,l\,\frac{\sqrt{}g+z\sqrt{}b}{\sqrt{}g-z\sqrt{}b} \text{ et}$$

$$\int\frac{2\partial z}{g+bzz} = \frac{2}{\sqrt{}bg}\text{ Arc. tang. }\frac{z\sqrt{}b}{\sqrt{}g},$$

habebimus hos casus:

$$\int\frac{\partial x}{\sqrt{}(a+bx)(f+gx)} = \frac{1}{\sqrt{}bg}\,l\,\frac{\sqrt{}g(a+bx)+\sqrt{}b(f+gx)}{\sqrt{}g(a+bx)-\sqrt{}b(f+gx)}+C$$

$$\int\frac{\partial x}{\sqrt{}(bx-a)(f+gx)} = \frac{1}{\sqrt{}bg}\,l\,\frac{\sqrt{}g(bx-a)+\sqrt{}b(f+gx)}{\sqrt{}g(bx-a)-\sqrt{}b(f+gx)}+C$$

$$\int\frac{\partial x}{\sqrt{}(bx-a)(gx-f)} = \frac{1}{\sqrt{}bg}\,l\,\frac{\sqrt{}g(bx-a)+\sqrt{}b(gx-f)}{\sqrt{}g(bx-a)-\sqrt{}b(gx-f)}+C$$

$$\int\frac{\partial x}{\sqrt{}(a-bx)(f-gx)} = \frac{-1}{\sqrt{}bg}\,l\,\frac{\sqrt{}g(a-bx)+\sqrt{}b(f-gx)}{\sqrt{}g(a-bx)-\sqrt{}b(f-gx)}+C$$

$$\int \frac{\partial x}{\sqrt{(a-bx)(f+gx)}} = \frac{2}{\sqrt{bg}} \text{Arc. tang.} \frac{\sqrt{b(f+gx)}}{\sqrt{g(a-bx)}} + C$$

$$\int \frac{\partial x}{\sqrt{(a-bx)(gx-f)}} = \frac{2}{\sqrt{bg}} \text{Arc. tang.} \frac{\sqrt{b(gx-f)}}{\sqrt{g(a-bx)}} + C.$$

Corollarium 3.

91. Harum sex integrationum quatuor priores omnes in casu Coroll. 1. continentur, binae autem postremae in hac formula $\partial y = \frac{\partial x}{\sqrt{(\alpha+\beta x-\gamma xx)}}$ continentur: sit enim pro penultima

$$af = \alpha,\ ag - bf = \beta,\ bg = \gamma,$$

unde colligitur

$$y = \frac{1}{\sqrt{\gamma}} \text{Arc. tang.} \frac{2\sqrt{\gamma(\alpha+\beta x-\gamma xx)}}{\beta - 2\gamma x};$$

si scilicet ille arcus duplicetur. Per cosinum autem erit

$$y = \frac{1}{\sqrt{\gamma}} \text{Arc. cos.} \frac{\beta - 2\gamma x}{\sqrt{(\beta\beta+4\alpha\gamma)}} + C;$$

cujus veritas ex differentiatione patet.

Scholion 1.

92. Ex solutione hujus problematis patet etiam, hanc formulam latius patentem $\frac{X\partial x}{\sqrt{(\alpha+\beta x+\gamma xx)}}$, si X fuerit functio rationalis quaecunque ipsius x, per praecepta capitis praecedentis integrari posse. Introducta enim loco x variabili z, qua formula radicalis rationalis redditur, etiam X abibit in functionem rationalem ipsius z. Idem adhuc generalius locum habet, si posito $\sqrt{(\alpha+\beta x+\gamma xx)} = u$, fuerit X functio quaecunque rationalis binarum quantitatum x et u, tum enim per substitutionem adhibitam, quia tam pro x quam pro u formulae rationales ipsius z scribuntur, prodibit formula differentialis rationalis. Hoc idem etiam ita enunciari potest, ut dicamus, formulae $X\partial x$, si functio X nullam aliam irrationalem praeter $\sqrt{(\alpha+\beta x+\gamma xx)}$ involvat, integrale assignari posse, propterea quod ea, ope substitutionis, in formulam differentialem rationalem transformari potest.

Scholion 2.

93. Proposita autem formula differentiali quaecunque irrationali, ante omnia videndum est, num ea ope cujuspiam substitutionis in rationalem transformari possit? quod si succedat, integratio per praecepta capitis praecedentis absolvi poterit: unde simul intelligitur, integrale nisi sit algebraicum, alias quantitates transcendentes non involvere praeter logarithmos et angulos. Quodsi autem nulla substitutio ad hoc idonea inveniri possit, ab integrationis labore est desistendum, quandoquidem integrale neque algebraice neque per logarithmos vel angulos exprimere valemus. Veluti si $X\partial x$ fuerit ejusmodi formula differentialis, quae nullo pacto ad rationalitatem reduci queat, ejus integrale $\int X\partial x$ ad novum genus functionum transcendentium erit referendum, in quo nihil aliud nobis relinquitur, nisi ut ejus valorem vero proxime assignare conemur. Admisso autem novo genere quantitatum transcendentium, innumerabiles aliae formulae eo reduci atque integrari poterunt. Imprimis igitur in hoc erit elaborandum, ut pro quolibet genere formula simplicissima notetur, qua concessa reliquarum formularum integralia definire liceat. Hinc deducimur ad quaestionem maximi momenti, quomodo integrationem formularum magis complicatarum ad simpliciores reduci oporteat. Quod antequam aggrediamur, alias ejusmodi formulas perpendamus, quae ope idoneae substitutionis ab irrationalitate liberari queant; quemadmodum jam ostendimus, quoties X fuerit functio rationalis quantitatum

$$x \text{ et } u = \sqrt{(\alpha + \beta x + \gamma xx)},$$

ita ut alia irrationalitas non ingrediatur praeter radicem quadratam hujusmodi formulae $\alpha + \beta x + \gamma xx$, toties formulam differentialem $X\partial x$ in rationalem transformari posse.

Problema 7.

94. Proposita formula differentiali $X\partial x (a + bx)^{\frac{\mu}{\nu}}$, in qua X denotet functionem quamcunque rationalem ipsius x, eam ab irrationalitate liberare.

Solutio.

Statuatur $a + bx = z^\nu$, ut fiat $(a + bx)^{\frac{\mu}{\nu}} = z^\mu$: tum quia $x = \frac{z^\nu - a}{b}$, facta hac substitutione, functio X abibit in functionem rationalem ipsius z, quae sit Z; et ob $\partial x = \frac{\nu}{b} z^{\nu-1} \partial z$, formula nostra differentialis induet hanc formam $\frac{\nu}{b} Zz^{\mu+\nu-1} \partial z$, quae cum sit rationalis, per caput superius integrari potest, et integrale, nisi sit algebraicum, per logarithmos et angulos exprimetur.

