DE MOTV CORPORIS AD DVO CENTRA VIRIVM FIXA ATTRACTI.

Auctore

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I.

Quaeſtionem hic tractare aggredior, quae ab omnibus, qui adhuc in determinatione motuum a viribus centripetis genitorum operam ſuam conſumſerunt, fruſtra tentata vires adeo Analyſeos ſuperare videbatur. Quanquam enim ſummo *Neutono* motus corporum ad vnicum virium centrum attractorum felici ſucceſſu eſt definitus, quippe quem in ſectione conica fieri demonſtrauit, ſi vis centripeta rationem duplicatam inuerſam diſtantiarum ſequatur, tamen ſtatim atque haec quaeſtio ad duo virium centra extenditur, nulla via patere eſt viſa ad ſolutionem perueniendi. Ad formulas quidem analyticas quaeſtio facile deducitur, quae cum differentialia ſecundi gradus inuoluant, primo per integrationem ad differentialia primi gradus eſt aſcendendum, quod negotium iam inſignibus difficultatibus implicatur; atque his etiam ſuperatis, tum demum

demum quantitatum variabilium multitudo et complicatio vix vllam ſpem ad earum ſeparationem pertingendi relinquit, vnde lineam curuam a corpore deſcriptam aſſignare liceret.

2. Cum autem ſemper cuiusquam problematis, quod a ſummis ingeniis fruſtra eſt tentatum, ſolutio maximi eſt momenti, tum vero haec quaeſtio ad eam Analyſeos partem pertinet, ex qua ſola nunc quidem omnia Aſtronomiae incrementa ſunt expectanda; quam ob cauſam, ſolutionem iſtius quaeſtionis eo confidentius iactare non dubito, quod ad eam nonniſi grauiſſimis impedimentis remotis tandem mihi quidem penetrare contigerit. Minime etiam eſt dubitandum, quin artificia analytica, quibus in hoc negotio ſum vſus, ad enodationem aliorum problematum huius generis, plurimum lucis ſint allatura.

3. Conſidero igitur duo centra virium, quorum vtrumque attrahat in ratione duplicata diſtantiae reciproca, quandoquidem aliae virium attrahentium hypotheſes omni vſu deſtituuntur: ac ſi corpus quodpiam initio vtcunque fuerit proiectum, eius motum ſecuturum viamque in qua incedet, inueſtigare conſtitui. Hic quidem primum obſeruo ſi motus directio ſemel fuerit cum binis virium centris in eodem plano, vniuerſum motum, in eodem plano abſolutum iri. Atque hunc caſum iam

 aliquot

aliquot ab hinc annis ita expediui, vt non solum pro curua a corpore descripta constructionem ope quadraturae curuae satis simplicis exhibuerim, verum etiam innumerabiles casus elicuerim, quibus haec curua adeo algebraica euadat. Nunc igitur vt huic quaestioni plene satisfaciam, inuestigationem meam quoque ad eos casus extendam, quibus corporis motus non in eodem plano cum binis centris virium absoluitur; vbi quidem nouae eaeque maximae difficultates successui aduersantur, quas superari oportet.

Status Quaestionis.

Tab. III. Fig. 1. 4. Sint igitur A et B centra virium, seu duo corpora in his punctis fixa, quae attrahant in ratione reciproca duplicata distantiarum secundum massas, quae iisdem litteris A et B denotentur; Statuaturque distantia $AB=a$.

Tum vero vtcunque aliud quoddam corpus initio fuerit proiectum, id nunc quidem elapso tempore $=\tau$ versetur in Z vnde ad planum quodpiam fixum per puncta A et B ductum demittatur perpendiculum ZY, ductaque ex Y ad rectam AB normali YX, vocentur lineae:

$$ZY=z;\ YX=y;\ AX=x;\ BX=t$$

ita vt sit $t+x=a$, ideoque $dt+dx=0$.

Ad

Ad motus igitur cognitionem requiritur, vt ad quoduis tempus hae quantitates determinari queant.

5. Vocentur porro diſtantiae $AZ = v$ et $BZ = u$ vt ſit $vv = xx + yy + zz$ et $uu = tt + yy + zz$. Iam quia corpus Z vrgetur ad centra virium A et B viribus $\frac{A}{vv}$ et $\frac{B}{uu}$; hinc motus principia ſequentes ſuppeditant formulas differentio-differentiales ſumto temporis elemento $d\tau$ conſtante:

I. $\frac{ddx}{d\tau^2} = -\frac{Ax}{v^3} + \frac{Bt}{u^3}$ ſeu $\frac{ddt}{d\tau^2} = +\frac{Ax}{v^3} - \frac{Bt}{u^3}$

II. $\frac{ddy}{d\tau^2} = -\frac{Ay}{v^3} - \frac{By}{u^3}$ et

III. $\frac{ddz}{d\tau^2} = -\frac{Az}{v^3} - \frac{Bz}{u^3}$,

ex quibus tribus aequationibus omnia motus phaenomena ſunt deducenda. Nihil autem inde concludere licet niſi ante omnia totidem aequationes differentiales primi gradus inde per integrationem eliciantur.

Inueſtigatio aequationum differentialium primi gradus motum corporis quaeſitum determinantium.

6. Ex aequationibus quidem II et III ſtatim vna aequatio differentialis primi gradus obtinetur; illa enim per z haec vero per y multiplicata, differentia dat:

$\frac{yddz - zddy}{d\tau^2} = 0$, vnde colligitur $d\tau = \alpha(ydz - zdy)$, ex qua discimus, si motus corporis in planum ad rectam AB normale proiiciatur, areas in hoc plano circa punctum quo a recta AB traiicitur descriptas esse tempori proportionales; quod quidem ex prima indole motuum a viribus centripetis oriundorum per se est manifestum.

7. Alia etiam aequatio integralis haud difficulter eruitur multiplicando aequationem I per $dx = -dt$; II per dy, et III per dz; tum enim his aequationibus in vnam summam collectis, ob

$$x\,dx + y\,dy + z\,dz = v\,dv \text{ et } t\,dt + y\,dy + z\,dz = u\,du$$

habebimus hanc aequationem:

$$\frac{dx\,ddx + dy\,ddy + dz\,ddz}{d\tau^2} = \frac{-A\,dv}{vv} - \frac{B\,du}{uu}$$

quae integrata praebet:

$$\frac{dx^2 + dy^2 + dz^2}{2\,d\tau^2} = \frac{A}{v} + \frac{B}{u} + \frac{C}{a}$$

ita vt iam duas aequationes differentiales primi gradus simus adepti; tertia autem qua adhuc indigemus, maiorem sagacitatem postulat.

8. Ex prima, quatenus duplici forma exhibetur, et secunda geminas aequationes elicimus istas:

$$\frac{x\,ddy - y\,ddx}{d\tau^2} = \frac{-By(x+f)}{u^3} = \frac{-Bay}{u^3} \text{ et}$$
$$\frac{t\,ddy - y\,ddt}{d\tau^2} = \frac{-Ay(x+f)}{u^3} = \frac{-Aay}{u^3}$$

simili

ſimili modo prima aequatio cum tertia combinata ſequentes nobis ſuppeditat:

$$\frac{x\,ddz - z\,ddx}{d\tau^2} = \frac{-Bz(x+t)}{u^3} = \frac{-Baz}{u^3}$$

$$\frac{t\,ddz - z\,ddt}{d\tau^2} = \frac{-Az(x+t)}{v^3} = \frac{-Aaz}{v^3}$$

vnde quidem parum lucri redundare videtur. Has autem quatuor aequationes ſequenti modo repraeſentari plurimum intererit:

$$\text{I. } \frac{d.(xdy - ydx)}{d\tau^2} = \frac{-Bay}{u^3};\quad \text{III. } \frac{d.(xdz - zdx)}{d\tau^2} = \frac{-Baz}{u^3}$$

$$\text{II. } \frac{d.(tdy - ydt)}{d\tau^2} = \frac{-Aay}{v^3};\quad \text{IV. } \frac{d.(tdz - zdt)}{d\tau^2} = \frac{-Aaz}{v^3}.$$

9. Ex harum prima et ſecunda formemus aequationem iſtam:

$$\frac{(tdy-ydt)d.(xdy-ydx)}{d\tau^2} + \frac{(xdy-ydx)d.(tdy-ydt)}{d\tau^2} = \frac{-Aay(xdy-ydx)}{v^3} - \frac{Bay(tdy-ydt)}{u^3}$$

ſimilique modo ex tertia et quarta hanc:

$$\frac{(tdz-zdt)d.(xdz-zdx)}{d\tau^2} + \frac{(xdz-zdx)d.(tdz-zdt)}{d\tau^2} = \frac{-Aaz(xdz-zdx)}{v^3} - \frac{Baz(tdz-zdt)}{u^3}$$

vbi prius membrum vtriusque aequationis eſt integrabile, ibi ſcilicet integrale eſt $= \frac{(xdy-ydx)(tdy-ydt)}{d\tau^2}$, hic vero $= \frac{(xdz-zdx)(tdy-ydt)}{d\tau^2}$; poſteriora vero membra in neutra integrationem admittunt. At vero hic commode vſu venit, vt ambo poſteriora membra in vnam ſummam collecta integrationem admittant.

