si jam pondus ipsius corporis, per cujus centrum inertiae axis gyrationis GG transit, ponatur ut ante $= M$, et vires quibus alter terminus cylindricus a planis, quibus in E et F incumbit, repellitur; secundum EG $= E$ et secundum FG $= F$, unde frictiones nascuntur secundum EM $= \delta E$ et secundum FL $= \delta F$, supra vidimus, hinc oriri vim verticaliter sursum tendentem $E \cos\zeta + F \cos\eta + \delta(E \sin\zeta - F \sin\eta)$, et vim horizontalem dextorsum directam $= E \sin\zeta - F \sin\eta - \delta(E \cos\zeta + F \cos\eta)$, quas ob binos terminos cylindricos duplicari oportet. Deinde ex pondere ipsius corporis habemus vim verticaliter deorsum nitentem $= M$ et ex onere elevando vim $= Q$. Ex vi sollicitante P vero oritur vis deorsum urgens $= P \sin\theta$, et vis horizontaliter sinistrorsum $= P \cos\theta$: quae vires cum se mutuo debeant destruere, obtinebimus has aequationes:

$$E \cos\zeta + F \cos\eta + \delta(E \sin\zeta - F \sin\eta) = \tfrac{1}{2}M + \tfrac{1}{2}Q + \tfrac{1}{2}P \sin\theta$$
$$E \sin\zeta - F \sin\eta - \delta(E \cos\zeta + F \cos\eta) = \tfrac{1}{2}P \cos\theta$$

unde colligimus

$$E \cos\zeta + F \cos\eta = \frac{M + \underline{Q} + P\sin\theta - \delta P\cos\theta}{2(1+\delta\delta)}$$
$$E \sin\zeta - F \sin\eta = \frac{M\delta + \underline{Q}\delta + P\delta\sin\theta + P\cos\theta}{2(1+\delta\delta)}$$

hincque porro

$$E = \frac{M(\sin\eta + \delta\cos\eta) + \underline{Q}(\sin\eta + \delta\cos\eta) + P(\sin\eta + \delta\cos\eta)\sin\theta + P(\cos\eta - \delta\sin\eta)\cos\theta}{2(1+\delta\delta)\sin(\zeta+\eta)}$$
$$F = \frac{(M + \underline{Q} + P\sin\theta)(\sin\zeta - \delta\cos\zeta) - P(\cos\zeta + \delta\sin\zeta)\cos\theta}{2(1+\delta\delta)\sin(\zeta+\eta)}.$$

Praeterea vero, quia motum uniformem desideramus, momenta virium respectu axis gyrationis se destruere debent. Est autem momentum accelerans $= Pr$, et momenta opposita $= 2\delta(E+F)f + Qs$, unde necesse est sit $Pr = 2\delta(E+F)f + Qs$, ideoque

$$Pr - Qs = \frac{\delta(M + \underline{Q} + P\sin\theta)(\sin\zeta + \sin\eta - \delta\cos\zeta + \delta\cos\eta) + \delta P(\cos\eta - \cos\zeta - \delta\sin\eta - \delta\sin\zeta)\cos\theta}{(1+\delta\delta)\sin(\zeta+\eta)} f$$

hincque vim sollicitantem P definire licet.

Quodsi jam ponamus terminos cylindricos in cavitatibus circularibus sustineri, ut contactus unico loco fiat, ubi scilicet tangens ad horizontem inclinetur angulo $= \zeta$: erit $F = 0$, ideoque

$$(M + Q + P\sin\theta - \delta P\cos\theta)\,\mathrm{tang}\,\zeta = \delta(M + Q + P\sin\theta) + P\cos\theta$$

et

$$\text{et } E = \frac{M + \underline{Q} + P\sin\theta - \delta P\cos\theta}{2(1+\delta\delta)\cos\zeta}.$$