Corollarium 1.

95. Hac substitutione generalius negotium confici poterit, si posito $(a + bx)^{\frac{1}{\nu}} = u$, littera V denotet functionem quamcunque rationalem binarum quantitatum x et u; cum enim posito $x = \frac{u^\nu - a}{b}$, fiat V functio rationalis ipsius u, formula $V \partial x = \frac{\nu}{b} V u^{\nu-1} \partial u$, erit rationalis.

Corollarium 2.

96. Quin etiam si binae irrationalitates ejusdem quantitatis $a + bx$, scilicet $(a + bx)^{\frac{1}{\nu}} = u$ et $(a + bx)^{\frac{1}{n}} = v$, ingrediantur in formulam $X\partial x$, posito $a + bx = z^{n\nu}$ fit $x = \frac{z^{n\nu} - a}{b}$, $u = z^n$, et $v = z^\nu$; unde cum X fiat functio rationalis ipsius z, et $\partial x = \frac{n\nu}{b} z^{n\nu-1} \partial z$, hac substitutione formula $X \partial x$ evadet rationalis.

Corollarium 3.

97. Eodem modo intelligitur, si posito

$$(a + bx)^{\frac{1}{\lambda}} = u, \ (a + bx)^{\frac{1}{\mu}} = v, \ (a + bx)^{\frac{1}{\nu}} = t \text{ etc.}$$

littera X denotet functionem quamcunque rationalem quantitatum x, u, v, t etc. formulam differentialem $X\partial x$ rationalem reddi facto $a+bx=z^{\lambda\mu\nu}$; erit enim

$$x=\frac{z^{\lambda\mu\nu}-a}{b};\quad u=z^{\mu\nu};\quad v=z^{\lambda\nu};\quad t=z^{\lambda\mu}\text{ etc. et}$$

$$\partial x=\frac{\lambda\mu\nu}{b}z^{\lambda\mu\nu-1}\partial z.$$

Exemplum.

98. Proposita hac formula $\partial y=\frac{x\partial x}{\sqrt[3]{(1+x)}-\sqrt{(1+x)}}$, facto $1+x=z^6$, reperitur $\partial y=-\frac{6z^3\partial z(1-z^6)}{1-z}$, seu

$$\partial y=-6\partial z(z^3+z^4+z^5+z^6+z^7+z^8):$$

hincque integrando

$$y=C-\tfrac{3}{2}z^4-\tfrac{6}{5}z^5-z^6-\tfrac{6}{7}z^7-\tfrac{3}{4}z^8-\tfrac{2}{3}z^9,$$

et restituendo

$$y=C-\tfrac{3}{2}\sqrt[3]{(1+x)^2}-\tfrac{6}{5}\sqrt[6]{(1+x)^5}-1-x-\tfrac{6}{7}(1+x)\sqrt[6]{(1+x)}$$
$$-\tfrac{3}{4}(1+x)\sqrt[3]{(1+x)}-\tfrac{2}{3}(1+x)\sqrt{(1+x)}$$

ita ut integrale adeo algebraice exhibeatur.

Problema 8.

99. Proposita formula differentiali $X\partial x\left(\frac{a+bx}{f+gx}\right)^{\frac{\mu}{\nu}}$, denotante X functionem rationalem quamcunque ipsius x, eam ab irrationalitate liberare.

Solutio.

Posito $\frac{a+bx}{f+gx}=z^\nu$, fit $\left(\frac{a+bx}{f+gx}\right)^{\frac{\mu}{\nu}}=z^\mu$, et

$$x=\frac{a-fz^\nu}{gz^\nu-b},\text{ atque }\partial x=\frac{\nu(bf-ag)z^{\nu-1}\partial z}{(gz^\nu-b)^2};$$

sicque loco X prodibit functio rationalis ipsius z, qua posita $= Z$, erit formula nostra differentialis

$$= \frac{\nu(bf - ag) Z z^{\mu+\nu-1} \partial z}{(gz^\nu - b)^2},$$

quae cum sit rationalis, per praecepta Cap. I. integrari poterit.

Corollarium 1.

100. Posito $\left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\nu}} = u$, si X fuerit functio quaecunque rationalis binarum quantitatum x et u, formula differentialis $X\partial x$ per substitutionem usurpatam in rationalem transformabitur, cujus propterea integratio constat.

Corollarium 2.

101. Si X fuerit functio rationalis tam ipsius x, quam quantitatum quotcunque hujusmodi

$$\left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\lambda}} = u, \quad \left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\mu}} = v, \quad \left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\nu}} = t$$

tum formula differentialis $X\partial x$ rationalis reddetur, adhibita substitutione $\frac{a+bx}{f+gx} = z^{\lambda\mu\nu}$, unde fit

$$x = \frac{a - fz^{\lambda\mu\nu}}{gz^{\lambda\mu\nu} - b}; \text{ et } u = z^{\mu\nu}; \; v = z^{\lambda\nu}; \; t = z^{\lambda\mu}.$$

Scholion 1.

102. His casibus reductio ad rationalitatem ideo succedit, etiamsi plures formulae irrationales insint, quod eae omnes simul per eandem substitutionem rationales efficiantur, indeque etiam ipsa quantitas x per novam variabilem z rationaliter exprimetur. Sin autem differentiale propositum duas ejusmodi formulas irrationales contineat, quae non ambae simul ope ejusdem substitutionis rationa-

les reddi queant, etiamsi hoc in utraque seorsim fieri possit, reductio locum non habet, nisi forte ipsum differentiale in duas partes dispesci liceat, quarum utraque unam tantum formulam irrationalem complectatur. Veluti si proposita sit haec formula differentialis

$$\partial y = \frac{\partial x}{\sqrt{(1+xx)} - \sqrt{(1-xx)}}$$

ejus numeratorem ac denominatorem per $\sqrt{(1+xx)} + \sqrt{(1-xx)}$ multiplicando, fit

$$\partial y = \frac{\partial x \sqrt{(1+xx)}}{2xx} + \frac{\partial x \sqrt{(1-xx)}}{2xx},$$

cujus utraque pars seorsim rationalis reddi et integrari potest. Reperitur autem:

$$y = C - \frac{\sqrt{(1-xx)} - \sqrt{(1+xx)}}{2x} + \tfrac{1}{2} l[x + \sqrt{(1+xx)}]$$
$$- \tfrac{1}{2} \text{Arc. tang.} \frac{x}{\sqrt{(1-xx)}}.$$

Commodissime autem ibi irrationalitas tollitur, si in parte priori ponatur $\sqrt{(1+xx)} = px$, in posteriori $\sqrt{(1-xx)} = qx$. Etsi enim hinc sit

$$x = \frac{1}{\sqrt{(pp-1)}} \text{ et } x = \frac{1}{\sqrt{(1+qq)}},$$

tamen oritur rationaliter

$$\partial y = \frac{-pp\partial p}{2(pp-1)} - \frac{qq\partial q}{2(1+qq)}.$$

Scholion 2.

