DE

# VSV NOVI ALGORITHMI IN PROBLEMATE PELLIANO SOLVENDO.

Auctore

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1.

Quicunque numeri integri pro litteris $l$, $m$ et $n$ assumantur, innumerabiles quoque numeri integri pro $x$ inueniri possunt, quibus haec formula:

$lxx + mx + n$ reddatur quadratum;

siquidem sequentes conditiones habeant locum:

1. vt $l$ sit numerus positiuus non quadratus.
2. vt pro $x$ vnus saltem valor sit cognitus.

Nam si $l$ est numerus, vel negatiuus, vel quadratus, manifestum est, infinitas solutiones in numeris integris exhiberi non posse, etiamsi vna innotuerit. Tum vero etiam euenire potest, vt formula $lxx + mx + n$ naturae quadrati prorsus aduersetur, vti sit hoc casu $3xx + 2$. Verum statim atque vnica solutio habetur, semper innumerabiles inuenire licet.

2. Quare si statuamus:

$$lxx + mx + n = yy,$$

vnus-

vnusque casus constet, quo huic conditioni satisfiat, ita vt posito $x = a$, prodeat

$$laa + ma + n = bb$$

sicque sumto $x = a$ obtineatur $y = b$; regula, cuius ope plures imo infinitae solutiones elici possunt, ita se habet:

Primo ex dato numero $l$ duo huiusmodi numeri $p$ et $q$ inuestigentur, vt sit

$$pp = lqq + 1, \text{ seu } p = \surd(lqq + 1)$$

quibus inuentis ex solutione primo cognita statim eruitur haec noua:

$$x = pa + qb + \frac{m(p-1)}{2l}, \text{ vnde fit}$$

$$y = pb + lqa + \frac{mq}{2}$$

ex qua deinceps simili modo aliae deriuantur. Si enim hos valores loco $a$ et $b$ substituamus, nascitur tertia solutio ista:

$$x = (2pp - 1)a + 2pqb + mqq, \text{ et}$$

$$y = (2pp - 1)b + 2lpqa + mpq$$

quae certe est in numeris integris, si forte praecedentes adhuc fuerint fracti.

3. Cum igitur hoc modo continuo nouae solutiones inueniri queant, ad calculi compendium plurimum iuuat notasse, continuos istos valores, tam ipsius $x$, quam ipsius $y$, secundum seriem recurrentem progredi, cuius singuli termini per binos prae-

 ceden-

cedentes certa et constante lege determinentur. Scilicet si fuerint valores hi continuo progredientes:

ipsius $x$ . . . . $a$ . . . . . P, Q, R, S etc.

ipsius $y$ . . . . $b$ . . . . . $\mathfrak{P}, \mathfrak{Q}, \mathfrak{R}, \mathfrak{S}$ etc.

erit per legem seriei recurrentis:

$$R = 2pQ - P + \frac{m(p-1)}{l}; \quad \mathfrak{R} = 2p\mathfrak{Q} - \mathfrak{P}$$

$$S = 2pR - Q + \frac{m(p-1)}{l}; \quad \mathfrak{S} = 2p\mathfrak{R} - \mathfrak{Q}.$$

Atque hinc isti valores expressionibus generalibus comprehendi possunt, quae ita se habent:

$$x = \frac{2la + m + 2b\sqrt{l}}{4l}(p + q\sqrt{l})^{\mu} + \frac{2la + m - 2b\sqrt{l}}{4l}(p - q\sqrt{l})^{\mu} - \frac{m}{2l}$$

$$y = \frac{2la + m + 2b\sqrt{l}}{4\sqrt{l}}(p + q\sqrt{l})^{\mu} - \frac{2la - m + 2b\sqrt{l}}{4\sqrt{l}}(p - q\sqrt{l})^{\mu}$$

vnde quicunque numeri integri exponenti $\mu$ tribuantur, semper valores rationales pro $x$ et $y$ resultant.

4. Haec autem inuestigatio multo latius ita potest extendi, vt proposita inter binos numeros $x$ et $y$ huiusmodi aequatione:

$$Axx + 2Bxy + Cyy + 2Dx + 2Ey + F = 0$$

omnes solutiones in numeris rationalibus et integris sint eruendae. Hic quidem pariter necesse est, vnam solutionem esse cognitam, quae sit $x = a$, et $y = b$, ita vt sit

$$Aaa + 2Bab + Cbb + 2Da + 2Eb + F = 0.$$

Tum vero quaerantur bini numeri $p$ et $q$, vt sit

$$pp = (BB - AC)qq + 1$$

quod

quod quidem fieri nequit, nisi sit $BB > AC$. Atque noua solutio ita erit comparata:

$$x = a(p + Bq) + bCq + Eq + \frac{BE - CD}{BB - AC}(p - 1)$$
$$y = b(p - Bq) - aAq - Dq + \frac{BD - AE}{BB - AC}(p - 1)$$

vnde per eandem legem continuo plures elicere licet.

5. Haec ideo in medium afferre est visum, vt intelligatur, ad omnes huius generis resolutiones id omnino requiri, vt proposito quocunque numero integro positiuo non quadrato $l$, eiusmodi binos numeros pariter integros $p$ et $q$ inueniri oporteat, vt fit $pp = lqq + 1$, seu $p = \sqrt{(lqq + 1)}$. Atque hoc est illud problema olim quidem maxime celebratum a solutionis ingeniosissimae auctore *Pellianum* vocatum, quo pro quouis huiusmodi numero $l$ numerus quadratus $qq$ requiritur, qui per $l$ multiplicatus adiuncta vnitate fiat quadratus. In fractis quidem haec quaestio nullam haberet difficultatem, cum sumto $q = \frac{2rs}{lss - rr}$, fiat $p = \frac{lss + rr}{lss - rr}$: verum quia numeri integri desiderantur, negotium iterum eo reuocatur, vt denominator $lss - rr$ in vnitatem abeat.

6. Etiamsi autem solutio *Pelliana* huius problematis sit elegantissima, tamen saepenumero tam operosis calculis implicatur, qui non minus taedii quam laboris creare solent. Cum igitur obseruassem, algorithmum illum nouum, cuius nuper indolem exposui, ad hos calculos, quibus hic est opus, contrahendos, insignia subsidia suppeditare, praeclarius certe

certe ſpecimen exhibere vix licebit, quo vſus iſtius algorithmi illuſtretur et commendetur. Vbi id imprimis notatu dignum occurrit, quod totum compendium inde ſubminiſtratum potiſſimum idoneorum ſignorum vſu contineatur.

7. Operationes, quibus *Pellius* eſt vſus, aliunde quidem ſatis ſunt notae, egoque iam eas alia occaſione fuſius deſcripſi; ex quo eo minus opus eſt, vt iis denuo explicandis hic immorer, cum totam Analyſin hic longe alia ratione ſim inſtituturus. Eius ſcilicet principium ex hoc fonte haurio, quod cum ſit $pp = lqq + 1$, proxime fiat $\frac{p}{q} = \sqrt{l}$, ex quo manifeſtum eſt, $\frac{p}{q}$ eiusmodi eſſe fractionem, quae valorem irrationalem $\sqrt{l}$ tam prope exprimat, ſeu eum tam parum excedat, vt id, niſi maioribus numeris adhibendis, accuratius fieri nequeat. Quod problema, olim feliciter a *Wallisio* ſolutum, equidem quoque iam dudum per fractiones continuas multo commodius expediui.

8. Quo ergo hoc argumentum luculentius et ordine pertractem, primum radicem quadratam ex quouis numero in fractionem continuam euoluere docebo, idque methodo quam minime moleſta. Deinde oſtendam, quomodo inde fractiones $\frac{p}{q}$ valorem irrationalem $\sqrt{l}$ proxime exprimentes formari debeant, in ſubſidium vocato Algorithmo nouo ſupra explicato. Tum vero facile patebit, quomodo hinc nume-

numeros $p$ et $q$ definiri oporteat, vt fiat $pp = lqq + 1$. Denique tabulam subiungam, in qua pro omnibus numeris $l$, centenarium non superantibus, numeri bini $p$ et $q$ exhibentur.

# De euolutione radicum quadratarum per fractiones continuas.

9. Operationes in hunc finem constituendae in exemplo facillime explicabuntur. Sit igitur proposita radix quadrata ex numero 13, et cum radix rationalis proxime minor sit 3, statuo $\sqrt{13} = 3 + \frac{1}{a}$. Hinc colligitur

$$a = \frac{1}{\sqrt{13} - 3} = \frac{\sqrt{13} + 3}{4}$$

cuius valor in integris proxime minor est 1, quod inde patet, si 3 loco $\sqrt{13}$ scribatur. Pono itaque

$$a = \frac{\sqrt{13} + 3}{4} = 1 + \frac{1}{b}, \text{ hincque}$$

$$b = \frac{4}{\sqrt{13} - 1} = \frac{4(\sqrt{13} + 1)}{12} = \frac{\sqrt{13} + 1}{3} = 1 + \frac{1}{c}$$

Ergo $c = \frac{3}{\sqrt{13} - 2} = \frac{3(\sqrt{13} + 2)}{9} = \frac{\sqrt{13} + 2}{3} = 1 + \frac{1}{d}$

Ergo $d = \frac{3}{\sqrt{13} - 1} = \frac{3(\sqrt{13} + 1)}{12} = \frac{\sqrt{13} + 1}{4} = 1 + \frac{1}{e}$

Ergo $e = \frac{4}{\sqrt{13} - 3} = \frac{4(\sqrt{13} + 3)}{4} = \sqrt{13} + 3 = 6 + \frac{1}{f}$

Ergo $f = \frac{1}{\sqrt{13} - 3} = \frac{\sqrt{13} + 3}{4} = 1 + \frac{1}{g}$

atque hic operationem abrumpere licet, quia va-

 lor

lor $f$ ipsi $a$ aequalis prodiit, ideoque sequentes eodem ordine repetuntur. Sicque inuenimus esse

$$\sqrt{13} = 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \text{etc.}}}}}}}}}}}$$