10. Cum enim ſit:

$$d.\frac{x}{v} = d.\frac{x}{\surd(xx+yy+zz)} = \frac{y(ydx - xdy) + z(zdx - xdz)}{v^3} \text{ et}$$

$$d.\frac{t}{u} = d.\frac{t}{\surd(tt+yy+zz)} = \frac{y(ydt - tdy) + z(zdt - tdz)}{u^3}$$

 euidens

euidens est, illorum posteriorum membrorum iunctim sumtorum integrale fore $= \frac{A\alpha x}{v} + \frac{B\alpha t}{u} +$ Const. Quocirca hinc sequentem aequationem integratam nanciscimur:

$$\frac{(xdy - ydx)(tdy - ydt) + (xdz - zdx)(tdz - zdt)}{d\tau^2} = \frac{A\alpha x}{v} + \frac{B\alpha t}{u} + D\alpha.$$

11. En ergo tres aequationes differentiales primi gradus, quibus motus corporis quaesitus continetur, scilicet:

$$\text{I.}\quad d\tau = \alpha(ydz - zdy)$$

$$\text{II.}\quad \frac{dx^2 + dy^2 + dz^2}{2d\tau^2} = \frac{A}{v} + \frac{B}{u} + \frac{C}{a}$$

$$\text{III.}\quad \frac{(xdy - ydx)(tdy - ydt) + (xdz - zdx)(tdz - zdt)}{d\tau^2} = \alpha\left(\frac{Ax}{v} + \frac{Bt}{u} + D\right)$$

in quas tres nouae constantes arbitrariae sunt ingressae. Hinc autem iam facile tempus τ exterminatur, quo facto via a corpore descripta binis sequentibus aequationibus determinatur:

$$\frac{dx^2 + dy^2 + dz^2}{2\alpha\alpha(ydz - zdy)^2} = \frac{A}{v} + \frac{B}{u} + \frac{C}{a}$$

$$\frac{(xdy - ydx)(tdy - ydt) + (xdz - zdx)(tdz - zdt)}{\alpha\alpha(ydz - zdy)^2} = \alpha\left(\frac{Ax}{v} + \frac{Bt}{u} + D\right)$$

ad quarum resolutionem totum negotium est perductum.

12. In his aequationibus ternae adhuc insunt variabiles x, y et z, dum reliquae t, v et u per eas dantur; quae autem tantopere sunt inter se permixtae, vt nulla methodus eas resoluendi tentari queat. Omnino autem necesse est, vt vna variabili elisa, ex his binis aequationibus vna eliciatur

duas

duas tantum variabiles inuoluens; ad hoc autem vtique idonea transformatio requiritur, cum satis sit perspicuum nullam harum trium x, y, z eliminari posse, praeter quam quod nulla foret ratio, cur vna potius quam reliquae ad hoc eligeretur.

Reductio aequationum inuentarum ad vnicam binas tantum variabiles continentem.

13. Ad hunc scopum perueniemus, si loco variabilium y et z in calculum introducamus rectam ZX a puncto Z ad rectam AB normaliter ductam vna cum angulo ZXY, quem haec recta cum plano assumto ABY facit. Sit ergo

recta $ZX = w$ et angulus $ZXY = \Phi$

erit $z = w \sin. \Phi$ et $y = w \cos. \Phi$, tum ob $yy + zz = ww$ habebitur $v = \sqrt{(xx + ww)}$ et $u = \sqrt{(tt + ww)}$.

Porro vero ob $dz = dw \sin. \Phi + w d\Phi \cos. \Phi$ et $dy = dw \cos. \Phi - w d\Phi \sin. \Phi$ hinc colligimus $y dz - z dy = ww\, d\Phi$, ita vt iam prodeat elementum temporis $d\tau = \alpha ww d\Phi$.

14. Deinde vero cum sit $dy^2 + dz^2 = dw^2 + ww d\Phi^2$, prior aequatio induit hanc formam:

$$\frac{dx^2 + dw^2 + ww\, d\Phi^2}{2\alpha\alpha\, w^4\, d\Phi^2} = \frac{A}{v} + \frac{B}{u} + \frac{C}{a}.$$

Pro altera vero aequatione euoluatur primo numerator membri antecedentis, et quia fit:

$(x dy$

$$(xdy-ydx)(tdy-ydt)=txdy^2-xydtdy-tydxdy+yydtdx$$
$$(xdz-zdx)(tdz-zdt)=txdz^2-xzdtdz-tzdxdz+zzdtdx$$

ob $dy^2+dz^2=dw^2+wwd\Phi^2$; $ydy+zdz=wdw$ et $yy+zz=ww$ erit hic numerator

$$tx(dw^2+wwd\Phi^2)-(xdt+tdx)wdw+wwdtdx$$

ex quo altera aequatio transformabitur in hanc:

$$\frac{txdw^2+txwwd\Phi^2-(xdt+tdx)wdw+wwdtdx}{\alpha\alpha w^4 d\Phi^2}=a\left(\frac{Ax}{v}+\frac{Bt}{u}+D\right).$$

15. Iam ex binis hisce aequationibus facile eliminatur elementum $d\Phi^2$, siquidem hac substitutione id sumus lucrati vt ipse angulus Φ excesserit. Prior autem aequatio dat:

$$d\Phi^2=(dx^2+dw^2):\left(2\alpha\alpha w^4\left(\frac{A}{v}+\frac{B}{u}+\frac{C}{a}\right)-ww\right)$$

posterior vero:

$$d\Phi^2=\frac{(txdw^2-(xdt+tdx)wdw+wwdtdx)}{\alpha\alpha a w^4\left(\frac{Ax}{v}+\frac{Bt}{u}+D\right)-txww}$$

vbi pro analogia obseruanda notetur ob $dt=-dx$ esse $dx^2=-dtdx$. His igitur valoribus inter se aequatis resultabit aequatio binas tantum variabiles x seu t et w inuoluens.

16. Ad calculum contrahendum statuamus:

$$\frac{A}{v}+\frac{B}{u}+\frac{C}{a}=\frac{P}{a} \quad \text{et} \quad \frac{Ax}{v}+\frac{Bt}{u}+D=Q$$

atque reperimus:

$$(-dxdt+dw^2)(\alpha\alpha aQww-tx)=$$
$$(txdw^2-(xdt+tdx)wdw+wwdtdx)\left(\frac{2\alpha\alpha}{a}Pww-1\right)$$

vnde

vnde euoluendo concludimus:

$$dt\,dx\left(\tfrac{2\alpha\alpha}{a}Pw^4 + \alpha\alpha a Q ww - tx - ww\right)$$
$$-(x\,dt + t\,dx)w\,dw\left(\tfrac{2\alpha\alpha}{a}Pww - 1\right)$$
$$+ ww\,dw^2\left(\tfrac{2\alpha\alpha}{a}Ptx - \alpha\alpha a Q\right) = 0$$

quae aequatio adhuc ingentibus difficultatibus laborat cum ob variabilium confusam permixtionem, tum vero ob valores irrationales quantitatum v et u.

Transformatio aequationis modo inuentae.

17. Cum introductio angulorum saepenumero calculos mirifice contrahere soleat, idem hic faciamus, vocemusque angulos:

$BAZ = \eta$ et $ABZ = \theta$, eritque hinc

$$AX = x = w\cot.\eta \text{ et } BX = t = w\cot.\theta$$

vnde ob $x + t = a$ deducimus statim:

$$w = \frac{a}{\cot.\eta + \cot.\theta} = \frac{a\sin.\eta\sin.\theta}{\sin.(\eta+\theta)} \text{ et differentiando}$$

$$dw = a\left(\frac{d\eta}{\sin.\eta^2} + \frac{d\theta}{\sin.\theta^2}\right) : (\cot.\eta + \cot.\theta)^2 = \frac{a(d\eta\sin.\theta^2 + d\theta\sin.\eta^2)}{\sin.(\eta+\theta)^2}$$

Deinde oritur $x = \frac{a\cot.\eta}{\cot.\eta + \cot.\theta} = \frac{a\cos.\eta\sin.\theta}{\sin.(\eta+\theta)}$ hincque

$$dx = a\left(\frac{d\theta\cot.\eta}{\sin.\theta^2} - \frac{d\eta\cot.\theta}{\sin.\eta^2}\right) : (\cot.\eta + \cot.\theta)^2 = \frac{a(d\theta\sin.\eta\cos.\eta - d\eta\sin.\theta\cos.\theta)}{\sin.(\eta+\theta)^2}$$

et $dx^2 + dw^2 = \frac{aa(d\eta^2\sin.\theta^2 + d\theta^2\sin.\eta^2 - 2d\eta\,d\theta\sin.\eta\sin.\theta\cos.(\eta+\theta))}{\sin.(\eta+\theta)^4}$

pro priore valore ipsius $d\Phi^2$.