103. Circa formulas generales, quae ab irrationalitate liberari queant, vix quicquam amplius praecipere licet: dummodo hunc casum addamus, quo functio X binas hujusmodi formulas radicales $\sqrt{(a+bx)}$ et $\sqrt{(f+gx)}$ complectitur. Posito enim $(a+bx) = (f+gx)tt$, fit $x = \frac{a-ftt}{gtt-b}$, atque

$$\sqrt{(a+bx)} = \frac{t\sqrt{(ag-bf)}}{\sqrt{(gtt-b)}}; \ \sqrt{(f+gx)} = \frac{\sqrt{(ag-bf)}}{\sqrt{(gtt-b)}};$$

et in formula differentiali unica tantum formula irrationalis erit $\sqrt{(gtt-b)}$, quae nova substitutione facile tolletur, per ea quae

Problemate 6. tradidimus. Ut igitur ad alia pergamus, imprimis considerari meretur haec formula differentialis

$$x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}},$$

cujus ob simplicitatem usus per universam analysin est amplissimus; ubi quidem sumimus litteras m, n, μ, ν numeros integros denotare, nisi enim tales essent, facile ad hanc formam reducerentur. Veluti si haberemus $x^{-\frac{1}{3}}\partial x(a+b\sqrt{x})^{\frac{\mu}{\nu}}$, statui oportet $x=u^6$, hinc $\partial x=6u^5\partial u$: unde prodit

$$6u^3\partial u(a+bu^3)^{\frac{\mu}{\nu}}.$$

Tum vero pro n valorem positivum assumere licet: si enim esset negativus, puta

$$x^{m-1}\partial x(a+bx^{-n})^{\frac{\mu}{\nu}},$$

ponatur $x=\frac{1}{u}$, fietque formula

$$-u^{-m-1}\partial u(a+bu^n)^{\frac{\mu}{\nu}},$$

similis principali; quae ergo quibus casibus ab irrationalitate liberari queat, investigemus.

Problema 9.

104. Definire casus, quibus formulam differentialem

$$x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}},$$

ad rationalitatem perducere liceat.

Solutio.

Primo patet, si fuerit $\nu=1$, seu $\frac{\mu}{\nu}$ numerus integer, formulam per se fore rationalem, neque substitutione opus esse. At si $\frac{\mu}{\nu}$ sit fractio, substitutione est utendum, eaque duplici.

I. Ponatur $a + bx^n = u^\nu$, ut fiat $(a + bx^n)^{\frac{\mu}{\nu}} = u^\mu$, erit

$$x^n = \frac{u^\nu - a}{b}, \text{ hinc } x^m = \left(\frac{u^\nu - a}{b}\right)^{\frac{m}{n}}, \text{ ideoque}$$

$$x^{m-1}\partial x = \frac{\nu}{nb} u^{\nu-1}\partial u \left(\frac{u^\nu - a}{b}\right)^{\frac{m-n}{n}};$$

unde formula nostra fiet

$$\frac{\nu}{nb} u^{\mu+\nu-1}\partial u \left(\frac{u^\nu - a}{b}\right)^{\frac{m-n}{n}}.$$

Hinc ergo patet, quoties exponens $\frac{m-n}{n}$ seu $\frac{m}{n}$ fuerit numerus integer sive positivus, sive negativus, hanc formulam esse rationalem.

II. Ponatur $a + bx^n = x^n z^\nu$, ut fiat

$$x^n = \frac{a}{z^\nu - b}, \text{ et } (a + bx^n)^{\frac{\mu}{\nu}} = \frac{a^{\frac{\mu}{\nu}} z^\mu}{(z^\nu - b)^{\frac{\mu}{\nu}}}; \text{ tum}$$

$$x^m = \frac{a^{\frac{m}{n}}}{(z^\nu - b)^{\frac{m}{n}}}, \text{ hinc } x^{m-1}\partial x = \frac{-\nu a^{\frac{m}{n}} z^{\nu-1}\partial z}{n(z^\nu - b)^{\frac{m}{n}+1}}.$$

Ideoque formula nostra erit

$$\frac{-\nu a^{\frac{m}{n}+\frac{\mu}{\nu}} z^{\mu+\nu-1}\partial z}{n(z^\nu - b)^{\frac{m}{n}+\frac{\mu}{\nu}+1}}.$$

Ex quo patet hanc formam fore rationalem, quoties $\frac{m}{n} + \frac{\mu}{\nu}$ fuerit numerus integer. Facile autem intelligitur, alias substitutiones huic scopo idoneas excogitari non posse.

Quare concludimus formulam irrationalem hanc

$$x^{m-1}\partial x(a + bx^n)^{\frac{\mu}{\nu}}$$

ab irrationalitate liberari posse, si fuerit vel $\frac{m}{n}$, vel $\frac{m}{n} + \frac{\mu}{\nu}$ numerus integer.

Corollarium 1.

105. Si sit $\frac{m}{n}$ numerus integer, casus per se est facilis; ponatur enim $m = in$, et sit $x^n = v$, erit $x^m = v^i$; ideoque formula nostra $\frac{i}{m} v^{i-1} \partial v (a + bv)^{\frac{\mu}{\nu}}$, quae per Problema 7. expeditur.

Corollarium 2.

106. At si $\frac{m}{n}$ non est numerus integer, ut reductio ad rationalitatem locum habeat, necesse est ut $\frac{m}{n} + \frac{\mu}{\nu}$ sit numerus integer: quod fieri nequit, nisi sit $\nu = n$, ideoque $m + \mu$ multiplum debet esse ipsius $n = \nu$.

Corollarium 3.

107. Quod si ergo haec formula

$$x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}},$$

ad rationalitatem reduci queat, etiam haec formula

$$x^{m \pm \alpha n - 1} \partial x (a + bx^n)^{\frac{\mu}{\nu} \pm \beta},$$

eandem reductionem admittet; quicunque numeri integri pro α et β assumantur. Unde ad casus reducibiles cognoscendos sufficit ponere $m < n$ et $\mu < \nu$.

