CALCULI INTEGRALIS
LIBER PRIOR.

PARS PRIMA,

SEU

METHODUS INVESTIGANDI FUNCTIONES UNIUS VARIABILIS EX DATA RELATIONE QUACUNQUE DIFFERENTIALIUM PRIMI GRADUS.

SECTIO PRIMA,

DE

INTEGRATIONE FORMULARUM DIFFERENTIALIUM.

CONSPECTUS

UNIVERSI OPERIS

DE

CALCULO INTEGRALI.

LIBER PRIOR: Tradit methodum investigandi functiones unius variabilis ex data quadam relatione differentialium, continetque duas partes:

Pars prior: Quando relatio illa data tantum differentialia primi gradus complectitur.

Pars posterior: Quando relatio illa data differentialia secundi altiorumve graduum complectitur.

LIBER POSTERIOR: Tradit methodum investigandi functiones duarum pluriumve variabilium ex data quadam relatione differentialium, continetque duas partes:

Pars prior: Seu Investigatio functionum duarum tantum variabilium ex data differentialium cujusvis gradus relatione.

Pars posterior: Seu Investigatio functionum trium variabilium ex data differentialium relatione.

CAPUT I.

DE
INTEGRATIONE FORMULARUM DIFFERENTIALIUM RATIONALIUM.

Definitio.

40.

Formula differentialis *rationalis* est, quando variabilis x, cujus functio quaeritur, differentiale ∂x multiplicatur in functionem rationalem ipsius x: seu si X designet functionem rationalem ipsius x, haec formula differentialis $X\,\partial x$ dicitur rationalis.

Corollarium 1.

41. In hoc ergo capite ejusmodi functio ipsius x quaeritur, quae si ponatur y, ut $\frac{\partial y}{\partial x}$ aequetur functioni rationali ipsius x seu posita tali functione $= X$, ut sit $\frac{\partial y}{\partial x} = X$.

Corollarium 2.

42. Hinc quaeritur ejusmodi functio ipsius x, cujus differentiale sit $= X\,\partial x$; hujus ergo integrale, quod ita indicari solet $\int X\,\partial x$, praebebit functionem quaesitam.

Corollarium 3.

43. Quodsi P fuerit ejusmodi functio ipsius x, ut ejus differentiale ∂P sit $= X\partial x$, quoniam quantitatis $P + C$ idem est differentiale, formulae propositae $X\partial x$ integrale completum est $P + C$.

Scholion 1.

44. Ad libri primi partem priorem hujusmodi referuntur quaestiones, quibus functiones solius variabilis x, ex data differentialium

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primi gradus relatione quaeruntur. Scilicet si functio quaesita $= y$ et $\frac{\partial y}{\partial x} = p$, id praestari oportet, ut proposita aequatione quacunque inter ternas quantitates x, y et p, inde indoles functionis y, seu aequatio inter x et y, elisa littera p, inveniatur. Quaestio autem sic in genere proposita vires analyseos adeo superare videtur, ut ejus solutio nunquam expectari queat. In casibus igitur simplicioribus vires nostrae sunt exercendae, inter quos primum occurrit casus, quo p functioni cuipiam ipsius x puta X aequatur, ut sit $\frac{\partial y}{\partial x} = X$, seu $\partial y = X \partial x$, ideoque integrale $y = \int X \partial x$ requiratur, in quo primam sectionem collocamus. Verum et hic casus pro varia indole functionis X latissime patet, ac plurimis difficultatibus implicatur: unde in hoc capite ejusmodi tantum quaestiones evolvere instituimus, in quibus ista functio X est rationalis: deinceps ad functiones irrationales atque adeo transcendentes progressuri. Hinc ista pars commode in duas sectiones subdividitur, in quarum altera integratio formularum simplicium, quibus $p = \frac{\partial y}{\partial x}$ functioni tantum ipsius x aequatur, est tradenda, in altera autem rationem integrandi doceri conveniet, cum proposita fuerit aequatio quaecunque ipsarum x, y et p. Et cum in his duabus sectionibus, ac potissimum priore, a Geometris plurimum sit elaboratum, eae maximam partem totius operis complebunt.

Scholion 2.

45. Prima autem integrationis principia ex ipso calculo differentiali sunt petenda, perinde ac principia divisionis ex multiplicatione, et principia extractionis radicum ex ratione evectionis ad potestates sumi solent. Cum igitur si quantitas differentianda ex pluribus partibus constet, ut $P + Q - R$, ejus differentiale sit $\partial P + \partial Q - \partial R$, ita vicissim si formula differentialis ex pluribus partibus constet, ut $P\partial x + Q\partial x - R\partial x$, integrale erit $\int P\partial x + \int Q\partial x - \int R\partial x$, singulis scilicet partibus seorsim integrandis. Deinde cum quantitatis aP differentiale sit $a\partial P$, formulae differentialis $aP\partial x$ integrale erit $a\int P\,\partial x$: scilicet

per quam quantitatem constantem formula differentialis multiplicatur, per eandem integrale multiplicari debet. Ita si formula differentialis sit $aP\partial x + bQ\partial x + cR\partial x$, quaecunque functiones ipsius x litteris P, Q, R designentur, integrale erit $a\int P\partial x + b\int Q\partial x + c\int R\partial x$: ita ut integratio tantum in singulis formulis $P\partial x$, $Q\partial x$ et $R\partial x$, sit instituenda. Hocque facto insuper adjici debet constans arbitraria C, ut integrale completum obtineatur.

Problema 1.

46. Invenire functionem ipsius x, ut ejus differentiale sit $= a x^n \partial x$, seu integrare formulam differentialem $a x^n \partial x$.

Solutio.

Cum potestatis x^m differentiale sit $m x^{m-1} \partial x$, erit vicissim:

$$\int m x^{m-1} \partial x = m \int x^{m-1} \partial x = x^m, \text{ ideoque } \int x^{m-1} \partial x = \tfrac{1}{m} x^m.$$

Fiat $m - 1 = n$, seu $m = n + 1$, erit:

$$\int x^n \partial x = \frac{1}{n+1} x^{n+1}, \text{ et } a\int x^n \partial x = \frac{a}{n+1} x^{n+1}.$$

Unde formulae differentialis propositae $a x^n \partial x$ integrale completum erit $\frac{a}{n+1} x^{n+1} + C$, cujus ratio vel inde patet, quod ejus differentiale revera fit $= a x^n \partial x$. Atque haec integratio semper locum habet, quicunque numerus exponenti n tribuatur, sive positivus sive negativus, sive integer sive fractus, sive etiam irrationalis.

Unicus casus hinc excipitur, quo est exponens $n = -1$, seu haec formula $\frac{a\partial x}{x}$ integranda proponitur. Verum in calculo differentiali jam ostendimus, si $l x$ denotet logarithmum hyperbolicum ipsius x, fore ejus differentiale $= \frac{\partial x}{x}$; unde vicissim concludimus esse $\int \frac{\partial x}{x} = lx$, et $\int \frac{a\partial x}{x} = a\, l\, x$. Quare adjecta constante arbitraria, erit formulae $\frac{a\partial x}{x}$ integrale completum $= alx + C = lx^a + C$: quod etiam pro C ponendo $l c$, ita exprimitur $l c x^a$.

Corollarium 1.

47. Formulae ergo differentialis $a x^n \partial x$ integrale semper est algebraicum, solo excepto casu quo $n = -1$, et integrale per logarithmos exprimitur, qui ad functionis transcendentes sunt referendi. Est scilicet $\int \frac{a \partial x}{x} = a l x + C = l c x^a$.

Corollarium 2.

48. Si exponens n numeros positivos denotet, sequentes integrationes utpote maxime obviae probe sunt tenendae:

$$\int a \partial x = a x + C;\ \int a x \partial x = \tfrac{a}{2} x x + C;\ \int a x^2 \partial x = \tfrac{a}{3} x^3 + C;$$
$$\int a x^3 \partial x = \tfrac{a}{4} x^4 + C;\ \int a x^4 \partial x = \tfrac{a}{5} x^5 + C;\ \int a x^5 \partial x = \tfrac{a}{6} x^6 + C; \text{ etc.}$$

Corollarium 3.

49. Si n sit numerus negativus, posito $n = -m$, fit

$$\int \frac{a \partial x}{x^m} = \frac{a}{1-m} x^{1-m} + C = \frac{-a}{(m-1) x^{m-1}} + C;$$

unde hi casus simpliciores notentur:

$$\int \frac{a \partial x}{x^2} = \frac{-a}{x} + C;\ \int \frac{a \partial x}{x^3} = \frac{-a}{2 x x} + C;\ \int \frac{a \partial x}{x^4} = \frac{-a}{3 x^3} + C;$$
$$\int \frac{a \partial x}{x^5} = \frac{-a}{4 x^4} + C;\ \int \frac{a \partial x}{x^6} = \frac{-a}{5 x^5} + C; \text{ etc.}$$

Corollarium 4.