18. Pro altero valore notetur esse numeratorem:

$$(t\,dw - w\,dt)(x\,dw - w\,dx)$$

 cuius

cuius valor facillime elicitur ex formulis:

$$\frac{t}{w} = \cot.\theta \quad \text{et} \quad \frac{x}{w} = \cot.\eta \text{ vnde fit}$$

$$\frac{t\,dw - w\,dt}{ww} = \frac{d\theta}{\sin.\theta^2} \quad \text{et} \quad \frac{x\,dw - w\,dx}{ww} = \frac{d\eta}{\sin.\eta^2}$$

sicque numerator ille fit:

$$\frac{w^4\,d\eta\,d\theta}{\sin.\eta^2\sin.\theta^2} = \frac{a^4\,d\eta\,d\theta\sin.\eta^2\sin.\theta^2}{\sin.(\eta+\theta)^4}.$$

Porro autem est $v = \frac{w}{\sin.\eta} = \frac{a\sin.\theta}{\sin.(\eta+\theta)}$ et $u = \frac{a\sin.\eta}{\sin.(\eta+\theta)}$.

Ergo $\frac{A}{v} + \frac{B}{u} + \frac{C}{a} = \frac{A\sin.(\eta+\theta)}{a\sin.\theta} + \frac{B\sin.(\eta+\theta)}{a\sin.\eta} + \frac{C}{a}$

et $\frac{Ax}{v} + \frac{Bt}{u} + D = A\cos.\eta + B\cos.\theta + D.$

19. Sit insuper $aa = \frac{m}{a}$, eritque pro priore valore ipsius $d\Phi^2$ denominator $= 2maa\left(\frac{A\sin.(\eta+\theta)}{\sin.\theta} + \frac{B\sin.(\eta+\theta)}{\sin.\eta} + C\right)\frac{\sin.\eta^4\sin.\theta^4}{\sin.(\eta+\theta)^4} - \frac{aa\sin.\eta^2\sin.\theta^2}{\sin.(\eta+\theta)^2}$

et pro posteriori $= m(A\cos.\eta + B\cos.\theta + D)\frac{a^4\sin.\eta^4\sin.\theta^4}{\sin.(\eta+\theta)^4} - \frac{a^4\sin.\eta^3\sin.\theta^3\cos.\eta\cos.\theta}{\sin.(\eta+\theta)^4}$.

Quocirca ambo valores ita se habebunt:

$$d\Phi^2 = \frac{d\eta^2\sin.\theta^2 + d\theta^2\sin.\eta^2 - 2d\eta\,d\theta\sin.\eta\cos.\theta\cos.(\eta+\theta)}{2m\sin.\eta^2\sin.\theta^3(A\sin.\eta\sin.(\eta+\theta) + B\sin.\theta\sin.(\eta+\theta) + C\sin.\eta\sin.\theta) - \sin.\eta^2\sin.\theta^2\sin.(\eta+\theta)^2}$$

$$d\Phi^2 = \frac{d\eta\,d\theta}{m\sin.\eta^2\sin.\theta^2(A\cos.\eta + B\cos.\theta + D) - \sin.\eta\sin.\theta\cos.\eta\cos.\theta}$$

His binis valoribus coaequatis, aequatio oritur primo quidem satis intricata, quae autem notis angulorum proprietatibus in subsidium vocatis, ad sequentem formam reducitur:

$$d\eta^2\sin.\theta^2 + d\theta^2\sin.\eta^2 = \frac{d\eta\,d\theta\sin.\eta\sin.\theta(2m(A\cos.\theta + B\cos.\eta + C\sin.\eta\sin.\theta + D\cos.(\eta+\theta)) - \cos.\eta^2 - \cos.\theta^2)}{m\sin.\eta\sin.\theta(A\cos.\eta + B\cos.\theta + D) - \cos.\eta\cos.\theta}$$

20. Parum equidem hac substitutione profecisse videor, cum in hac aequatione non solum binae varia-

variabiles η et θ maxime sint inter se permixtae, sed etiam adhuc necesse sit per radicis extractionem ipsam differentialium $d\eta$ et $d\theta$ rationem elicere, quo quidem ad formulam irrationalem maxime intricatam peruenitur. Quo minus ergo resolutio huius aequationis speranda videtur, eo magis mirari oportet sequentem substitutionem, cuius beneficio adeo ad aequationem, in qua variabiles admittunt separationem deducimur.

Constructio huius aequationis per separationem variabilium.

21. Singulari prorsus substitutione inopinato hunc scopum sum assecutus, posui enim tang. $\frac{1}{2}\eta = \sqrt{pq}$ et tang. $\frac{1}{2}\theta = \frac{\sqrt{p}}{\sqrt{q}}$, vnde sequentes promanant determinationes:

$$\sin.\eta = \frac{2\sqrt{pq}}{1+pq};\quad \cos.\eta = \frac{1-pq}{1+pq};\quad \sin.\theta = \frac{2\sqrt{pq}}{p+q};\quad \cos.\theta = \frac{q-p}{p+q}$$

$$\sin.(\eta+\theta) = \frac{2(1-p)(1+q)\sqrt{pq}}{(p+q)(1+pq)};\quad \cos.(\eta+\theta) = \frac{(q-p)(1-pq)+4pq}{(p+q)(1+pq)}$$

$$x = \frac{a(1-pq)}{(1-p)(1+q)};\quad t = \frac{a(q-p)}{(1-p)(1+q)};\quad w = \frac{2a\sqrt{pq}}{(1-p)(1+q)};$$

$$v = \frac{a(1+pq)}{(1-p)(1+q)};\quad u = \frac{a(p+q)}{(1-p)(1+q)};$$

hincque differentiando elicimus:

$$d\eta = \frac{p\,dq + q\,dp}{(1+pq)\sqrt{pq}};\quad d\theta = \frac{q\,dp - p\,dq}{(p+q)\sqrt{pq}}$$

$$d\eta\,\sin.\theta = \frac{2(p\,dq + q\,dp)}{(p+q)(1+pq)};\quad d\theta\,\sin.\eta = \frac{2(q\,dp - p\,dq)}{(p+q)(1+pq)}$$

$$d\eta^2\,\sin.\theta^2 + d\theta^2\,\sin.\eta^2 = \frac{4(qq\,dp^2 + pp\,dq^2)}{(p+q)^2(1+pq)^2};\quad d\eta\,d\theta\,\sin.\eta\,\sin.\theta = \frac{4(qq\,dp^2 - pp\,dq^2)}{(p-q)^2(1+pq)^2}$$

 22. Si

22. Si iam has expressiones in binis pro $d\Phi^2$ inuentis valoribus substituamus, reperiemus:

$$\text{I. } d\Phi^2 = \frac{(p+q)(1+pq)}{4ppqq} \cdot \frac{q\,dp^2(1+q)^2 + p\,dq^2(1-p)^2}{8mpq\left(\frac{A(1-p)(1+q)}{1+pq} + \frac{B(1-p)(1+q)}{p+q} + C\right) - (1-p)^2(1+q)^2}$$

$$\text{II. } d\Phi^2 = \frac{(p+q)(1+pq)}{4ppqq} \cdot \frac{qq\,dp^2 - pp\,dq^2}{4mpq\left(\frac{A(1-pq)}{1+pq} + \frac{B(q-p)}{p+q} + D\right) - (q-p)(1-pq)}$$

quibus inuicem coaequatis per operosos admodum ac taediosos calculos tandem adipiscimur hanc aequationem:

$$dp^2(4m(B-A)q(1-qq) + 8mCqq - 4mDq(1+q)^2 - (1-qq)^2) = dq^2(4m(A+B)p(1-pp) + 8mCpp + 4mDp(1-p)^2 - (1-pp)^2)$$

in qua ambae variabiles p et q manifesto a se inuicem separari possunt.

