CONSIDERATIONES DE MOTV CORPORVM COELESTIVM.

Auctore

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1.

Etsi nullum est dubium, quin leges motus corporum coelestium a *Keplero* obseruatae atque a *Neutono* confirmatae, Astronomiae maxima incrementa attulerint, tamen nunc quidem certissimum est, nullum in coelo reperiri corpus, quod leges istas in motu suo perfecte sequatur, cum potius in omnibus haud leues aberrationes ab istis legibus deprehendantur. Vera scilicet omnium motuum coelestium causa in mutua horum corporum attractione est posita, qua vnumquodque ad singula reliqua vrgetur viribus rationem compositam ex directa simplici massarum, et inuersa duplicata distantiarum tenentibus. Semper autem commode vsu venit, vt inter has vires vna prae reliquis maxime emineat, ideoque motus proxime regulis *Keplerianis* conformis euadat; sicque effectus a reliquis oriundus veluti minimus per methodos appropinquandi definiri possit. Quod nisi eueniret, in maxima adhuc ignoratione motuum coelestium versaremur, cum nulla methodus adhuc sit inuenta, cuius ope trium saltem corporum se mutuo attrahentium motus assignari queat; nisi forte vna vis caeteras plurimum superet.

2. Verum etiam hic casus, in quo soli Geometrae operam suam non omnino frustra consumserunt, neutiquam pro confecto haberi potest, cum ipsa methodus appropinquandi, qua Geometrae vti solent, plurimis difficultatibus adhuc sit inuoluta, atque infinita minorum perturbationum multitudo negligatur, quo fit vt haec ipsa approximatio negotium minime conficiat, sed ad eam perficiendam plurima adhuc adminicula desiderentur. Quare etsi motus Lunae ex hac Theoria satis accurate est definitus, id tamen potius singularibus circumstantiis, quae in Luna locum inueniunt, est tribuendum, quam cuipiam perfectioni, ad quam Theoria euecta censeri queat; si enim Luna bis vel ter longius a terra abesset, vel eius orbita magis esset excentrica, omnes labores adhuc exantlati omni fructu caruissent, ac ne nunc quidem eius motum obiter saltem ad certam quandam regulam reuocare liceret.

3. Plurimum igitur is in Theoria Astronomiae praestitisse esset censendus, qui in hypothesi ficta, qua Luna multo longius a terra abesset, eius motum assignare valuerit, cum inde maxima adiumenta in hanc scientiam certo essent redundatura. Si quidem Luna centies longius a terra esset remota, nullum est dubium, quin leges motus planetae principalis esset secutura, neque amplius, tanquam satelles terrae, spectari posset. Sin autem decies tantum magis distaret, eius motus ita foret comparatus, vt in dubio relinqueretur, vtrum planetis primariis, an secundariis, esset accensenda. Tantopere certe ab omnibus motibus in coelo obseruatis discre-

discreparet, vt vix intelligi possit, quemadmodum saltem sideam motus medii constitui conueniat. Innumerabiles forsitan obseruationes legem quandam aperuissent, ex qua in posterum eius loca quodammodo praedicere licuisset; nequaquam autem patet, quomodo Theoria ad huiusmodi motum explicandum accommodari potuisset. Imbecillitati nostrae sapientissimus creator consuluisse videtur, quod nulla corpora in coelo ita collocauerit, vt eorum motus, neque ad legem planetarum principalium, neque satellitum, referri posset.

4. Huiusmodi inuestigationem, quae vires ingenii humani tantum non transcendere videtur, certe non subito suscipi conueniet, sed potius conatus nostros pedetentim eo dirigi oportebit. Generale ergo problema trium corporum se mutuo attrahentium ita commodissime restringetur, vt vnius massa prae binis reliquis quasi euanescat, quo pacto id commodi assequemur, vt duo corpora, maiora scilicet, secundum leges *Keplerianas* moueantur, omnisque perturbatio in motu tertii consumatur, cuius situs et motus si ab initio ita fuerit comparatus, vt ad ambo maiorum aequa vi quasi attrahatur, habebimus eiusmodi casum, cuius inuestigatio nouam plane methodum postulat. Plurimum abest, vt hoc problema aggredi ausim, vt potius, frustra in eo euoluendo desudasse, fateri cogar; verum tamen casum obseruaui omnino singularem, ac simplicitate memorabilem, quo Lunae eiusmodi motus imprimi potuisset, vt perpetuo Soli, vel coniuncta, vel opposita, appatitura fuisset, cuius casus consideratio, cum forte vsum in hoc

hoc difficillimo negotio non destituatur; haud displicitura videtur.

