OBSERVATIONES ANALYTICAE.

Auctore

L. EVLERO.

Conſideranti poteſtates, quae ex eleuatione huius formae trinomiae $1 + x + xx$ naſcuntur, termini medii maximis coefficientibus numericis deprehenduntur affecti, quorum ordo progreſſionis cum non parum ſit reconditus, omni attentione dignus videtur; praecipue quoniam huiusmodi ſpeculationes plerumque fructum haud ſpernendum in Analyſi afferre ſolent. Primum ergo harum poteſtatum ſimpliciores conſpectui exponam:

Exponens poteſtatis	Poteſtates euolutae
0	1
1	$1 + x + x^2$
2	$1 + 2x + 3x^2 + 2x^3 + x^4$
3	$1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$
4	$1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8$
5	$1 + 5x + 15x^2 + 30x^3 + 45x^4 + 51x^5 + 45x^6 + 30x^7 + 15x^8 + 5x^9 + x^{10}$
.	etc.

hinc ſi termini medii ex ſingulis poteſtatibus ordine repraeſententur, haec exoritur progreſſio:

$$1,\ 1x,\ 3x^2,\ 7x^3,\ 19x^4,\ 51x^5,\ 141x^6 \text{ etc.}$$

qui

qui numeri, quanam lege progrediantur, haud immerito indagari videtur, vt non solum inde terminus generalis seu coefficiens dignitati indefinitae x^n conueniens innotescat, sed etiam insignes huius seriei proprietates explorentur. Hunc in finem sequentia problemata proponam, quorum resolutio deinceps ad alias speculationes non parum curiosas manuducet.

Problema I.

1. Euoluta hac potestate indefinita $(1+x+xx)^n$ coefficientem termini medii seu dignitatis x^n definire.

Solutio.

Potestas proposita ita sub forma binomii repraesentetur $(x(1+x)+1)^n$, quae more solito euoluta praebet:

$$x^n(1+x)^n+\tfrac{n}{1}x^{n-1}(1+x)^{n-1}+\tfrac{n(n-1)}{1.2}x^{n-2}(1+x)^{n-2}$$
$$+\tfrac{n(n-1)(n-2)}{1.2.3}x^{n-3}(1+x)^{n-3} \text{ etc.}$$

ex cuius singulis membris, si vlterius euoluantur, terminos formae x^n elici oportet. Ac primum quidem membrum praebet x^n, cum reliquae potestates omnes ipsius x ex eius euolutione ortae futurae sint altiores. Ex secundo autem membro pro hac dignitate x^n oritur:

$$\tfrac{n}{1}x^{n-1}.\tfrac{n-1}{1}x = \tfrac{n(n-1)}{1.1}x^n$$

 ex

ex tertio membro fimili modo confequimur:

$$\frac{n(n-1)}{1.\,2}x^{n-2}\cdot\frac{(n-2)(n-3)}{1.\,2}x^2 = \frac{n(n-1)(n-2)(n-3)}{1.\,2.\,1.\,2}x^n$$

quas cunctas partes fi in vnam fummam colligamus obtinetur dignitatis x^n coefficiens quaefitus:

$$1 + \frac{n(n-1)}{1.\,1} + \frac{n(n-1)(n-2)(n-3)}{1.\,2.\,1.\,2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.\,2.\,3.\,1.\,2.\,3} \text{ etc.}$$

Coroll. 1.

2. Haec ergo feries, quae quoties n eft numerus integer abrumpitur, coefficientem praebet dignitatis x^n pro ferie propofita $1 + x + 3x^2 + 7x^3 + 19x^4 +$ etc. ficque eius ope terminus quantumuis ab initio remotus ftatim fine praecedentibus inueniri poteft.

Coroll. 2.

3. Quod fi pro n fucceffiue numeros 1, 2, 3 etc. fubftituamus, fequentes valores reperiuntur:

n	coefficiens ipfius x^n
0	1
1	1
2	$1 + 2 = 3$
3	$1 + 6 = 7$
4	$1 + 12 + 6 = 19$
5	$1 + 20 + 30 = 51$
6	$1 + 30 + 90 + 20 = 141$
7	$1 + 42 + 210 + 140 = 393$

n | coe-

n	coefficiens ipsius x^n
8	$1+56+420+560+70=1107$
9	$1+72+756+1680+630=3139$
10	$1+90+1260+4200+3150+252=8953$
11	$1+110+1980+9240+11550+2772=25653$
12	$1+132+2970+18480+34650+16632+924=73789$
	etc.