Inde colligitur $P = \dfrac{(M + \underline{Q})(\delta - \mathrm{tang}\,\zeta)}{(\sin\theta - \delta\cos\theta)\,\mathrm{tang}\,\zeta - \cos\theta - \delta\sin\theta}$

quo valore in postrema aequatione, quae fit $Pr - Qs = 2\delta Ef$, substituto prodet

$$(M + Q)\,\delta f \cos\theta = (M + Q)\,r(\sin\zeta - \delta\cos\zeta) + Qs((\sin\theta - \delta\cos\theta)\sin\zeta - (\cos\theta + \delta\sin\theta)\cos\zeta)$$

ubi si ponamus $\delta = \mathrm{tang}\,\lambda$, haec aequatio erit

$$(M + Q)f \sin\lambda \cos\theta = (M + Q)\,r \sin(\zeta - \lambda) - Qs \cos(\zeta + \theta - \lambda)$$

unde angulus ζ erui debet; quo invento erit

$$P = \frac{(M + \underline{Q})\sin(\zeta - \lambda)}{\cos(\zeta + \theta - \lambda)}$$

$$\text{seu } P = \frac{Qs}{r} + \frac{(M + \underline{Q})f \sin\lambda \cos\theta}{r\cos(\zeta + \theta - \lambda)}.$$

COROLL. 1.

1012. Si terminus cylindricus unico loco incumbat in alveolo cavo, pro P substituto valore prodit pressio in eo loco

$$E = \frac{(M + \underline{Q})\cos\theta}{2(1+\delta\delta)\cos\lambda\cos(\zeta + \theta - \lambda)} = \frac{(M + \underline{Q})\cos\lambda\cos\theta}{2\cos(\zeta + \theta - \lambda)}$$

posito $\delta = \mathrm{tang}\,\lambda$. Haec ergo pressio evanescit casu $\cos\theta = 0$, nisi simul fiat $\cos(\zeta + \theta - \lambda) = 0$.

COROLL. 2.

1013. Posito autem $\theta = 90^\circ$, erit

$$(M + Q)\,r \sin(\zeta - \lambda) + Qs \sin(\zeta - \lambda) = 0$$

quo ergo casu fit $\zeta = \lambda$ seu $\mathrm{tang}\,\zeta = \delta$, et $P = \dfrac{Qs}{r} + \dfrac{(M + \underline{Q})f\sin\lambda}{r}$.

3. Cum autem sit $E = \dfrac{M + \underline{Q} + P}{2(1+\delta\delta)\cos\lambda}$, erit $Pr - Qs = \dfrac{\delta(M + \underline{Q} + P)f}{(1+\delta\delta)\cos\lambda}$

$$= (M + Q + P)f \sin\lambda \text{ et } P = \frac{Qs + (M + \underline{Q})f\sin\lambda}{r - f\sin\lambda}.$$

CO

COROLL. 3.

1014. Si ponamus $\theta = -90°$, primo ob $F = 0$ habemus

$$(M + Q - P)\, tang\, \zeta = \delta\, (M + Q - P)$$

tum vero cum sit $E = \dfrac{M + Q - P}{2(1 + \delta\delta) cos\, \zeta}$, erit

$$Pr - Qs = \frac{\delta f (M + Q - P)}{(1 + \delta\delta) cos\, \zeta}.$$

Quare si capiatur $P = M + Q$, pressio ideoque et frictio evanescit, sumique oportet $r = \dfrac{Qs}{M + Q}$.

COROLL. 4.

1015. Nisi autem hoc casu $\theta = -90°$ statuatur $P = M + Q$, erit $tang\, \zeta = \delta$, et $Pr - Qs = \dfrac{\delta f (M + Q - P)}{r (1 + \delta\delta)} = f (M + Q - P) \sin \lambda$ hincque $P = \dfrac{Qs + (M + Q) f \sin \lambda}{r + f \sin \lambda}$. At r ita sumi oportet, ut valor ipsius E ne fiat negativus. Hoc enim casu sustentatio ex opposito fieret, ibique frictio orietur.

SCHOLION. 1.