50. Quin etiam si n denotet numeros fractos, integralia hinc obtinentur. Sit primo $n = \frac{m}{2}$, erit

$$\int a \partial x \sqrt{x^m} = \frac{2a}{m+2} x \sqrt{x^m} + C.$$

Unde casus notentur:

$$\int a \partial x \sqrt{x} = \tfrac{2a}{3} x \sqrt{x} + C;\ \int a x \partial x \sqrt{x} = \tfrac{2a}{5} x^2 \sqrt{x} + C;$$
$$\int a x x \partial x \sqrt{x} = \tfrac{2a}{7} x^3 \sqrt{x} + C;\ \int a x^3 \partial x \sqrt{x} = \tfrac{2a}{9} x^4 \sqrt{x} + C; \text{ etc.}$$

Corollarium 5.

51. Ponatur etiam $n = \frac{-m}{2}$, et habebitur

$$\int \frac{a\partial x}{\sqrt{x^m}} = \frac{2a}{2-m} \cdot \frac{x}{\sqrt{x^m}} + C = \frac{-2a}{(m-2)\sqrt{x^{m-2}}} + C.$$

Unde hi casus notentur:

$$\int \frac{a\partial x}{\sqrt{x}} = 2a\sqrt{x} + C; \int \frac{a\partial x}{x\sqrt{x}} = \frac{-2a}{\sqrt{x}} + C;$$

$$\int \frac{a\partial x}{xx\sqrt{x}} = \frac{-2a}{3x\sqrt{x}} + C; \int \frac{a\partial x}{x^3\sqrt{x}} = \frac{-2a}{5x^2\sqrt{x}} + C; \text{ etc.}$$

Corollarium 6.

52. Si in genere ponamus $n = \frac{\mu}{\nu}$, fiet:

$$\int a x^{\frac{\mu}{\nu}} \partial x = \frac{\nu a}{\mu+\nu} x^{\frac{\mu+\nu}{\nu}} + C, \text{ seu per radicalia}$$

$$\int a\partial x \sqrt[\nu]{x^\mu} = \frac{\nu a}{\mu+\nu} \sqrt[\nu]{x^{\mu+\nu}} + C.$$

Sin autem ponatur $n = \frac{-\mu}{\nu}$ habebitur:

$$\int \frac{a\partial x}{x^{\frac{\mu}{\nu}}} = \frac{\nu a}{\nu-\mu} x^{\frac{\nu-\mu}{\nu}} + C, \text{ seu per radicalia:}$$

$$\int \frac{a\partial x}{\sqrt[\nu]{x^\mu}} = \frac{\nu a}{\nu-\mu} \sqrt[\nu]{x^{\nu-\mu}} + C.$$

Scholion 1.

53. Quanquam in hoc capite functiones tantum rationales tractare institueram, tamen istae irrationalitates tam sponte se obtulerunt, ut perinde ac rationales tractari possint. Caeterum hinc quoque formulae magis complicatae integrari possunt, si pro x functiones alius cujuspiam variabilis z statuantur. Veluti si ponamus $x = f + gz$, erit $\partial x = g\partial z$: quare si pro a scribamus $\frac{a}{g}$, habebitur:

$$\int a\,\partial z\,(f+gz)^n = \frac{a}{(n+1)g}(f+gz)^{n+1} + C.$$

Casu autem singulari, quo $n = -1$:

$$\int \frac{a\,\partial z}{f+gz} = \frac{a}{g}\,l\,(f+gz) + C.$$

Tum si sit $n = -m$, fiet:

$$\int \frac{a\,\partial z}{(f+gz)^m} = \frac{-a}{(m-1)g(f+gz)^{m-1}} + C.$$

Ac posito $n = \frac{\mu}{\nu}$, prodit:

$$\int a\,\partial z\,(f+gz)^{\frac{\mu}{\nu}} = \frac{\nu a}{(\nu+\mu)g}(f+gz)^{\frac{\mu}{\nu}+1} + C.$$

Posito autem $n = -\frac{\mu}{\nu}$, obtinetur,

$$\int \frac{a\,\partial z}{(f+gz)^{\frac{\mu}{\nu}}} = \frac{\nu a\,(f+gz)}{(\nu-\mu)g\,(f+gz)^{\frac{\mu}{\nu}}} + C.$$

Scholion 2.

54. Caeterum hic insignis proprietas annotari meretur. Cum hic quaeratur functio y, ut sit $\partial y = ax^n \partial x$, si ponamus $\frac{\partial y}{\partial x} = p$, haec habebitur relatio $p = ax^n$, ex qua functio y investigari debet. Quoniam igitur est

$$y = \frac{a}{n+1} x^{n+1} + C,$$

ob $ax^n = p$, erit quoque $y = \frac{px}{n+1} + C$: sicque casum habemus, ubi relatio differentialium per aequationem quandam inter x, y et p proponitur, cuique jam novimus satisfieri per aequationem $y = \frac{a}{n+1} x^{n+1} + C$. Verum haec non amplius erit integrale completum pro relatione in aequatione $y = \frac{px}{n+1} + C$ contenta, sed tantum particulare, quoniam integrale illud non involvit novam constantem, quae in relatione differentiali non insit. Integrale autem completum est $y = \frac{aD}{n+1} x^{n+1} + C$: novam constantem D involvens: hinc enim fit $\frac{\partial y}{\partial x} = aDx^n = p$, ideoque $y = \frac{px}{n+1} + C$. Etsi hoc non ad praesens institutum pertinet, tamen notasse juvabit.

Problema 2.

55. Invenire functionem ipsius x, cujus differentiale sit $= X\partial x$, denotante X functionem quamcunque rationalem integram ipsius x, seu definire integrale $\int X\,\partial x$.

Solutio.

Cum X sit functio rationalis integra ipsius x, in hac forma contineatur necesse est:

$$X = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \text{etc.}$$

unde per problema praecedens integrale quaesitum est

$$\int X\partial x = C + \alpha x + \tfrac{1}{2}\beta x^2 + \tfrac{1}{3}\gamma x^3 + \tfrac{1}{4}\delta x^4 + \tfrac{1}{5}\varepsilon x^5 + \tfrac{1}{6}\zeta x^6 + \text{etc.}$$

Atque in genere si sit $X = \alpha x^\lambda + \beta x^\mu + \gamma x^\nu + \text{etc.}$ erit

$$\int X\partial x = C + \frac{\alpha}{\lambda+1} x^{\lambda+1} + \frac{\beta}{\mu+1} x^{\mu+1} + \frac{\gamma}{\nu+1} x^{\nu+1} + \text{etc.}$$

ubi exponentes λ, μ, ν etc. etiam numeros tam negativos quam fractos significare possunt; dummodo notetur, si fuerit $\lambda = -1$, fore $\int \frac{\alpha\partial x}{x} = \alpha l x$, qui est unicus casus ad ordinem transcendentium referendus.

Problema 3.

56. Si X denotet functionem quamcunque rationalem fractam ipsius x, methodum describere, cujus ope formulae $X\,\partial x$ integrale investigari conveniat.

Solutio.

Sit igitur $X = \frac{M}{N}$, ita ut M et N futurae sint functiones integrae ipsius x, ac primo dispiciatur, num summa potestas ipsius x in numeratore M tanta sit, vel etiam major quam in denominatore N? quo casu ex fractione $\frac{M}{N}$ partes integrae per divisionem eliciantur, quarum integratio, cum nihil habeat difficultatis, totum negotium reducitur ad ejusmodi fractionem $\frac{M}{N}$, in cujus numeratore M summa potestas ipsius x minor sit quam denominatore N.

Tum quaerantur omnes factores ipsius denominatoris N, tam simplices si fuerint reales, quam duplices reales, vicem scilicet binorum simplicium imaginariorum gerentes; simulque videndum est, utrum hi factores omnes sint inaequales nec ne? pro factorum enim aequalitate alio modo resolutio fractionis $\frac{M}{N}$ in fractiones simplices est instituenda, quandoquidem ex singulis factoribus fractiones partiales nascuntur, quarum aggregatum fractioni propositae $\frac{M}{N}$ aequatur. Scilicet ex factore simplici $a + bx$ nascitur fractio $\frac{A}{a + bx}$; si bini sint aequales, seu denominator N factorem habeat $(a + bx)^2$, hinc nascuntur fractiones $\frac{A}{(a + bx)^2} + \frac{B}{a + bx}$; ex hujusmodi autem factore $(a + bx)^3$ hae tres fractiones

$$\frac{A}{(a + bx)^3} + \frac{B}{(a + bx)^2} + \frac{C}{a + bx}$$

et ita porro.

Factor autem duplex, cujus forma est $aa - 2abx\cos.\zeta + bbxx$, nisi alius ipsi fuerit aequalis, dabit fractionem partialem $\frac{A + Bx}{aa - 2abx\cos.\zeta + bbxx}$: si autem denominator N duos hujusmodi factores aequales involvat, inde nascuntur binae hujusmodi fractiones partiales:

$$\frac{A + Bx}{(aa - 2abx\cos.\zeta + bbxx)^2} + \frac{C + Dx}{aa - 2abx\cos.\zeta + bbxx}:$$

at si cubus adeo $(aa - 2abx\cos.\zeta + bbxx)^3$ fuerit factor denominatoris N, ex eo oriuntur hujusmodi tres fractiones partiales:

$$\frac{A + Bx}{(aa - 2abx\cos.\zeta + bbxx)^3} + \frac{C + Dx}{(aa - 2abx\cos.\zeta + bbxx)^2} + \frac{E + Fx}{aa - 2abx\cos.\zeta + bbxx}$$

et ita porro.