Coroll. 3.

4. Series horum numerorum ita est comparata, vt quisque terminus cum triplo praecedentis commode conferri posse videatur, ex qua comparatione sequentes differentiae nascuntur:

$$\begin{array}{l} 1,\ 1,\ 3,\ 7,\ 19,\ 51,\ 141,\ 393,\ 1107,\ 3139 \text{ etc.} \\ \quad 3,\ 3,\ 9,\ 21,\ 57,\ 153,\ 423,\ 1179,\ 3321 \\ \hline \quad 2,\ 0,\ 2,\ 2,\ 6,\ 12,\ 30,\ 72,\ 182 \text{ etc.} \end{array}$$

Scholion 1.

5. Si has differentias accuratius contemplemur, non sine ratione euenire videtur, quod eae sint numeri pronici, seu trigonales duplicati in forma $mm+m$ contenti, ac si ad istorum pronicorum numerorum radices spectemus, quae hanc seriem constituunt: Exemplum memorabile inductionis fallacis.

$$1,\ 0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13 \text{ etc.}$$

ea manifesto est recurrens, cuius quisque terminus est summa binorum praecedentium. Qui ordo cum in in decem primoribus terminis deprehendatur, quis dubitauerit eundem vniuerſae ſeriei tribuere? ſaepe profecto inductiones minus certae ſucceſſu non fuerunt deſtitutae. Operae ergo pretium erit hanc rationem accuratius perpendere, ſcilicet cum numerus 13 conueniat ſeriei termino x^9, in genere dignitati x^n reſpondebit numerus:

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}$$

cuius numerus pronicus eſt:

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-2} + \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{2n-4} + \frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^{2n-4} - \frac{2}{5}(-1)^{n-2}.$$

Quare ſi in ſerie propoſita bini termini contigui generatim ita exhibeantur:

$$1 + x + 3x^2 + 7x^3 + 10x^4 \ldots Px^n + Qx^{n+1}$$

erit $3P - Q =$

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} + \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{2n-2} + \frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^{2n-2} - \frac{2}{5}(-1)^{n-1}$$

vnde concluditur fore:

$$P = \frac{3^n + (-1)^n}{10} + \frac{1}{5}\left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{3-\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

ita vt ipſa quoque ſeries propoſita foret recurrens ſcala relationis exiſtente:

$$6, -8, -8, 13, 4, -3$$

ſecun-

secundum quam erit:

$3139 = 6.1107 - 8.393 - 8.141 + 14.51 + 4.9 - 3.7.$

Scholion 2.

6. Verum quantumuis probabili inductione haec lex progressionis inniti videatur, dum adeo in decem primoribus terminis locum habet; tamen ea fallax deprehenditur, dum iam in termino vndecimo 8953 fallit; hoc enim a triplo praecedentis 9417 sublato residuum 464 ne numerus quidem pronicus est, multo minus radicem pronicam habet $21 = 13 + 8$, est enim $21^2 + 21 = 462$, qui numerus binario deficit ab eo 464, qui secundum legem obseruatam resultare debebat. Quam ob caussam nunc quidem in veram progressionis legem huius seriei sum inquisiturus, vt pateat quomodo quisque terminus per aliquot praecedentes reuera determinetur.

Problema 2.

7. Pro serie proposita

$$1, x, 3x^2, 7x^3, 19x^4, 51x^5 \text{ etc.}$$

inuestigare legem, qua quisque terminus per aliquot praecedentes determinatur.

Solutio.

Considerentur generatim aliquot huius seriei termini se mutuo sequentes:

$$1, x, 3x^2, 7x^3 \ldots\ldots Px^n, Qx^{n+1}, Rx^{n+2}$$

et quoniam in problemate praecedente vidimus esse:

$$P = 1 + \frac{n(n-1)}{1.\ 1} + \frac{n(n-1)(n-2)(n-3)}{1.\ 1.\ 2.\ 2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.\ 1\ 2.\ 2\ 3.\ 3}\ \text{etc.}$$

erit simili modo:

$$Q = 1 + \frac{(n+1)n}{1.\ 1} + \frac{(n+1)(n-1)(n-2)}{1.\ 1.\ 2.\ 2} + \frac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{1.\ 1.\ 2.\ 2.\ 3.\ 3}\ \text{etc.}$$