23. Cum autem huc per tantas ambages peruenerim, dubium est nullum, quin via planior eodem perueniendo pateat, quam quidem sine hac praeuia euolutione vix suspicari licuisset. Nunc autem cum substitutiones hic in vsum vocatae praebeant $v+u = \frac{a(1+p)}{1-p}$ et $v-u = \frac{a(1-q)}{1+q}$, facile colligitur, si $v+u$ et $v-u$ pro binis nouis variabilibus introducantur, inde tandem aequationem, in qua variabiles a se inuicem sint separatae, resultare debere. Cum igitur pulchrius sit vnica substitutione hunc scopum attingere, iam totum negotium sequenti modo sum expediturus.

Metho-

X 2

Methodus ſuccinctior ad aequationem ſeparabilem perueniendi.

24. Poſito igitur ſtatim $\alpha\alpha = \frac{m}{a}$, et introducendo rectam $ZX = w$ cum angulo $YXZ = \Phi$, vt ſit $y = w \cos.\Phi$ et $z = w \sin.\Phi$, habemus primo elementum temporis $d\tau = ww\,d\Phi\sqrt{\frac{m}{a}}$; et aequationes reſoluendae erunt:

I. $dx^2 + dw^2 + ww\,d\Phi^2 = \frac{2m}{aa} w^4 d\Phi^2 \left(\frac{A\alpha}{v} + \frac{B\beta}{u} + C\right)$

II. $(x\,dw - w\,dx)(t\,dw - w\,dt) + txww\,d\Phi^2 = m w^4 d\Phi^2 \left(\frac{A\alpha}{v} + \frac{B\beta}{u} + D\right)$.

Nunc ergo ſequenti ſubſtitutione vtamur:

$$v = \tfrac{1}{2}a(r + s) \quad \text{et} \quad u = \tfrac{1}{2}a(r - s)$$

vt ſit $v + u = ar$ et $v - u = as$

indeque colligimus:

$$x = \frac{aa + vv - uu}{2a} = \tfrac{1}{2}a(1 + rs) \text{ et } t = \tfrac{1}{2}a(1 - rs)$$

atque $w = \sqrt{(vv - xx)} = \tfrac{1}{2}a\sqrt{(rr - 1)(1 - ss)}$.

25. Differentialibus ergo ſumendis reperimus:

$$dx = \tfrac{1}{2}a(r\,ds + s\,dr); \quad dt = -\tfrac{1}{2}a(r\,ds + s\,dr)$$

atque $dw = \frac{a(r\,dr(1 - ss) - s\,ds(rr - 1))}{2\sqrt{(rr - 1)(1 - ss)}}$

vnde fit:

$$dx^2 + dw^2 = \frac{aa(rr - ss)(dr^2(1 - ss) + ds^2(rr - 1))}{4(rr - 1)(1 - ss)}$$

ſeu $\frac{dx^2 + dw^2}{ww} = \frac{(rr - ss)(dr^2(1 - ss) + ds^2(rr - 1))}{(rr - 1)^2(1 - ss)^2}$

 Tum

Tum vero:

$$x\,dw - w\,dx = \frac{aa(r+s)(dr(1-ss)+ds(rr-1))}{4\sqrt{(rr-1)(1-ss)}}$$

$$t\,dw - w\,dt = \frac{aa(r-s)(dr(1-ss)+ds(rr-1))}{4\sqrt{(rr-1)(1-ss)}} \text{ hincque}$$

$$\frac{(x\,dw - w\,dx)(t\,dw - w\,dt)}{ww} = \frac{aa(rr-ss)(dr^2(1-ss)^2 - ds^2(rr-1)^2)}{4(rr-1)^2(1-ss)^2}$$

26. His valoribus substitutis nostrae aequationes in sequentes transformabuntur:

$$\text{I. } \frac{(rr-ss)(dr^2(1-ss)+ds^2(rr-1))}{(rr-1)^2(1-ss)^2} + d\Phi^2 = \frac{2m}{aa}w^2\,d\Phi^2\left(\frac{2A}{r+s}+\frac{2B}{r-s}+C\right)$$

$$\text{II. } \frac{aa(rr-ss)(dr^2(1-ss)^2-ds^2(rr-1)^2)}{4(rr-1)^2(1-ss)^2} + \tfrac{1}{4}aa(1-rrss)\,d\Phi^2 = m w^2\,d\Phi^2\left(\frac{A(1+rs)}{r+s}+\frac{B(1-rs)}{r-s}+D\right)$$

seu vtramque adhuc per $ww = \frac{1}{4}aa(rr-1)(1-ss)$ diuidendo:

$$\text{I. } \frac{2(rr-ss)(dr^2(1-ss)+ds^2(rr-1))}{(rr-1)^3(1-ss)^3} = m\,d\Phi^2\left(\frac{2A}{r+s}+\frac{2B}{r-s}+C\right) - \frac{2\,d\Phi^2}{(rr-1)(1-ss)}$$

$$\text{II. } \frac{(rr-ss)(dr^2(1-ss)^2-ds^2(rr-1)^2)}{(rr-1)^3(1-ss)^3} = m\,d\Phi^2\left(\frac{A(1+rs)}{r+s}+\frac{B(1-rs)}{r-s}+D\right) - \frac{(1-rrss)\,d\Phi^2}{(rr-1)(1-ss)}.$$

27. Hinc iam vtriusque differentialis dr et ds ratio ad $d\Phi$ definiri potest; scilicet

haec combinatio I. $(rr-1)$ + II. 2 dat

$$\frac{2(rr-ss)^2(1-ss)dr^2}{(rr-1)^3(1-ss)^3} = m\,d\Phi^2(2Ar+2Br+C(rr-1)+2D) - \frac{2rr\,d\Phi^2}{rr-1}$$

haec vero I. $(1-ss)$ + II. 2 dat

$$\frac{2(rr-ss)^2(rr-1)ds^2}{(rr-1)^3(1-ss)^3} = m\,d\Phi^2(-2As+2Bs+C(1-ss)-2D) - \frac{2ss\,d\Phi^2}{1-ss}.$$

Nunc illam per hanc diuidendo efficitur:

$$\frac{(1-ss)dr^2}{(rr-1)ds^2} = \frac{2m(B+A)r+mC(rr-1)+2mD-\frac{2rr}{rr-1}}{2m(B-A)s+mC(1-ss)-2mD-\frac{2ss}{1-ss}}$$

Consequenter aequatio separata ita erit comparata:

$$\frac{dr}{\sqrt{m(rr-1)(2(B+A)r+C(rr-1)+2D)-2rr}}=\frac{ds}{\sqrt{m(1-ss)(2(B-A)s+C(1-ss)-2D)-2ss}}$$

vnde per quadraturas relatio inter binas variabiles r et s definiri potest.

28. Statuamus breuitatis gratia:

$$\sqrt{(m(rr-1)((B+A)r+\tfrac{1}{2}C(rr-1)+D)-rr)}=R$$
$$\sqrt{(m(1-ss)((B-A)s+\tfrac{1}{2}C(1-ss)-D)-ss)}=S$$

vt aequatio nostra separata sit; $\frac{dr}{R}=\frac{ds}{S}$, et binae praecedentes aequationes fient:

$$\frac{(rr-ss)^2\,dr^2}{(rr-1)^2(1-ss)^2}=RR\,d\Phi^2 \text{ seu } d\Phi=\frac{(rr-ss)\,dr}{(rr-1)(1-ss)R}$$
$$\frac{(rr-ss)^2\,ds^2}{(rr-1)^2(1-ss)^2}=SS\,d\Phi^2 \text{ seu } d\Phi=\frac{(rr-ss)\,ds}{(rr-1)(1-ss)S},$$

hincque elementum temporis ob

$$aww=\tfrac{1}{4}a\sqrt{am}.(rr-1)(1-ss)$$

ita exprimetur:

$$d\tau=\tfrac{1}{4}a\sqrt{ma}.\frac{(rr-ss)dr}{R} \text{ vel } d\tau=\tfrac{1}{4}a\sqrt{ma}.\frac{(rr-ss)ds}{S}$$

29. Vt etiam angulus Φ per formulas simpliciter integrales exhiberi possit, binarum formularum differentialium pro $d\Phi$ inuentarum prior multiplicetur per $\frac{1-ss}{rr-ss}$, posterior vero per $\frac{rr-1}{rr-ss}$, et productorum summa dabit:

$$d\Phi=\frac{dr}{(rr-1)R}+\frac{ds}{(1-ss)S} \text{ ideoque } \Phi=\int\frac{dr}{(rr-1)R}+\int\frac{ds}{(1-ss)S}$$

simili modo ob $\frac{dr}{R}=\frac{ds}{S}$ pro tempore fiet:

$$d\tau=\frac{a\sqrt{ma}}{4}\left(\frac{rr\,dr}{R}-\frac{ss\,ds}{S}\right) \text{ ideoque } \tau=\frac{a\sqrt{ma}}{4}\left(\int\frac{rr\,dr}{R}-\int\frac{ss\,ds}{S}\right)$$

quae

quae est problematis solutio aeque simplex ac perfecta, cum hinc ad quoduis tempus τ ex constructa aequatione $\int\frac{dr}{R} = \int\frac{ds}{S}$ non solum vtraque variabilis r et s, indeque distantiae v et u sed etiam angulus Φ assignari possint.