Cum igitur hoc modo fractio proposita $\frac{M}{N}$ in omnes suas fractiones simplices fuerit resoluta, omnes continebuntur in alterutra harum formarum,

$$\text{vel } \frac{A}{(a+bx)^n}, \text{ vel } \frac{A+Bx}{(aa-2abx\cos.\zeta+bbxx)^n},$$

ac singulos jam per ∂x multiplicatos integrari oportet, erit omnium horum integralium aggregatum valor functionis quaesitae $\int X\partial x = \int \frac{M}{N}\partial x$.

Corollarium 1.

57. Pro integratione ergo omnium hujusmodi formularum $\frac{M}{N}\partial x$, totum negotium reducitur ad integrationem hujusmodi binarum formularum:

$$\int \frac{A\partial x}{(a+bx)^n} \text{ et } \int \frac{(A+Bx)\partial x}{(aa-2abx\cos.\zeta+bbxx)^n}$$

dum pro n successive scribuntur numeri 1, 2, 3, 4 etc.

Corollarium 2.

58. Ac prioris quidem formae integrale jam supra (53) est expeditum, unde patet fore:

$$\int \frac{A\partial x}{a+bx} = \frac{A}{b}\, l\,(a+bx) + \text{Const.}$$

$$\int \frac{A\partial x}{(a+bx)^2} = \frac{-A}{b(a+bx)} + \text{Const.}$$

$$\int \frac{A\partial x}{(a+bx)^3} = \frac{-A}{2b(a+bx)^2} + \text{Const.}$$

et generatim:

$$\int \frac{A\partial x}{(a+bx)^n} = \frac{-A}{(n-1)b(a+bx)^{n-1}} + \text{Const.}$$

Corollarium 3.

59. Ad propositum ergo absolvendum nihil aliud superest, nisi ut integratio hujus formulae

$$\int \frac{(A+Bx)\partial x}{(aa-2abx\cos.\zeta+bbxx)^n}$$

doceatur, primo quidem casu $n = 1$, tum vero casibus $n = 2$, $n = 3$, $n = 4$, etc.

Scholion 1.

60. Nisi vellemus imaginaria evitare, totum negotium ex jam traditis confici posset: denominatore enim N in omnes suos factores simplices resoluto, sive sint reales sive imaginarii, fractio proposita semper resolvi poterit in fractiones partiales hujus formae $\frac{A}{a + bx}$, vel hujus $\frac{A}{(a + bx)^n}$, quarum integralia cum sint in promptu, totius formae $\frac{M}{N} \partial x$ integrale habetur. Tum autem non parum molestum foret binas partes imaginarias ita conjungere, ut expressio realis resultaret, quod tamen rei natura absolute exigit.

Scholion 2.

61. Hic utique postulamus, resolutionem cujusque functionis integrae in factores nobis concedi, etiamsi algebra neutiquam adhuc eo sit perducta, ut haec resolutio actu institui possit. Hoc autem in Analysi ubique postulari solet, ut quo longius progrediamur, ea quae retro sunt relicta, etiamsi non satis fuerint explorata, tanquam cognita assumamus: sufficere scilicet hic potest, omnes factores per methodum approximationum quantumvis prope assignari posse. Simili modo cum in calculo integrali longius processerimus, integralia omnium hujusmodi formularum $X \partial x$, quaecunque functio ipsius x littera X significetur, tanquam cognita spectabimus; plurimumque nobis praestitisse videbimur, si integralia magis abscondita ad eas formas reducere valuerimus: atque hoc etiam in usu practico nihil turbat, cum valores talium formularum $\int X \partial x$, quantumvis prope assignare liceat, uti in sequentibus ostendemus. Caeterum ad has integrationes, resolutio denominatoris N in suos factores absolute est necessaria, propterea quod singuli hi factores in expressionem integralis ingrediuntur: paucissimi sunt casus, iique maxime obvii, quibus ista resolutione carere possumus: veluti si proponatur haec

formula $\frac{x^{n-1}\partial x}{1+x^n}$, statim patet, posito $x^n = v$, eam abire in $\frac{\partial v}{n(1+v)}$, cujus integrale est $\frac{1}{n}l(1+v) = \frac{1}{n}l(1+x^n)$; ubi resolutione in factores non fuerat opus. Verum hujusmodi casus per se tam sunt perspicui, ut eorum tractatio nulla peculiari explicatione indigeat.

Problema 4.

62. Invenire integrale hujus formulae:

$$y = \int \frac{(A+Bx)\partial x}{aa - 2abx\cos.\zeta + bbxx}.$$

Solutio.

Cum numerator duabus constet partibus $A\partial x + Bx\partial x$, haec posterior $Bx\partial x$ sequenti modo tolli poterit. Cum sit

$$l(aa - 2abx\cos.\zeta + bbxx) = \int \frac{-2ab\partial x\cos.\zeta + 2bbx\partial x}{aa - 2abx\cos.\zeta + bbxx},$$

multiplicetur haec aequatio per $\frac{B}{2bb}$, et a proposita auferatur: sic enim prodibit

$$y - \frac{B}{2bb}l(aa - 2abx\cos.\zeta + bbxx) = \int \frac{\left(A + \frac{Ba\cos.\zeta}{b}\right)\partial x}{aa - 2abx\cos.\zeta + bbxx}:$$

ita ut haec tantum formula integranda supersit. Ponatur brevitatis gratia $A + \frac{Ba\cos.\zeta}{b} = C$, ut habeatur haec formula:

$$\int \frac{C\partial x}{aa - 2abx\cos.\zeta + bbxx},$$

quae ita exhiberi potest

$$\int \frac{Cdx}{aa\sin.\zeta^2 + (bx - a\cos.\zeta)^2}.$$

Statuatur $bx - a\cos.\zeta = av\sin.\zeta$, hincque $\partial x = \frac{a\partial v\sin.\zeta}{b}$: unde formula nostra erit:

$$\int \frac{Ca\partial v\sin\zeta : b}{aa\sin.\zeta^2(1+vv)} = \frac{C}{ab\sin.\zeta}\int \frac{\partial v}{1+vv}.$$

Ex calculo autem differentiali novimus esse:

$$\int \frac{\partial v}{1+vv} = \text{Arc. tang. } v = \text{Arc. tang. } \frac{bx - a\cos.\zeta}{a\sin.\zeta};$$

unde ob $C = \frac{Ab + Ba\cos.\zeta}{b}$, erit nostrum integrale

$$\frac{Ab + Ba\cos.\zeta}{abb\sin.\zeta} \text{ Arc. tang. } \frac{bx - a\cos.\zeta}{a\sin.\zeta}.$$

Quocirca formulae propositae $\frac{(A + Bx)\partial x}{aa - 2abx\cos.\zeta + bbxx}$ integrale est:

$$\frac{B}{2bb} l(aa - 2abx\cos.\zeta + bbxx) + \frac{Ab + Ba\cos.\zeta}{abb\sin.\zeta} \text{ Arc. tang. } \frac{bx - a\cos.\zeta}{a\sin.\zeta},$$

quod ut fiat completum, constans arbitraria C insuper addatur.

Corollarium 1.

63. Si ad Arc. tang. $\frac{bx - a\cos.\zeta}{a\sin.\zeta}$ addamus Arc tang. $\frac{\cos.\zeta}{\sin.\zeta}$, quippe qui in constante addenda contentus concipiatur, prodibit Arc. tang. $\frac{bx\sin.\zeta}{a - bx\cos.\zeta}$, sicque habebimus:

$$\int \frac{(A + Bx)\partial x}{aa - 2abx\cos.\zeta + bbxx} = \frac{B}{2bb} l(aa - 2abx\cos.\zeta + bbxx)$$
$$+ \frac{Ab + Ba\cos.\zeta}{abb\sin.\zeta} \text{ Arc. tang. } \frac{bx\sin.\zeta}{a - bx\cos.\zeta}$$

adjecta constante C.

Corollarium 2.

64. Si velimus ut integrale hoc evanescat, posito $x = 0$, constans C sumi debet $= -\frac{B}{2bb} laa$, sicque fiet:

$$\int \frac{(A + Bx)\partial x}{aa - 2abx\cos.\zeta + bbxx} = \frac{B}{bb} l \frac{\sqrt{(aa - 2abx\cos.\zeta + bbxx)}}{a}$$
$$+ \frac{Ab + Ba\cos.\zeta}{abb\sin.\zeta} \text{ Arc. tang. } \frac{bx\sin.\zeta}{a - bx\cos.\zeta}.$$

Pendet ergo hoc integrale partim a logarithmis, partim ab arcubus circularibus seu angulis.

Corollarium 3.

65. Si littera B evanescat, pars a logarithmis pendens evanescit, fitque

$$\int \frac{A\partial x}{aa - 2abx\cos.\zeta + bbxx} = \frac{A}{ab\sin.\zeta} \text{ Arc. tang. } \frac{bx\sin.\zeta}{a - bx\cos.\zeta} + C$$

sicque per solum angulum definitur.

Corollarium 4.