10. Quo indoles harum operationum melius perspiciatur, aliud exemplum, prolixiorem calculum postulans, adiungam. Proposita scilicet sit $\sqrt{61}$, cuius valor proxime minor cum sit 7, pono $\sqrt{61} = 7 + \frac{1}{a}$, et operationes sequenti modo erunt instituendae:

I. $a = \frac{1}{\sqrt{61}-7} = \frac{\sqrt{61}+7}{12} = 1 + \frac{1}{b}$

II. $b = \frac{12}{\sqrt{61}-5} = \frac{12(\sqrt{61}+5)}{36} = \frac{\sqrt{61}+5}{3} = 4 + \frac{1}{c}$

III. $c = \frac{3}{\sqrt{61}-7} = \frac{3(\sqrt{61}+7)}{12} = \frac{\sqrt{61}+7}{4} = 3 + \frac{1}{d}$

IV. $d = \frac{4}{\sqrt{61}-5} = \frac{4(\sqrt{61}+5)}{36} = \frac{\sqrt{61}+5}{9} = 1 + \frac{1}{e}$

V. $e = \frac{9}{\sqrt{61}-4} = \frac{9(\sqrt{61}+4)}{45} = \frac{\sqrt{61}+4}{5} = 2 + \frac{1}{f}$

VI $f = \frac{5}{\sqrt{61}-6} = \frac{5(\sqrt{61}+6)}{25} = \frac{\sqrt{61}+6}{5} = 2 + \frac{1}{g}$

VII. $g = \frac{5}{\sqrt{61}-4} = \frac{5(\sqrt{61}+4)}{45} = \frac{\sqrt{61}+4}{9} = 1 + \frac{1}{h}$

VIII.

VIII. $h = \frac{9}{\sqrt{61}-5} = \frac{9(\sqrt{61}+5)}{36} = \frac{\sqrt{61}+5}{4} = 3 + \frac{1}{i}$

IX. $i = \frac{4}{\sqrt{61}-7} = \frac{4(\sqrt{61}+7)}{12} = \frac{\sqrt{61}+7}{3} = 4 + \frac{1}{k}$

X. $k = \frac{3}{\sqrt{61}-5} = \frac{3(\sqrt{61}+5)}{36} = \frac{\sqrt{61}+5}{12} = 1 + \frac{1}{l}$

XI. $l = \frac{12}{\sqrt{61}-7} = \frac{12(\sqrt{61}+7)}{12} = \sqrt{61}+7 = 14 + \frac{1}{m}$

XII. $m = \frac{1}{\sqrt{61}-7}$ ergo $m = a$, hincque porro $n = b$, $o = c$. etc. Ex quo indices pro fractione continua erunt:

7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, etc. neque opus est, ipsam fractionem continuam hic exhibere.

11. Adhuc aliud exemplum adiecisse iuuabit, vbi indicum numerus, antequam iidem recurrunt, fit impar. Esto hoc exemplum: $\sqrt{67} = 8 + \frac{1}{a}$; et operationes sequentes institui oportebit:

I. $a = \frac{1}{\sqrt{67}-8} = \frac{\sqrt{67}-8}{3} = 5 + \frac{1}{b}$

II. $b = \frac{3}{\sqrt{67}-7} = \frac{3(\sqrt{67}+7)}{18} = \frac{\sqrt{67}+7}{6} = 2 + \frac{1}{c}$

III. $c = \frac{6}{\sqrt{67}-5} = \frac{6(\sqrt{67}+5)}{42} = \frac{\sqrt{67}+5}{7} = 1 + \frac{1}{d}$

IV. $d = \frac{7}{\sqrt{67}-2} = \frac{7(\sqrt{67}+2)}{63} = \frac{\sqrt{67}+2}{9} = 1 + \frac{1}{e}$

V. $e = \frac{9}{\sqrt{67}-7} = \frac{9(\sqrt{67}+7)}{18} = \frac{\sqrt{67}+7}{2} = 7 + \frac{1}{f}$

VI. $f = \frac{2}{\sqrt{67}-7} = \frac{2(\sqrt{67}+7)}{18} = \frac{\sqrt{67}+7}{9} = 1 + \frac{1}{g}$

VII. $g = \frac{9}{\sqrt{67}-2} = \frac{9(\sqrt{67}+2)}{63} = \frac{\sqrt{67}+2}{7} = 1 + \frac{1}{h}$

VIII. $h = \frac{7}{\sqrt{67}-5} = \frac{7(\sqrt{67}+5)}{42} = \frac{\sqrt{67}+5}{6} = 2 + \frac{1}{i}$

IX. $i = \frac{6}{\sqrt{67} - 7} = \frac{6(\sqrt{67} + 7)}{18} = \frac{\sqrt{67} + 7}{3} = 5 + \frac{1}{k}$

X. $k = \frac{3}{\sqrt{67} - 8} = \frac{3(\sqrt{67} + 8)}{3} = \sqrt{67} + 8 = 16 + \frac{1}{l}$

XI. $l = \frac{1}{\sqrt{67} - 8}$, ergo $l = a$, indeque indices $b, c, d$ etc. ordine recurrunt: quare indices fractionis continuae quaesitae sunt:

8, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16 etc.

12. His exemplis probe perpensis, poterimus iam in genere operationes describere, quibus pro cuiusuis numeri radice quadrata fractio continua ipsi aequalis, seu indices eam constituentes, inueniuntur. Sit scilicet numerus propositus $= z$, eiusque radix integra proxime minor $= v$, vera autem hac fractione continua exprimatur:

$$\sqrt{z} = v + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}}$$

cuius indices $a$, $b$, $c$, $d$ etc. post primum $v$, per se cognitum, sequentibus operationibus reperiuntur:

| Capiatur | tum vero | eritque |
|---|---|---|
| I. $A = v$; | $\alpha = z - AA = z - vv$ | $a < \frac{v + A}{\alpha}$ |
| II. $B = \alpha a - A$ | $\beta = \frac{z - BB}{\alpha} = 1 + a(A - B)$ | $b < \frac{v + B}{\beta}$ |
| III. $C = \beta b - B$ | $\gamma = \frac{z - CC}{\beta} = \alpha + b(B - C)$ | $c < \frac{v + C}{\gamma}$ |
| IV. $D = \gamma c - C$ | $\delta = \frac{z - DD}{\gamma} = \beta + c(C - D)$ | $d < \frac{v + D}{\delta}$ |
| V. $E = \delta d - D$ | $\varepsilon = \frac{z - EE}{\delta} = \gamma + d(D - E)$ | $e < \frac{v + E}{\varepsilon}$ |

etc.

vbi

vbi in postrema columna signum $\lessdot$ indicat, pro litteris $a$, $b$, $c$, $d$ etc. sumi debere numeros integros proxime minores fractionibus adiectis, nisi hae fractiones ipsae in numeros integros abeant, quo casu hi ipsi erunt indices.

13. Pro indicibus igitur $a$, $b$, $c$, $d$ etc. eliciendis binas alias numerorum series inuestigari oportet, quarum priorem litteris maiusculis A, B, C, D etc. posteriorem vero graecis $\alpha$, $\beta$, $\gamma$, $\delta$ etc. designaui. Circa priores numeros obseruo, eos numerum $v$ nunquam superare posse, eorum quidem primus est $A = v$, at cum sit $a \lessdot \frac{v + A}{\alpha}$, erit $a\alpha - A \lessdot v$, ideoque $B \lessdot v$, vel ad summum $B = v$, quo casu fit $\beta = 1$, et $b = 2v$. Deinde ob $b \lessdot \frac{v + \beta}{\beta}$ est $\beta b - B = C \lessdot v$, similique modo $D \lessdot v$, $E \lessdot v$ etc. ita vt horum numerorum nullus ipso $v$ maior prodire possit. Deinde patet, praeter casus, quibus graecarum litterarum quaepiam fit vnitas, indices $a$, $b$, $c$, etc. omnes ipso $v$ maiores esse non posse, quandoquidem in fractionibus $\frac{v + B}{\beta}$, $\frac{v + C}{\gamma}$ etc. numeratores non excedere possunt $2v$, denominatores vero ad minimum sint $= 2$. Denique cum fuerit perventum ad indicem $= 2v$, sequentes iterum prodeunt $a$, $b$, $c$, $d$ etc.

14. Illustremus etiam has operationes nonnullis exemplis.

I. Sit $z = 31$, erit $v = 5$

| | | |
|---|---|---|
| $A = 5$ | $\alpha = 6$ | $a < \frac{10}{6} = 1$ |
| $B = 6 - 5 = 1$; | $\beta = 1 + 1.4 = 5$; | $b < \frac{6}{5} = 1$ |
| $C = 5 - 1 = 4$; | $\gamma = 6 - 1.3 = 3$; | $c < \frac{9}{3} = 3$ |
| $D = 9 - 4 = 5$; | $\delta = 5 - 3.1 = 2$; | $d < \frac{10}{2} = 5$ |
| $E = 10 - 5 = 5$; | $\varepsilon = 3 - 5.0 = 3$; | $e < \frac{10}{3} = 3$ |
| $F = 9 - 5 = 4$; | $\zeta = 2 + 3.1 = 5$; | $f < \frac{9}{5} = 1$ |
| $G = 5 - 4 = 1$; | $\eta = 3 + 1.3 = 6$; | $g < \frac{6}{6} = 1$ |
| $H = 6 - 1 = 5$; | $\theta = 5 - 1.4 = 1$; | $h < \frac{10}{1} = 10$. |