Corollarium 4.

108. Si $m = 0$, haec formula $\frac{\partial x}{x} (a + bx^n)^{\frac{\mu}{\nu}}$, semper per casum primum ad rationalitatem reducitur, ponendo

$$x^n = \frac{u^\nu - a}{b};$$

transformatur enim in hanc

$$\frac{\nu b u^{\mu + \nu - 1} \partial u}{n (u^\nu - a)}$$

Scholion 1.

109. Quoniam formula $x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$, quoties est $m=in$, denotante i numerum integrum sive positivum sive negativum quemcunque, semper ad rationalitatem reduci potest, hicque casus per se sunt perspicui, reliquos casus hanc reductionem admittentes accuratius contemplari operae pretium videtur. Quem in finem statuamus $\nu=n$ et $m<n$, item $\mu<n$, ac necesse est ut sit $m+\mu=n$: unde sequentes formae in genere suo simplicissimae, quae quidem ad rationalitatem reduci queant, obtinentur.

I. $\partial x\,(a+bx^2)^{\frac{1}{2}}$;

II. $\partial x\,(a+bx^3)^{\frac{2}{3}}$; $x\partial x\,(a+bx^3)^{\frac{1}{3}}$;

III. $\partial x\,(a+bx^4)^{\frac{3}{4}}$; $xx\partial x\,(a+bx^4)^{\frac{1}{4}}$;

IV. $\partial x\,(a+bx^5)^{\frac{4}{5}}$; $x\partial x\,(a+bx^5)^{\frac{3}{5}}$; $x^2\partial x\,(a+bx^5)^{\frac{2}{5}}$; $x^3\partial x\,(a+bx^5)^{\frac{1}{5}}$;

V. $\partial x\,(a+bx^6)^{\frac{5}{6}}$; $x^4\partial x\,(a+bx^6)^{\frac{1}{6}}$;

unde etiam hae reductionem admittent:

$x^{\pm 2\alpha}\partial x\,(a+bx^2)^{\frac{1}{2}\pm\beta}$;

$x^{\pm 3\alpha}\partial x\,(a+bx^3)^{\frac{2}{3}\pm\beta}$; $x^{1\pm 3\alpha}\partial x\,(a+bx^3)^{\frac{1}{3}\pm\beta}$;

$x^{\pm 4\alpha}\partial x\,(a+bx^4)^{\frac{3}{4}\pm\beta}$; $x^{2\pm 4\alpha}\partial x\,(a+bx^4)^{\frac{1}{4}\pm\beta}$;

$x^{\pm 5\alpha}\partial x\,(a+bx^5)^{\frac{4}{5}\pm\beta}$; $x^{1\pm 5\alpha}\partial x\,(a+bx^5)^{\frac{3}{5}\pm\beta}$;

$x^{2\pm 5\alpha}\partial x\,(a+bx^5)^{\frac{2}{5}\pm\beta}$; $x^{3\pm 5\alpha}\partial x\,(a+bx^5)^{\frac{1}{5}\pm\beta}$;

$x^{\pm 6\alpha}\partial x\,(a+bx^6)^{\frac{5}{6}\pm\beta}$; $x^{4\pm 6\alpha}\partial x\,(a+bx^6)^{\frac{1}{6}\pm\beta}$.

Scholion 2.

110. Verum etiamsi formula $x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}}$, ab irrationalitate liberari nequeat, tamen semper omnium harum formularum $x^{m\pm n\alpha-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$, integrationem ad eam reducere licet, ita ut illius integrali tanquam cognito spectato, etiam harum integralia assignari queant. Quae reductio cum in Analysi summam afferat utilitatem, eam hic exponere necesse erit. Caeterum hic affirmare haud dubitamus, praeter eos casus, quos reductionem ad rationalitatem admittere hic ostendimus, nullos alios existere, qui ulla substitutione adhibita ab irrationalitate liberari queant. Proposita enim hac formula $\frac{\partial x}{\sqrt{(a+bx^3)}}$, nulla functio rationalis ipsius z loco x poni potest, ut $a+bx^3$ extractionem radicis quadratae admittat: objici quidem potest, scopo satisfieri posse, etiamsi loco x functio irrationalis ipsius z substituatur, dummodo similis irrationalitas in denominatore $\sqrt{(a+bx^3)}$ contineatur, qua illa numeratorem ∂x afficiens destruatur: quemadmodum fit in hac formula $\frac{\partial x}{\sqrt[3]{(a+bx^3)}}$, adhibendo substitutionem:

$$x=\frac{\sqrt[3]{a}}{\sqrt[3]{(z^3-b)}},$$

verum quod hic commode usu venit, nullo modo perspicitur, quomodo idem illo casu evenire possit. Hoc tamen minime pro demonstratione haberi volo.

Problema 10.

111. Integrationem formulae

$$\int x^{m+n-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}},$$

perducere ad integrationem hujus formulae: $\int x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}}$.

Solutio.

Consideretur functio $x^m (a + bx^n)^{\frac{\mu}{\nu} + 1}$, cujus differentiale cum sit

$$\left(max^{m-1} \partial x + mbx^{m+n-1} \partial x + \frac{n(\mu+\nu)}{\nu} bx^{m+n-1} \partial x\right)(a+bx^n)^{\frac{\mu}{\nu}},$$

erit

$$x^m (a + bx^n)^{\frac{\mu}{\nu}+1} = ma \int x^{m-1} \partial x\, (a + bx^n)^{\frac{\mu}{\nu}}$$
$$+ \frac{(m\nu + n\mu + n\nu) b}{\nu} \int x^{m+n-1} \partial x\, (a + bx^n)^{\frac{\mu}{\nu}};$$

unde elicitur

$$\int x^{m+n-1} \partial x\, (a + bx^n)^{\frac{\mu}{\nu}} = \frac{\nu x^m (a + bx^n)^{\frac{\mu}{\nu}+1}}{(m\nu + n\mu + n\nu)\, b}$$
$$- \frac{m\nu a}{(m\nu + n\mu + n\nu)\, b} \int x^{m-1} \partial x\, (a + bx^n)^{\frac{\mu}{\nu}}.$$

Corollarium 1.