$$R = 1 + \frac{(n+2)(n+1)}{1.\ 1} + \frac{(n+2)(n+1)n(n-1)}{1.\ 1.\ 2.\ 2} + \frac{(n+2)(n+1)n(n-1)(n-2)(n-3)}{1.\ 1.\ 2.\ 2.\ 3.\ 3}\ \text{etc.}$$

vnde quamlibet a sequente subtrahendo colligimus:

$$Q - P = \frac{2n}{1} + \frac{2n(n-1)(n-2)}{1.\ 1.\ 2.} + \frac{2n(n-1)(n-2)(n-3)(n-4)}{1.\ 1.\ 2.\ 2.\ 3}\ \text{etc.}$$

$$R - Q = \frac{2(n+1)}{1} + \frac{2(n+1)n(n-1)}{1.\ 1.\ 2} + \frac{2(n+1)n(n-1)(n-2)(n-3)}{1.\ 1.\ 2.\ 2\ 3.}\ \text{etc.}$$

hinc capiamus hanc formam:

$$\frac{n+2}{n+1}(R-Q) = \frac{2(n+2)}{1} + \frac{2(n+2)n(n-1)}{1.\ 1.\ 2} + \frac{2(n+2)n(n-1)(n-2)(n-3)}{1.\ 1.\ 2.\ 2.\ 3}\ \text{etc.}$$

a qua illa $Q - P$ subtrahatur vt fiat:

$$\frac{n+2}{n+1}(R-Q)-(Q-P) = 4 + \frac{4n(n-1)}{1.\ 1} + \frac{4n(n-1)(n-2)(n-3)}{1.\ 1.\ 2.\ 2} + \text{etc.}$$

quae series cum sit $= 4P$, habebimus:

$$R = Q + \frac{(n+1)(Q-P)}{n+2} + \frac{4(n+1)P}{n+2}\ \text{seu}$$

$$R = \frac{(2n+3)Q + 3(n+1)P}{n+2}.$$

Coroll. 1.

8. En ergo legem qua quisque seriei terminus per binos praecedentes determinatur, quae ita se habet:

$$R = Q + \frac{n+1}{n+2}(Q + 3P)$$

vnde etiam ex binis sequentibus Q et R praecedens P ita definitur:

$$P = \frac{(n+2)R - (2n+3)Q}{3(n+1)};$$

Coroll.

Coroll. 2.

9. Quo appareat, quomodo haec lex in ſerie propoſita locum habeat, per caſus aliquot eam illuſtremus:

ſi $n=0$, $3=1+\frac{1}{2}(1+3.1)$
ſi $n=1$, $7=3+\frac{2}{3}(3+3.1)$
ſi $n=2$, $19=7+\frac{3}{4}(7+3.3)$
ſi $n=3$, $51=19+\frac{4}{5}(19+3.7)$
ſi $n=4$, $141=51+\frac{5}{6}(51+3.19)$
etc.

Coroll. 3.

10. Quoniam igitur in relationem, quae inter ternos terminos contiguos intercedit, ipſe exponens n ingreditur, hinc facile colligitur hanc ſeriem non ad genus recurrentium eſſe referendam.

Coroll. 4.

11. Inter quatuor autem continuos terminos P, Q, R et S relatio ab exponente n libera exhiberi poteſt, cum enim ex ternis prioribus ſit
$n=\frac{2R-3Q-3P}{3P+2Q-R}$
erit ſimili modo $n+1=\frac{2S-3R-3Q}{3Q+2R-S}$
vnde concludimus fore
$$S=R+Q+\frac{3P(Q+R)+2QR}{6P+3Q-R}$$

R 2 quae

quae eſt relatio conſtans, qua per ternos quosque terminos contiguos ſequens definitur.

Scholion 1.