II. Sit $z = 46$, erit $v = 6$

| | | |
|---|---|---|
| $A = 6$; | $\alpha = 10$ | $a = \frac{12}{10} = 1$ |
| $B = 10 - 6 = 4$; | $\beta = 1 + 1.2 = 3$; | $b = \frac{10}{3} = 3$ |
| $C = 9 - 4 = 5$; | $\gamma = 10 - 3.1 = 7$; | $c < \frac{11}{7} = 1$ |
| $D = 7 - 5 = 2$; | $\delta = 3 + 1.3 = 6$; | $d < \frac{8}{6} = 1$ |
| $E = 6 - 2 = 4$; | $\varepsilon = 7 - 1.2 = 5$; | $e < \frac{10}{5} = 2$ |
| $F = 10 - 4 = 6$; | $\zeta = 6 - 2.2 = 2$; | $f < \frac{12}{2} = 6$ |
| $G = 12 - 6 = 6$; | $\eta = 5 - 6.0 = 5$; | $g < \frac{12}{5} = 2$ |
| $H = 10 - 6 = 4$; | $\theta = 2 + 2.2 = 6$; | $h < \frac{10}{6} = 1$ |
| $I = 6 - 4 = 2$; | $\iota = 5 + 1.2 = 7$; | $i < \frac{8}{7} = 1$ |
| $K = 7 - 2 = 5$: | $\kappa = 6 - 1.3 = 3$; | $k < \frac{11}{3} = 3$ |
| $L = 9 - 5 = 4$; | $\lambda = 7 + 3.1 = 10$; | $l < \frac{10}{10} = 1$ |
| $M = 10 - 4 = 6$; | $\mu = 3 - 1.2 = 1$; | $m < \frac{12}{1} = 12$. |

III. Sit $z = 54$, erit $v = 7$;

| | | |
|---|---|---|
| $A = 7$; | $\alpha = 5$; | $a < \frac{14}{5} = 2$ |
| $B = 10 - 7 = 3$; | $\beta = 1 + 2.4 = 9$; | $b < \frac{10}{9} = 1$ |
| $C = 9 - 3 = 6$; | $\gamma = 5 - 1.3 = 2$; | $c < \frac{13}{2} = 6$ |
| $D = 12 - 6 = 6$; | $\delta = 9 + 6.0 = 9$; | $d < \frac{15}{9} = 1$ |
| $E = 9 - 6 = 3$; | $\varepsilon = 2 + 1.3 = 5$; | $e < \frac{10}{5} = 2$ |
| $F = 10 - 3 = 7$; | $\zeta = 9 - 2.4 = 1$; | $f < \frac{14}{1} = 14$. |

15. Tabulam ergo hic subiungam, pro singulorum numerorum radicibus quadratis indices continentem, ex quibus fractiones continuae ipsis aequales formari queant. Simul vero litterarum graecarum singulis conuenientium valores subscripti reperiuntur.

| Numeri surdi. | Indices. |
|---|---|
| $\sqrt{2}$ | 1, 2, 2, 2 etc.<br>1 1 1 1 |
| $\sqrt{3}$ | 1, 1, 2, 1, 2, 1, 2 etc.<br>1 2 1 2 1 2 1 |
| $\sqrt{5}$ | 2, 4, 4, 4 etc.<br>1 1 1 1 |
| $\sqrt{6}$ | 2, 2, 4, 2, 4, 2, 4 etc.<br>1 2 1 2 1 2 1 |
| $\sqrt{7}$ | 2, 1, 1, 1, 4, 1, 1, 1, 4 etc.<br>1 3 2 3 1 3 2 3 1 |
| $\sqrt{8}$ | 2, 1, 4, 1, 4, 1, 4 etc.<br>1 4 1 4 1 4 1 |
| $\sqrt{10}$ | 3, 6, 6, 6 etc.<br>1 1 1 1 |
| $\sqrt{11}$ | 3, 3, 6, 3, 6, 3, 6 etc.<br>1 2 1 2 1 2 1 |
| $\sqrt{12}$ | 3, 2, 6, 2, 6, 2, 6 etc.<br>1 3 1 3 1 3 1 |
| $\sqrt{13}$ | 3, 1, 1, 1, 1, 6, 1, 1, 1, 4, 6 etc.<br>1 4 3 3 4 1 4 3 3 4 1 |
| $\sqrt{14}$ | 3, 1, 2, 1, 6, 1, 2, 1, 6 etc.<br>1 5 2 5 1 5 2 5 1 |
| $\sqrt{15}$ | 3, 1, 6, 1, 6, 1, 6 etc.<br>1 6 1 6 1 6 1 |
| $\sqrt{17}$ | 4, 8, 8, 8, 8 etc.<br>1 1 1 1 1 |
| $\sqrt{18}$ | 4, 4, 8, 4, 8, 4, 8, 4, 8 etc.<br>1 2 1 2 1 2 1 2 1 |

4, 2,
1 3

| Numeri surdi. | Indices. |
|---|---|
| √19 | 4, 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8 etc.<br>1 3 5 2 5 3 1 3 5 2 5 3 1 |
| √20 | 4, 2, 8, 2, 8, 2, 8, 2, 8 etc.<br>1 4 1 4 1 4 1 4 1 |
| √21 | 4, 1, 1, 2, 1, 1, 8, 1, 1, 2, 1, 1, 8 etc.<br>1 5 4 3 4 5 1 5 4 3 4 5 1 |
| √22 | 4, 1, 2, 4, 2, 1, 8, 1, 2, 4, 2, 1, 8 etc.<br>1 6 3 2 3 6 1 6 3 2 3 6 1 |
| √23 | 4, 1, 3, 1, 8, 1, 3, 1, 8 etc.<br>1 7 2 7 1 7 2 7 1 |
| √24 | 4, 1, 8, 1, 8, 1, 8 etc.<br>1 8 1 8 1 8 1 |
| √26 | 5, 10, 10, 10 etc.<br>1 1 1 1 |
| √27 | 5, 5, 10, 5, 10, 5, 10 etc.<br>1 2 1 2 1 2 1 |
| √28 | 5, 3, 2, 3, 10, 3, 2, 3, 10 etc.<br>1 3 4 3 1 3 4 3 1 |
| √29 | 5, 2, 1, 1, 2, 10, 2, 1, 1, 2, 10 etc.<br>1 4 5 5 4 1 4 5 5 4 1 |
| √30 | 5, 2, 10, 2, 10, 2, 10, 2, 10 etc.<br>1 5 1 5 1 5 1 5 1 |
| √31 | 5, 1, 1, 3, 5, 3, 1, 1, 10 etc.<br>1 6 5 3 2 3 5 6 1 |
| √32 | 5, 1, 1, 1, 10, 1, 1, 1, 10 etc.<br>1 7 4 7 1 7 4 7 1 |
| √33 | 5, 1, 2, 1, 10, 1, 2, 1, 10 etc.<br>1 8 3 8 1 8 3 8 1 |
| √34 | 5, 1, 4, 1, 10, 1, 4, 1, 10 etc.<br>1 9 2 9 1 9 2 9 1 |
| √35 | 5, 1, 10, 1, 10, 1, 10 etc.<br>1 10 1 10 1 10 1 |
| √37 | 6, 12, 12, 12 etc.<br>1 1 1 1 |
| √38 | 6, 6, 12, 6, 12, 6, 12 etc.<br>1 2 1 2 1 2 1 |

6, 4,
1 2

| Numeri surdi. | Indices. |
|---|---|
| $\sqrt{39}$ | 6, 4, 12, 4, 12, 4, 12 etc.<br>1 3 1 3 1 3 1 |
| $\sqrt{40}$ | 6, 3, 12, 3, 12, 3, 12 etc.<br>1 4 1 4 1 4 1 |
| $\sqrt{41}$ | 6, 2, 2, 12, 2, 2, 12 etc.<br>1 5 5 1 5 5 1 |
| $\sqrt{42}$ | 6, 2, 12, 2, 12, 2, 12 etc.<br>1 6 1 6 1 6 1 |
| $\sqrt{43}$ | 6, 1, 1, 3, 1, 5, 1, 3, 1, 1, 12 etc.<br>1 7 6 3 9 2 9 3 6 7 1 |
| $\sqrt{44}$ | 6, 1, 1, 1, 2, 1, 1, 1, 12 etc.<br>1 8 5 7 4 7 5 8 1 |
| $\sqrt{45}$ | 6, 1, 2, 2, 2, 1, 12, 1, 2, 2, 2, 1, 12 etc.<br>1 9 4 5 4 9 1 9 4 5 4 9 1 |
| $\sqrt{46}$ | 6, 1, 3, 1, 1, 2, 6, 2, 1, 1, 3, 1, 12 etc.<br>1 10 3 7 6 5 2 5 6 7 3 10 1 |
| $\sqrt{47}$ | 6, 1, 5, 1, 12, 1, 5, 1, 12 etc.<br>1 11 2 11 1 11 2 11 1 |
| $\sqrt{48}$ | 6, 1, 12, 1, 12, 1, 12 etc.<br>1 12 1 12 1 12 1 |
| $\sqrt{50}$ | 7, 14, 14, 14 etc.<br>1 1 1 1 |
| $\sqrt{51}$ | 7, 7, 14, 7, 14, 7, 14 etc.<br>1 2 1 2 1 2 1 |
| $\sqrt{52}$ | 7, 4, 1, 2, 1, 4, 14, 4, 1, 2, 1, 4, 14 etc.<br>1 3 9 4 9 3 1 3 9 4 9 7 1 |
| $\sqrt{53}$ | 7, 3, 1, 1, 3, 14, 3, 1, 1, 3, 14 etc.<br>1 4 7 7 4 1 4 7 7 4 1 |
| $\sqrt{54}$ | 7, 2, 1, 6, 1, 2, 14, 2, 1, 6, 1, 2, 14 etc.<br>1 5 9 2 9 5 1 5 9 2 9 5 1 |
| $\sqrt{55}$ | 7, 2, 2, 2, 14, 2, 2, 2, 14, 2, 2, 2, 14 etc.<br>1 6 5 6 1 6 5 6 1 6 5 6 1 |
| $\sqrt{56}$ | 7, 2, 14, 2, 14, 2, 14 etc.<br>1 7 1 7 1 7 1 |
| $\sqrt{57}$ | 7, 1, 1, 4, 1, 1, 14 etc.<br>1 8 7 3 7 8 1 |