112. Cum inde quoque sit

$$\int x^{m-1} \partial x\, (a + bx^n)^{\frac{\mu}{\nu}} = \frac{x^m (a + bx^n)^{\frac{\mu}{\nu}+1}}{ma}$$
$$- \frac{(m\nu + n\mu + n\nu)\, b}{m\nu a} \int x^{m+n-1} \partial x\, (a + bx^n)^{\frac{\mu}{\nu}}$$

loco m scribamus $m - n$, et habebimus hanc reductionem:

$$\int x^{m-n-1} \partial x\, (a + bx^n)^{\frac{\mu}{\nu}} = \frac{x^{m-n} (a + bx^n)^{\frac{\mu}{\nu}+1}}{(m-n)\, a}$$
$$- \frac{(m\nu + n\mu)\, b}{(m-n)\, \nu a} \int x^{m-1} \partial x\, (a + bx^n)^{\frac{\mu}{\nu}}.$$

Corollarium 2.

113. Concesso ergo integrali $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$, etiam harum formularum $\int x^{m\pm n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$, similique modo ulterius progrediendo omnium harum formularum

$$\int x^{m\pm\alpha n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$

integralia exhiberi possunt.

Problema 11.

114. Integrationem formulae $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1}$ ad integrationem hujus $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$ perducere.

Solutio.

Functionis $x^m(a+bx^n)^{\frac{\mu}{\nu}+1}$ differentiale hoc modo exhiberi potest

$$\left(ma-\frac{(m\nu+n\mu+n\nu)a}{\nu}\right)x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$
$$+\frac{m\nu+n\mu+n\nu}{\nu}x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1},$$

unde concluditur

$$x^m(a+bx^n)^{\frac{\mu}{\nu}+1}=-\frac{(n\mu+n\nu)a}{\nu}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$
$$+\frac{m\nu+n\mu+n\nu}{\nu}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1},$$

quocirca habebimus:

$$\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1}=\frac{\nu x^m(a+bx^n)^{\frac{\mu}{\nu}+1}}{m\nu+n(\mu+\nu)}$$
$$+\frac{n(\mu+\nu)a}{m\nu+n(\mu+\nu)}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}.$$

Corollarium 1.

115. Deinde ex eadem aequatione elicimus:

$$\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}} = \frac{-\nu x^m\,(a+bx^n)^{\frac{\mu}{\nu}+1}}{n(\mu+\nu)a}$$
$$+\frac{m\nu+n(\mu+\nu)}{n(\mu+\nu)a}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1}.$$

Scribamus jam $\mu-\nu$ loco μ, ut nasciscamur hanc reductionem

$$\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{-\nu x^m\,(a+bx^n)^{\frac{\mu}{\nu}}}{n\mu a}$$
$$+\frac{m\nu+n\mu}{n\mu a}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}.$$

Corollarium 2.

116. Concesso ergo integrali $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$, etiam harum formularum $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}\pm 1}$, et ulterius progrediendo, harum $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$ integralia exhiberi possunt, denotante β numerum integrum quemcunque.

Corollarium 3.

117. His cum praecedentibus conjunctis, ad integrationem $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$, omnia haec integralia

$$\int x^{m\pm\alpha n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$$

revocari possunt, quae ergo omnia ab eadem functione transcendente pendent.

Scholion 1.

118. Ex formae $x^m (a + bx^n)^{\frac{\mu}{\nu}}$ differentiali ita disposito

$$m x^{m-1} \partial x (a + b x^n)^{\frac{\mu}{\nu}} + \frac{n\mu}{\nu} b x^{m+n-1} \partial x (a + b x^n)^{\frac{\mu}{\nu} - 1}$$

deducimus hanc reductionem:

$$\int x^{m+n-1} \partial x (a + b x^n)^{\frac{\mu}{\nu} - 1} = \frac{\nu x^m (a + b x^n)^{\frac{\mu}{\nu}}}{n \mu b} - \frac{m\nu}{n\mu b} \int x^{m-1} \partial x (a + b x^n)^{\frac{\mu}{\nu}}:$$

ac praeterea hanc inversam, pro m et μ scribendo $m - n$ et $\mu + \nu$:

$$\int x^{m-n-1} \partial x (a + b x^n)^{\frac{\mu}{\nu} + 1} = \frac{x^{m-n} (a + b x^n)^{\frac{\mu}{\nu} + 1}}{m - n} - \frac{n(\mu + \nu) b}{\nu (m - n)} \int x^{m-1} \partial x (a + b x^n)^{\frac{\mu}{\nu}}.$$

Hinc scilicet una operatione absolvitur reductio, cum superiores formulae duplicem reductionem exigant; ex quo sex reductiones sumus nacti, omnino memorabiles, quas idcirco conjunctim conspectui exponamus.

$$\text{I. } \int x^{m+n-1} \partial x (a + b x^n)^{\frac{\mu}{\nu}} = \frac{\nu x^m (a + b x^n)^{\frac{\mu}{\nu} + 1}}{[m\nu + n(\mu + \nu)] b} - \frac{m \nu a}{[m\nu + n(\mu + \nu)] b} \int x^{m-1} \partial x (a + b x^n)^{\frac{\mu}{\nu}}$$

$$\text{II. } \int x^{m-n-1} \partial x (a + b x^n)^{\frac{\mu}{\nu}} = \frac{x^{m-n} (a + b x^n)^{\frac{\mu}{\nu} + 1}}{(m - n) a} - \frac{(m\nu + n\mu) b}{(m - n) \nu a} \int x^{m-1} \partial x (a + b x^n)^{\frac{\mu}{\nu}}$$

$$\text{III. } \int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1} = \frac{\nu x^m(a+bx^n)^{\frac{\mu}{\nu}+1}}{m\nu+n(\mu+\nu)}$$
$$+\frac{n(\mu+\nu)a}{m\nu+n(\mu+\nu)}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{IV. } \int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{-\nu x^m(a+bx^n)^{\frac{\mu}{\nu}}}{n\mu a}$$
$$+\frac{m\nu+n\mu}{n\mu a}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{V. } \int x^{m+n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{\nu x^m(a+bx^n)^{\frac{\mu}{\nu}}}{n\mu b}$$
$$-\frac{m\nu}{n\mu b}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{VI. } \int x^{m-n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1} = \frac{x^{m-n}(a+bx^n)^{\frac{\mu}{\nu}+1}}{m-n}$$
$$-\frac{n(\mu+\nu)b}{\nu(m-n)}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}.$$

Scholion 2.