66. Si angulus ζ sit rectus, ideoque $\cos.\zeta = 0$, et $\sin.\zeta = 1$, habebitur:

$$\int \frac{(A + Bx)\partial x}{aa + bbxx} = \frac{B}{bb}\, l \frac{\sqrt{(aa + bbxx)}}{a} + \frac{A}{ab} \text{ Arc. tang. } \frac{bx}{a} + C.$$

Si angulus ζ sit 60°, ideoque $\cos.\zeta = \frac{1}{2}$ et $\sin.\zeta = \frac{\sqrt{3}}{2}$, erit:

$$\int \frac{(A + Bx)\partial x}{aa - abx + bbxx} = \frac{B}{bb}\, l \frac{\sqrt{(aa - abx + bbxx)}}{a} + \frac{2Ab + Ba}{abb\sqrt{3}} \text{ Arc. tang. } \frac{bx\sqrt{3}}{2a - bx}.$$

At si $\zeta = 120^{\circ}$, ideoque $\cos.\zeta = -\frac{1}{2}$ et $\sin.\zeta = \frac{\sqrt{3}}{2}$ erit:

$$\int \frac{(A + Bx)\partial x}{aa + abx + bbxx} = \frac{B}{bb}\, l \frac{\sqrt{(aa + abx + bbxx)}}{a} + \frac{2Ab - Ba}{abb\sqrt{3}} \text{ Arc. tang. } \frac{bx\sqrt{3}}{2a + bx}.$$

Scholion 1.

67. Omnino hic notatu dignum evenit, quod casu $\zeta = 0$, quo denominator $aa - 2abx + bbxx$ fit quadratum, ratio anguli ex integrali discedat. Posito enim angulo ζ infinite parvo, erit $\cos.\zeta = 1$ et $\sin.\zeta = \zeta$; unde pars logarithmica fit $\frac{B}{bb}\, l \frac{a - bx}{a}$, et altera pars:

$$\frac{Ab + Ba}{abb\zeta} \text{ Arc. tang. } \frac{bx\zeta}{a - bx} = \frac{(Ab + Ba)x}{ab(a - bx)}$$

quia arcus infinite parvi $\frac{bx\zeta}{a - bx}$ tangens ipsi est aequalis, sicque haec pars fit algebraica. Quocirca erit

$$\int \frac{(A + Bx)\partial x}{(a - bx)^2} = \frac{B}{bb}\, l \frac{a - bx}{a} + \frac{(Ab + Ba)x}{ab(a - bx)} + \text{Const.}$$

cujus veritas ex praecedentibus est manifesta: est enim

$$\frac{A + Bx}{(a - bx)^2} = -\frac{B}{b(a - bx)} + \frac{Ab + Ba}{b(a - bx)^2}.$$

Jam vero est

$$\int \frac{-B\partial x}{b(a - bx)} = \frac{B}{bb}\, l(a - bx) - \frac{B}{bb}\, l\, a = \frac{B}{bb}\, l \frac{a - bx}{a},$$

$$\int \frac{(Ab + Ba)\partial x}{b(a - bx)^2} = \frac{Ab + Ba}{bb(a - bx)} - \frac{(Ab + Ba)}{abb} = \frac{(Ab + Ba)x}{ab(a - bx)},$$

siquidem utraque integratio ita determinetur ut, casu $x = 0$, integralia evanescant.

Scholion 2.

68. Simili modo, quo hic usi sumus, si in formula differentiali fracta $\frac{M\partial x}{N}$, summa potestas ipsius x, in numeratore M, uno gradu minor sit quam in denominatore N, etiam is terminus tolli poterit. Sit enim

$$M = Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \text{etc. et}$$
$$N = \alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \text{etc.}$$

ac ponatur $\frac{M\partial x}{N} = \partial y$: Cum jam sit

$$\partial N = n\alpha x^{n-1}\partial x + (n-1)\beta x^{n-2}\partial x + (n-2)\gamma x^{n-3}\partial x + \text{etc.}$$

erit:

$$\frac{A\partial N}{n\alpha N} = \frac{\partial x}{N}\left(Ax^{n-1} + \frac{(n-1)A\beta}{n\alpha}x^{n-2} + \frac{(n-2)A\gamma}{n\alpha}x^{n-3} + \text{etc.}\right)$$

quo valoro inde subtracto remanebit:

$$\partial y - \frac{A\partial N}{n\alpha N} = \frac{\partial x}{N}\left[\left(B - \frac{(n-1)A\beta}{n\alpha}\right)x^{n-2} + \left(C - \frac{(n-2)A\gamma}{n\alpha}\right)x^{n-3} + \text{etc.}\right]$$

Quare si brevitatis gratia ponatur:

$$B - \frac{(n-1)A\beta}{n\alpha} = \mathfrak{B};\ C - \frac{(n-2)A\gamma}{n\alpha} = \mathfrak{C};\ D - \frac{(n-3)A\delta}{n\alpha} = \mathfrak{D};\ \text{etc.}$$

obtinebitur:

$$y = \frac{A}{n\alpha}lN + \int\frac{\partial x(\mathfrak{B}x^{n-2} + \mathfrak{C}x^{n-3} + \mathfrak{D}x^{n-4} + \text{etc.})}{\alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \text{etc.}} = \int\frac{M\partial x}{N}.$$

Hoc igitur modo omnes formulae differentiales fractae eo reduci possunt, ut summa potestas ipsius x in numeratore duobus pluribusve gradibus minor sit quam in denominatore.

Problema 5.

69. Formulam integralem $\int\frac{(A + \beta x)\partial x}{(aa - 2abx\cos.\zeta + bbxx)^{n+1}}$ ad aliam similem reducere, ubi potestas denominatoris sit uno gradu inferior.

Solutio.

Sit brevitatis gratia $aa - 2abx\cos.\zeta + bbxx = X$, ac ponatur $\int \frac{(A+Bx)\partial x}{X^{n+1}} = y$. Cum ob $\partial X = -2ab\partial x\cos.\zeta + 2bbx\partial x$, sit:

$$\partial \frac{C+Dx}{X^n} = -\frac{n(C+Dx)\partial X}{X^{n+1}} + \frac{D\partial x}{X^n}$$

adeoque:

$$\frac{C+Dx}{X^n} = \int \frac{2nb(C+Dx)(a\cos.\zeta - bx)\partial x}{X^{n+1}} + \int \frac{D\partial x}{X^n},$$

habebimus:

$$y + \frac{C+Dx}{X^n} = \int \frac{\partial x[A+2nCab\cos.\zeta + x(B+2nDab\cos.\zeta - 2nCbb) - 2nDbbxx]}{X^{n+1}} + \int \frac{D\partial x}{X^n}.$$

Jam in formula priori litterae C et D ita definiantur, ut numerator per X fiat divisibilis. Oportet ergo sit $= -2nDX\partial x$, unde nanciscimur:

$A + 2nCab\cos.\zeta = -2nDaa$, et

$B + 2nDab\cos.\zeta - 2nCbb = 4nDab\cos.\zeta$,

seu $B - 2nCbb = 2nDab\cos.\zeta$; hincque

$2nDa = \frac{B - 2nCbb}{b\cos.\zeta}$.

At ex priori conditione est

$2nDa = \frac{-A - 2nCab\cos.\zeta}{a}$, quibus aequatis fit:

$Ba + Ab\cos.\zeta - 2nCabb\sin.\zeta^2 = 0$, seu

$C = \frac{Ba + Ab\cos.\zeta}{2nabb\sin.\zeta^2}$: unde

$B - 2nCbb = \frac{Ba\sin.\zeta^2 - Ba - Ab\cos.\zeta}{a\sin.\zeta^2} = \frac{-Ab\cos.\zeta - Ba\cos.\zeta^2}{a\sin.\zeta^2}$,

ita ut reperiatur $D = \frac{-Ab - Ba\cos.\zeta}{2naab\sin.\zeta^2}$. Sumtis ergo litteris

$C = \frac{Ba + Ab\cos.\zeta}{2naabb\sin.\zeta^2}$ et $D = \frac{-Ab - Ba\cos.\zeta}{2naab\sin.\zeta^2}$, erit

$$y + \frac{C + Dx}{X^n} = \int \frac{-2nD\partial x}{X^n} + \int \frac{D\partial x}{X^n} = -(2n-1)D\int\frac{\partial x}{X^n}.$$

ideoque

$$\int\frac{(A+Bx)\partial x}{X^{n+1}} = \frac{-C - Dx}{X^n} - (2n-1)D\int\frac{\partial X}{X^n}, \text{ sive}$$

$$\int\frac{(A+Bx)\partial x}{X^{n+1}} = \frac{-Baa - Aab\cos.\zeta + (Abb + Bab\cos.\zeta)x}{2naabb\sin.\zeta^2 X^n}$$
$$+ \frac{(2n-1)(Ab + Ba\cos.\zeta)}{2naab\sin.\zeta^2}\int\frac{\partial x}{X^n}.$$

Quare si formula $\int\frac{\partial x}{X^n}$ constet, etiam integrale hoc $\int\frac{(A+Bx)\partial x}{X^{n+1}}$ assignari poterit.

Corollarium 1.