12. Inuenta lege, qua noſtrae progreſſionis quisque terminus a binis praecedentibus pendet, iam multo facilius hanc progreſſionem quousque lubuerit continuare licet. Ita cum dignitates x^{11} et x^{12} affectae ſint numeris 25653 et 73789, ſequentis x^{13} ob $n=11$ coefficiens erit:

$$73789+\tfrac{12}{13}(73789+3.25653)=212941$$

et dignitatis x^{14}

$$212941+\tfrac{13}{14}(212941+3.73789)=616227$$

vnde noſtra progreſſio ad dignitatem viceſimam vsque continuata ita ſe habebit:

1	
$1x$	$25653x^{11}$
$3x^2$	$73789x^{12}$
$7x^3$	$212941x^{13}$
$19x^4$	$616227x^{14}$
$51x^5$	$1787607x^{15}$
$141x^6$	$5196627x^{16}$
$393x^7$	$15134931x^{17}$
$1107x^8$	$44152809x^{18}$
$3139x^9$	$128996853x^{19}$
$8953x^{10}$	$377379369x^{20}$

circa

circa quos numeros obſeruo nullum eorum eſſe per 5 diuiſibilem, dignitatum vero $x^{5\alpha+2}$ coefficientes eſſe per 3. diuiſibiles, dignitatum $x^{7\alpha+2}$ per 7. neque vero hinc quicquam circa indolem horum numerorum concludere licet. Verum ex lege progreſſionis hic inuenta eius ſummam, ſiquidem in infinitum continuetur, definire poterimus, cui fini ſequens problema deſtinatur.

Scholion 2.

13. Si noſtrae progreſſionis quilibet terminus a triplo antecedentis ſubtrahatur, differentiae talem progreſſionem conſtituunt:

1.2; 2.1; 3.2; 4.3; 5.6; 6.12; 7.26; 8.58; 9.134; 10.317; 11.766; 12.1883; 13.4698; 14.11871; 15.30330; 16.78249; 17.203622; 18.533955

pro qua generatim ſtatuamus:

$$mp;\ (m+1)q;\ (m+2)r$$

vbi primum notatu dignum occurrit, quod horum terminorum factores priores in ſerie numerorum naturali progrediantur, poſteriores vero ita ſint comparati, vt quilibet ex binis praecedentibus hoc modo conficiatur:

$$r = \frac{3mp + 2(m+1)q}{m+4}.$$

Problema 3.

14. Si series nostra

$$1 + x + 3x^2 + 7x^3 + 19x^4 + \text{etc.}$$

in infinitum continuetur, eius summam inuestigare.

Solutio.

Cum relatio cuiusque termini ad binos antecedentes sit definita, statuamus:

$$s = 1 + x + 3x^2 + \ldots + Px^n + Qx^{n+1} + Rx^{n+2} + \text{etc.}$$

vbi notetur esse $(n+2)R - (2n+3)Q - 3(n+1)P = 0$ cui conditioni vt satisfaciamus, sumamus differentiale:

$$\frac{ds}{dx} = 1 + 6x + \ldots + nPx^{n-1} + (n+1)Qx^n + (n+2)Rx^{n+1} \text{ etc.}$$

quod multiplicatum per $1 - 2x - 3xx$ praebet:

$$\frac{ds}{dx}(1 - 2x - 3xx) =$$

$$\begin{array}{llllll} 1 + 6x + 21xx \ldots & + nPx^{n-1} & + (n+1)Qx^n & + (n+2)Rx^{n+1} \\ \;\;\; - 2 \;\; - 12 & & - 2nP & - (2n+2)Q \\ \;\;\;\;\;\;\;\;\; - 3 & & & - 3nP \end{array}$$

quae series reducitur ad hanc:

$$1 + 4x + 6xx \ldots (Q + 3P)x^{n+1} + \text{etc.}$$

At ipsa series proposita per $1 + 3x$ multiplicata dat

$$s(1 + 3x) = 1 + 4x + 6xx \ldots (Q + 3P)x^{n+1}$$

vnde manifestum est fore:

$$\frac{ds}{dx}(1 - 2x - 3xx) = s(1 + 3x) \text{ ideoque}$$

$\frac{ds}{s}$

$\frac{ds}{s} = \frac{dx(1+3x)}{1-2x-3xx}$, cuius integratio praebet

$s = \frac{1}{\sqrt{(1-2x-3xx)}} = \frac{1}{\sqrt{(1+x)(1-3x)}}$

quae eſt ipſa ſumma ſeriei propoſitae in infinitum continuatae.

Coroll. 1.

15. Liquet ergo ſeriei huius ſummam eſſe imaginariam niſi ſumatur $x < \frac{1}{3}$, caſu autem $x = \frac{1}{3}$ fieri infinitam. At ipſi x valores negatiuos tribuendo, puta $x = -y$, ſumma fit finita ſumendo $y < 1$, at caſu $y > 1$ imaginaria euadit. Ita ſtatuendo $x = -\frac{1}{2}$ fit

$\frac{2}{\sqrt{5}} = 1 - \frac{1}{2} + \frac{3}{4} - \frac{7}{8} + \frac{19}{16} - \frac{51}{32} + \frac{141}{64} - \text{etc.}$

Coroll. 2.