5. Motum igitur tam Solis, quam Lunae, ex terra visum in plano ecliptico fieri assumens, terram quiescentem in T, et post aliquod tempus elapsum Solem in S, Lunam vero in L versari pono, et ducta recta fixa TA ad principium arietis directa, statuo angulos ATS $= \theta$, ATL $= \Phi$, et STL $= \Phi - \theta = \eta$, tum vero distantias TS $= u$, TL $= v$ et LS $= \sqrt{(uu - 2uv \operatorname{cos}. \eta + vv)} = z$. Sit porro longitudo Solis media $= \zeta$, eiusque distantia media a terra $= a$, hisque positis pro motu Solis vtpote regulari habebimus Tab. XX. Fig. 13.

$$\frac{2\, du\, d\theta + u\, dd\theta}{d\zeta^2} = 0 \text{ et } \frac{ddu - u\, d\theta^2}{d\zeta^2} + \frac{a^3}{uu} = 0$$

pro motu autem Lunae:

$$\frac{2\, dv\, d\Phi + v\, dd\Phi}{d\zeta^2} - \frac{a^3}{uu}\left(1 - \frac{u^3}{z^3}\right) \operatorname{sin}. \eta = 0$$

$$\frac{ddv - v\, d\Phi^2}{d\zeta^2} + \frac{nnc^3}{vv} + \frac{a^3 v}{z^3} + \frac{a^3}{uu}\left(1 - \frac{u^3}{z^3}\right) \operatorname{cos}. \eta = 0,$$

vbi c est distantia media, ad quam Luna, a sola vi terrae sollicitata, pari motu medio reuolueretur, existente $n : 1$ ratione motus medii Lunae ad motum medium Solis. Caeterum circa differentialia secundi gradus hic est monendum, elementum $d\zeta$ constans esse sumtum.

6. Tota ergo difficultas in resolutione harum duarum aequationum consistit, vt scilicet inde ad quoduis tempus, seu longitudinem Solis mediam ζ, tam distantia v, quam angulus Φ, definiatur. Quod cum in genere fieri nequeat, Geometrae adhuc in eo laborauerunt, vt saltem pro casu, quo distantia v, prae u, est

eſt vehementer parua, ſimulque n numerus mediocriter magnus, idoneas approximationes eruerent, in quo tamen negotio plurimum adhuc iure deſideratur. Hic autem binas iſtas aequationes in genere ſpecto, ſine vllo reſpectu ad Lunam habito, et quosdam caſus ſum euoluturus, quibus iis abſolute ſatisfieri queat. Eiusmodi ſcilicet motus in coelo locum habere poſſe oſtendam, quos perfecte cognoſcere in noſtra ſit poteſtate, etiamſi eorum ratio maxime a motu regulari abhorreat.

7. Primum igitur obſeruo, has duas aequationes abſolutam reſolutionem admittere caſu $\eta = 0$, ſeu $\Phi = \theta$; ita vt tum Luna perpetuo in coniunctione cum Sole eſſet apparitura. Cum enim ſit ſin. $\eta = 0$, et coſ. $\eta = 1$, erit $z = u - v$, noſtrae aequationes has induent formas:

$$\frac{2dvd\theta + vdd\theta}{d\zeta^2} = 0, \text{ et } \frac{ddv - vd\theta^2}{d\zeta^2} + \frac{nnc^3}{vv} + \frac{a^3 v}{(u-v)^3}$$
$$+ \frac{a^3}{uu} \cdot \frac{-3uuv + 3uv^2 - v^3}{(u-v)^3} = 0$$

ſeu $\frac{ddv - vd\theta^2}{d\zeta^2} + \frac{nnc^3}{vv} - \frac{a^3 v(3uu - 3uv + vv)}{uu(u-v)^3} = 0$

quae cum formulis, pro motu Solis datis, comparatae ſtatim dant $v = \alpha u$, quippe quo pacto prioribus aequationibus ſatisfit. Hinc altera aequatio pro Luna erit

$$\frac{\alpha(ddu - ud\theta^2)}{d\zeta^2} + \frac{nnc^3}{\alpha\alpha uu} - \frac{\alpha a^3(3 - 3\alpha + \alpha\alpha)}{(1-\alpha)^3 uu} = 0.$$