70. Cum igitur manente

$$X = aa - 2abx\cos.\zeta + bbxx, \text{ fit}$$

$$\int\frac{\partial x}{X} = \frac{1}{ab\sin.\zeta}\text{ Arc. tang. }\frac{bx\sin.\zeta}{a - bx\cos.\zeta} + \text{Const. erit:}$$

$$\int\frac{(A+Bx)\partial x}{X^2} = \frac{-Baa - Aab\cos.\zeta + (Abb + Bab\cos.\zeta)x}{2aabb\sin.\zeta^2 X}$$
$$+ \frac{Ab + Ba\cos.\zeta}{2a^3bb\sin.\zeta^3}\text{ Arc. tang. }\frac{bx\sin.\zeta}{a - bx\cos.\zeta} + \text{Const.}$$

Ideoque posito $B = 0$ et $A = 1$, fiet

$$\int\frac{\partial x}{X^2} = \frac{-a\cos.\zeta + bx}{2aab\sin.\zeta^2 X} + \frac{1}{2a^3b\sin.\zeta^3}\text{ Arc. tang. }\frac{bx\sin.\zeta}{a - bx\cos.\zeta} + \text{Const.}$$

Integrale ergo $\int\frac{(A+Bx)\partial x}{X^2}$ logarithmos non involvit.

Corollarium 2.

71. Hinc ergo cum sit:

$$\int\frac{\partial x}{X^3} = \frac{-a\cos.\zeta + bx}{4aab\sin.\zeta^2.X^2} + \frac{3}{4aa\sin.\zeta^2}\int\frac{\partial x}{X^2} + \text{Const.}$$

erit illum valorem substituendo:

$$\int\frac{\partial x}{X^3} = \frac{-a\cos.\zeta + bx}{4aab\sin.\zeta^2.X^2} + \frac{3(-a\cos.\zeta + bx)}{2.4a^4b\sin\zeta^4.X}$$
$$+ \frac{1.3}{2.4a^5b\sin.\zeta^5}\text{ Arc. tang. }\frac{bx\sin.\zeta}{a - bx\cos.\zeta}.$$

Hincque porro concluditur:

$$\int \frac{\partial x}{X^4} = \frac{-a\cos.\zeta + bx}{6aab\sin.\zeta^2.X^3} + \frac{5(-a\cos.\zeta + bx)}{4.6a^4b\sin\zeta^4.X^2} + \frac{3.5(a-\cos.\zeta + bx)}{2.4.6a^6b\sin.\zeta^6.X}$$
$$+ \frac{1.3.5}{2.4.6a^7b\sin.\zeta^7}\text{ Arc. tang. }\frac{bx\sin.\zeta}{a-bx\cos.\zeta}.$$

Corollarium 3.

72. Sic ulterius progrediendo, omnium hujusmodi formularum integralia obtinebuntur:

$$\int \frac{\partial x}{X}, \int \frac{\partial x}{X^2}, \int \frac{\partial x}{X^3}, \int \frac{\partial x}{X^4}, \text{ etc.}$$

quorum primum arcu circulari solo exprimitur, reliqua vero praeterea partes algebraicas continent.

Scholion.

73. Sufficit autem integralia $\int \frac{\partial x}{X^{n+1}}$ nosse, quia formula $\int \frac{(A + Bx)\partial x}{X^{n+1}}$ facile eo reducitur: ita enim repraesentari potest

$$\frac{1}{2bb}\int \frac{2Abb\partial x + 2Bbbx\partial x - 2Bab\partial x\cos.\zeta + 2Bab\partial x\cos.\zeta}{X^{n+1}},$$

quae ob $2bbx\partial x - 2ab\partial x\cos.\zeta = \partial X$, abit in hanc

$$\frac{1}{2bb}\int \frac{B\partial X}{X^{n+1}} + \frac{1}{b}\int \frac{(Ab + Ba\cos.\zeta)\partial x}{X^{n+1}}.$$

At $\int \frac{\partial X}{X^{n+1}} = -\frac{1}{nX^n}$, unde habebitur:

$$\int \frac{(A + Bx)\partial x}{X^{n+1}} = \frac{-B}{2nbbX^n} + \frac{Ab + Ba\cos.\zeta}{b}\int \frac{\partial x}{X^{n+1}};$$

unde tantum opus est nosse integralia $\int \frac{\partial X}{X^{n+1}}$, quae modo exhibuimus. Atque haec sunt omnia subsidia quibus indigemus ad omnes formulas fractas $\frac{M}{N}\partial x$ integrandas, dummodo M et N sunt functiones integrae ipsius x. Quocirca in genere integratio omnium hujusmodi formularum $\int V\partial x$, ubi V est functio rationalis ipsius x quaecunque, est in potestate: de quibus notandum est, nisi integralia fuerint algebraica, semper vel per logarithmos vel angulos exhiberi posse. Nihil aliud igitur superest, nisi ut hanc methodum aliquot exemplis illustremus.

Exemplum 1.

74. *Proposita formula differentiali* $\frac{(A + Bx)\partial x}{\alpha + \beta x + \gamma xx}$, *definire ejus integrale.*

Cum in numeratore variabilis x pauciores habeat dimensiones, quam in denominatore, haec fractio nullas partes integras complectitur. Hinc denominatoris indoles perpendatur, utrum habeat duos factores simplices reales nec ne? ac priori casu num factores sint aequales? ex quo tres habebimus casus evolvendos.

I. Habeat denominator ambos factores aequales, sitque $= (a + bx)^2$, et fractio $\frac{A + Bx}{(a + bx)^2}$ resolvitur in duas,

$$\frac{Ab - Ba}{b(a + bx)^2} + \frac{B}{b(a + bx)}, \text{ unde fit:}$$

$$\int \frac{(A + Bx)\partial x}{(a + bx)^2} = \frac{Ba - Ab}{bb(a + bx)} + \frac{B}{bb} l(a + bx) + \text{Const.}$$

Si integrale ita determinetur, ut evanescat posito $x = 0$, reperitur

$$\int \frac{(A + Bx)\partial x}{(a + bx)^2} = \frac{(Ab - Ba)x}{ab(a + bx)} + \frac{B}{bb} l \frac{a + bx}{a}.$$

II. Habeat denominator duos factores inaequales, sitque proposita haec formula $\frac{A + Bx}{(a + bx)(f + gx)} \cdot \partial x$, et haec fractio resolvitur in has partiales:

$$\frac{Ab - Ba}{bf - ag} \cdot \frac{\partial x}{a + bx} + \frac{Ag - Bf}{ag - bf} \cdot \frac{\partial x}{f + gx},$$

unde obtinetur integrale quaesitum:

$$\int \frac{(A + Bx)\partial x}{(a + bx)(f + gx)} = \frac{Ab - Ba}{b(bf - ag)} l \frac{a + bx}{a} + \frac{Ag - Bf}{g(ag - bf)} l \frac{f + gx}{f} + \text{Const.}$$

Ponatur:

$$\frac{Ab - Ba}{b(bf - ag)} = m + n \text{ et } \frac{Bf - Ag}{g(bf - ag)} = m - n,$$

ut integrale fiat:

$$m\,l\frac{(a+bx)(f+gx)}{af} + n\,l\frac{f(a+bx)}{a(f+gx)}, \text{ erit:}$$

$$2m = \frac{B(bf - ag)}{bg(bf - ag)} = \frac{B}{bg}, \text{ et}$$

$$2n = \frac{2Abg - Bag - Bbf}{bg(bf - ag)}. \text{ Erit ergo:}$$

$$\int\frac{(A + Bx)\partial x}{(a+bx)(f+gx)} = \frac{B}{2bg}\,l\frac{(a+bx)(f+gx)}{af} + \frac{2Abg - B(ag+bf)}{2bg(bf - ag)}\,l\frac{f(a+bx)}{a(f+gx)}.$$

III. Sint denominatoris factores simplices ambo imaginarii, quo casu formam habebit $aa - 2abx\cos.\zeta + bbxx$; qui casus cum supra jam sit tractatus, erit:

$$\int\frac{(A + Bx)\partial x}{aa - 2abx\cos.\zeta + bbxx} = \frac{B}{bb}\,l\frac{\sqrt{(aa - 2abx\cos.\zeta + bbxx)}}{a}$$
$$+ \frac{Ab + Ba\cos.\zeta}{abb\sin.\zeta}\text{ Arc. tang. }\frac{bx\sin.\zeta}{a - bx\cos.\zeta}.$$

Corollarium 1.

75. Casu secundo, quo $f = a$, et $g = -b$, erit

$$\int\frac{(A + Bx)\partial x}{aa - bbxx} = \frac{-B}{2bb}\,l\frac{aa - bbxx}{aa} + \frac{A}{2ab}\,l\frac{a+bx}{a-bx}.$$

Hinc seorsim sequitur:

$$\int\frac{A\partial x}{aa - bbxx} = \frac{A}{2ab}\,l\frac{a+bx}{a-bx} + C \text{ et}$$

$$\int\frac{Bx\partial x}{aa - bbxx} = \frac{-B}{2bb}\,l\frac{aa - bbxx}{aa} = \frac{B}{bb}\,l\frac{a}{\sqrt{(aa - bbxx)}} + C$$

Corollarium 2.

76. Casu tertio, si ponamus $\cos.\zeta = 0$, habemus

$$\int\frac{(A + Bx)\partial x}{aa + bbxx} = \frac{B}{bb}\,l\frac{\sqrt{(aa + bbxx)}}{a} + \frac{A}{ab}\text{ Arc. tang. }\frac{bx}{a} + C,$$

hincque sigillatim:

$$\int\frac{A\partial x}{aa + bbxx} = \frac{A}{ab}\text{ Arc. tang. }\frac{bx}{a} + C, \text{ et}$$

$$\int\frac{Bx\partial x}{aa + bbxx} = \frac{B}{bb}\,l\frac{\sqrt{(aa + bbxx)}}{a} + C.$$

Exemplum 2.