16. Nunc ergo nouimus ſeriem noſtram quoque reſultare ſi formula irrationalis $(1-2x-3xx)^{-\frac{1}{2}}$ more ſolito in ſeriem euoluatur: quae formula cum ita repraeſentari poſſit $s = ((1-x)^2 - 4xx)^{-\frac{1}{2}}$, prodit:

$s = \frac{1}{1-x} + \frac{2x}{1} \cdot \frac{xx}{(1-x)^3} + \frac{2 \cdot 6}{1 \cdot 2} \cdot \frac{x^4}{(1-x)^5} + \frac{2 \cdot 6 \cdot 10}{1 \cdot 2 \cdot 3} \cdot \frac{x^6}{(1-x)^7} + \text{etc.}$

ex cuius vlteriori euolutione oritur:

$$\begin{array}{lccccccccccc} s = & 1 & +x & +x^2 & +x^3 & +x^4 & +x^5 & +x^6 & +x^7 & +x^8 & +x^9 & +x^{10} \\ & & & +2.1 & +2.3 & +2.6 & +2.10 & +2.15 & +2.21 & +2.28 & +2.36 & +2.45 \\ & & & & & +6.1 & +6.5 & +6.15 & +6.35 & +6.70 & +6.126 & +6.210 \\ & & & & & & & +20.1 & +20.7 & +20.28 & +20.84 & +20.210 \\ & & & & & & & & & +70.1 & +70.9 & +70.45 \\ & & & & & & & & & & & +252.1 \end{array}$$

Coroll.

Coroll. 3.

17. Hinc colligimus in genere dignitatis x^n coefficientem numericum ita expressum iri:

$$+\frac{2}{1}\cdot\frac{n(n-1)}{1\cdot 2}+\frac{2\cdot 6}{1\cdot 2}\cdot\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}$$

quae forma non discrepat ab ea, quam problemate primo inuenimus.

Scholion.

18. Quodsi formam huius summae accuratius perpendamus, haud difficulter inde methodum multo latius patentem elicimus, cuius ope adeo haec potestas generalior $(a+bx+cxx)^n$ ita pertractari poterit, vt non solum termini medii singularum potestatum sed etiam termini a mediis vtrinque aequidistantes assignari queant. Hanc ergo methodum in sequente problemate sum expositurus.

Problema 4.

19. Si trinomii $a+bx+cxx$ singulae potestates euoluantur, indeque tam termini medii, quam a mediis aequidistantes seorsim in series disponantur, singularum harum serierum naturam et summam inuestigare.

Solutio.

Consideretur formula ista $\frac{1}{1-y(a+bx+cxx)}$ quae euoluta praebet:

$$1+y(a+bx+cxx)+yy(a+bx+cxx)^2+y^3(a+bx+cxx)^3\text{ etc.}$$

vbi

vbi cum trinomii propositi singulae potestates occurrant, a explicatis orietur:

$$\begin{gathered}
1 \\
y(a + bx + cxx) \\
y^2(a^2 + 2abx + 2acx^2 + 2bcx^3 + ccx^4) \\
+ bb \\
y^3(a^3 + 3a^2bx + 3a^2cx^2 + 6abcx^3 + 3bbcx^4 + 3bccx^5 + c^3x^6) \\
+ 3ab^2 + b^3 + 3acc \\
\text{etc.}
\end{gathered}$$

hinc si primo termini medii, tum vero termini a mediis vtrinque aequidistantes capiantur, nascentur sequentes series:

$$\begin{aligned}
&1 + bxy + (2ac + bb)xxyy + (6abc + b^3)x^3y^3 + \text{etc.} \\
&y(a + cxx)(1 + 2bxy + (3ac + 3bb)xxyy + \text{etc.}) \\
&y^2(a^2 + c^2x^4)(1 + 3bxy + \text{etc.}) \\
&y^3(a^3 + c^3x^6)(1 + 4bxy + \text{etc.}) \\
&y^4(a^4 + c^4x^8)(1 + 5bxy + \text{etc.}) \\
&\qquad \text{etc.}
\end{aligned}$$