7, 1,
1 8

| Numeri surdi. | Indices. |
|---|---|
| $\sqrt{58}$ | 7, 1, 1, 1, 1, 1, 1, 14 etc.<br>1 9 6 7 7 6 9 1 |
| $\sqrt{59}$ | 7, 1, 2, 7, 2, 1, 14 etc.<br>1 10 5 2 5 10 1 |
| $\sqrt{60}$ | 7, 1, 2, 1, 14 etc.<br>1 11 4 11 1 |
| $\sqrt{61}$ | 7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14 etc.<br>1 12 3 4 9 5 5 9 4 3 12 1 |
| $\sqrt{62}$ | 7, 1, 6, 1, 14 etc.<br>1 13 2 13 1 |
| $\sqrt{63}$ | 7, 1, 14, 1, 14 etc.<br>1 14 1 14 1 |
| $\sqrt{65}$ | 9, 16, 16 etc.<br>1 1 1 |
| $\sqrt{66}$ | 8, 8, 16, 8, 16 etc.<br>1 2 1 2 1 |
| $\sqrt{67}$ | 8, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16 etc.<br>1 3 6 7 9 2 9 7 6 3 1 |
| $\sqrt{68}$ | 8, 5, 16, 4, 16 etc.<br>1 4 1 4 1 |
| $\sqrt{69}$ | 8, 3, 3, 1, 4, 1, 3, 3, 16 etc.<br>1 5 4 11 3 11 4 5 1 |
| $\sqrt{70}$ | 8, 2, 1, 2, 1, 2, 16 etc.<br>1 6 9 5 9 6 1 |
| $\sqrt{71}$ | 8, 2, 2, 1, 7, 1, 2, 2, 16 etc.<br>1 7 5 11 2 11 5 7 1 |
| $\sqrt{72}$ | 8, 2, 16, 2, 16 etc.<br>1 8 1 8 1 |
| $\sqrt{73}$ | 8, 1, 1, 5, 5, 1, 1, 16 etc.<br>1 9 8 3 3 8 9 1 |
| $\sqrt{74}$ | 8, 1, 1, 1, 1, 16 etc.<br>1 10 7 7 10 1 |
| $\sqrt{75}$ | 8, 1, 1, 1, 16 etc.<br>1 11 6 11 1 |
| $\sqrt{76}$ | 8, 1, 2, 1, 1, 5, 4, 5, 1, 1, 2, 1, 16 etc.<br>1 12 5 8 9 3 4 3 9 8 5 12 1 |

8, 1,
1 12

| Numeri surdi. | Indices. |
|---|---|
| $\sqrt{77}$ | 8, 1, 3, 2, 3, 1, 16 etc.<br>1 13 4 7 4 13 1 |
| $\sqrt{78}$ | 8, 1, 4, 1, 16 etc.<br>1 14 3 14 1 |
| $\sqrt{79}$ | 8, 1, 7, 1, 16 etc.<br>1 15 2 15 1 |
| $\sqrt{80}$ | 8, 1, 16, 1, 16 etc.<br>1 16 1 16 1 |
| $\sqrt{82}$ | 9, 18, 18, 18 etc.<br>1 1 1 1 |
| $\sqrt{83}$ | 9, 9, 18, 9, 18 etc.<br>1 2 1 2 1 |
| $\sqrt{84}$ | 9, 6, 18, 6, 18 etc.<br>3 3 1 3 1 |
| $\sqrt{85}$ | 9, 4, 1, 1, 4, 18 etc.<br>1 4 9 9 4 1 |
| $\sqrt{86}$ | 9, 3, 1, 1, 1 8, 1, 1, 1, 3, 18 etc.<br>1 5 10 7 11 2 11 7 10 5 1 |
| $\sqrt{87}$ | 9, 3, 18, 3, 18 etc.<br>1 6 1 6 1 |
| $\sqrt{88}$ | 9, 2, 1, 1, 1, 2, 18 etc.<br>1 7 9 8 9 7 1 |
| $\sqrt{89}$ | 9. 2, 3, 3, 2, 18 etc.<br>1 8 5 5 8 1 |
| $\sqrt{90}$ | 9, 2, 18, 2, 18 etc.<br>1 9 1 9 1 |
| $\sqrt{91}$ | 9, 1, 1, 5, 1, 5, 1, 1, 18 etc.<br>1 10 9 3 14 3 9 10 1 |
| $\sqrt{92}$ | 9, 1, 1, 2, 4, 2, 1, 1, 18 etc.<br>1 11 8 7 4 7 8 11 1 |
| $\sqrt{93}$ | 9, 1, 1, 1, 4, 6 4, 1, 1, 1, 18 etc.<br>1 12 7 11 4 3 4 11 7 12 1 |
| $\sqrt{94}$ | 9, 1, 2, 3, 1, 1, 5, 1, 8, 1, 5, 1, 1, 3, 2, 1, 18 etc.<br>1 13 6 5 9 10 3 15 2 15 3 10 9 5 6 13 1 |
| $\sqrt{95}$ | 9, 1, 2, 1, 18 etc.<br>1 14 5 14 1 |



9, 1,
1 16

| Numer' furdi. | Indices. |
|---|---|
| √96 | 9, 1, 3, 1, 18 etc.<br>1 15 4 15 1 |
| √97 | 9, 1, 5, 1, 1, 1, 1, 1, 1, 5, 1, 18 etc.<br>1 16 3 11 8 9 9 8 11 3 16 1 |
| √98 | 9, 1, 8, 1, 18 etc.<br>1 17 2 17 1 |
| √99 | 9, 1, 18, 1, 18 etc.<br>1 18 1 18 1 |
| √101 | 10, 20, 20 etc.<br>1 1 1 |
| √102 | 10, 10, 20, 10, 20 etc.<br>1 2 1 2 1 |
| √103 | 10, 6, 1, 2, 1, 1, 9, 1, 1, 2, 1, 6, 20 etc.<br>1 3 13 6 9 11 2 11 9 6 13 3 1 |
| √104 | 10, 5, 20, 5, 20 etc.<br>1 4 1 4 1 |
| √105 | 10, 4, 20, 4, 20 etc.<br>1 5 1 5 1 |
| √106 | 10, 3, 2, 1, 1, 1, 1, 2, 3, 20 etc.<br>1 6 7 10 9 9 10 7 6 1 |
| √107 | 10, 2, 1, 9, 1, 2, 20 etc.<br>1 7 13 2 13 7 1 |
| √108 | 10, 2, 1, 1, 4, 1, 1, 2, 20 etc.<br>1 8 9 11 4 11 9 8 1 |
| √109 | 10, 2, 3, 1, 2, 4, 1, 6, 6 1, 4, 2, 1, 3, 2, 20 etc.<br>1 9 5 12 7 4 15 3 3 15 4 7 12 5 9 1 |
| √110 | 10, 2, 20, 2, 20 etc.<br>1 10 1 10 1 |
| √111 | 10, 1, 1, 6, 1, 1, 20 etc.<br>1 11 10 3 10 11 1 |
| √112 | 10, 1, 1, 2, 1, 1, 20 etc.<br>1 12 9 7 9 12 1 |
| √113 | 10, 1, 1, 1, 2, 2, 1, 1, 1, 20 etc.<br>1 13 8 11 7 7 11 8 13 1 |
| √114 | 10, 1, 2, 10, 2, 1, 20 etc.<br>1 14 7 2 7 14 1 |

10, 1,
1 15

| Numeri surdi. | Indices. |
|---|---|
| $\sqrt{115}$ | 10, 1, 2, 1, 1, 1, 1, 1, 2, 1, 20 etc. |
| | 1 15 6 11 9 10 9 11 6 15 1 |
| $\sqrt{116}$ | 10, 1, 3, 2, 1, 4, 1, 2, 3, 1, 20 etc. |
| | 1 16 5 7 13 4 13 7 5 16 1 |
| $\sqrt{117}$ | 10, 1, 4, 2, 4, 1, 20 etc. |
| | 1 17 4 9 4 17 1 |
| $\sqrt{118}$ | 10, 1, 6, 3, 2, 10, 2, 3, 6, 1, 20 etc. |
| | 1 18 3 6 9 2 9 6 3 18 1 |
| $\sqrt{119}$ | 10, 1, 9, 1, 20 etc. |
| | 1 19 2 19 1 |
| $\sqrt{120}$ | 10, 1, 20, 1, 20 etc. |
| | 1 20 1 20 1 |

16. In omnibus his indicum ſeriebus periodi deprehenduntur modo ſtrictiores modo largiores, quae indicibus iis, qui primo duplo ſunt maiores, includuntur, atque hae periodi eo clarius in oculos incidunt, ſi primi indices cuiusque ſeriei duplicantur. Deinde in qualibet periodo idem indicum ordo, ſiue antrorſum, ſiue retrorſum, obſeruatur: ex quo in qualibet periodo vel vnus datur index medius, vel duo, prout terminorum numerus fuerit par vel impar. In litteris vero etiam graecis ſimiles periodi obſeruantur, vbi imprimis animaduertendum, pro omnibus indicibus 2 v. litteram graecam in vnitatem abire. Hanc proprietatem inſignem, quae in ipſis operationibus facilius perſpicitur, quam verborum ambage demonſtratur, probe notaſſe in ſequentibus plurimum intererit.