Quare cum altera aequatio pro Sole ſit

$$\frac{ddu - ud\theta^2}{d^2} + \frac{a^3}{uu} = 0$$

neceſſe eſt ſit: $\alpha a^3 = \frac{nnc^3}{\alpha\alpha} - \frac{\alpha a^3(3 - 3\alpha + \alpha\alpha)}{(1-\alpha)^3}$

ſeu $\frac{nnc^3}{\alpha\alpha a^3} = \frac{3\alpha - 3\alpha\alpha + \alpha^3}{(1-\alpha)^3}$ vbi

vbi cum ſit $\frac{nnc^3}{a^3}$ quantitas conſtans, ponatur breuitatis cauſa: $\frac{nnc^3}{a^3} = m$, eritque $m(1-\alpha)^2 = \alpha\alpha(3\alpha - 3\alpha^2 + \alpha^3)$ ſeu $m(1-\alpha)^2 = \alpha\alpha - \alpha\alpha(1-\alpha)^3$. Poſito ergo $2-\alpha = x$, fit $mxx = (1-x^3)(1-x)^2$, ſeu

$$1 - 2x + xx - mxx - x^3 + 2x^4 - x^5 = 0.$$

8. Pendet ergo determinatio numeri α vel x ab aequatione quinti gradus, pro cuius reſolutione notari oportet, eſſe m fractionem quam minimam; quare cum ſit

$$m(1-\alpha)^2 = 3\alpha^3 - 3\alpha^4 + \alpha^5$$

euidens eſt quoque, α minimum eſſe habiturum valorem, et quam proxime fore $\alpha = \sqrt[3]{\frac{m}{3}} = \frac{c}{a}\sqrt[3]{\frac{nn}{3}}$, accuratius autem $\alpha = \sqrt[3]{\frac{m}{3}} - \frac{1}{3}\sqrt[3]{\frac{mm}{9}} - \frac{1}{27}m + \frac{1}{81}m\sqrt[3]{\frac{m}{3}}$. Primus autem terminus ſufficit, ſicque eſt $v = \frac{cu}{a}\sqrt[3]{\frac{nn}{3}}$ vnde cum ſit proxime $u = a$, et $nn = 175$, erit circiter $v = 4c$; ſeu ſi Luna fere quater longius a nobis eſſet remota, eiusmodi motum habere poſſet, vt Soli perpetuo iuncta appareret. Talis Luna aequo iure tanquam Satelles terrae ac planeta primarius ſpectari poſſet, et vterque motus maxime foret regularis, hoc tantum a regulis *Keplerianis* recedens, quod Soli propior, quam terra, pari tamen tempore reuoluatur, ob vim ſcilicet terrae vis Solis tantum imminuitur, vt cum maiori tempore periodico conſiſtere poſſit. Hinc diſtantiam a terra quaſi quadruplo maiorem, quam Luna reuera inde diſtat, tanquam limitem ſpectare licet, vt corpora magis remota pro planetis primariis, propiora vero pro ſatellitibus terrae sint habenda. Similes limites circa reliquos planetas constitui poterunt.

9. Quemadmodum casus euolutus in perpetua coniunctione cum Sole constat, ita etiam perpetua oppositio similem casum suppeditat. Pro quo ponamus $\eta = 180°$, vt sit $\sin.\eta = 0$, $\cos.\eta = -1$ et $\Phi = 180° + \theta$, ideoque $d\Phi = d\theta$, atque $z = u + v$. Aequationes ergo pro motu Lunae sequentes induent formas:

$$\frac{2\,dv\,d\theta + v\,dd\theta}{d\zeta^2} = 0, \quad \text{et} \quad \frac{ddv - v\,d\theta^2}{d\zeta^2} + \frac{nnc^3}{v^2} + \frac{a^3 v}{(u+v)^3}$$
$$- \frac{a^3}{uu}\left(1 - \frac{u^3}{(u+v)^3}\right) = 0$$

quae posterior reducitur ad hanc:

$$\frac{ddv - v\,d\theta^2}{d\zeta^2} + \frac{nnc^3}{vv} - \frac{a^3}{uu} + \frac{a^3}{(u+v)^2} = 0.$$