Omissis ergo his multiplicatoribus, quia in seriebus ipsis adsunt potestates producti xy, ponamus $xy = z$ et indicemus istas series hoc modo:

$$\begin{aligned}
1 + bz + (2ac + bb)zz + (6abc + b^3)z^3 &= P \\
1 + 2bz + (3ac + 3bb)zz + \text{etc.} &= Q \\
&= R \\
1 + 3bz + \text{etc.} &= S \\
1 + 4bz + \text{etc.} & \\
\text{etc.} &
\end{aligned}$$

 ita

ita vt iam ob $y = \frac{z}{x}$, habeamus:

$$\frac{1}{1 - bz - z(\frac{a}{x} + cx)} = P + z(\frac{a}{x} + cx)Q + zz(\frac{aa}{xx} + ccxx)R$$
$$+ z^3(\frac{a^3}{x^3} + c^3x^3)S + \text{etc.}$$

multiplicetur vtrinque per $1 - bz - z(\frac{a}{x} + cx)$, et quoniam quantitates P, Q, R etc. a ſola z pendent, omnia membra ſecundum poteſtates ipſius x tam poſitiua quam negatiua diſponantur: quo facto obtinebimus:

$$\begin{array}{llll} 1 = P(1-bz) & + Qz(1-bz)cx & + Rzz(1-bz)c^2x^2 & + Sz^3(1-bz)c^3x^3 \\ -2Qaczz & - Pz.cx & - Qzz.ccx^2 & - Rz^3c^3x^3 \\ & - Rz^3.accx & - Sz^4.ac^3x^2 & - Tz^5ac^4x^3 \\ & + Qz(1-bz)\frac{a}{x} & + Rzz(1-bz)\frac{a^2}{x^2} & + Sz^3(1-bz)\frac{a^3}{x^3} \\ & - Pz.\frac{a}{x} & - Qzz.\frac{aa}{xx} & - Rz^3.\frac{a^3}{x^3} \\ & - Rz^3ac.\frac{a}{x} & - Sz^4ac.\frac{aa}{xx} & - Tz^5ac.\frac{a^3}{x^3} \end{array}$$

vbi euidens eſt poteſtates negatiuas ipſius x iisdem conditionibus ad nihilum redigi ac poſitiuas. Hinc erga ſequentes determinationes adipiſcimur:

$$Q = \frac{P(1 - bz) - 1}{2aczz}$$
$$R = \frac{Q(1 - bz) - P}{aczz}$$
$$S = \frac{R(1 - bz) - Q}{aczz}$$
$$T = \frac{S(1 - bz) - R}{aczz}$$

etc.

Vide-

Videmus ergo quantitates P, Q, R, S etc. secundum seriem recurrentem progredi, cuius scala relationis est:

$$\frac{1-bz}{aczz}\,; \quad -\frac{1}{aczz}$$

hinc si illis quantitatibus indices tribuantur:

$$\overset{0}{P},\ \overset{1}{Q},\ \overset{2}{R},\ \overset{3}{S}\ \ldots\ldots\ \overset{n}{Z}$$

ita vt Z sit ea, quae indici n conuenit, ex natura recurrentiae erit:

$$Z = A\left(\frac{1-bz-\sqrt{((1-bz)^2-4aczz)}}{2aczz}\right)^n + B\left(\frac{1-bz+\sqrt{((1-bz)^2-4aczz)}}{2aczz}\right)^n$$

vbi cum constet quantitatem Z eiusmodi serie exprimi, vt sit:

$$Z = 1 + (n+1)bz + \ldots zz + \ldots z^3 + \ldots z^4 \text{ etc.}$$

vnde manifestum est necessario esse debere $B = 0$, quia alioquin termini ex posteriori membro orierentur potestatibus negatiuis ipsius z affecti. Facto ergo $B = 0$, erit:

$$Z = A\left(\frac{1-bz-\sqrt{(1-2bz+(bb-4ac)zz)}}{2aczz}\right)^n$$

Iam fiat $n = 0$, ac necesse est prodire $A = P$, posito autem $n = 1$, effici debet:

$$A.\ \frac{1-bz-\sqrt{(1-2bz+(bb-4ac)zz)}}{2aczz} = Q.$$

Cum igitur sit $A = P$ et $2aczzQ + 1 = P(1-bz)$ sequitur fore:

$$P(1-bz) - P\sqrt{(1-2bz+(bb-4ac)zz)} = P(1-bz) - 1$$

 ideoque

ideoque $P = \frac{1}{\surd(1-2bz+(bb-4ac)zz)}$.