17. Ex his autem exemplis formas quasdam generales colligere licet, quae ita ſe habent:

I. Si $z = nn + 1$ erunt indices $\frac{n}{1}, \frac{2n}{1}, \frac{2n}{1}, \frac{2n}{1}$ etc.

II. Si $z = nn + 2$ erunt indices $\frac{n}{1}\ \frac{n}{2}\ \frac{2n}{1}\ \frac{n}{2}\ \frac{2n}{1}$ etc.

III. Si $z = nn + n$ erunt indices $\frac{n}{1}, \frac{2}{n}, \frac{2n}{1}, \frac{2}{n}, \frac{2n}{1}$ etc.

IV. Si $z = nn + 2n - 1$ erunt indices $\frac{n}{1}, \frac{1}{2n-1}, \frac{n-1}{2}, \frac{1}{2n-1}\ \frac{2n}{1}$ etc.

V. Si $z = nn + 2n$ erunt indices $\frac{n}{1}, \frac{1}{2n}, \frac{2n}{1}\ \frac{1}{2n}, \frac{2n}{1}$ etc.

Ac fractionum quidem continuarum ex his indicibus formatarum valor in genere facile definitur, idemque, quem hic aſſignauimus, deprehenditur. Tum vero etiam patet

VI. Si fit $z = 4nn + 4$ fore indices $\frac{2n}{1}, \frac{n}{4}, \frac{4n}{1}, \frac{n}{4}, \frac{4n}{1}$ etc.

VII. Si fit $z = 9nn + 3$ fore indices $\frac{2n}{1}, \frac{2n}{3}, \frac{6n}{1}, \frac{2n}{3}, \frac{6n}{1}$ etc.

VIII. Si fit $z = 9nn + 6$ fore indices $\frac{3n}{1}, \frac{n}{6}, \frac{6n}{1}\ \frac{n}{6}, \frac{6n}{1}$ etc.

## De reſolutione formulae $p = \sqrt{(lqq + 1)}$ in numeris integris.

18. Inuentis indicibus pro radice quadrata numeri cuiusuis $z$, ea hoc modo per fractionem continuam exprimitur:

$$\sqrt{z} = v + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}}$$

atque

atque ex his indicibus $v$, $a$, $b$, $c$, $d$ etc. fractiones $\frac{x}{y}$ formari possunt, quae tam prope ad $\sqrt{z}$ accedunt, vt nonnisi maioribus numeris adhibendis, eius valor accuratius exhiberi possit. Hae fractiones autem ita formantur:

Indices $v,\ a,\quad b\quad c\quad \ldots . m,\ n,$

$$\frac{x}{y}\qquad \frac{1}{0},\ \frac{v}{1},\ \frac{av+1}{a},\ \frac{(ab+1)v+b}{ab+1}\ \ldots .\ \frac{M}{P},\ \frac{N}{Q},\ \frac{nN+M}{nQ+P}$$

quae continuo propius valorem irrationalem $\sqrt{z}$ exprimunt.

19. Nouus autem algorithmus succinctum modum suppeditat, has fractiones commode per indices repraesentandi, quae ita se habent:

$$\frac{1}{0},\ \frac{(v)}{1},\ \frac{(v,a)}{(a)},\ \frac{(v,a,b)}{(a,b)},\ \frac{(v,a,b,c)}{(a,b,c)},\ \frac{(v,a,b,c,d)}{(a,b,c,d)}\ \text{etc.}$$

vbi cum ex natura progressionis sit:

$$(v,a)=a(v)+1;\ (v,a,b)=b(v,a)+(v);\ (v,a,b,c)=c(v,a,b)+(v,a);$$

$$(a)=a1+10;\ (a,b)=b(a)+1;\ (a,b,c)=c(a,b)+(a);$$

erit etiam, ex natura harum formularum:

$$(v,a)=v(a)+1;\ (v,a,b(=v(a,b,)+b;\ (v,a,b,c)=v(a,b,c)+(b,c)$$

deinde etiam sequentes transformationes demonstraui:

$$(v,a,b,c,d,e)=v(a,b,c,d,e)\ +(b,c,d,e)$$
$$(v,a,b,c,d,e)=(v,a)(b,c,d,e)+v(c,d,e)$$
$$(v,a,b,c,d,e)=(v,a,b)(c,d,e)+(v;a,)(d,e)$$
$$(v,a,b,c,d,e)=(v,a,b,c)(d,e)+(v,a,b)(e)$$

quas probe notasse in sequentibus plurimum iuuabit.

20. Videamus iam, quam prope singulae istae fractiones ad valorem $\sqrt{z}$ accedant, quod pro instituto nostro luculentissime inde patebit, si ex quaque fractione $\frac{x}{y}$ valorem $xx-zyy$ colligamus, quippe qui quo minor fuerit prae ipsis numeris $x$ et $y$, eo exactius fractio $\frac{x}{y}$ valori $\sqrt{z}$ aequabitur. Ac primo quidem si $\frac{x}{y}=\frac{1}{0}$, erit $xx-zyy=1$. Deinde sumto $\frac{x}{y}=\frac{v}{1}$, fit $xx-zyy=vv-z$, quae differentia per operationes supra expositas (12) prima littera graeca negatiue sumta $-\alpha$ designatur. Porro posito $\frac{x}{y}=\frac{(v,a)}{(a)}=\frac{va+1}{a}$, colligitur

$$xx-zyy=(vv-z)aa+2va+1=-\alpha aa+2va+1$$

ergo $xx-zyy=1+a(2v-\alpha a)=1+a(A-B)=\beta$

ob $v=A$, et $\alpha a=A+B$.

Quocirca hoc casu fit $xx-zyy=\beta$.

21. Cum igitur nacti simus:

$$vv-z=-\alpha \text{ et } (v,a)^2-z(a)^2=\beta$$

hinc vlterius progredi poterimus. Sit igitur

$$\frac{x}{y}=\frac{(v,a,b)}{(a,b)}=\frac{b(v,a)+v}{b(a)+1}$$

atque adhibitis illis reductionibus obtinebimus

$$xx-zyy=\beta bb+2vb(v,a)-2zb(a)-\alpha.$$

ergo ob $(v,a)=v(a)+1$, erit

$xx-zyy=\beta bb-2\alpha ab+2vb-\alpha=-\alpha-b(2\alpha a-\beta b-2v)$

at est $v=A$, $\alpha a=A+B$ et $\beta b=B+C$

ideoque $xx-zyy=-\alpha-b(B-C)=-\gamma$

ita vt sit $(v,a,b)^2-z(a,b)^2=-\gamma$.

22.

22. Confideremus nunc fractionem fequentem:

$$\frac{x}{y}=\frac{(v,a,b,c)}{(a,b,c)}=\frac{c(v,a,b)+(v,a)}{c(a,b)+a}$$

ex qua colligitur:

$$xx-zyy=-\gamma cc+2c(v,a,b)(v,a)+\beta$$
$$-2zca(a,b)$$

cuius pars media reducitur ad

$$2c(\beta b-\alpha a+v)$$

vnde ob $v=A$, $\alpha a=A+B$; $\beta b=B+C$, $\gamma c=C+D$ refultat $xx-zyy=\beta+c(C-D)=\delta$ ita vt fit $(v,a,b,c)^2-z(a,b,c)^2=\delta$, vnde per inductionem fequentes valores facile colliguntur.

23. Ne autem hic inductioni nimium videar tribuiffe, fequenti modo haec inueftigatio inftitui poteft:

fit $(v)^2 \quad -z1^2_d \quad =\mathfrak{A}$
$(v,a)^2 \quad -z(a)^2 \quad =\mathfrak{B}$
$(v,a,b)^2 \quad -z(a,b)^2 \quad =\mathfrak{C}$
$(v,a,b,c)^2 \quad -z(a,b,c)^2 \quad =\mathfrak{D}$
$(v,a,b,c,d)^2-z(a,b,c,d)^2=\mathfrak{E}$
etc.

vbi quidem iam vidimus effe $\mathfrak{A}=-\alpha$, $\mathfrak{B}=\beta$, $\mathfrak{C}=-\gamma$ etc. Cum vero fit

$(v,a) \quad =a(v)+1 \qquad (a) \quad =a$
$(v,a,b) \quad =b(v,a)+(v) \qquad (a,b) \quad =b(a)+1$
$(v,a,b,c)=c(v,a,b)+(v,a)$; $(a,b,c)=c(a,b,)+(a)$
$(v,a,b,c,d)=d(v,a,b,c)+(v,a,b)$; $(a,b,c,d)=d,(a,b,c)+(a,b)$
etc.

 habe-

habebimus:

$\mathfrak{B} = \mathfrak{A}aa + 1 + 2a, v$

$\mathfrak{C} = \mathfrak{B}bb + \mathfrak{A} + 2b((v,a)(v) - z(a))$

$\mathfrak{D} = \mathfrak{C}cc + \mathfrak{B} + 2c((v,a,b)(v,a) - z((a,b)(a))$

$\mathfrak{E} = \mathfrak{D}dd + \mathfrak{C} + 2d((v,a,b,c)(v,a,b) - z(a,b,c)(a,b))$

$\mathfrak{F} = \mathfrak{E}ee + \mathfrak{D} + 2e((v,a,b,c,d)(v,a,b,c) - z(a,b,c,d)(a,b,c))$

etc.

24. Statuamus iam breuitatis gratia:

$\mathfrak{B} = 1 + \mathfrak{A}aa + 2a.O$

$\mathfrak{C} = \mathfrak{A} + \mathfrak{B}bb + 2b.P$

$\mathfrak{D} = \mathfrak{B} + \mathfrak{C}cc + 2c.Q$

$\mathfrak{E} = \mathfrak{C} + \mathfrak{D}dd + 2d.R$

$\mathfrak{F} = \mathfrak{D} + \mathfrak{E}ee + 2e.S$

etc.

et ex ſuperioribus reductionibus colligemus:

$P - O = a(v)^2 - za = \mathfrak{A}a$

$Q - P = b(v,a)^2 - zb(a)^2 = \mathfrak{B}b$

$R - Q = c(v,a,b)^2 - cz(a,b)^2 = \mathfrak{C}c$

$S - R = d(v,a,b,c)^2 - dz(a,b,c)^2 = \mathfrak{D}d$

etc.

ſicque fiet

$O = v$

$P = v + \mathfrak{A}a$

$Q = v + \mathfrak{A}a + \mathfrak{B}b$

$R = v + \mathfrak{A}a + \mathfrak{B}b + \mathfrak{C}c$

$S = v + \mathfrak{A}a + \mathfrak{B}b + \mathfrak{C}c + \mathfrak{D}d$

etc.

25.