Prior cum motu Solis collata praebet statim $v = \alpha u$, vnde posterior fit

$$\frac{\alpha(ddu - u\,d\theta^2)}{d\zeta^2} + \frac{nnc^3}{\alpha\alpha uu} - \frac{a^3}{uu} + \frac{a^3}{(1+\alpha)^2 uu} = 0.$$

At ex motu Solis est $\frac{ddu - u\,d\theta^2}{d\zeta^2} = -\frac{a^3}{uu}$, ex quo fit $-\alpha a^3 + \frac{nnc^3}{\alpha\alpha} - a^3 + \frac{a^3}{(1+\alpha)^2} = 0$, seu

$$\frac{nnc^3}{a^3} - \alpha\alpha(1+\alpha) + \frac{\alpha\alpha}{(1+\alpha)^2} = 0$$

et posito breuitatis gratia $\frac{nnc^3}{a^3} = m$, erit

$$m(1+\alpha)^2 = \alpha\alpha(1+\alpha)^3 - \alpha\alpha.$$

quae ex superiori nascitur, sumendis m et α negatiuis. Quamobrem hinc colligitur

$$\alpha = \sqrt[3]{\frac{m}{3}} + \frac{1}{3}\sqrt[3]{\frac{mm}{9}} - \frac{1}{27}m - \frac{1}{81}m\sqrt[3]{\frac{m}{3}}$$

satis

satis autem exacte est $a = \sqrt[3]{\frac{m}{3}}$ et $v = \frac{c\,u}{c}\sqrt[3]{\frac{n\,n}{3}}$ vt ante.

10. Casus hi eo magis sunt notatu digni, quod sine vlla approximatione absolute expediri possunt, etiamsi ambae vires Solis et terrae ad motum producendum concurrant, id quod nullo alio casu praestare licet. Tali autem motu simplici corpus re vera moueretur, si ipsi in distantia assignata, dum Soli, vel coniunctum, vel oppositum, ex terra appareret, eiusmodi motus imprimeretur, vt cum terra pari passu in plano eclipticae ingredi inciperet. Sin autem motus impressus tantillum ab hac lege discrepet, non quidem perpetuo Soli, vel coniunctum, vel oppositum, maneret, sed exiguas excursiones hinc inde quasi oscillando conficeret. Quo casu cum motus minime ab inuenta ratione esset discrepaturus, more solito, approximando etiam, eiusmodi motum definire licebit; in quo cum quasi initium motuum irregularium, quos nullo etiamnum modo ad calculum reuocare licet, conspiciatur, vsu certe non carebit, si in naturam istiusmodi motuum accuratius inquisiuero.

11. Cum autem haec inuestigatio haud leuibus difficultatibus implicetur, statim ab initio aequationes nostras tractatu faciliores reddi conueniet; quod, cum distantia v prae u vehementer sit exigua, commode per approximationem fieri potest. Scilicet ob $z = \sqrt{(uu - 2uv\cos.\eta + vv)}$ eliciemus proxime $\frac{1}{z^3} = \frac{1}{u^3} + \frac{3v\cos.\eta}{u^4} - \frac{3vv}{2u^5} + \frac{15vv\cos.\eta^2}{2u^5}$, ideoque $1 - \frac{u^3}{z^3} = -\frac{3v}{u}\cos.\eta + \frac{3vv}{2uu}(1 - 5\cos.\eta^2)$, ex quo aequationes no-

noſtrae, pro motu Lunae inuentae, in ſequentes formas tranſibunt:

I. $\frac{2dvd\Phi+vdd\Phi}{d\zeta^2}+\frac{3a^3v}{u^3}\sin.\eta\cos.\eta-\frac{3a^3vv}{2u^4}\sin.\eta(1-5\cos.\eta^2)=0$

II. $\frac{ddv-vd\Phi^2}{d\zeta^2}+\frac{nnc^3}{vv}+\frac{a^3v}{u^3}(1-3\cos.\eta^2+\frac{3a^3vv}{2u^4}(3\cos.\eta-5\cos.\eta^3)=0.$

Deinde etiam calculus non parum ſubleuabitur, ſi motum Solis, vt vniformem, ſpectemus, vt ſit $u=a$, et $\vartheta=\zeta$, ideoque $\eta=\Phi-\zeta$, ſeu $\Phi=\eta+\zeta$, vnde ſequentes emergunt aequationes:

I. $\frac{2dvd\eta+vdd\eta}{d\zeta^2}+\frac{2dv}{d\zeta}+3v\sin.\eta\cos.\eta-\frac{3vv}{2a}\sin.\eta(1-5\cos.\eta^2)=0$

II. $\frac{ddv}{d\zeta^2}-v(1+\frac{d\eta}{d\zeta})^2+v(1\,3\cos.\eta^2)+\frac{nnc^3}{vv}+\frac{3vv}{2a}\cos.\eta(3-5\cos.\eta^2)=0,$

vbi etiam poſtrema membra facile omitti poſſunt, quia fractio $\frac{v}{a}$ eſt vehementer parua, etiamſi diſtantia Lunae quadruplo maior ſtatuatur.