Quocirca feriei noftrae P, Q, R, S . . . Z terminus generalis eft:

$$Z = \frac{1}{\surd(1-2bz+(bb-4ac)zz)}\left(\frac{1-bz-\surd(1-2bz+(bb-4ac)zz)}{2aczz}\right)^n.$$

Pofito ergo $y=1$ vt fit $x=z$, fi omnes poteftates trinomii $a+bz+czz$ euoluantur, feries terminorum intermediorum $1+bz+(2ac+bb)zz$ etc. erit $= P$

terminorum autem a mediis, n locis in antecedentia remotorum fumma eft $=a^n Z$, totidem vero locis in confequentia remotorum fumma $=c^n z^n Z$. At omnium harum ferierum iunctim fumtarum fumma eft $= \frac{1}{1-a-bz-czz}$.

Coroll. 1.

20 Quantitates ergo P, Q, R, S etc. progreffionem geometricam conftituunt, cuius primus terminus eft $P = \frac{1}{\surd(1-2bz+(bb-4ac)zz)}$, et denominator progreffionis:

$$\frac{1-bz-\surd(1-2bz+(bb-4ac)zz)}{2aczz}.$$

Coroll. 2.

21. Si fumamus $a=1$, $b=1$ et $c=1$, prodir cafus ante tractatus quo poteftates trinomii

$1+z$

$1+z+zz$ conſiderauimus, quarum termini medii ſeriem conſtituunt, cuius ſumma eſt $=\frac{1}{\sqrt{(1-2z-3zz)}}$, vti ſupra inuenimus.

Problema 5.

22. Formulam in praecedente problemate inventam:

$$\frac{1}{\sqrt{(1-2bz+(bb-4ac)zz)}}\left(\frac{1-bz-\sqrt{(1-2bz+(bb-4ac)zz)}}{2aczz}\right)^n.$$

in ſeriem conuertere, cuius termini ſecundum dignitates ipſius z procedant.

Solutio.

Sit breuitatis gratia $bb-4ac=e$, ac ponatur

$$s=\frac{1}{\sqrt{(1-2bz+ezz)}}\left(\frac{1-bz-\sqrt{(1-2bz+ezz)}}{2aczz}\right)^n$$

quam relationem inter z et s per differentiationem ab irrationalitate liberari oportet. Hunc in finem ſtatuatur:

$\frac{1-bz-\sqrt{(1-2bz+ezz)}}{2aczz}=v$ vt ſit $acvvzz-(1-bz)v+1=0$

vnde differentiando fit:

$dv(2acvzz-1+bz)+vdz(2acvz+b)=0$ ſeu

$dv\sqrt{(1-2bz+ezz)}=\frac{vdz}{z}(1-\sqrt{(1-2bz+ezz)})$

ideoque $\frac{dv}{v}=\frac{dz}{z\sqrt{(1-2bz+ezz)}}-\frac{dz}{z}$.

Hinc illa aequatione logarithmice differentiata prodit

$$\frac{ds}{s}=\frac{dz(b-ez)}{1-2bz+ezz}-\frac{ndz}{z}+\frac{ndz}{z\sqrt{(1-2bz+ezz)}}.$$

 Pona-

Ponamus tantisper $\frac{dt}{t} = \frac{ds}{s} + \frac{n\,dz}{z} - \frac{dz(b-ez)}{1-2bz+ezz}$, vt ſit $\frac{dt}{t} = \frac{n\,dz}{z\sqrt{(1-2bz+ezz)}}$, vnde quadrata ſumendo colligimus: $zz\,dt^2(1-2bz+ezz) = nn\mathrm{H}\,dz^2$, quae aequatio denuo differentiata poſito elemento dz conſtante dat:

$$zz\,ddt(1-2bz+ezz) + z\,dt\,dz(1-3bz+2ezz) = nnt\,dz^2$$

ſeu $\frac{ddt}{t} + \frac{dz(1-3bz+2ezz)}{z(1-2bz+ezz)} \cdot \frac{dt}{t} - \frac{nn\,dz^2}{zz(1-2bz+ezz)} = 0.$