30. Eo maiore autem haec solutio attentione digna est, quod non solum motum corporis in eodem plano, quem casum equidem iam ante aliquot annos euolui, complectatur, sed etiam ad omnes plane motus, qui quidem in corpus pro motus ipsi initio impressi ratione cadere possunt extendatur. Quare quo vis huius solutionis clarius perspiciatur, primo quidem eam ad casum, quo motus corporis in eodem plano absoluitur, in quo bina virium centra sunt sita, accommodabo; vbi simul insignia quaedam huius motus phaenomena annotabo, dum fieri potest, vt corpus adeo in sectione conica, circa bina virium centra tanquam focos descripta moveatur, de quo quidem casu non dubito, quin iam ab aliis sit obseruatus. Deinde vero imprimis docebo simile phaenomenon etiam in motu non ad idem planum relato, contingere posse, vt corpus in superficie sphaeroidis elliptici vel conoidis hyperbolici circumferatur.

I. Casus, quo corpus in eodem plano motum suum peragit.

31. Eiusmodi scilicet hic planum intelligitur, in quo simul ambo virium centra A et B sint sita. Cum igitur angulus Φ sit constans, ideoque nihilo aequalis assumi possit, vt motus in ipso tabulae plano fieri concipiatur, sitque perpendiculum ZY $= z = 0$; ob $d\Phi = 0$ necesse est vt tam R quam S fiat quantitas infinita. Ne autem simul temporis elementum $d\tau$ euanescat, manifestum est quantitatem m infinitam statui debere. Si enim altera constantium C et D infinita sumeretur, celeritas corporis fieret infinite magna, quem casum vtique hinc excludi conuenit.

32. Hunc ergo casum adipiscimur ponendo $m = \infty$ tum vero posito ad abbreuiandum:

$$\surd(rr-1)((B+A)r+\tfrac{1}{2}C(rr-1)+D)=P$$
$$\surd(1-ss)((B-A)s+\tfrac{1}{2}C(1-ss)-D)=Q$$

erit aequatio nostra principalis $\int\frac{dr}{P}=\int\frac{ds}{Q}$, qua natura curuae a corpore descriptae determinatur, et quae non discrepat ab ea, quam iam pridem inueneram. Pro ipso autem motu cognoscendo prodit haec temporis determinatio; $\tau=\frac{a\surd a}{4}(\int\frac{rr\,dr}{P}-\int\frac{ss\,ds}{Q})$: quam quidem formulam tum temporis per plures ambages eram consecutus. Ibi quidem innumerabiles casus, quibus curua adeo fit algebraica, assignavi, quibus hic idcirco non immoror, verum hic casus multo simpliciores expendam, quibus corpus adeo vel in ellipsi vel hyperbola promouetur.

De motu corporis in ellipsi.

33. Cum pertigerimus ad hanc aequationem $\frac{dr}{P} = \frac{ds}{Q}$ seu $Qdr - Pds = 0$, euidens est huic aequationi satisfieri si sit r eiusmodi quantitas constans, quae simul reddat $P = 0$. Cum autem P binas constantes arbitrarias C et D contineat, pro lubitu ipsi r valor constans puta $r = n$ tribui potest, indeque statui oportet:

$$(nn-1)((B+A)n + \tfrac{1}{2}C(nn-1) + D) = 0$$

vnde constans D ita definitur, vt sit:

$$D = -(B+A)n - \tfrac{1}{2}C(nn-1)$$

altera autem C adhuc indeterminata manere videtur, cum tamen hoc casu, quo curua per aequationem $r = n$ seu $v + u = na$ determinatur, etiam in motu nihil indeterminati relinqui possit.

34. Iam dudum autem obseruaui, huiusmodi aequatione differentiali $Qdr - Mds(r-n)^{\lambda} = 0$, proposita, valorem $r = n$ minime pro integrale haberi posse, nisi exponens λ sit vnitati aequalis vel ea maior. Quare vt nostro casu valor $r = n$ locum habeat, non sufficit vt formula

$(rr-1)$

$$(rr-1)((B+A)r+\tfrac{1}{2}C(rr-1)+D)$$

factorem habeat $r-n$ quia inde quantitatis P factor tantum esset $(r-n)^{\frac{1}{2}}$, sed necesse est, vt etiam $(r-n)^{2}$ eius sit factor, seu quod idem est, vt illius formulae differentiale:

$$2rdr((B+A)r+\tfrac{1}{2}C(rr-1)+D)+dr(rr-1)(B+A+Cr)$$

factorem habeat $r-n$. Hinc igitur facto $r=n$ fit $C=-\frac{2}{n}(B+A)$, ideoque $D=-\frac{(B+A)(nn+1)}{2n}$.

35. His iam valoribus substitutis concluditur:

$$Q=\sqrt{(1-ss)}\left((B-A)s-\frac{(B+A)(1-ss)}{2n}+\frac{(B+A)(nn+1)}{2n}\right) \text{ seu}$$

$$Q=\sqrt{(1-ss)}\left((B-A)s+\frac{(B+A)(nn+ss)}{2n}\right)$$

vnde pro motu corporis in ellipsi elementum temporis ita exprimitur, vt sit:

$$d\tau=\frac{+a\sqrt{a}}{4}\cdot\frac{(nn-ss)ds\sqrt{2n}}{\sqrt{(1-ss)(2n(B-A)s+(B+A)(nn+ss))}}$$

quod cum elemento curuae comparatum:

$$\sqrt{(dx^2+dw^2)}=\frac{a\,ds\sqrt{(nn-ss)}}{2\sqrt{(1-ss)}}$$

praebet celeritatem corporis in puncto Z:

$$\frac{\sqrt{(dx^2+dw^2)}}{d\tau}=\frac{2\sqrt{(2n(B-A)s+(B+A)(nn+ss))}}{\sqrt{2na(nn-ss)}} \text{ seu}$$

$$\frac{\sqrt{(dx^2+dw^2)}}{d\tau}=\frac{2\sqrt{(A(n-s)^2+B(n+s)^2)}}{\sqrt{2na(nn-ss)}}.$$

36. Curua igitur a corpore descripta est ellipsis, cuius bini foci in ipsis virium centris A et B existunt, et axis transuersus est $v+u=na$, et coniugatus $=a\sqrt{(nn-1)}$. Atque in huiusmodi ellipsi

lipſi corpus ad vtrumque focum attractum libere moueri poteſt, dummodo ipſi initio eiusmodi celeritas fuerit impreſſa, vt deinceps pro quouis loco Z eius celeritas prodeat $= \frac{2\sqrt{(A(n-s)^2+B(n+s)^2)}}{\sqrt{2na(nn-ss)}}$, ſeu $= \sqrt{\frac{2}{a}} \cdot \frac{Auu+Bvv}{vu(v+u)} = \sqrt{\frac{2}{a}} \cdot \frac{vu}{v+u}\left(\frac{A}{vv}+\frac{B}{uu}\right)$.

In vertice ergo foco A propiore, vbi $s=1$, eſt celeritas $= \sqrt{\frac{2}{na}}\left(\frac{n-1}{n+1}A+\frac{n+1}{n-1}B\right)$, in altero vero vertice foco B propiori vbi $s=-1$, eſt celeritas $= \sqrt{\frac{2}{na}}\left(\frac{n+1}{n-1}A+\frac{n-1}{n+1}B\right)$.

Ad axem vero coniugatum celeritas eſt $= \frac{2n\sqrt{(A+B)}}{n\sqrt{2na}} = \sqrt{\frac{2(A+B)}{na}}$.

De motu corporis in Hyperbola.