119. Circa has reductiones primo observandum est, formulam priorem algebraice esse integrabilem, si coëfficiens posterioris evanescat. Ita sit

pro I. si $m=0\ldots\int x^{n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}} = \frac{\nu(a+bx^n)^{\frac{\mu}{\nu}+1}}{n(\mu+\nu)b}$

pro II. si $\frac{\mu}{\nu}=\frac{-m}{n}\ldots\int x^{m-n-1}\partial x\,(a+bx^n)^{\frac{-m}{n}} = \frac{x^{m-n}(a+bx^n)^{\frac{-m}{n}+1}}{(m-n)a}$

pro IV. si $\frac{\mu}{\nu} = \frac{-m}{n} \ldots \int x^{m-1}\,\partial x\,(a+bx^n)^{\frac{-m}{n}-1} = \frac{x^m\,(a+bx^n)^{\frac{-m}{n}}}{m\,a}$

pro V. si $m = 0 \ldots \int x^{n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{\nu\,(a+bx^n)^{\frac{\mu}{\nu}}}{n\,\mu\,b}$

Deinde etiam casus notari merentur, quibus coëfficiens postremae formulae fit infinitus; tum enim reductio cessat, et prior formula peculiare habet integrale seorsim evolvendum.

In prima hoc evenit si $\frac{\mu+\nu}{\nu} = \frac{-m}{n}$, et formula

$$\int x^{m+n-1}\,\partial x\,(a+bx^n)^{\frac{-m}{n}-1},$$

posito $a+bx^n = x^n z^n$, seu $x^n = \frac{a}{z^n-b}$, abit in $-\frac{z^{-m-1}\partial z}{z^n-b}$, cujus integrale per caput primum definiri debet.

In secunda evenit si $m = n$, et formula $\int \frac{\partial x}{x}\,(a+bx^n)^{\frac{\mu}{\nu}}$, posito $a+bx^n = z^\nu$, seu $x^n = \frac{z^\nu - a}{b}$, abit in $\frac{\nu z^{\mu+\nu-1}\,\partial z}{n\,(z^\nu - a)}$.

In tertia evenit, si $\frac{\mu}{\nu} = \frac{-m}{n} - 1$, et formula

$$\int x^{m-1}\,\partial x\,(a+bx^n)^{\frac{-m}{n}},$$

posito $a+bx^n = x^n z^n$, seu $x^n = \frac{a}{z^n-b}$, abit in $\int \frac{-z^{-m-n-1}\partial z}{z^n-b}$, seu posito $z = \frac{1}{u}$, in

$$\int \frac{u^{m+2n-1}\,\partial u}{1-b\,u^n} = \frac{-u^{m+n}}{(m+n)\,b} - \frac{u^m}{m\,b\,b} + \frac{1}{b\,b}\int \frac{u^{m-1}\,\partial u}{a-b\,u^n}.$$

In quarta evenit, si $\mu = 0$, et formula $\int \frac{x^{m-1}\,\partial x}{a+bx^n}$ per se est rationalis.

In quinta idem evenit, si $\mu = 0$.

In sexta autem, si $m = n$, et formula $\int \frac{\partial x}{x} (a + b x^n)^{\frac{\mu}{\nu} + 1}$, posito $a + b x^n = z^\nu$, abit in $\frac{\nu}{n} \int \frac{z^{\mu + 2\nu - 1} \partial z}{z^\nu - a}$.

Exemplum 1.

120. *Invenire integrale hujus formulae* $\int \frac{x^{m-1} \partial x}{\sqrt{(1 - xx)}}$, *pro numeris positivis exponenti* m *datis.*

Hic ob $a = 1$, $b = -1$, $n = 2$, $\mu = -1$, $\nu = 2$, prima reductio dat:

$$\int \frac{x^{m+1} \partial x}{\sqrt{(1 - xx)}} = \frac{-x^m \sqrt{(1 - xx)}}{m + 1} + \frac{m}{m+1} \int \frac{x^{m-1} \partial x}{\sqrt{(1 - xx)}}:$$

hinc prout pro m sumantur numeri vel impares vel pares, obtinebimus.

Pro numeris imparibus:

$$\int \frac{xx \partial x}{\sqrt{(1 - xx)}} = -\tfrac{1}{2} x \sqrt{(1 - xx)} + \tfrac{1}{2} \int \frac{\partial x}{\sqrt{(1 - xx)}}$$

$$\int \frac{x^4 \partial x}{\sqrt{(1 - xx)}} = -\tfrac{1}{4} x^3 \sqrt{(1 - xx)} + \tfrac{3}{4} \int \frac{x^2 \partial x}{\sqrt{(1 - xx)}}$$

$$\int \frac{x^6 \partial x}{\sqrt{(1 - xx)}} = -\tfrac{1}{6} x^5 \sqrt{(1 - xx)} + \tfrac{5}{6} \int \frac{x^4 \partial x}{\sqrt{(1 - xx)}}$$

Pro numeris paribus:

$$\int \frac{x^3 \partial x}{\sqrt{(1 - xx)}} = -\tfrac{1}{3} x^2 \sqrt{(1 - xx)} + \tfrac{2}{3} \int \frac{x \partial x}{\sqrt{(1 - xx)}}$$

$$\int \frac{x^5 \partial x}{\sqrt{(1 - xx)}} = -\tfrac{1}{5} x^4 \sqrt{(1 - xx)} + \tfrac{4}{5} \int \frac{x^3 \partial x}{\sqrt{(1 - xx)}}$$

$$\int \frac{x^7 \partial x}{\sqrt{(1 - xx)}} = -\tfrac{1}{7} x^6 \sqrt{(1 - xx)} + \tfrac{1}{7} \int \frac{x^5 \partial x}{\sqrt{(1 - xx)}}$$

etc.

Cum nunc sit $\int \frac{\partial x}{\sqrt{(1-xx)}} = \text{Arc. sin. } x$, et

$$\int \frac{x \partial x}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)},$$

habebimus sequentia integralia.

Pro ordine priore:

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \text{Arc. sin. } x$$

$$\int \frac{xx \partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{2} x \sqrt{(1-xx)} + \tfrac{1}{2} \text{Arc. sin. } x$$

$$\int \frac{x^4 \partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{4} x^3 + \tfrac{1.3}{2.4} x\right) \sqrt{(1-xx)} + \tfrac{1.3}{2.4} \text{Arc. sin. } x$$

$$\int \frac{x^6 \partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{6} x^5 + \tfrac{1.5}{4.6} x^3 + \tfrac{1.3.5}{2.4.6} x\right) \sqrt{(1-xx)} + \tfrac{1.3.5}{2.4.6} \text{Arc. sin. } x$$

$$\int \frac{x^8 \partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{8} x^7 + \tfrac{1.7}{6.8} x^5 + \tfrac{1.5.7}{4.6.8} x^3 + \tfrac{1.3.5.7}{2.4.6.8} x\right) \sqrt{(1-xx)} + \tfrac{1.3.5.7}{2.4.6.8} \text{Arc. sin. } x.$$

Pro ordine posteriore:

$$\int \frac{x \partial x}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)}$$

$$\int \frac{x^3 \partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{3} x^2 + \tfrac{2}{3}\right) \sqrt{(1-xx)}$$

$$\int \frac{x^5 \partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{5} x^4 + \tfrac{1.4}{3.5} x^2 + \tfrac{2.4}{3.5}\right) \sqrt{(1-xx)}$$

$$\int \frac{x^7 \partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{7} x^6 + \tfrac{1.6}{5.7} x^4 + \tfrac{1.4.6}{3.5.7} x^2 + \tfrac{2.4.6}{3.5.7}\right) \sqrt{(1-xx)}.$$

Corollarium 1.