Iam cum ſit $\frac{ddt}{t} = d.\frac{dt}{t} + \frac{dt^2}{tt}$ erit

$$\frac{ddt}{t} = \frac{dds}{s} - \frac{ds^2}{ss} - \frac{n\,dz^2}{zz} + \frac{dz^2(e-2bb+2bez-eezz)}{(1-2bz+ezz)^2} + \frac{nn\,dz^2}{zz} - \frac{2n\,dz^2(b-ez)}{z(1-2bz+ezz)}$$
$$+\frac{ds^2}{ss} + \frac{2n\,dz\,ds}{sz} - \frac{2\,dz\,ds(b-ez)}{s(1-2bz+ezz)} + \frac{dz^2(bb-2bez+eezz)}{(1-2bz+ezz)^2}.$$

Facta ergo ſubſtitutione ſuperior aequatio in hanc abit formam:

$$\frac{dds}{s} + \frac{2n\,dz}{z}\cdot\frac{ds}{s} - \frac{2\,dz(b-ez)}{1-2bz+ezz}\cdot\frac{ds}{s} + \frac{n(n-1)dz^2}{zz} - \frac{2n\,dz^2(b-ez)}{z(1-2bz+ezz)} + \frac{dz^2(e-bb)}{(1-2bz+ezz)^2}$$
$$+\frac{dz(1-3bz+2ezz)}{z(1-2bz+ezz)}\cdot\frac{ds}{s} + \frac{n\,dz^2(1-3bz+2ezz)}{zz(1-2bz+ezz)} - \frac{dz^2(b-(e+3bb)z+5bezz-2eez^3)}{z(1-2bz+ezz)^2}$$
$$-\frac{nn\,dz^2}{zz(1-2bz+ezz)} = 0$$

vbi ſi termini per $(1-2bz+ezz)^2$ diuiſi in vnam ſummam colligantur, fractio per $1-2bz+ezz$ deprimi poterit, vnde facta reductione adipiſcimur:

$$\frac{dds}{s} + \frac{2n\,dz}{z}\cdot\frac{ds}{s} + \frac{dz(1-5bz+4ezz)}{z(1-2bz+ezz)}\cdot\frac{ds}{s} - \frac{nn\,dz^2(2b-ez)}{z(1-2bz+ezz)} - \frac{3n\,dz^2(b-ez)}{z(1-2bz+ezz)}$$
$$-\frac{dz^2(b-2ez)}{z(1-2bz+ezz)} = 0$$

quae ordinata euadit:

$$z\,dds(1-2bz+ezz) + dz\,ds(2n+1-(4n+5)bz+2(n+2)ezz)$$
$$-s\,dz^2((n+1)(2n+1)b-(n+1)(n+2)zz) = 0.$$

Cum

Cum nunc conſtet poſito $z=0$ fieri $s=1$, fingamus hanc ſeriem :

$$s=1+Az+Bzz+Cz^3+Dz^4+Ez^5+\text{etc.}$$

qua ſerie ſubſtituta ſequens forma ad nihilum eſt redigenda :

$$\begin{array}{rrrrr}
 & 2Bz+6Czz & +12Dz^3 & +20Ez^4 & \\
 & -4Bb & -12Cb & -24Db & \\
 & & +2Be & +\ 6Ce & \\
(2n+1)A & +2(2n+1)B & +3(2n+1)C & +4(2n+1)D & +5(2n+1)E \\
 & -(4n+5)A & -2(4n+5)Bb & -3(4n+5)Cb & -4(4n+5)Db \\
 & & +2(n+2)Ae & +4(n+2)Be & +6(n+2)Ce \\
-(n+1)(2n+1)b & -(n+1)(2n+1)Ab & -(n+1)(2n+1)Bb & -(n+1)(2n+1)Cb & -(n+1)(2n+1)De \\
 & +(n+1)(n+2)e & +(n+1)(n+2)Ae & +(n+1)(n+2)Be & +(n+1)(n+2)Ce
\end{array}$$

vnde colligimus has determinationes :

$$A=(n+1)b$$

$$B=\frac{(n+2)((2n+3)Ab-(n+1)e)}{2(2n+2)}$$

$$C=\frac{(n+3)((2n+5)Bb-(n+2)Ae)}{3(2n+3)}$$

$$D=\frac{(n+4)((2n+7)Cb-(n+3)Be)}{4(2n+4)}$$

$$E=\frac{(n+5)((2n+9)Db-(n+4)Ce)}{5(2n+5)}$$

etc

vbi notetur eſſe $e=bb-4ac$. Sicque ſeriei quaeſitae ſinguli termini per binos praecedentes determinantur.

DE