12. Vt iam hinc caſum memoratum, quo Luna circa Solem motu quaſi oſcillatiorio nutare videbitur, eliciamus, angulum η quam minimum concipiamus, vt ſit $\sin.\eta=\eta$, et $\cos.\eta=1-\frac{1}{2}\eta\eta$, et habebimus:

I. $\frac{2dvd\eta+vdd\eta}{d\zeta^2}+\frac{2dv}{d\zeta}+3v\eta\eta=0$

II. $\frac{ddv}{d\zeta^2}-v(1+\frac{d\eta}{d\zeta})^2+\frac{nnc^3}{vv}-2v+3v\eta\eta=0.$

Deinde quia diſtantia v parum immutatur, ponamus $v=b(1+x)$, vt x ſit quantitas minima, tum vero ſit breuitatis gratia $\frac{nnc^3}{b^3}=m$, eritque:

I. $\frac{2dxd\eta+xdd\eta}{d\zeta^2}+\frac{dd\eta}{d\zeta^2}+\frac{2dx}{d\zeta}+3\eta+3x\eta=0$

II. $\frac{ddx}{d\zeta^2}-3-3x-\frac{2d\eta}{d\zeta}-\frac{2xd\eta}{d\zeta}-\frac{d\eta^2}{d\zeta^2}-\frac{xd\eta^2}{d\zeta^2}+3\eta\eta$
$+3x\eta\eta+m-2mx+3mxx=0,$

vnde

vnde pro quouis angulo ζ valores quantitatum x et η definiri oportet.

13. Cum angulus η fit minimus, alternatimque positiuus euadat et negatiuus; quoniam Luna vltro citroque a Sole digredi conspicietur: facile colligere licet, eum per quempiam angulum ω ad ζ datam rationem tenente ita definiri, vt fit

$$\eta = A \operatorname{fin.} \omega + B \operatorname{fin.} 2\omega + C \operatorname{fin.} 3\omega \text{ etc.}$$

atque $d\omega = \alpha d\zeta$. Quo posito erit

$$\frac{d\eta}{d\zeta} = \alpha A \operatorname{cof.} \omega + 2\alpha B \operatorname{cof.} 2\omega + 3\alpha C \operatorname{cof.} 3\omega \text{ et}$$

$$\frac{dd\eta}{d\zeta^2} = -\alpha^2 A \operatorname{fin.} \omega - 4\alpha^2 B \operatorname{fin.} 2\omega - 9\alpha\alpha C \operatorname{fin.} 3\omega.$$

Quare cum aequatio prima in hanc formam transfundatur

$$\frac{2dx}{1+x} + \frac{dd\eta + 3\eta d\zeta^2}{d\eta + d\zeta} = 0$$

fiet $2l(1+x) + l\left(1 + \frac{d\eta}{d\zeta}\right) + 3\int \frac{\eta d\zeta}{1 + \frac{d\eta}{d\zeta}} = \text{Conft.}$

seu ob x et $\frac{d\eta}{d\zeta}$ minima

$$2x - xx + \tfrac{2}{3}x^3 + \frac{d\eta}{d\zeta} - \frac{d\eta^2}{2d\zeta^2} + \frac{d\eta^3}{3d\zeta^3} + 3\int \eta d\zeta$$
$$- \tfrac{3}{2}\eta\eta + 3\int \frac{\eta d\eta^2}{d\zeta} - 3\int \frac{\eta d\eta^3}{d\zeta^2} = \text{Conft.}$$