25. Formulae autem ſupra vſurpatae praebent

$$A = v$$
$$B = -v + \alpha a$$
$$C = v - \alpha a + \beta b$$
$$D = -v + \alpha a - \beta b + \gamma c$$
$$E = v - \alpha a + \beta b - \gamma c + \delta c$$

vnde patet eſſe $O = A$ et $P = -B$ ob $\mathfrak{A} = -\alpha$. Cum iam ſit $\mathfrak{B} = 1 - \alpha a a + 2 a v = 1 + a(A - B)$ erit vtique $\mathfrak{B} = \beta$, hincque $Q = C$, ex quo porro colligitur:

$$\mathfrak{C} = -\alpha + \beta b b - 2 b B = -\alpha - b(2B - \beta b) = -\alpha - b(B - C)$$

ſicque eſt $\mathfrak{C} = -\gamma$ et $R = -D$; ſimili modo

$$\mathfrak{D} = \beta - \gamma c c + 2 c C = \beta + c(2C - \gamma c) = \beta + c(C - D)$$

ideoque eſt $\mathfrak{D} = \delta$ et $S = E$. Tum vero porro

$$\mathfrak{E} = -\gamma + \delta d d - 2 d D = -\gamma - d(2D - \delta d)$$
$$= \gamma - d(D - E)$$

ac propterea $\mathfrak{E} = -\varepsilon$, vnde ſuperior inductio ſatis confirmatur.

26. Pro fractionibus ergo $\frac{x}{y}$ formulae radicali $\sqrt{z}$ proxime aequalibus ſequentes adipiſcimur relationes:

Si ſumatur

$\begin{cases} x = 1 \\ y = 0 \end{cases}$ erit $xx = zyy + 1$

$\begin{cases} x = (v) \\ y = 1 \end{cases}$ erit $xx = zyy - \alpha$

 $\begin{cases} x = (v, a) \\ y = (a) \end{cases}$

$$\left\{\begin{array}{l} x = (v, a) \\ y = (a) \end{array}\right. \text{ erit } xx = zyy + \beta$$

$$\left\{\begin{array}{l} x = (v, a, b) \\ y = (a, b) \end{array}\right. \text{ erit } xx = zyy - \gamma$$

$$\left\{\begin{array}{l} x = (v, a, b, c) \\ y = (a, b, c) \end{array}\right. \text{ erit } xx = zyy + \delta$$

$$\left\{\begin{array}{l} x = (v, a, b, c, d) \\ y = (a, b, c, d) \end{array}\right. \text{ erit } xx = zyy - \varepsilon$$

etc.

vnde problema *Pellianum* ſoluetur, quoties litterarum graecarum per ſaltum excerptarum $\beta$, $\delta$, $\zeta$ etc. quaepiam in vnitatem abit.

27. Vidimus autem ſupra, nonniſi iis indicibus, qui ſunt $2v$, reſpondere litteram graecam in vnitatem abeuntem; cum igitur quaelibet periodorum, quas in indicum ordine obſeruauimus, indice $2v$ inchoetur, perſpicuum eſt, ſi numeros $x$ et $y$ per indices primae periodi definiamus, fore vel $xx = zyy - 1$, vel $xx = zyy + 1$; ac prius quidem euenit, ſi indicum ſingulas periodos conſtituentium numerus fuerit impar, poſterius vero ſi is fuerit par. Hoc igitur caſu ſtatim habetur ſolutio problematis *Pelliani*, quo requiritur, vt ſit $pp = zqq + 1$, quandoquidem capi oportet $p = x$ et $q = y$.

28. At ſi ex prima periodo prodeat $xx = zyy - 1$, quod euenit ſi indicum numerus eſt impar, tum indices vsque ad initium tertiae periodi ad definiendos nume-

numeros $x$ et $y$ capi possent, quorum numerus cum sit par, hoc modo idonei numeri pro $p$ et $q$ obtinerentur. Verum casu inuento, quo sit $xx = zyy - 1$, multo facilius inde numeri $p$ et $q$ reperiri possunt, vt sit: $pp = zqq + 1$. Sumatur enim $p = 2xx + 1$ et $q = 2xy$, eritque $pp - zqq = 4x^4 + 4xx + 1 - 4zxxyy = 1 + 4xx(xx - zyy + 1)$, at $xx - zyy + 1 = 0$, ideoque $pp - zqq = 1$ seu $pp = zqq + 1$, quemadmodum problema *Pellianum* postulat. Videamus igitur, quomodo pro quouis numero $z$ ex indicibus inde natis numeri $p$ et $q$ sint definiendi, vt fiat $pp = zqq + 1$, vbi quidem casus secundum periodos percurramus.

## I. Casus, quo pro numero $z$ indices sunt $v$, $2v$, $2v$, etc.

29. Hic singulae periodi vnicum indicem continent, sumto ergo $x = (v)$ et $y = 1$, erit $xx = zyy - 1$. Quamobrem, vt fiat $pp = zqq + 1$, capiatur:

$$p = 2xx + 1 = 2vv + 1 \text{ et } q = 2xy = 2v.$$

Hic casus, vt supra vidimus, locum habet, si sit $z = vv + 1$, seu quo numerus $z$ vnitate superat quadratum; tum igitur capi debet $p = 2vv + 1$ seu $p = 2z - 1$ et $q = 2v$, quo pacto problemati *Pelliano* satisfit, vt sit $p = \sqrt{(zqq + 1)}$. Ita si sit

$z = 2$ erit $p = 3$ et $q = 2$ sicque $p = \sqrt{(2qq + 1)}$
si $z = 5$ erit $p = 9$ et $q = 4$ sicque $p = \sqrt{(5qq + 1)}$
si $z = 10$ erit $p = 19$ et $q = 6$ sicque $p = \sqrt{(10qq + 1)}$
si $z = 17$ erit $p = 33$ et $q = 8$ sicque $p = \sqrt{(17qq + 1)}$
etc.

 II.

## II. Casus, quo pro numero $z$ indices sunt $v, a, 2v, a, 2v$ etc.

30. Prima periodus constat binis numeris $v, a$, vnde sumtis $x = (v, a) = va + 1$ et $y = (a) = a$, habebitur $xx = zyy + 1$. Vt igitur pro problemate *Pelliano* fiat $pp = zqq + 1$, capi oportet:

$$p = va + 1 \text{ et } q = a.$$

Ex indicibus autem patet, hunc casum locum habere, quoties fuerit numerus $z = vv + \frac{2v}{a}$, vnde intelligitur, hunc casum in integris, de quibus hic agitur, existere non posse, nisi sit $a$ diuisor ipsius $2v$, vbi duo casus sunt considerandi:

1. si $a = 2n$, erit $v = mn$ et $\frac{2v}{a} = m$
2. si $a = 2n + 1$, erit $v = m(2n + 1)$ et $\frac{2v}{a} = 2m$.

## III. Casus, quo pro numero $z$ indices sunt $v, a, a, 2v, a, a, 2v$ etc.

31. Ex prima periodo, sumtis numeris $x$ et $y$, ita vt sit $x = (v, a, a)$ et $y = (a, a)$, erit $xx = zyy - 1$; vnde vt fiat $pp = zqq + 1$, sumi debet

$$p = 2xx + 1 \text{ et } q = 2xy.$$

Hic vero est $y = aa + 1$, et $x = vy + a$, vnde numeri $p$ et $q$ facillime definiuntur. Ex indicibus autem numerus $z$ eiusmodi habebit formam:

$$z = vv + u, \text{ existente } u = \frac{2av + 1}{aa + 1}$$

vnde

vnde patet numerum $a$ esse debere parem. Si ergo statuatur $a = 2n$, necesse est sit

$v = n + m(4nn + 1)$ tumque fit $u = 1 + 4mn$

## IV. Casus, quo pro numero $z$ indices sunt $v, a, b, a, 2v, a, b, a, 2v$ etc.

32. Quia numerus indicum in quaque periodo est par, si sumatur:

$x = (v, a, b, a)$ et $y = (a, b, a)$

erit $xx = zyy + 1$, ideoque $p = x$ et $q = y$.
Per transformationes autem supra ostensas duplicatio indicum tolli potest hoc modo:

$x = (a)(v, a, b) + (v, a)$ et $y = (a)(a, b) + (a)$.

Hinc si ex indicibus $v, a, b$ sequentes fractiones formentur:

indic. $v$, $a$, $b$

fract. $\frac{1}{0}$, $\frac{\mathfrak{A}}{\mathfrak{a}}$, $\frac{\mathfrak{B}}{\mathfrak{b}}$, $\frac{\mathfrak{C}}{\mathfrak{c}}$

ob $\mathfrak{A} = (v)$, $\mathfrak{B} = (v, a)$, $\mathfrak{C} = (v, a, b)$

et $\mathfrak{a} = 1$, $\mathfrak{b} = (a)$, $\mathfrak{c} = (a, b)$

erit $x = \mathfrak{b}\mathfrak{C} + \mathfrak{a}\mathfrak{B}$ et $y = \mathfrak{b}\mathfrak{c} + \mathfrak{a}\mathfrak{b}$.
Ex indicibus autem fit $z = vv + u$, existente

$2v = m(a, b, a) - b(a, b)$

et $u = m(a, b) - b(b)$

## V. Casus, quo pro numero $z$ indices sunt $v, a, b, b, a, 2v$ etc.

33. Ob indicum cuiusque periodi numerum imparem, si capiamus

$$x=(v,a,b,b,a,) \text{ et } y=(a,b,b,a)$$

erit $xx=zyy-1$; hinc pro problemate *Pelliano*, vt fiat $pp=zqq+1$, statui oportet:

$$p=2xx+1 \text{ et } q=2xy.$$

Quo autem numeri $x$ et $y$ facilius inueniri queant, sequentes transformationes instituantur:

$$x=(a,b)(v,a,b)+(a)(v,a) \text{ et}$$
$$y=(a,b)\ (a,b)\ +(a)(a)$$

qui ergo per solos indices $v$, $a$, $b$ fractionibus inde formandis difinientur:

Ind. $v$, $a$, $b$

Fract. $\frac{1}{0}$, $\frac{\mathfrak{A}}{\mathfrak{a}}$, $\frac{\mathfrak{B}}{\mathfrak{b}}$, $\frac{\mathfrak{C}}{\mathfrak{c}}$

vbi $\mathfrak{A}=v$, $\mathfrak{B}=a\mathfrak{A}+1$; $\mathfrak{C}=b\mathfrak{B}+\mathfrak{A}$

et $\mathfrak{a}=1$, $\mathfrak{b}=a\mathfrak{a}+0$; $\mathfrak{c}=b\mathfrak{b}+\mathfrak{a}$

tum enim capi oportet

$$x=\mathfrak{c}\mathfrak{C}+\mathfrak{b}\mathfrak{B} \text{ et } y=\mathfrak{c}\mathfrak{c}+\mathfrak{b}\mathfrak{b}.$$

Hic autem casus locum habet, quoties posito $z=vv+u$ fuerit

$$2v=m(a,b,b,a)+(b,b)(a,b,b) \text{ et}$$
$$u=m(a,b,b)+(b,b)(b,b)$$

## VI. Casus, quo pro numero $z$ indices sunt $v$, $a$, $b$, $c$, $b$, $a$, $2v$ etc.

34. Quoniam hic numerus indicum in qualibet periodo est par, si sumamus

$$x=(v,a,b,c,b,a) \text{ et } y=(a,b,c,b,a)$$

erit

erit $xx = zyy + 1$, ideoque pro *Pelliano* problemate statim habetur $p = x$ et $q = y$. Facilius autem numeri $x$ et $y$ his transformationibus adhibitis inuenientur:

$$x = (a,b)(v,a,b,c) + (a)(v,a,b) \text{ et}$$
$$y = (a,b)(a,b,c) + (a)(a,b)$$

vnde si ex indicibus, $v$, $a$, $b$, $c$, more exposito, fractiones formentur:

Ind. $v$, $a$, $b$, $c$

Fract. $\frac{1}{0}$, $\frac{\mathfrak{A}}{\mathfrak{a}}$, $\frac{\mathfrak{B}}{\mathfrak{b}}$, $\frac{\mathfrak{C}}{\mathfrak{c}}$, $\frac{\mathfrak{D}}{\mathfrak{d}}$

sumi oportet

$$x = \mathfrak{c}\mathfrak{D} + \mathfrak{b}\mathfrak{C} \text{ et } y = \mathfrak{c}\mathfrak{d} + \mathfrak{b}\mathfrak{c}.$$

At hic casus locum habet, quoties posito $z = vv + u$ fuerit

$$2v = m(a,b,c,b,a) - (b,c,b)(a,b,e,b) \text{ et}$$
$$u = m(a,b,c,b) - (b,c,b)(b,c,b)$$

## VII. Casus, quo pro numero $z$ indices sunt $v$, $a$, $b$, $c$, $c$, $b$, $a$, $2v$ etc.

35. Hic iterum indicum numerus in qualibet periodo est impar, ideoque si ponamus

$$x = (v,a,b,c,c,b,a) \text{ et}$$
$$y = (a,b,c,c,b,a)$$

erit $xx = zyy - 1$, ex quo vt fiat $pp = zqq + 1$, sumi oportet $p = 2xx + 1$, et $q = 2xy$.

Pro faciliori autem numerorum $x$ et $y$ inuentione ex indicibus $v$, $a$, $b$, $c$ formentur fractiones:

Ind. $v$, $a$, $b$, $c$

Fract. $\frac{1}{0}$, $\frac{\mathfrak{A}}{\mathfrak{a}}$, $\frac{\mathfrak{B}}{\mathfrak{b}}$, $\frac{\mathfrak{C}}{\mathfrak{c}}$, $\frac{\mathfrak{D}}{\mathfrak{d}}$

hincque erit

$$x = \mathfrak{d}\mathfrak{D} + \mathfrak{c}\mathfrak{C} \text{ et } y = \mathfrak{d}\mathfrak{d} + \mathfrak{c}\mathfrak{c}.$$

At hic casus locum habebit, quoties posito $z = vv + u$ fuerit:

$$2v = m(a,b,c,c,b,a) + (b,c,c,b)(a,b,c,c,b) \text{ et}$$
$$u = m(a,b,c,c,b) + (b,c,c,b)(b,c,c,b)$$

## VIII. Casus, quo pro numero $z$ indices sunt $v$, $a$, $b$, $c$, $d$, $c$, $b$, $a$, $2v$.

36. Hic quaelibet periodus octo continet indices, ideoque si ponamus:

$$x = (v,a,b,c,d,c,b,a) \text{ et}$$
$$y = (a,b,c,d,c,b,a)$$

erit $xx = zyy + 1$, et problemate *Pelliano* capi oportet $p = x$, et $q = y$, vt fiat $pp = zqq + 1$. Transformationibus autem adhibitis, numeros $x$ et $y$ per solos indices $v$, $a$, $b$, $c$, $d$ definire licet. Formatis enim inde fractionibus:

Ind. $v$, $a$, $b$, $c$, $d$

Fract. $\frac{1}{0}$, $\frac{\mathfrak{A}}{\mathfrak{a}}$, $\frac{\mathfrak{B}}{\mathfrak{b}}$, $\frac{\mathfrak{C}}{\mathfrak{c}}$, $\frac{\mathfrak{D}}{\mathfrak{d}}$, $\frac{\mathfrak{E}}{\mathfrak{e}}$

fiet: $x = \mathfrak{d}\mathfrak{E} + \mathfrak{c}\mathfrak{D}$, et $y = \mathfrak{d}\mathfrak{e} + \mathfrak{c}\mathfrak{d}$.

Hic vero casus locum habet, quoties posito $z = vv + u$ fuerit:

$$2v = m(a,b,c,d,c,b,a) - (b,c,d,c,b)(a,b,c,d,c,b)$$
et $$u = m(a,b,c,d,c,b) - (b,c,d,c,b)(b,c,d,c,b).$$

Ex-

## Expositio calculi pro quolibet numero $z$, vt fiat $pp = zqq + 1$.

37. Primum igitur methodo supra exposita pro numero $z$, ex eius radice quadrata indices investigari oportet, quam operationem autem vlterius continuari non est opus, quam donec indices ordine retrogrado prodire incipiant, quo pacto semissi laboris supra explicati supersedere poterimus. Cum autem in prima periodo vel vnus index medius occurrat, vel bini, hi casus probe sunt distinguendi, cum si vnicus medius affuerit, inuentio numerorum $p$ et $q$ modo in casibus II, IV, VI et VIII tradito institui debeat: sin autem bini fuerint medii, eo modo, qui in casibus I, III, V et VII est descriptus. Scilicet si prius eueniat, numeri $p$ et $q$ numeris $x$ et $y$ aequales sumuntur, sin autem posterius, vti vidimus, statui oportet $p = 2xx + 1$, et $q = 2xy$, ita vt his casibus numeri $p$ et $q$ caeteris paribus multo grandiores reperiantur.

38. En igitur exempla prioris generis, quo in qualibet periodo vnus datur index medius:

I. Si $z = 6$, sunt indices 2, 2, 4, hinc operatio

$$\begin{matrix} & 2 & 2 \\ \frac{1}{0}, & \frac{2}{1}, & \frac{5}{2}, \end{matrix} \quad \frac{x}{y} = \frac{1.5 + 0.2}{1.2 + 0.1} \text{ ergo } \frac{p}{q} = \frac{5}{2}$$

II. Si $z = 14$, sunt indices 3, 1, 2, 1, 6

$$\begin{matrix} & 3 & 1 & 2 \\ \frac{1}{0}, & \frac{3}{1}, & \frac{4}{1}, & \frac{11}{3}; \end{matrix} \quad \frac{x}{y} = \frac{1.11 + 1.4}{1.3 + 1.1} \text{ ergo } \frac{p}{q} = \frac{15}{4}$$

III. Si $z = 19$, ſunt indices $4, 2, 1, 3, 1, 2, 8$

$4 \quad 2 \quad 1 \quad 3$

$\frac{1}{0}, \frac{4}{1}, \frac{9}{2}, \frac{13}{3}, \frac{48}{11}; \frac{x}{y} = \frac{3.48 + 2.13}{3.11 + 2.3}$ ergo $\frac{p}{q} = \frac{170}{39}$

IV. Si $z = 31$, ſunt indices $5, 1, 1, 3, 5, 3, 1, 1, 10$

$5 \quad 1 \quad 1 \quad 3 \quad 5$

$\frac{1}{0}, \frac{5}{1}, \frac{6}{1}, \frac{11}{2}, \frac{39}{7}, \frac{206}{37}$

hinc $x = 7.206 + 2.39$ ergo $p = 1520$
$\quad y = 7.\ 37 + 2.\ 7 \qquad q = 273$

V. Si $z = 44$, ſunt indices $6, 1, 1, 1, 2, 1, 1, 1, 12$

$6 \quad 1 \quad 1 \quad 1 \quad 2$

$\frac{1}{0}, \frac{6}{1}, \frac{7}{1}, \frac{13}{2}, \frac{20}{3}, \frac{53}{8}$

hinc $x = 3.53 + 2.20$ ergo $p = 199$
$\quad y = 3.\ 8 + 2.\ 3 \qquad q = 30$

VI. Si $z = 55$, ſunt indices $7, 2, 2, 2, 14$

$7 \quad 2 \quad 2$

$\frac{1}{0}, \frac{7}{1}, \frac{15}{2}, \frac{37}{5}$

hinc $x = 2.37 + 1.15$ ergo $p = 89$
$\quad y = 2.\ 5 + 1.\ 2 \qquad q = 12$

39. Alterius vero generis, quo bini dantur indices medii in qualibet periodo, haec adiungo exempla.

I. Si $z = 13$, ſunt indices $3, 1, 1, 1, 1, 6$

$3 \quad 1 \quad 1$

$\frac{1}{0}, \frac{3}{1}, \frac{4}{1}, \frac{7}{2}$ hinc $\frac{x}{y} = \frac{2.7 + 1.4}{2.2 + 1.1} = \frac{18}{5}$

Ergo $p = 2xx + 1 = 649$
$\quad q = 2xy \qquad = 180$

II.