37. Hyperbola oritur, ſi s conſtans aſſumatur. Sit ergo $s=n$ ſeu $v-u=na$, atque vt hic valor ſatisfaciat tam haec formula:

$$(1-ss)((B-A)s+\tfrac{1}{2}C(1-ss)-D)$$

quam eius differentialis, ita debet eſſe comparata vt poſito $s=n$, in nihilum abeat. Primo ergo erit $(B-A)-Cn=0$ ſeu $C=\frac{B-A}{n}$, tum vero $D=(B-A)n+\frac{(B-A)(1-nn)}{2n}$ ſeu $D=\frac{(B-A)(1+nn)}{2n}$; vnde conficitur $P=\sqrt{(rr-1)}\left((B+A)r+\frac{(B-A)(nn+rr)}{2n}\right)=\sqrt{(rr-1)}.\frac{B(r+n)^2+A(r-n)^2}{2n}$ hincque $d\tau=\frac{a\sqrt{a}}{4}\int\frac{(rr-nn)dr\sqrt{2n}}{\sqrt{(rr-1)}(A(r-n)^2+B(r+n)^2)}$ vnde motus ratio facile colligitur, quoniam haec formu-

la

la ex praecedente nafcitur fi loco n et r fcribantur $-n$ et $-s$.

Axis quidem transuerfus huius hyperbolae eft $u-v = na$ exiftente $n < 1$, et puncto A et B funt eius foci.

II. Cafus quo corpus non in eodem plano mouetur.

Ac primo quidem de motu in fphaeroide elliptico.

38. Simili modo circa formulas generales fupra inuentas eft operandum; ac primo aequationi $\frac{dr}{R} = \frac{ds}{S}$ fatisfieri poteft ponendo $r = n$, vt fit $v+u = na$, ideoque punctum Z perpetuo verfetur in fuperficie fphaeroidis elliptici ex rotatione ellipfis circa axem maiorem, in quo puncta A et B funt foci geniti. Tum autem facto $r = n$ non folum quantitas R fed etiam eius differentiale euanefcere debet, vnde fequentes duae aequationes nafcuntur.

$$\text{I. } m(nn-1)((B+A)n+\tfrac{1}{2}C(nn-1)+D)-nn=0$$

$$\text{II. } 2mn((B+A)n+\tfrac{1}{2}C(nn-1)+D) + m(nn-1)((B+A)+Cn)-2n=0$$

quae altera ex prima per eliminationem quantitatis D praebet:

$$\frac{2n^3}{nn-1}-2n+m(nn-1)(A+B+Cn)=0$$

feu $\frac{2n}{m(nn-1)^2}+A+B+Cn=0$.

 Ergo

Ergo $C = -\frac{1}{n}(A+B) - \frac{2}{m(nn-1)^2}$; tum vero eſt

$$D = -\frac{(A+B)(nn+1)}{2n} + \frac{nn+1}{m(nn-1)}.$$

39. Definitis his binis conſtantibus C et D et facta ſubſtitutione tandem nanciſcimur:

$$S = \sqrt{\left(\frac{m(1-ss)}{2n}\left(A(n-s)^2 + B(n+s)^2\right) - \frac{(nn-ss)^2}{(nn-1)^2}\right)}$$

ex quo angulus $ZXY = \Phi$ ex hac formula differentiali definiri debet $d\Phi = \frac{(nn-ss)ds}{(nn-1)(1-ss)S}$ ſiue hac

$$d\Phi = \frac{ds}{(nn-1)(1-ss)} : \sqrt{\left(\frac{m(1-ss)}{2n}\left(\frac{A}{(n+s)^2} + \frac{B}{(n-s)^2}\right) - \frac{1}{(nn-1)^2}\right)} \text{ ſeu}$$

$$d\Phi = \frac{ds}{1-ss} : \sqrt{\left(\frac{m(nn-1)^2(1-ss)}{2n}\left(\frac{A}{(n+s)^2} + \frac{B}{(n-s)^2}\right) - 1\right)}.$$

Pro tempore autem habetur:

$$d\tau = \tfrac{1}{4}a\sqrt{ma}.\frac{(nn-ss)ds}{S} \quad \text{ſeu}$$

$$d\tau = \tfrac{1}{4}a\sqrt{ma}.(nn-1)ds : \sqrt{\left(\frac{m(nn-1)^2(1-ss)}{2n}\left(\frac{A}{(n+s)^2} + \frac{B}{(n-s)^2}\right) - 1\right)}.$$

De motu ſuper conoide hyperbolico.

40. Hic caſus ex formulis generalibus deducitur ponendo $s = n$ vt ſit $v - u = na$, proditque hyperbola circa ambos focos A et B deſcripta cuius axis transuerſus eſt $= na$, exiſtente $n < 1$, tum vero haec hyperbola circa axem reuoluta id conoides generabit, in cuius ſuperficie corpus Z mouebitur: Perſpicuum autem eſt formulas praecedentes ad hunc caſum transferri, ſi ibi loco n et s ſcribatur $-n$ et $-r$. Quare poſito

$$R = \sqrt{\left(\frac{m(rr-1)}{2n}\left(A(r-n)^2 + B(r+n)^2\right) - \frac{(rr-nn)^2}{(1-nn)^2}\right)}$$

pro

pro angulo Φ obtinetur:

$$d\Phi = \frac{(rr - nn)dr}{(1 - nn)(rr - nn)R}$$

pro tempore vero τ:

$$d\tau = \frac{a\sqrt{ma}}{4} \cdot \frac{(rr - nn)dr}{R}.$$

41. Etsi autem his duobus casibus superficies, in qua corpus motum suum peragit facillime assignatur, eius tamen vera via, quam percurret, vix algebraice definiri posse videtur, propterea quod anguli Φ sinum ex formulis inuentis algebraice exprimere non licet: neque tamen adhuc asseuerare ausim, hoc nonnisi transcendenter praestari posse. Pro motu quidem in elliptoide si ponamus $\frac{an}{m(nn-1)} = M$, angulus Φ ita simplicius exprimitur vt sit:

$$d\Phi = \frac{ds(nn - ss)\sqrt{M}}{(1 - ss)\sqrt{(A(1 - ss)(n - s)^2 + B(1 - ss)(n + s)^2 - M(nn - ss)^2)}}$$

vbi quidem obseruo si sit $B = 0$, quo quidem casu integrale aliunde patet fore:

$$\frac{A}{M}\cos.(\zeta - \Phi)^2 + (nn - 1)\sin.(\zeta - \Phi)^2 = \frac{(n + s)^2}{1 - ss} \quad \text{seu}$$

$$\cos. 2(\zeta - \Phi) = \frac{M((nn + 1)(1 + ss) + 4ns) - A(1 - ss)}{(A - (nn - 1)M)(1 - ss)} \quad \text{siue}$$

$$\cos.(\zeta - \Phi) = \frac{(1 + ns)\sqrt{M}}{\sqrt{(A - (nn - 1)M)(1 - ss)}} \text{ et } \sin.(\zeta - \Phi) = \frac{\sqrt{(A(1 - ss) - M(n + s)^2)}}{\sqrt{(A - (nn - 1)M)(1 - ss)}}.$$

42. In genere etiam, si fuerit vel $A = 0$ vel $B = 0$, quoniam motus corporis ad vnicum centrum virium attracti algebraice assignari potest, necesse est vt formulae nostrae integrationem admittant, etiamsi ratio integrandi minus perspiciatur. Quam ob rem hic ipse casus haud contemnendum vsum in

in Analyſi praeſtare eſt cenſendus, cum inde formularum, quae primo intuitu integrationi vehementer aduerſari videntur, integralia tamen commode exhiberi queant. Haud ergo abs re fore arbitror ſi hos caſus accuratius examini ſubiecero.

Applicatio formularum inuentarum ad caſum, quo altera virium centripetarum euaneſcit.

43. Statuamus ergo $B = 0$, ita vt ſit:

$$R = \sqrt{(m(rr-1)(Ar+\tfrac{1}{2}C(rr-1)+D)-rr)} \text{ et}$$
$$S = \sqrt{(m(1-ss)(-As+\tfrac{1}{2}C(1-ss)-D)-ss)}$$

atque ad motum corporis definiendum ſequentes aequationes reſolui oportet:

$$\text{I.} \quad \frac{dr}{R} = \frac{ds}{S}$$
$$\text{II.} \quad d\Phi = \frac{(rr-ss)dr}{(rr-1)(1-ss)R} \text{ ſeu } \Phi = \int \frac{dr}{(rr-1)R} + \int \frac{ds}{(1-ss)S}$$
$$\text{III.} \quad d\tau = \frac{a\sqrt{ma}}{4} \cdot \frac{(rr-ss)dr}{R} \text{ ſeu } \tau = \frac{a\sqrt{ma}}{4}\left(\int \frac{rrdr}{R} - \int \frac{ssds}{S}\right)$$

vbi certe nulla methodus eſt obuia has aequationes tractandi, cum tamen aliunde earum integralia innoteſcant, quae igitur conſideraſſe plurimum intererit.