77. *Proposita formula differentiali* $\frac{x^{m-1}\partial x}{1+x^n}$, *siquidem exponens* m — 1 *minor sit quam* n, *integrale definire.*

In capite ultimo Institut. Calculi Differential. invenimus fractiones simplices, in quas haec fractio $\frac{x^m}{1+x^n}$ resolvitur, sumto π pro mensura duorum angulorum rectorum, in hac forma generali contineri:

$$\frac{2\sin\frac{(2k-1)\pi}{n}\sin.\frac{m(2k-1)\pi}{n}-2\cos.\frac{m(2k-1)\pi}{n}\left(x-\cos.\frac{(2k-1)\pi}{n}\right)}{n\left(1-2x\cos.\frac{(2k-1)\pi}{n}+xx\right)}:$$

ubi pro k successive omnes numeros 1, 2, 3, etc. substitui convenit, quoad $2k-1$ numerum n superare incipiat. Hac ergo forma in ∂x ducta, et cum generali nostra

$$\frac{(A+Bx)\partial x}{aa-2abx\cos.\zeta+bbxx} \text{ comparata, fit}$$

$$a=1,\ b=1,\ \zeta=\frac{(2k-1)\pi}{n};\ \text{et}$$

$$A=\frac{2}{n}\sin.\frac{(2k-1)\pi}{n}\sin.\frac{m(2k-1)\pi}{n}+\frac{2}{n}\cos.\frac{(2k-1)\pi}{n}\cos.\frac{m(2k-1)\pi}{n}:$$

seu $A=\frac{2}{n}\cos.\frac{(m-1)(2k-1)\pi}{n}$, et

$$B=-\frac{2}{n}\cos.\frac{m(2k-1)\pi}{n},\ \text{unde fit}$$

$$Ab+B.a\cos.\zeta=\frac{2}{n}\sin.\frac{(2k-1)\pi}{n}\sin.\frac{m(2k-1)\pi}{n}:$$

ac propterea hujus partis integrale erit =

$$-\frac{2}{n}\cos.\frac{m(2k-1)\pi}{n}\,l\sqrt{\left(1-2x\cos.\frac{(2k-1)\pi}{n}+xx\right)}$$

$$+\frac{2}{n}\sin.\frac{m(2k-1)\pi}{n}\,\text{Arc. tang.}\frac{x\sin.\frac{(2k-1)\pi}{n}}{1-x\cos.\frac{(2k-1)\pi}{n}}.$$

Ac si n numerus impar, praeterea accedit fractio $\frac{\pm\partial x}{n(1+x)}$ cujus integrale est $\pm\frac{1}{n}l(1+x)$: ubi signum superius valet, si m impar, inferius vero, si m par. Quocirea integrale quaesitum $\int \frac{x^{m-1}\partial x}{1+x^n}$, sequenti modo exprimetur:

$$-\frac{2}{n}\cos.\frac{m\pi}{n}l\surd(1-2x\cos.\frac{\pi}{n}+xx)+\frac{2}{n}\sin.\frac{m\pi}{n}\text{Arc. tang. }\frac{x\sin.\frac{\pi}{n}}{1-x\cos.\frac{\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{3m\pi}{n}l\surd(1-2x\cos.\frac{3\pi}{n}+xx)+\frac{2}{n}\sin.\frac{3m\pi}{n}\text{Arc. tang. }\frac{x\sin.\frac{3\pi}{n}}{1-x\cos.\frac{3\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{5m\pi}{n}l\surd(1-2x\cos.\frac{5\pi}{n}+xx)+\frac{2}{n}\sin.\frac{5m\pi}{n}\text{Arc. tang. }\frac{x\sin.\frac{5\pi}{n}}{1-x\cos.\frac{5\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{7m\pi}{n}l\surd(1-2x\cos.\frac{7\pi}{n}+xx)+\frac{2}{n}\sin.\frac{7m\pi}{n}\text{Arc. tang. }\frac{x\sin.\frac{7\pi}{n}}{1-x\cos.\frac{7\pi}{n}}$$

etc.

secundum numeros impares ipso n minores, sicque totum obtinetur integrale si n fuerit numerus par; sin autem n sit numerus impar, insuper accedit haec pars $\pm\frac{1}{n}l(1+x)$, prout m sit numerus vel impar vel par: unde si $m=1$, accedit insuper $+\frac{1}{n}l(1+x)$.

Corollarium 1.

78. Sumamus $m=1$, ut habeatur forma $\int\frac{\partial x}{1+x^n}$, et pro variis casibus ipsius n adipiscimur:

I. $\int\frac{\partial x}{1+x}=l(1+x)$

II. $\int\frac{\partial x}{1+x^2}=\text{Arc. tang }x$

III. $\int\frac{\partial x}{1+x^3}=-\frac{2}{3}\cos.\frac{\pi}{3}l\surd(1-2x\cos.\frac{\pi}{3}+xx)+\frac{2}{3}\sin.\frac{\pi}{3}\text{Arc.tang. }\frac{x\sin.\frac{\pi}{3}}{1-x\cos.\frac{\pi}{3}}$

$+\frac{1}{3}l(1+x)$

$$\text{IV.}\int\frac{\partial x}{1+x^4}=\begin{cases}-\frac{2}{4}\cos.\frac{\pi}{4}\,l\sqrt{(1-2x\cos.\frac{\pi}{4}+xx)}+\frac{2}{4}\sin.\frac{\pi}{4}\,\text{Arc.tang.}\,\dfrac{x\sin.\frac{\pi}{4}}{1-x\cos.\frac{\pi}{4}}\\[2ex]-\frac{2}{4}\cos.\frac{3\pi}{4}\,l\sqrt{(1-2x\cos.\frac{3\pi}{4}+xx)}+\frac{2}{4}\sin.\frac{3\pi}{4}\,\text{Arc.tang.}\,\dfrac{x\sin.\frac{3\pi}{4}}{1-x\cos.\frac{3\pi}{4}}\end{cases}$$

$$\text{V.}\int\frac{\partial x}{1+x^5}=\begin{cases}-\frac{2}{5}\cos.\frac{\pi}{5}\,l\sqrt{(1-2x\cos.\frac{\pi}{5}+xx)}+\frac{2}{5}\sin.\frac{\pi}{5}\,\text{Arc.tang.}\,\dfrac{x\sin.\frac{\pi}{5}}{1-x\cos\frac{\pi}{5}}\\[2ex]-\frac{2}{5}\cos.\frac{3\pi}{5}\,l\sqrt{(1-2x\cos.\frac{3\pi}{5}+xx)}+\frac{2}{5}\sin.\frac{3\pi}{5}\,\text{Arc.tang.}\,\dfrac{x\sin.\frac{3\pi}{5}}{1-x\cos.\frac{3\pi}{5}}\\[2ex]+\frac{1}{5}\,l\,(1+x)\end{cases}$$

$$\text{VI.}\int\frac{\partial x}{1+x^6}=\begin{cases}-\frac{2}{6}\cos.\frac{\pi}{6}\,l\sqrt{(1-2x\cos.\frac{\pi}{6}+xx)}+\frac{2}{6}\sin.\frac{\pi}{6}\,\text{Arc.tang.}\,\dfrac{x\sin.\frac{\pi}{6}}{1-x\cos\frac{\pi}{6}}\\[2ex]-\frac{2}{6}\cos.\frac{3\pi}{6}\,l\sqrt{(1-2x\cos.\frac{3\pi}{6}+xx)}+\frac{2}{6}\sin.\frac{3\pi}{6}\,\text{Arc.tang}\,\dfrac{x\sin\frac{3\pi}{6}}{1-x\cos.\frac{3\pi}{6}}\\[2ex]-\frac{2}{6}\cos.\frac{5\pi}{6}\,l\sqrt{(1-2x\cos.\frac{5\pi}{6}+xx)}+\frac{2}{6}\sin.\frac{5\pi}{6}\,\text{Arc.tang}\,\dfrac{x\sin.\frac{5\pi}{6}}{1-x\cos.\frac{5\pi}{6}}\end{cases}$$

Corollarium 2.

79. Loco sinuum et cosinuum valores, ubi commode fieri potest, substituendo, obtinemus:

$$\int\frac{\partial x}{1+x^3}=-\tfrac{1}{3}\,l\sqrt{(1-x+xx)}+\tfrac{1}{\sqrt{3}}\,\text{Arc. tang.}\,\frac{x\sqrt{3}}{2-x}+\tfrac{1}{3}\,l(1+x)$$

seu

$$\int\frac{\partial x}{1+x^3}=\tfrac{1}{3}\,l\,\frac{1+x}{\sqrt{(1-x+xx)}}+\tfrac{1}{\sqrt{3}}\,\text{Arc. tang.}\,\frac{x\sqrt{3}}{2-x}.$$

Deinde ob $\sin.\frac{\pi}{4}=\cos.\frac{\pi}{4}=\frac{1}{\sqrt{2}}=\sin.\frac{3\pi}{4}=-\cos.\frac{3\pi}{4}$, fit

$$\int\frac{\partial x}{1+x^4}=+\frac{1}{2\sqrt{2}}\,l\,\frac{\sqrt{(1+x\sqrt{2}+xx)}}{\sqrt{(1-x\sqrt{2}+xx)}}+\frac{1}{2\sqrt{2}}\,\text{Arc. tang.}\,\frac{x\sqrt{2}}{1-xx},$$

tum vero

$$\int\frac{\partial x}{1+x^6}=\frac{1}{2\sqrt{3}}\,l\,\frac{\sqrt{(\;+x\sqrt{3}+xx)}}{\sqrt{(1-x\sqrt{3}+xx)}}+\tfrac{1}{6}\,\text{Arc. tang.}\,\frac{3x(1-xx)}{1-4xx+x^4}.$$

Exemplum 3.