II. Si $z = 29$, ſunt indices $5, 2, 1, 1, 2, 10$

$$\begin{matrix} & 5 & 2 & 1 \\ \frac{1}{0}, & \frac{5}{1}, & \frac{11}{2}, & \frac{16}{3} \end{matrix} \text{ hinc } \frac{x}{y} = \frac{3.16 + 2.11}{3.\;3 + 2.\;2} = \frac{70}{13}.$$

Ergo $p = 2xx + 1 = 9801$

$q = 2xy \qquad = 1820$

III. Si $z = 58$, ſunt indices $7, 1, 1, 1, 1, 1, 1, 14$

$$\begin{matrix} & 7 & 1 & 1 & 1 \\ \frac{1}{0}, & \frac{7}{1}, & \frac{8}{1}, & \frac{15}{2}, & \frac{23}{3} \end{matrix} \text{ hinc } \frac{x}{y} = \frac{3.23 + 2.15}{3.\;3 + 2.\;2} = \frac{99}{13}.$$

Ergo $p = 2xx + 1 = 19603$

$q = 2xy \qquad = 2574$

IV. Si $z = 61$, indices ſunt $7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14$

$$\begin{matrix} & 7 & 1 & 4 & 3 & 1 & 2 \\ \frac{1}{0}, & \frac{7}{1}, & \frac{8}{1}, & \frac{39}{5}, & \frac{125}{16}, & \frac{164}{21}, & \frac{453}{58} \end{matrix}$$

hinc fit $x = 58.453 + 21.164 = 29718$

$y = 58.\;58 + 21.\;21 = 3805.$

Ergo $p = 2xx + 1 = 1766319049$

$q = 2xy \qquad = 226153980.$

40. Quodſi pro maioribus numeris $z$, quam ante ſunt euoluti, quaeri debeant numeri $p$ et $q$, vt ſit $pp = zqq + 1$, primum methodo ſupra expoſita (§. 12.) indices $v$, $a$, $b$, $c$, $d$ etc. quaeri oportet, quos autem vlterius continuari non eſt opus, quam donec ad indicem medium, vel binos medios, primae periodi perueniatur; tum vero ex iis per ope-

operationes hic descriptas primo numeri $x$ et $y$, tum vero ipsi quaesiti $p$ et $q$ determinabuntur. Id quod aliquibus exemplis illustrari conueniet.

## I. Quaerantur numeri $p$ et $q$, vt sit $pp = 157qq + 1$.

41. Cum hic sit $z = 157$, erit $v = 12$, et $\alpha = 13$, vnde indicum inuentio ita se habebit:

$$\begin{array}{lll} A = 12, & \alpha = 13, & a = 1 \\ B = 1, & \beta = 12, & b = 1 \\ C = 11, & \gamma = 3, & c = 7 \\ D = 10, & \delta = 19, & d = 1 \\ E = 9, & \varepsilon = 4, & e = 5 \\ F = 11, & \zeta = 9, & f = 2 \\ G = 7, & \eta = 12, & g = 1 \\ H = 5, & \theta = 11, & h = 1 \\ I = 6, & \iota = 11, & i = 1 \end{array} \Big\} \text{ medii.}$$

Hinc ob binos medios exemplum ad genus secundum pertinet, et operationes ita sunt instituendae:

$$\begin{array}{cccccccccc} 12 & 1 & 1 & 7 & 1 & 5 & 2 & 1 & 1 & \\ \frac{1}{0}, & \frac{12}{1}, & \frac{13}{1}, & \frac{25}{2}, & \frac{188}{15}, & \frac{213}{17}, & \frac{1253}{100}, & \frac{2719}{217}, & \frac{3972}{317}, & \frac{6691}{534} \end{array}$$

Hinc erit $x = 534.6691 + 317.3972 = 4832118$

et $y = 534.534 + 317.317 = 385645$.

Quocirca $p = 2xx + 1 = 46698728731849$

et $q = 2xy \quad = 3726964292220$

atque hi adeo sunt minimi numeri integri formulae $p = \sqrt{(157qq + 1)}$ satisfacientes.

II.

## II. Quaerantur numeri $p$ et $q$, vt fit

$$pp = 367qq + 1.$$

42. Hic ergo eft $z = 367$, $v = 19$, hincque

$A = 19$, $\alpha = 6$, $a = 6$
$B = 17$, $\beta = 13$, $b = 2$
$C = 9$, $\gamma = 22$, $c = 1$
$D = 13$, $\delta = 9$, $d = 9$
$E = 14$, $\varepsilon = 19$, $e = 1$
$F = 5$, $\zeta = 18$, $f = 1$
$G = 13$, $\eta = 11$, $g = 2$
$H = 9$, $\theta = 26$, $h = 1$
$I = 17$, $\iota = 3$, $i = 12$
$K = 19$, $\varkappa = 2$, $k = 19$ medius
$L = 19$, $\lambda = 3$, $l = 12$

hoc ergo exemplum ad genus primum pertinet,

$$\begin{matrix} & 19 & 6 & 2 & 1 & 9 & 1 & 1 & 2 \\ \frac{1}{0}, & \frac{19}{1}, & \frac{115}{6}, & \frac{249}{13}, & \frac{364}{19}, & \frac{3525}{184}, & \frac{3889}{203}, & \frac{7414}{387} & \end{matrix}$$

$$\begin{matrix} 1 & 12 & 19 & \\ \frac{18717}{977}, & \frac{26131}{1364}, & \frac{332289}{17345}, & \frac{6339623}{330919} \end{matrix}$$

Hinc erit $x = 17345 \cdot 6339622 + 1364 \cdot 332289$
et $y = 17345 \cdot 330919 + 1364 \cdot 17345$

ex quo minimi numeri fatisfacientes funt:

$p = 110413985786$
$q = 5763448635.$

Tabula

*Tabula numerorum* p *et* q, *quibus fit* $pp = lqq + 1$ *pro omnibus valoribus numeri* l *vsque ad* 100.

| *l* | *q* | *p* | *l* | *q* | *p* |
|---|---|---|---|---|---|
| 2 | 2 | 3 | 30 | 2 | 11 |
| 3 | 1 | 2 | 31 | 273 | 1520 |
| 5 | 4 | 9 | 32 | 3 | 17 |
| 6 | 2 | 5 | 33 | 4 | 23 |
| 7 | 3 | 8 | 34 | 6 | 35 |
| 8 | 1 | 3 | 35 | 1 | 6 |
| 10 | 6 | 19 | 37 | 12 | 73 |
| 11 | 3 | 10 | 38 | 6 | 37 |
| 12 | 2 | 7 | 39 | 4 | 25 |
| 13 | 180 | 649 | 40 | 3 | 19 |
| 14 | 4 | 15 | 41 | 320 | 2049 |
| 15 | 1 | 4 | 42 | 2 | 13 |
| 17 | 8 | 33 | 43 | 531 | 3482 |
| 18 | 4 | 17 | 44 | 30 | 199 |
| 19 | 39 | 170 | 45 | 24 | 161 |
| 20 | 2 | 9 | 46 | 3588 | 24335 |
| 21 | 12 | 55 | 47 | 7 | 48 |
| 22 | 42 | 197 | 48 | 1 | 7 |
| 23 | 5 | 24 | 50 | 14 | 99 |
| 24 | 1 | 5 | 51 | 7 | 50 |
| 26 | 10 | 51 | 52 | 90 | 649 |
| 27 | 5 | 26 | 53 | 9100 | 33125 |
| 28 | 24 | 127 | 54 | 66 | 485 |
| 29 | 1820 | 9801 | 55 | 12 | 89 |

| l | q | p | l | q | p |
|---|---|---|---|---|---|
| 56 | 2 | 15 | 78 | 6 | 53 |
| 57 | 20 | 151 | 79 | 9 | 80 |
| 58 | 2574 | 19603 | 80 | 1 | 9 |
| 59 | 69 | 530 | 82 | 18 | 163 |
| 60 | 4 | 31 | 83 | 9 | 32 |
| 61 | 226153980 | 1766319049 | 84 | 6 | 55 |
| 62 | 8 | 63 | 85 | 30906 | 285769 |
| 63 | 1 | 8 | 86 | 1122 | 10405 |
| 65 | 16 | 129 | 87 | 3 | 28 |
| 66 | 8 | 65 | 88 | 21 | 197 |
| 67 | 5967 | 48842 | 89 | 53000 | 500901 |
| 68 | 4 | 33 | 90 | 2 | 19 |
| 69 | 936 | 7775 | 91 | 165 | 1574 |
| 70 | 30 | 251 | 92 | 120 | 1151 |
| 71 | 413 | 3480 | 93 | 1260 | 12151 |
| 72 | 2 | 17 | 94 | 221064 | 2143295 |
| 73 | 267000 | 2281249 | 95 | 4 | 39 |
| 74 | 430 | 3699 | 96 | 5 | 49 |
| 75 | 3 | 26 | 97 | 6377352 | 62809633 |
| 76 | 6630 | 57799 | 98 | 10 | 99 |
| 77 | 40 | 351 | 99 | 1 | 10 |

Exempla denique quaedam numerorum maiorum pro $l$ assumtorum adiungam:

si erit

$l = 103 \begin{cases} p = 227528 \\ q = 22419 \end{cases}$

si erit

$l = 109 \begin{cases} p = 158070671986249 \\ q = 15140424455100 \end{cases}$

$l = 113 \begin{cases} p = 1204353 \\ q = 113296 \end{cases}$

$l = 157 \begin{cases} p = 46698728731849 \\ q = 3726964292220 \end{cases}$

$l = 367 \begin{cases} p = 110413985786 \\ q = 5763448635 \end{cases}$

PROPRIE-