Nunc vero ob $d\zeta = \frac{d\omega}{\alpha}$ est

$$\int \eta d\zeta = -\frac{A}{\alpha} \operatorname{cof.} \omega - \frac{B}{2\alpha} \operatorname{cof.} 2\omega - \frac{C}{3\alpha} \operatorname{cof.} 3\omega$$

$$\eta\eta = \tfrac{1}{2}AA + AB \operatorname{cof.} \omega - \tfrac{1}{2}AA \operatorname{cof.} 2\omega - AB \operatorname{cof.} 3\omega$$
$$+ \tfrac{1}{2}BB \qquad\qquad + AC$$

$$\frac{d\eta^2}{d\zeta^2} = \tfrac{1}{2}\alpha\alpha AA + 2\alpha\alpha AB\cos\omega + \tfrac{1}{2}\alpha\alpha AA\cos 2\omega + 2\alpha\alpha AB\cos 3\omega$$
$$+ 2\alpha\alpha BB \qquad + 3\alpha\alpha AC \qquad + \alpha^2 AB\cos 2\omega$$

$$\frac{d\eta^3}{d\zeta^3} = + \alpha^3 AAB + \tfrac{3}{4}\alpha^3 A^3\cos\omega + [illegible]$$
$$+ 4\alpha^3 ABB$$

vbi ob literas A, B, C, minimas altiores potestates merito negligimus.

14. Cum ergo sit

$$\frac{\eta\, d\eta^2}{d\zeta^2} = \tfrac{1}{4}\alpha\alpha A^3\sin\omega + \tfrac{3}{2}\alpha\alpha AAB\sin 2\omega + \tfrac{1}{4}\alpha\alpha A^3\sin 3\omega$$
$$+ 3\alpha\alpha ABB + 2\alpha\alpha BBB \qquad + \tfrac{5}{8}\alpha\alpha A^2 C$$
$$- \tfrac{5}{8}\alpha\alpha A^2 C \qquad + \alpha\alpha ABB$$

ob $d\zeta = \frac{d\omega}{\alpha}$ habebimus integrando

$$\int\frac{\eta\, d\eta^2}{d\zeta} = -\tfrac{1}{4}\alpha A^3\cos\omega - \tfrac{3}{4}\alpha AAB\cos 2\omega - \tfrac{1}{12}\alpha A^3\cos 3\omega$$
$$- 3\alpha ABB \quad - \alpha B^3 \qquad - \tfrac{5}{8}\alpha A^2 C$$
$$+ \tfrac{5}{8}\alpha A^2 C \qquad - \tfrac{1}{3}\alpha ABB$$

vbi cum series A, B, C maxime decrescat, plura membra omitti possunt. Deinde cum sit

$$\frac{\eta\, d\eta^3}{d\zeta^3} = \tfrac{7}{8}\alpha^3 A^3 B\sin\omega + \tfrac{3}{8}\alpha^3 A^4\sin 2\omega + \tfrac{7}{8}\alpha^3 A^3 B\sin 3\omega$$

erit integrando

$$\int\frac{\eta\, d\eta^3}{d\zeta^2} = -\tfrac{7}{8}\alpha^2 A^3 B\cos\omega - \tfrac{3}{16}\alpha^2 A^4\cos 2\omega + \tfrac{7}{24}\alpha^2 A^3 B\cos 3\omega$$

atque ex his tandem conficitur haec aequatio omissis partibus constantibus:

$$\begin{array}{llll}
2x - xx + \frac{2}{3}x^3 & + \alpha A \cos.\omega & + 2\alpha B \cos.2\omega & + 3\alpha C \cos.3\omega = 0 \\
 & - \alpha\alpha AB & - \frac{1}{4}\alpha\alpha AA & - \alpha\alpha AB \\
 & + \frac{1}{4}\alpha^3 A^3 & - \frac{3}{2}\alpha\alpha AC & - \frac{3C}{3\alpha} \\
 & - \frac{3A}{\alpha} & + \frac{1}{3}\alpha^3 AAB & + \frac{3}{2}AB \\
 & - \frac{3}{2}AB & - \frac{3B}{2\alpha} & - \frac{1}{4}\alpha A^3 \\
 & - \frac{3}{4}\alpha A^3 & + \frac{3}{4}AA & + \frac{1}{4}\alpha^3 A^3 \\
 & & - \frac{3}{2}AC & \\
 & & - \frac{3}{8}\alpha AAB. &
\end{array}$$