80. *Proposita formula differentiali* $\frac{x^{m-1}\,dx}{1-x^n}$, *siquidem exponens* m − 1 *sit minor quam* n, *ejus integrale definire.*

Functionis fractae $\frac{x^{m-1}}{1-x^n}$ pars, ex factore quocunque oriunda, hac forma continetur:

$$\frac{2\sin.\frac{2k\pi}{n}\sin.\frac{2mk\pi}{n}-2\cos.\frac{2mk\pi}{n}\left(x-\cos.\frac{2k\pi}{n}\right)}{n\left(1-2x\cos.\frac{2k\pi}{n}+xx\right)}$$

quae cum forma nostra $\frac{A+Bx}{aa-2abx\cos.\zeta+bbxx}$ comparata, dat $a=1$, $b=1$, $\zeta=\frac{2k\pi}{n}$;

$$A=\frac{2}{n}\sin.\frac{2k\pi}{n}\sin.\frac{2mk\pi}{n}+\frac{2}{n}\cos.\frac{2k\pi}{n}\cos.\frac{2mk\pi}{n},$$
$$B=-\frac{2}{n}\cos.\frac{2mk\pi}{n}\text{: hincque}$$
$$Ab+Ba\cos.\zeta=\frac{2}{n}\sin.\frac{2k\pi}{n}\sin.\frac{2mk\pi}{n}.$$

Ex quo integrale hinc oriundum erit =

$$-\frac{2}{n}\cos.\frac{2km\pi}{n}\,l\sqrt{\left(1-2x\cos.\frac{2k\pi}{n}+xx\right)}$$
$$+\frac{2}{n}\sin.\frac{2km\pi}{n}\text{ Arc. tang. }\frac{x\sin.\frac{2k\pi}{n}}{1-x\cos.\frac{2k\pi}{n}}:$$

ubi pro k successive omnes numeri 0, 1, 2, 3, etc. substitui debent, quamdiu $2k$ non superat n. At casu $k=0$ fit integralis pars $=-\frac{1}{n}l(1-x)$: et quando n est numerus par, ultima pars oritur ex $2k=n$, quae ergo erit

$$-\frac{2}{n}\cos.m\pi\,l\sqrt{(1+2x+xx)}=-\frac{\cos.m\pi}{n}\,l(1+x):$$

ergo si m est par, erit $\cos.m\pi=+1$, at si m impar, fit $\cos.m\pi=-1$. Quocirca integrale $\int\frac{x^{m-1}\partial x}{1-x^n}$, hoc modo exprimitur:

$$-\tfrac{1}{n}\, l\,(1-x)$$

$$-\tfrac{2}{n}\cos.\tfrac{2m\pi}{n}\, l\sqrt{(1-2x\cos.\tfrac{2\pi}{n}+xx)}$$

$$+\tfrac{2}{n}\sin.\tfrac{2m\pi}{n}\ \text{Arc. tang.}\ \frac{x\sin.\frac{2\pi}{n}}{1-x\cos.\frac{2\pi}{n}}$$

$$-\tfrac{2}{n}\cos.\tfrac{4m\pi}{n}\, l\sqrt{(1-2x\cos.\tfrac{4\pi}{n}+xx)}$$

$$+\tfrac{2}{n}\sin.\tfrac{4m\pi}{n}\ \text{Arc. tang.}\ \frac{x\sin.\frac{4\pi}{n}}{1-x\cos.\frac{4\pi}{n}}$$

$$-\tfrac{2}{n}\cos.\tfrac{6m\pi}{n}\, l\sqrt{(1-2x\cos.\tfrac{6\pi}{n}+xx)}$$

$$+\tfrac{2}{n}\sin.\tfrac{6m\pi}{n}\ \text{Arc. tang.}\ \frac{x\sin.\frac{6\pi}{n}}{1-x\cos.\frac{6\pi}{n}}$$

etc.

Corollarium.

81. Sit $m=1$, et pro n successive numeri 1, 2, 3, etc. substituantur, ut nanciscamur sequentes integrationes:

I. $$\int\frac{\partial x}{1-x}=-l\,(1-x)$$

II. $$\int\frac{\partial x}{1-xx}=-\tfrac{1}{2}l\,(1-x)+\tfrac{1}{2}l(1+x)=\tfrac{1}{2}l\,\frac{1+x}{1-x}$$

III. $$\int\frac{\partial x}{1-x^3}=\begin{cases}-\tfrac{1}{3}l(1-x)-\tfrac{2}{3}\cos.\tfrac{2}{3}\pi\, l\sqrt{(1-2x\cos.\tfrac{2}{3}\pi+xx)}\\ \qquad+\tfrac{2}{3}\sin.\tfrac{2}{3}\pi\ \text{Arc. tang.}\ \dfrac{x\sin.\frac{2}{3}\pi}{1-x\cos.\frac{2}{3}\pi}\end{cases}$$

IV. $$\int\frac{\partial x}{1-x^4}=\begin{cases}-\tfrac{1}{4}l(1-x)-\tfrac{2}{4}\cos.\tfrac{2}{4}\pi\, l\sqrt{(1-2x\cos.\tfrac{2}{4}\pi+xx)}\\ \qquad+\tfrac{2}{4}\sin.\tfrac{2}{4}\pi\ \text{Arc. tang.}\ \dfrac{x\sin.\frac{2}{4}\pi}{1-x\cos.\frac{2}{4}\pi}\\ +\tfrac{1}{4}l\,(1+x)\end{cases}$$

$$\text{V. } \int \frac{\partial x}{1-x^5} = \begin{cases} -\frac{1}{5} l(1-x) - \frac{2}{5}\cos.\frac{2}{5}\pi\, l\sqrt{(1-2x\cos.\frac{2}{5}\pi + xx)} \\ \quad + \frac{2}{5}\sin.\frac{2}{5}\pi \text{ Arc. tang. } \frac{x\sin.\frac{2}{5}\pi}{1-x\cos.\frac{2}{5}\pi} \\ \quad - \frac{2}{5}\cos.\frac{4}{5}\pi\, l\sqrt{(1-2x\cos.\frac{4}{5}\pi + xx)} \\ \quad + \frac{2}{5}\sin.\frac{4}{5}\pi \text{ Arc. tang. } \frac{x\sin.\frac{4}{5}\pi}{1-x\cos.\frac{4}{5}\pi} \end{cases}$$

$$\text{VI. } \int \frac{\partial x}{1-x^6} = \begin{cases} -\frac{1}{6} l(1-x) - \frac{2}{6}\cos.\frac{2}{6}\pi\, l\sqrt{(1-2x\cos.\frac{2}{6}\pi + xx)} \\ \quad + \frac{2}{6}\sin.\frac{2}{6}\pi \text{ Arc. tang. } \frac{x\sin.\frac{2}{6}\pi}{1-x\cos.\frac{2}{6}\pi} \\ + \frac{1}{6} l(1+x) - \frac{2}{6}\cos.\frac{4}{6}\pi\, l\sqrt{(1-2x\cos.\frac{4}{6}\pi + xx)} \\ \quad + \frac{2}{6}\sin.\frac{4}{6}\pi \text{ Arc. tang. } \frac{x\sin.\frac{4}{6}\pi}{1-x\cos.\frac{4}{6}\pi} \end{cases}$$

Exemplum 4.

82. *Proposita formula differentiali* $\frac{(x^{m-1} + x^{n-m-1})\,\partial x}{1+x^n}$, *existente* n > m − 1, *ejus integrale definire.*

Ex exemplo 2do patet, integralis partem quamcunque in genere esse, sumto i pro numero quocunque impare non majore quam n,

$$-\frac{2}{n}\cos.\frac{im\pi}{n}\, l\sqrt{\left(1-2x\cos.\frac{i\pi}{n}+xx\right)}$$

$$+\frac{2}{n}\sin.\frac{im\pi}{n}\text{ Arc. tang. }\frac{x\sin.\frac{i\pi}{n}}{1-x\cos.\frac{i\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{i(n-m)\pi}{n}\, l\sqrt{\left(1-2x\cos.\frac{i\pi}{n}+xx\right)}$$

$$+\frac{2}{n}\sin.\frac{i(n-m)\pi}{n}\text{ Arc. tang. }\frac{x\sin.\frac{i\pi}{n}}{1-x\cos.\frac{i\pi}{n}};$$

Verum est

$$\cos.\frac{i(n-m)\pi}{n} = \cos.\left(i\pi - \frac{im\pi}{n}\right) = -\cos.\frac{im\pi}{n}, \text{ et}$$

$$\sin.\frac{i(n-m)\pi}{n} = \sin.\left(i\pi - \frac{im\pi}{n}\right) = +\sin.\frac{im\pi}{n};$$

unde partes logarithmicae se destruent, eritque pars integralis in genere,

$$+\frac{4}{n}\sin.\frac{im\pi}{n}\text{ Arc. tang. }\frac{x\sin.\frac{i\pi}{n}}{1-x\cos.\frac{i\pi}{n}}.$$

Ponatur commoditatis ergo angulus $\frac{\pi}{n}=\omega$, eritque

$$\int\frac{(x^{m-1}+x^{n-m-1})\partial x}{1+x^n}=+\frac{4}{n}\sin.\ m\omega\text{ Arc. tang. }\frac{x\sin.\omega}{1-x\cos.\omega}$$
$$+\frac{4}{n}\sin.3m\omega\text{ Arc. tang. }\frac{x\sin.3\omega}{1-x\cos.3\omega}$$
$$+\frac{4}{n}\sin.5m\omega\text{ Arc. tang. }\frac{x\sin.5\omega}{1-x\cos.5\omega}$$
$$\vdots$$
$$+\frac{4}{n}\sin.im\omega\text{ Arc. tang. }\frac{x\sin.i\omega}{1-x\cos.i\omega}:$$

sumto pro i maximo numero impare, exponentem n non excedente. Si ipse numerus n sit impar, pars ex positione $i=n$ oriunda, ob $\sin.m\pi=0$, evanescet. Notetur ergo, hic totum integrale per meros angulos exprimi.