121. In genere ergo formula $\int \frac{x^{2i}\,\partial x}{\sqrt{(1-xx)}}$, si ponamus brevitatis gratia $\frac{1.3.5\ldots\ldots(2i-1)}{2.4.6\ldots\ldots 2i} = J$, habebimus hoc integrale

$$\int \frac{x^{2i}\,\partial x}{\sqrt{(1-xx)}} = J \text{ Arc. sin. } x$$
$$-J\left(x+\tfrac{2}{3}x^3+\tfrac{2.4}{3.5}x^5+\tfrac{2.4.6}{3.5.7}x^7\ldots+\tfrac{2.4.6\ldots(2i-2)}{3.5.7\ldots(2i-1)}x^{2i-1}\right)\sqrt{(1-xx)}.$$

Corollarium 2.

122. Simili modo pro formula $\int \frac{x^{2i+1}\,\partial x}{\sqrt{(1-xx)}}$, si ponanus brevitatis ergo $\frac{2.4.6\ldots 2i}{3.5.7\ldots\ldots(2i+1)} = K$, habebimus hoc integrale:

$$\int \frac{x^{2i+1}\,\partial x}{\sqrt{(1-xx)}} = K$$
$$-K\left(1+\tfrac{1}{2}x^2+\tfrac{1.3}{2.4}x^4+\tfrac{1.3.5}{2.4.6}x^6+\ldots\tfrac{1.3.5\ldots(2i-1)}{2.4.6\ldots 2i}x^{2i}\right)\sqrt{(1-xx)}$$

ut integrale evanescat posito $x = 0$.

Exemplum 2.

123. *Invenire integrale formulae* $\int \frac{x^{m-1}\,\partial x}{\sqrt{(1-xx)}}$, *casibus quibus pro* m *numeri negativi assumuntur.*

Hic utendum est secunda reductione quae dat:

$$\int \frac{x^{m-3}\,\partial x}{\sqrt{(1-xx)}} = \frac{x^{m-2}\sqrt{(1-xx)}}{m-2} + \frac{m-1}{m-2}\int \frac{x^{m-1}\,\partial x}{\sqrt{(1-xx)}}:$$

unde patet si $m = 1$, fore $\int \frac{\partial x}{xx\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{x}$. Deinde si $m = 2$, formula $\int \frac{\partial x}{x\sqrt{(1-xx)}}$, facta substitutione $1-xx=zz$, abit in $\frac{-\partial z}{1-zz}$ cujus integrale est

$$-\tfrac{1}{2}\,l\,\frac{1+z}{1-z} = -\tfrac{1}{2}\,l\,\frac{1+\sqrt{(1-xx)}}{1-\sqrt{(1-xx)}} = -l\,\frac{1+\sqrt{(1-xx)}}{x};$$

unde duplicem seriem integrationum elicimus:

$$\int \frac{\partial x}{x\sqrt{(1-xx)}} = -l\frac{1+\sqrt{(1-xx)}}{x} = l\,\frac{1-\sqrt{(1-xx)}}{x};$$

$$\int \frac{\partial x}{x^3\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{2xx} + \tfrac{1}{2}\int \frac{\partial x}{x\sqrt{(1-xx)}}$$

$$\int \frac{\partial x}{x^5\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{4x^4} + \tfrac{3}{4}\int \frac{\partial x}{x^3\sqrt{(1-xx)}};$$

$$\int \frac{\partial x}{x^7\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{6x^6} + \tfrac{5}{6}\int \frac{\partial x}{x^5\sqrt{(1-xx)}}.$$

etc.

$$\int \frac{\partial x}{xx\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{x};$$

$$\int \frac{\partial x}{x^5\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{3x^3} + \tfrac{2}{3}\int \frac{\partial x}{xx\sqrt{(1-xx)}};$$

$$\int \frac{\partial x}{x^6\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{5x^5} + \tfrac{4}{5}\int \frac{\partial x}{x^4\sqrt{(1-xx)}}.$$

etc.

Hinc erit, ut in binis praecedentibus corrollariis

$$\int \frac{\partial x}{x^{2i+1}\sqrt{(1-xx)}} = lJ.\frac{1-\sqrt{(1-xx)}}{x} - J\left[\frac{1}{xx} + \frac{2}{3x^4} + \frac{2.4}{3.5x^6} + \ldots \right.$$
$$\left. \ldots\ldots + \frac{2.4\ldots.(2i-2)}{3.5\ldots\ldots(2i-1)x^{2i}}\right]\sqrt{(1-xx)};$$

$$\int \frac{\partial x}{x^{2i}\sqrt{(1-xx)}} = C - K\left[\frac{1}{x} + \frac{1}{2x^3} + \frac{1.3}{2.4x^5} + \ldots\ldots \right.$$
$$\left. \ldots\ldots + \frac{1.3\ldots(2i-1)}{2.4\ldots.2i.x^{2i+1}}\right]\sqrt{(1-xx)}.$$

Scholion 1.

124. Hinc jam facile integralia formularum

$$\int x^{m-1}\partial x\,(1-xx)^{\frac{\mu}{2}}$$

tam pro omnibus numeris m, quam pro imparibus μ assignari poterunt. Reductiones autem nostrae generales ad hunc casum accommodatae sunt:

$$\text{I. } \int x^{m+1}\,\partial x\,(1-xx)^{\frac{\mu}{2}} = \frac{-x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{m}{m+\mu+2}\int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}};$$

$$\text{II. } \int x^{m-3}\,\partial x\,(1-xx)^{\frac{\mu}{2}} = \frac{x^{m-2}(1-xx)^{\frac{\mu}{2}+1}}{m-2} + \frac{m+\mu}{m-2}\int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}};$$

$$\text{III. } \int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}+1} = \frac{x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{\mu+2}{m+\mu+2}\int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}};$$

$$\text{IV. } \int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}-1} = \frac{-x^m(1-xx)^{\frac{\mu}{2}}}{\mu} + \frac{m+\mu}{\mu}\int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}};$$

$$\text{V. } \int x^{m+1}\,\partial x\,(1-xx)^{\frac{\mu}{2}-1} = \frac{-x^m(1-xx)^{\frac{\mu}{2}}}{\mu} + \frac{m}{\mu}\int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}};$$

$$\text{VI. } \int x^{m-3}\,\partial x\,(1-xx)^{\frac{\mu}{2}+1} = \frac{x^{m-2}(1-xx)^{\frac{\mu}{2}+1}}{m-2} + \frac{\mu+2}{m-2}\int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}}.$$