44. Quoniam vis centripeta in B euaneſcit, tam rectae AB poſitio, quam quantitas $AB = a$, precario in calculum inducitur, et corpus mouebitur

in

in plano per centrum virium A transeunte. Sit ergo recta AP intersectio huius plani cum plano tabulae; et quia punctum Z in illo existit plano, si ab Y ad rectam AP ducatur normalis YP, iungaturque recta ZP pariter ad AP normalis angulus ZPY erit inclinatio orbitae descriptae ad planum tabulae, ideoque constans. Statuamus ergo angulos constantes BAP $= \gamma$ et ZPY $= \delta$. Tum sit orbitae semiparameter $= b$, et excentricitas $= e$, ob AZ $= v$ erit ex natura sectionum, $v = \frac{b}{1 + e \cos. w}$ denotante w anomaliam veram. Sit ergo AE linea absidum et angulus PAE $= \zeta$ erit $w = \mathrm{PAZ} - \zeta$, ideoque $\cos. w = \frac{\mathrm{AP}.\cos.\zeta + \mathrm{PZ}\sin.\zeta}{v}$ vnde fit $v + e.\mathrm{AP}\cos.\zeta + e.\mathrm{PZ}\sin.\zeta = b$, hincque Tab. III. Fig. 2.

$$\mathrm{AP} = \frac{(b-v)\cos.\zeta + \sin.\zeta.\sqrt{(ee\,v^2\,(b-v)^2)}}{e}$$

et $\mathrm{PZ} = \frac{(b-v)\sin.\zeta - \cos.\zeta.\sqrt{(ee\,v\,v-(b-v)^2)}}{e}$.

45. His binis valoribus inuentis ex posteriori cum angulo ZPY $= \delta$ erit

$$\mathrm{PY} = \mathrm{PZ}\cos.\delta \text{ et } \mathrm{ZY} = \mathrm{PZ}\sin.\delta$$

deinde ex illo cum angulo BAP $= \gamma$ colligitur:

$$\mathrm{AX} = \mathrm{AP}\cos.\gamma + \mathrm{PY}\sin.\gamma \text{ et } \mathrm{XY} = \mathrm{AP}\sin.\gamma - \mathrm{PY}\cos.\gamma,$$

ideoque ob angulum ZXY $= \Phi$ erit:

$$\mathrm{tang.}\,\Phi = \frac{\mathrm{PZ}\sin.\delta}{\mathrm{AP}\sin.\gamma - \mathrm{PY}\cos.\gamma} = \frac{\mathrm{PZ}\sin.\delta}{\mathrm{AP}\sin.\gamma - \mathrm{PZ}\cos.\gamma\cos.\delta}.$$

Temporis autem elementum $d\tau$ est proportionale formulae:

$$\mathrm{AP}.d.\mathrm{PZ} - \mathrm{PZ}.d.\mathrm{AP} = \frac{-b\,v\,d\,v}{\sqrt{(ee\,v\,v - (b-v)^2)}}$$

 ita

ita vt fit $d\tau = \frac{\alpha b v\, dv}{\sqrt{(eevv - (b-v)^2)}}$

at pro angulo Φ habetur:

$$\text{tang.}\,\Phi = \frac{(b-v)\,\text{fin.}\,\delta\,\text{fin.}\,\zeta - \text{fin.}\,\delta\,\text{cof.}\,\zeta\,\sqrt{(eevv-(b-v)^2)}}{(b-v)(\text{fin.}\,\gamma\,\text{cof.}\,\zeta - \text{cof.}\,\gamma\,\text{fin.}\,\zeta\,\text{cof.}\,\delta) + (\text{fin.}\,\gamma\,\text{fin.}\,\zeta + \text{cof.}\,\gamma\,\text{cof.}\,\zeta\,\text{cof.}\,\delta)\sqrt{(eevv-(b-v)^2)}}$$

vnde etiam valor ipfius $d\Phi$ per dv exprimi poteft.

46. Porro ob $AX = AP\,\text{cof.}\,\gamma + PZ\,\text{fin.}\,\gamma\,\text{cof.}\,\delta$ erit:

$$BX = a - AP\,\text{cof.}\,\gamma - PZ\,\text{fin.}\,\gamma\,\text{cof.}\,\delta$$

hincque $BZ^2 = uu = BX^2 + vv - AX^2 = vv + a(BX - AX)$ feu $uu - vv = aa - 2a(AP\,\text{cof.}\,\gamma + PZ\,\text{fin.}\,\gamma\,\text{cof.}\,\delta)$ hincque

$$\frac{e}{2a}(aa + vv - uu) = (b-v)(\text{cof.}\,\gamma\,\text{cof.}\,\zeta + \text{fin.}\,\gamma\,\text{fin.}\,\zeta\,\text{cof.}\,\delta) + (\text{cof.}\,\gamma\,\text{fin.}\,\zeta - \text{fin.}\,\gamma\,\text{cof.}\,\zeta\,\text{cof.}\,\delta)\sqrt{(eevv - (b-v)^2)}$$

fit iam $v = \frac{1}{2}a(r+s)$ et $u = \frac{1}{2}a(r-s)$ eritque

$$\tfrac{1}{2}ea(1+rs) = (b - \tfrac{1}{2}a(r+s))(\text{cof.}\,\gamma\,\text{cof.}\,\zeta + \text{fin.}\,\gamma\,\text{fin.}\,\zeta\,\text{cof.}\,\delta) + (\text{cof.}\,\gamma\,\text{fin.}\,\zeta - \text{fin.}\,\gamma\,\text{cof}\,\zeta\,\text{cof.}\,\delta)\sqrt{(\tfrac{1}{4}eeaa(r+s)^2 - (b - \tfrac{1}{2}a(r+s))^2)}$$

47. Quo hanc aequationem facilius euoluamus, ponamus breuitatis gratia:

$$\text{cof.}\,\gamma\,\text{cof.}\,\zeta + \text{fin.}\,\gamma\,\text{fin.}\,\zeta\,\text{cof.}\,\delta = \mu \text{ et } \text{cof.}\,\gamma\,\text{fin.}\,\zeta - \text{fin.}\,\gamma\,\text{cof.}\,\zeta\,\text{cof.}\,\delta = \nu$$

et aequatio ad rationalitatem perducta huiusmodi formam induet:

$$E + 2Frs + Grrss + 2H(r+s) + 2Irs(r+s) + K(rr+ss) = 0$$

exiftente:

$$E = \tfrac{1}{4}eeaa - \mu eab + (\mu\mu + \nu\nu)bb$$
$$F = \tfrac{1}{4}(\mu\mu + \nu\nu)aa - \tfrac{1}{2}\mu eab$$

G $= \frac{1}{4}$

$$G = \tfrac{1}{4}eeaa$$
$$H = \tfrac{1}{4}\mu eaa - \tfrac{1}{2}(\mu\mu + \nu\nu)\,ab$$
$$I = \tfrac{1}{4}\mu eaa$$
$$K = \tfrac{1}{4}(\mu\mu + \nu\nu)aa - \tfrac{1}{4}\nu\nu eeaa$$

et haec aequatio pro integrali completo formulae differentialis $\frac{dr}{R} = \frac{ds}{S}$ haberi debet.

48. Mirum omnino hoc videretur, nisi jam dudum huiusmodi aequationum differentialium integralia inueniendi methodum tradidissem. Methodus quidem, qua hoc praestiti nondum ita est perfecta, vt a priori procedat, sed proposita eiusmodi aequatione differentiali $\frac{dr}{R} = \frac{ds}{S}$, vbi quidem formulae irrationales R et S simili modo per r et s determinantur, assumere cogor aequationem algebraicam generalem eiusdem formae ac supra exposita, in qua litteras constantes E, F, G etc. deinceps ita desinio, vt inde ipsa aequatio differentialis conficiatur. Istam ergo methodum ad hunc casum insignem, qui praeter expectationem hic se offert, accommodabo; siquidem inde eadem integratio, quae iam aliunde constat, obtinebitur.