Corollarium.

83. Simili modo sequens integrale elicitur, ubi soli logarithmi relinquentur, manente $\frac{\pi}{n}=\omega$:

$$\int\frac{(x^{m-1}-x^{n-m-1})\partial x}{1+x^n}=-\frac{4}{n}\cos.\ m\omega l\sqrt{(1-2x\cos.\ \omega+xx)}$$
$$-\frac{4}{n}\cos.3m\omega l\sqrt{(1-2x\cos.3\omega+xx)}$$
$$-\frac{4}{n}\cos.5m\omega l\sqrt{(1-2x\cos.5\omega+xx)}$$
$$\vdots$$
$$-\frac{4}{n}\cos.im\omega l\sqrt{(1-2x\cos.i\omega+xx)};$$

donec scilicet numerus impar i non superet exponentem n.

Exemplum 5.

84. *Proposita formula differentiali* $\frac{(x^{m-1} - x^{n-m-1})\,\partial x}{1 - x^n}$; *existente* $n > m - 1$, *ejus integrale definire.*

Ex exemplo 3tio integralis pars quaecunque concluditur, siquidem brevitatis gratia $\frac{\pi}{n} = \omega$ statuamus:

$$\begin{aligned}
&- \tfrac{2}{n} \cos. 2 k m \omega \, l \sqrt{(1 - 2x \cos. 2k\omega + xx)} \\
&\quad + \tfrac{2}{n} \sin. 2 k m \omega \text{ Arc. tang. } \tfrac{x \sin. 2k\omega}{1 - x \cos. 2k\omega} \\
&+ \tfrac{2}{n} \cos. 2k (n - m) \omega \, l \sqrt{(1 - 2x \cos. 2k\omega + xx)} \\
&\quad - \tfrac{2}{n} \sin. 2k (n - m) \omega \text{ Arc. tang. } \tfrac{x \sin. 2k\omega}{1 - x \cos. 2k\omega}.
\end{aligned}$$

At est:

$$\cos. 2k(n-m)\omega = \cos. (2k\pi - 2km\omega) = \cos. 2km\omega, \text{ et}$$
$$\sin. 2k(n-m)\omega = \sin. (2k\pi - 2km\omega) = -\sin. 2km\omega:$$

unde ista pars generalis abit in: $\frac{4}{n} \sin. 2km\omega \text{ Arc. tang. } \frac{x \sin. 2k\omega}{1 - x \cos. 2k\omega}$. Quare hinc ista integratio colligitur:

$$\begin{aligned}
\int \frac{(x^{m-1} - x^{n-m-1})\,\partial x}{1 - x^n} = &+ \tfrac{4}{n} \sin. 2m\omega \text{ Arc. tang. } \tfrac{x \sin. 2\omega}{1 - x \cos. 2\omega} \\
&+ \tfrac{4}{n} \sin. 4m\omega \text{ Arc. tang. } \tfrac{x \sin. 4\omega}{1 - x \cos. 4\omega} \\
&+ \tfrac{4}{n} \sin. 6m\omega \text{ Arc. tang. } \tfrac{x \sin. 6\omega}{1 - x \cos. 6\omega} \\
&\text{etc.}
\end{aligned}$$

numeris paribus tamdiu ascendendo, quoad exponentem n non superent.

Corollarium.

85. Indidem etiam haec integratio absolvitur, manente $\frac{\pi}{n} = \omega$:

$$\int \frac{(x^{m-1} + x^{n-m-1})\,\partial x}{1 - x^n} = -\tfrac{2}{n}\, l\,(1 - x)$$
$$-\tfrac{4}{n}\cos.\, 2m\omega\, l\sqrt{(1 - 2x\cos.2\omega + xx)}$$
$$-\tfrac{4}{n}\cos.\, 4m\omega\, l\sqrt{(1 - 2x\cos 4\omega + xx)}$$
$$-\tfrac{4}{n}\cos.\, 6m\omega\, l\sqrt{(1 - 2x\cos.6\omega + xx)}$$
etc.

ubi etiam numeri pares non ultra terminum n sunt continuandi.

Exemplum 6.

86. *Proposita formula differentiali* $\partial y = \frac{\partial x}{x^3(1+x)(1-x^4)}$ *ejus integrale invenire.*

Functio fracta per ∂x affecta secundum denominatoris factores est $\frac{1}{x^3(1+x)^2(1-x)(1+xx)}$, quae in has fractiones simplices resolvitur:

$$\frac{1}{x^3} - \frac{1}{x^2} + \frac{1}{x} - \frac{1}{4(1+x)^2} - \frac{9}{8(1+x)} + \frac{1}{8(1-x)} + \frac{1+x}{4(1+xx)} = \frac{\partial y}{\partial x},$$

unde per integrationem elicitur:

$$y = -\frac{1}{2xx} + \frac{1}{x} + lx + \frac{1}{4(1+x)} - \tfrac{9}{8}l(1+x) - \tfrac{1}{8}l(1-x)$$
$$+ \tfrac{1}{8}l(1+xx) + \tfrac{1}{4}\,\text{Arc. tang.}\, x,$$

quae expressio in hanc formam transmutatur

$$y = C - \frac{2+2x+5xx}{4xx(1+x)} - l\frac{1+x}{x} + \tfrac{1}{8}l\frac{1+xx}{1-xx} + \tfrac{1}{4}\,\text{Arc. tang.}\, x.$$

Scholion.

87. Hoc igitur caput ita pertractare licuit, ut nihil amplius in hoc genere desiderari possit. Quoties ergo ejusmodi functio y ipsius x quaeritur, ut $\frac{\partial y}{\partial x}$ aequetur functioni rationali ipsius x, toties integratio nihil habet difficultatis, nisi forte ad denominatoris singu-

fos factores eliciendos Algebrae praecepta non sufficiant: verum tum defectus ipsi Algebrae, non vero methodo integrandi, quam hic tractamus, est tribuendus. Deinde etiam potissimum notari convenit, semper, cum $\frac{\partial y}{\partial x}$ functioni rationali ipsius x aequale ponitur, functionem y, nisi sit algebraica, alias quantitates transcendentes non involvere praeter logarithmos et angulos: ubi quidem observandum est, hic perpetuo logarithmos hyperbolicos intelligi oportere, cum ipsius lx differentiale non sit $= \frac{\partial x}{x}$, nisi logarithmus hyperbolicus sumatur: at horum reductio ad vulgares est facillima, ita ut hinc applicatio calculi ad praxin nulli impedimento sit obnoxia. Quare progrediamur ad eos casus, quibus formula $\frac{\partial y}{\partial x}$ functioni irrationali ipsius x aequatur, ubi quidem primo notandum est, quoties ista functio per idoneam substitutionem ad rationalitatem perduci poterit, casum ad hoc caput revolvi. Veluti si fuerit $\partial y = \frac{(1+\sqrt{x}-\sqrt[3]{xx})\partial x}{1+\sqrt[3]{x}}$, evidens est, ponendo $x = z^6$, unde fit $\partial x = 6z^5\partial z$, fore

$$\partial y = \frac{(1+z^3-z^4)}{1+zz} \cdot 6z^5\partial z, \text{ ideoque}$$

$$\frac{\partial y}{\partial z} = -6z^7 + 6z^6 + 6z^5 - 6z^4 + 6zz - 6 + \frac{6}{1+zz},$$

unde integrale

$$y = -\tfrac{3}{4}z^8 + \tfrac{6}{7}z^7 + z^6 - \tfrac{6}{5}z^5 + 2z^3 - 6z + 6 \text{ Arc. tang. } x,$$

et restituto valore

$$y = -\tfrac{3}{4}x\sqrt[3]{x} + \tfrac{6}{7}x\sqrt[6]{x} + x - \tfrac{6}{5}\sqrt[6]{x^5} + 2\sqrt{x} - 6\sqrt[6]{x}$$
$$+ 6 \text{ Arc. tang. } \sqrt[6]{x} + C.$$