Posito enim $\mu = -1$, quatuor posteriores dant:

$$\int x^{m-1}\partial x\sqrt{(1-xx)} = \frac{x^m\sqrt{(1-xx)}}{m+1} + \frac{1}{m+1}\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}};$$

$$\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)^3}} = \frac{x^m}{\sqrt{(1-xx)}} - (m-1)\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}};$$

$$\int\frac{x^{m+1}\partial x}{\sqrt{(1-xx)^3}} = \frac{x^m}{\sqrt{(1-xx)}} - m\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}};$$

$$\int x^{m-3}\partial x\sqrt{(1-xx)} = \frac{x^{m-2}\sqrt{(1-xx)}}{m-2} + \frac{1}{m-2}\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}};$$

unde integrationes pro casibus $\mu = 1$ et $\mu = -3$ eliciuntur, indeque porro reliqui.

Scholion 2.

125. Pro aliis formulis irrationalibus magis complicatis vix regulas dare licet, quibus ad formam simpliciorem reduci queant: et quoties ejusmodi formulae occurrant, reductio, si quam admittunt, plerumque sponte se offert. Veluti si formula fuerit hujusmodi $\int\frac{P\partial x}{Q^{n+1}}$, sive n sit numerus integer sive fractus, semper ad aliam hujus formae $\int\frac{S\partial x}{Q^n}$, quae utique simplicior aestimatur, reduci potest. Cum enim sit

$$\partial\frac{R}{Q^n} = \frac{Q\partial R - nR\partial Q}{Q^{n+1}},\ \text{posito}\ \int\frac{P\partial x}{Q^{n+1}} = y,\ \text{erit}$$

$$y + \frac{R}{Q^n} = \int\frac{P\partial x + Q\partial R - nR\partial Q}{Q^{n+1}}.$$

Jam definiatur R ita, ut $P\partial x + Q\partial R - nR\partial Q$ per Q fiat divisibile, vel quia $Q\partial R$ jam factorem habet Q, ut fiat $P\partial x - nR\partial Q = QT\partial x$, prodibitque

$$y + \frac{R}{Q^n} = \int \frac{\partial R + T\partial x}{Q^n}, \text{ seu}$$

$$\int \frac{P\partial x}{Q^{n+1}} = -\frac{R}{Q^n} + \int \frac{\partial R + T\partial x}{Q^n}.$$

At semper functionem R ita definire licet, ut $P\partial x - nR\partial Q$ factorem Q obtineat, quod etsi in genere praestari nequit, tamen rem in exemplis tentando, mox perspicietur negotium semper succedere. Assumo autem hic P et Q esse functiones integras, ac talis quoque semper pro R erui poterit. Si forte eveniat, ut $\partial R + T\partial x = 0$, formula proposita algebraicum habebit integrale, quod hoc modo reperietur; contra autem haec forma ulterius reduci poterit in alias; ubi denominatoris exponens continuo unitate diminuatur; ac si n sit numerus integer, negotium tandem reducitur ad hujusmodi formam $\frac{V\partial x}{Q}$, quae sine dubio est simplicissima. Quamobrem cum in hoc capite vix quicquam amplius proferri possit, ad integrationem formularum irrationalium juvandam, methodum easdem integrationes per series infinitas perficiendi exponamus.

ADDITAMENTUM.

Problema.

Proposita formula $\partial y = [x + \sqrt{(1 + xx)}]^n \partial x$, *invenire ejus integrale.*

Solutio.

Posito $x + \sqrt{(1 + xx)} = u$, fit $x = \frac{uu - 1}{2u}$, et $\partial x = \frac{\partial u (uu + 1)}{2uu}$: unde formula nostra

$$\partial y = \tfrac{1}{2} u^{n-2} \partial u \, (uu + 1),$$

deoque ejus integrale

$$y = \frac{u^{n+1}}{2(n+1)} + \frac{u^{n-1}}{2(n-1)} + \text{Const.}$$

quod ergo semper est algebraicum nisi sit vel $n=1$, vel $n=-1$.

Corollarium 1.

Patet etiam hanc formam latius patentem

$$\partial y = [x + \surd(1 + xx)]^n X \partial x$$

hoc modo integrari posse, dummodo X fuerit functio rationalis ipsius x. Posito enim $x = \frac{uu-1}{2u}$, pro X prodit functio rationalis ipsius u, quae sit $=$ U, hineque fit

$$\partial y = \tfrac{1}{2} U u^{n-2} \partial u (uu + 1),$$

quae formula vel est rationalis, si n sit numerus integer, vel ad rationalitatem facile reducitur, si n sit numerus fractus.

Corollarium 2.

Cum sit $\surd(1 + xx) = \frac{uu+1}{2u}$; posito $\surd(1 + xx) = v$, etiam haec formula

$$\partial y = [x + \surd(1 + xx)]^n X \partial x$$

integrabitur, si X fuerit functio rationalis quaecunque quantitatum x et v. Facto enim $x = \frac{uu-1}{2u}$, functio X abit in functionem rationalem ipsius u, qua posita $=$ U, habebitur ut ante $\partial y = \frac{1}{2} U u^{n-2} \partial u (uu + 1)$.

Exemplum.

Proposita sit formula

$$\partial y = [ax + b\surd(1 + xx)] [x + \surd(1 + xx)]^n \partial x.$$

Posito $x = \frac{uu-1}{2u}$, fit

$$\partial y = \left(\frac{a(uu-1) + b(uu+1)}{2u}\right) \times \tfrac{1}{2} u^{n-2} \partial u (uu + 1):$$

seu

$$\partial y = \tfrac{1}{4} u^{n-3} \partial u \left[a (u^4 - 1) + b (u^4 + 2uu + 1)\right],$$

cujus integrale est:

$$y = \frac{a+b}{4(n+2)} u^{n+2} + \frac{b}{2n} u^n + \frac{b-a}{4(n-2)} u^{n-2} + \text{Const.}$$

quae est algebraica, nisi sit vel $n = 2$, vel $n = -2$, vel etiam $n = 0$.