49. Primum igitur ex aequatione algebraica assumta desinio valorem ipsarum s et r seorsim:

$$s = \frac{-Irr - Fr - H + \surd((Irr + Fr + H)^2 - (Grr + 2Ir + K)(Krr + 2Hr + E))}{Grr + 2Ir + K}$$

$$r = \frac{-Iss - Fs - H\surd((Iss + Fs + H)^2 - (Gss + 2Is + K)(Kss + 2Hs + E))}{Gss + 2Is + K}$$

 atque

atque formularum radicalium illam ipsi R hanc vero ipsi S ex §. 43. aequalem facio, vnde fit:

r^4 $II - GK = \frac{1}{2} m C$

r^3 $2FI - 2IK - 2GH = mA$

r^2 $2HI + FF - EG - 4HI - KK = -mC + mD - 1$

r^1 $2FH - 2EI - 2HK = -mA$

r^0 $HH - EK = \frac{1}{2} m C - mD.$

50. Hinc aequationum secundae et quartae summa dat:

$$FI + FH - IK - EI - GH - HK = 0$$

vnde fit $\frac{H}{I} = \frac{E+K-F}{F-K-G}$. Statuatur ergo

$$H = n(E + K - F) \text{ et } I = -n(G + K - F)$$

qui valores in alterutra substituti praebent:

$$\frac{mA}{2n} = (K - F)^2 - EG$$

Porro primae, tertiae et quintae summa suppeditat:

$$(I - H)^2 + FF - (E + K)(G + K) = -1 \text{ seu}$$

$$nn(E + G + 2K - 2F)^2 = (E + K)(G + K) - FF - 1.$$

Hoc autem modo calculus fit nimis intricatus; ex quo expediet determinationem coefficientium in genere suscipere.

51 Considerentur ergo sequentes formulae:

I. $II - GK = a = \frac{1}{2} m C$

II. $I(F - K) - GH = b = \frac{1}{2} m A$

III. $FF - KK - EG - 2HI = c = -mC + mD - 1$

IV. $H(F - K) - EI = d = -\frac{1}{2} m A$

V. $HH - EK = e = \frac{1}{2} m C - mD,$

vnde.

vnde ex prima et quinta ſtatim elicimus:

$$G = \frac{II - a}{K} \quad \text{et} \quad E = \frac{HH - e}{K}.$$

Deinde ex II et IV eſt

primo $bH - dI = EII - GHH = \frac{aHH - eII}{O}$

tum $bEI - dGH = (EII - GHH)(F - K = \frac{(F-K)(aHH - eII)}{K}$

ſeu $-HI(bH - dI) + adH - beI = (F - K)(aHH - eII)$

vnde colligitur:

$$K = \frac{aHH - eII}{bH - dI} \quad \text{et} \quad F = K + \frac{HI(bH - dI) + adH - beI}{aHH - eII}.$$

Ergo $G = \frac{(bH - dI)(II - a)}{aHH - eII}$ et $E = \frac{(bH - dI)(HH - e)}{aHH - eII}$.

§ 52. Cum igitur ſit:

$$F = K + \frac{bI(HH - e) - dH(II - a)}{aHH - eII}$$

facta ſubſtitutione in tertia fiet

$$\begin{array}{l|l}
FF & KK + \frac{2bI(HH - e) - 2dH(II - a)}{bH - dI} + \frac{bbII(HH - e)^2 + ddHH(II - a)^2}{(aHH - eII)^2} \\
 & \qquad - \frac{2bdHI(HH - e)(II - a)}{(aHH - eII)} \\
-KK & -KK \\
-2HI & -2HI \\
-EG & - \frac{(bbHH + ddII)(II - a)(HH - e)}{(aHH - eII)^2} + \frac{2bdHI(HH - e)(II - a)}{(aHH - eII)^2} \\
= & = \\
c & c
\end{array}$$

quae aequatio reducitur ad hanc formam:

$$c = \frac{2adH - 2beI}{bH - dI} + \frac{bb(HH - e) - dd(II - a)}{aHH - eII}.$$

ex qua binas litteras H et I definiri oportet, ita vt altera maneat indeterminata, atque aequatio assumta vna littera abundet prae differentiali, quod est criterium aequationis integralis completae.

53. Quo haec aequatio facilius resoluatur, ratio inter binas litteras H et I nostro arbitrio relinquatur, sitque $I = nH$ vnde obtinemus:

$$c = \frac{2ad - 2ben}{b - dn} + \frac{bb - ddnn}{a - enn} + \frac{add - bbe}{a - enn} \cdot \frac{1}{HH}$$

sicque littera H definitur per simplicem extractionem radicis quadratae; qua inuenta erit $I = nH$; et reliquae litterae:

$$K = \frac{a - enn}{b - dn} \cdot H;\quad F = K + \frac{n(b - dn)}{a - enn} H + \frac{ad - ben}{a - enn} \cdot \frac{1}{H}$$

$$G = \frac{nnHH - a}{K} \quad \text{et} \quad E = \frac{HH - e}{K}.$$

54. Pro casu ergo oblato, si loco litterarum a, b, c, d, e valores debiti substituantur, quantitatem H ex hac aequatione definiri oportet:

$$-mC + mD - 1 = \frac{-mC - mn(C - 2D)}{1 + n} + \frac{\frac{1}{2}mAA(1 - nn)}{C(1 - nn) + 2Dnn} + \frac{\frac{1}{2}mmAAD}{C(1 - nn) + 2Dnn} \cdot \frac{1}{HH}$$

$$\text{seu } -1 + \frac{mD(1 - n)}{1 + n} - \frac{\frac{1}{2}mAA(1 - nn)}{C(1 - nn) + 2Dnn} = \frac{\frac{1}{2}mmAAD}{C(1 - nn) + 2Dnn} \cdot \frac{1}{HH}$$

vnde colligitur:

$$H = mA : \sqrt{\left(2\left(\frac{C}{D}(1 - nn) + 2nn\right)\left(\frac{mD(1 - n)}{1 + n} - 1\right) - \frac{mAA(1 - nn)}{D}\right)}$$

hincque

hincque porro:

$$\mathrm{I} = n\mathrm{H};\ \mathrm{K} = \frac{\mathrm{C}(1 - nn) + 2\,\mathrm{D}nn}{\mathrm{A}(1+n)}\,\mathrm{H};$$

$$\mathrm{F} = \mathrm{K} + \frac{n\mathrm{HH}}{\mathrm{K}} - \frac{\tfrac{1}{2}m\mathrm{A}(\mathrm{C}(1+n) - 2\,\mathrm{D}n)}{\mathrm{C}(1-nn) + 2\,\mathrm{D}nn}\cdot\frac{1}{\mathrm{H}}$$

$$\mathrm{G} = \frac{nn\mathrm{HH}}{\mathrm{K}} - \frac{m\mathrm{C}}{2\,\mathrm{K}} \text{ et } \mathrm{E} = \frac{\mathrm{HH}}{\mathrm{K}} - \frac{m(\mathrm{C} - 2\,\mathrm{D})}{2\,\mathrm{K}}.$$

55. His iam valoribus definitis aequationis nostrae primae $\frac{dr}{\mathrm{R}} = \frac{ds}{\mathrm{S}}$ aequatio integralis completa est:

$$\mathrm{E} + 2\mathrm{F}rs + \mathrm{G}rrss + 2\mathrm{H}(r+s) + 2\,\mathrm{I}rs(r+s) + \mathrm{K}(rr+ss) = 0$$

in qua littera n est constans per integrationem ingressa.

Deinde vero est:

$$s = \frac{-\mathrm{I}rr - \mathrm{F}r - \mathrm{H} + \mathrm{R}}{\mathrm{G}rr + 2\mathrm{I}r + \mathrm{K}} \text{ vel } r = \frac{-\mathrm{I}ss - \mathrm{F}s - \mathrm{H} - \mathrm{E}}{\mathrm{G}ss + 2\mathrm{I}s + \mathrm{K}}$$

vnde habemus:

$$\mathrm{R} = \mathrm{G}rrs + 2\mathrm{I}rs + \mathrm{K}s + \mathrm{I}rr + \mathrm{F}r + \mathrm{H} \text{ et}$$

$$\mathrm{S} = -\mathrm{G}rss - 2\mathrm{I}rs - \mathrm{K}r - \mathrm{I}ss - \mathrm{F}s - \mathrm{H}$$

tum enim aequatio nostra differentiata dat:

$$2ds.\ \mathrm{R} - 2dr.\ \mathrm{S} = 0 \text{ seu } \frac{dr}{\mathrm{R}} = \frac{ds}{\mathrm{S}},$$

quae erat prima aequatio in §. 43. integranda.

56. Cum

56. Cum nunc inuenta ſit aequatio inter r et s vnde ob $x = \frac{1}{2}a(1+rs)$ et $v = \frac{1}{2}a(r+s)$ relatio algebraica inter x et v facile elicitur; hinc calculo quidem prolixo tam angulus Φ quam tempus τ, artificia alibi expoſita in ſubſidium vocando, definiri poterit; quam euolutionem, cum ad quaeſtionem hic tractatam minus pertineat, praetermitto, contentus viam eo perueniendi indicaſſe.

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