15. Ad valorem ipsius x hinc definiendum ponamus breuitatis gratia

$$\left(\alpha - \frac{3}{\alpha}\right)A - \left(\alpha\alpha + \frac{3}{2}\right)AB + \frac{1}{4}\alpha(\alpha\alpha - 3)A^3 = \mathfrak{A}$$

$$\frac{4\alpha\alpha - 3}{2\alpha}B - \frac{(\alpha\alpha - 3)}{4}AA = \mathfrak{B}$$

$$\frac{3\alpha\alpha - 1}{\alpha}C - \frac{(2\alpha\alpha - 3)}{2}AB + \frac{1}{4}\alpha(\alpha\alpha - 1)A^3 = \mathfrak{C}$$

vt sit $2\,l(1 + x) + \mathfrak{A}\cos.\omega + \mathfrak{B}\cos.2\omega + \mathfrak{C}\cos.3\omega = 0$

et $1 + x = e^{-\frac{1}{2}\mathfrak{A}\cos.\omega - \frac{1}{2}\mathfrak{B}\cos.2\omega - \frac{1}{2}\mathfrak{C}\cos.3\omega}$

vnde concludimus fore

$$\begin{array}{llll}
x = & -\frac{1}{2}\mathfrak{A}\cos.\omega & - \frac{1}{2}\mathfrak{B}\cos.2\omega & - \frac{1}{2}\mathfrak{C}\cos.3\omega \\
 & + \frac{1}{8}\mathfrak{A}\mathfrak{B} & + \frac{1}{16}\mathfrak{A}\mathfrak{A} & + \frac{1}{8}\mathfrak{A}\mathfrak{B} \\
 & + \frac{1}{64}\mathfrak{A}^3 & & - \frac{1}{192}\mathfrak{A}^3
\end{array}$$

Verum ne in calculos nimis taediosos immergamur, rem aliquanto minus curate expediamus, neglectoque angulo triplo, vt sit $\eta = A\sin.\omega + B\sin.2\omega$, habebimus

$$x = -\frac{(\alpha\alpha - 3)}{2\alpha}A\cos.\omega - \frac{(4\alpha\alpha - 3)}{4\alpha}B\cos.2\omega$$
$$+ \frac{3(\alpha\alpha - 1)(\alpha\alpha - 3)}{16\alpha\alpha}AA$$

vbi breuitatis gratia ſcribamus:

$$x = \mathrm{E}\,\mathrm{coſ}\,\omega + \mathrm{F}\,\mathrm{coſ}.\,2\omega\text{, vt ſit}$$

$$\mathrm{E} = \frac{3-\alpha\alpha}{2\alpha}\mathrm{A} \text{ et } \mathrm{F} = \frac{3-4\alpha\alpha}{4\alpha}\mathrm{B} + \frac{3(\alpha\alpha-1)(\alpha\alpha-3)}{16\alpha\alpha}\mathrm{AA}.$$

16. Hi iam valores in aequatione ſecunda ſubſtituantur, atque reperiemus:

$$\begin{array}{lllll}
\frac{ddx}{d\zeta^2} & = & & -\alpha\alpha\mathrm{E}\,\mathrm{coſ}.\,\omega & -4\alpha\alpha\mathrm{F}\,\mathrm{coſ}.\,2\omega \\
-3-3x & = -3 & & -3\mathrm{E} & -3\mathrm{F} \\
-\frac{2d\eta}{d\zeta} & = -\alpha\mathrm{AE} & & -2\alpha\mathrm{A} & -4\alpha\mathrm{B} \\
 & & & -2\alpha\mathrm{BE} & -\alpha\mathrm{AE} \\
-\frac{2x\,d\eta}{d\zeta} & = & & -\alpha\mathrm{AF} & \\
-\frac{d\eta^2}{d\zeta^2} & = -\frac{1}{2}\alpha\alpha\mathrm{AA} & & +2\alpha\alpha\mathrm{AB} & -\frac{1}{2}\alpha\alpha\mathrm{AA} \\
+3\eta\eta & = +\frac{3}{2}\mathrm{AA} & & +3\mathrm{AB}\,\mathrm{coſ}.\,\omega & -\frac{3}{2}\mathrm{AA}\,\mathrm{coſ}.\,2\omega \\
+m & = +m & & & \\
-2mx & = & & -2m\mathrm{E} & -2m\mathrm{F} \\
+3mxx & = +\frac{3}{2}m\mathrm{EE} & & +3m\mathrm{EF} & +\frac{3}{2}m\mathrm{EE}
\end{array}$$

vnde primo concludimus:

$$m(1+\tfrac{3}{2}\mathrm{EE}) = 3 + \alpha\mathrm{AE} + \tfrac{1}{2}\alpha\alpha\mathrm{AA} - \tfrac{3}{2}\mathrm{AA} = 3$$

$$\text{ideoque } m = 3 - \tfrac{9}{2}\mathrm{EE}$$

pro determinatione numeri m indeque diſtantia b. Manifeſtum autem eſt, eſſe proxime $m = 3$, ideoque $b = c\sqrt[3]{\frac{nn}{3}}$. Porro autem fit

$$-\alpha\alpha\mathrm{E} - 9\mathrm{E} - 2\alpha\mathrm{A} - 2\alpha\mathrm{BE} - \alpha\mathrm{AF} + (2\alpha\alpha+3)\mathrm{AB}$$

$$+9\mathrm{EF} = 0;$$

vnde neglectis terminis minimis ob $\frac{\mathrm{E}}{\mathrm{A}} = \frac{3-\alpha\alpha}{2\alpha}$, erit $(\alpha\alpha+9)(3-\alpha\alpha)+4\alpha\alpha = 0$, ſeu

$$\alpha^4 + 2\alpha\alpha - 27 = 0\text{, hincque } \alpha\alpha = \sqrt{28} - 1.$$

Tertia

Tertia denique aequatio dat

$$-(4\alpha\alpha+9)F-4\alpha B-\alpha AE-\tfrac{1}{2}\alpha\alpha AA-\tfrac{3}{2}AA+\tfrac{9}{2}EE=0$$

ideoque:

$$\left.\begin{array}{l} \frac{(4\alpha\alpha-3)(4\alpha\alpha+9)}{4\alpha}B-\frac{3(\alpha\alpha-1)(\alpha\alpha-3)(4\alpha\alpha+9)}{16\alpha\alpha}AA \\ -4\alpha B \qquad +\frac{(\alpha\alpha-3)}{2}AA \\ \qquad\qquad -\frac{(\alpha\alpha+3)}{2}AA \\ \qquad\qquad +\frac{9(\alpha\alpha-3)^2}{8\alpha\alpha}AA \end{array}\right\}=0$$

vnde colligitur:

$$B(16\alpha^4+8\alpha\alpha-27)=\tfrac{1}{2}AA\alpha(13-7\alpha\alpha+2\alpha^4)$$

seu ob $27=\alpha^4+2\alpha\alpha$

$$3B(5\alpha\alpha+2)=\frac{3AA}{2\alpha}(13-7\alpha\alpha+2\alpha^4)$$

ideoque $B=\frac{13-7\alpha\alpha+2\alpha^4}{2\alpha(5\alpha\alpha+2)}AA=\frac{67-11\alpha\alpha}{2\alpha(5\alpha\alpha+2)}AA$

$$F=\frac{291-94\alpha\alpha-23\alpha^4}{8\alpha\alpha(5\alpha\alpha+2)}AA=-\frac{165-24\alpha\alpha}{4\alpha\alpha(5\alpha\alpha+2)}AA.$$

17. Quantitas ergo A arbitrio nostro relinquitur, a qua digressiones a linea syzygiarum pendent, pro ea autem valde paruam fractionem assumi oportet, quae si fuerit tam exigua, vt eius quadratum nullius sit momenti, primi termini sufficiunt. Pro distantia ergo $v=b(1+x)$ erit $b=c\sqrt[3]{\frac{nn}{3}}$, et angulus ω ita definitur, vt sit $\omega=\alpha\zeta+\beta$ existente $\alpha\alpha=\sqrt{28}-1$, hincque $\alpha\alpha=4,291502$ et $\alpha=2,071594$. Tum vero erit $\eta=A\,\text{sin.}\,\omega$ et $v=b(1-\frac{(\alpha\alpha-3)}{8\alpha}A\,\text{cos.}\,\omega)$ seu $v=b(1-0,311717\,A\,\text{cos.}\,\omega)$.

 Excur-

Excursiones fiunt maximae, si angulus ω fit 90°. 270°. etc. ergo ab vna digressione maxima ad sequentem est $\alpha\zeta = 180^\circ$ et $\zeta = 86^\circ, 53\frac{1}{2}'$: at in digressionibus maximis est $v = b$. Verum etiam huiusmodi librationis, si maior existeret, determinatio insignibus laborat difficultatibus, vt, quo accuratius omnes variationes definire vellemus, eo minus certi de reliquis neglectis redderemur.