1016. Hoc ergo modo frictio penitus tolli posset, vim P ita applicando, ut cum pondere corporis M et onere Q aequilibrium constituat. Verum hic casus in praxi parum utilitatis haberet, quia termini cylindrici intra alveos suos, quos ipsis ampliores esse oportet, hinc inde vacillarent, quo incommodo motus magis quam frictione impediretur. Deinde vero pleraeque hujus generis machinae ita disponi solent, ut vis sollicitans P multo sit minor quam onus elevandum Q, ideoque multo magis $P < M + Q$. Si enim vim oneri aequalem impendere vellemus, negotium sine machina absolvi posset, unde non mirum hoc casu frictionis lucrum obtineri posse. Ac si vis P pro data sumatur, ex nostris formulis elicitur r, pro loco applicationis: unde si celeritas angularis machinae sit $= \varepsilon$, onus elevabitur celeritate εs, vis vero sollicitans aget celeritate $= \varepsilon r$. Nisi ergo frictio motum impediret, foret $P\varepsilon r = Q\varepsilon s$; nunc autem ob frictionem erit $P\varepsilon r - Q\varepsilon s = 2\delta\varepsilon Ef$: ubi observari convenit, denotare $P\varepsilon r$ actionem vis sollicitantis, $Q\varepsilon s$ vero quantitatem effectus uno minuto secundo producti, cum εr et εs sint spatia uno minuto secundo confecta. Verum haec ad Theoriam machinarum sunt referenda, quam seorsim pertractari convenit.

SCHOLION. 2.

1017. Si vis sollicitans P cum angulo θ fuerit data, quaeraturque distantia applicationis seu longitudo vectis $GR = r$, ex prima aequatione statim colligitur angulus ζ seu punctum E, ubi in cavitate fiet contactus, scilicet:

$$tang\, \zeta = \frac{\delta (M + Q + P \sin \theta) + P cos\, \theta}{M + Q + P \sin \theta - \delta P cos\, \theta};$$

ad quem cognoscendum statuantur duo anguli λ et ξ ut sit

$$tang\, \lambda = \delta \text{ et } tang\, \xi = \frac{P cos\, \theta}{M + Q + P \sin \theta},$$

eritque $tang\, \zeta = \dfrac{tang\, \lambda + tang\, \xi}{1 - tang\, \lambda\, tang\, \xi}$ ideoque $\zeta = \lambda + \xi$.

Unde patet fore $\zeta > \lambda$, si est $\theta > 0$, hoc est, si recta GR sursum vergat; sin autem deorsum dirigatur, fore $\zeta < \lambda$, quo casu fieri potest, ut contactus fiat in infimo puncto, si scilicet fuerit $P = \dfrac{\delta (M + Q)}{-cos\, \theta - \delta \sin \theta}$.

Tum vero habebitur pressio $E = \dfrac{(M + Q + P \sin \theta - \delta P cos\, \theta) cos\, \lambda^2}{2 cos (\lambda + \xi)}$ seu

$$E = \frac{P cos\, \lambda\, cos\, \theta}{2 \sin \xi} = \tfrac{1}{2} cos\, \lambda \sqrt{((M + Q)^2 + 2P (M + Q) \sin \theta + PP)}$$

hincque tandem concluditur longitudo vectis

$$GR = r = \frac{Qs}{P} + \frac{f \sin \lambda}{P} \sqrt{((M + Q)^2 + 2P (M + Q) \sin \theta + PP)}.$$

Ut igitur pro eadem vi sollicitante P pressio E ideoque et frictio fiat minima, angulum θ esse oportet $= -90°$, seu vectem GR in ipso radio GS capi convenit, quo casu fit, ut jam vidimus, $\xi = 0$, hincque $\zeta = \lambda$, et $E = \frac{1}{2} (M + Q - P)\, cos\, \lambda$

atque $GR = r = \dfrac{Qs}{P} + \dfrac{f (M + Q - P) \sin \lambda}{P}$.

Investigemus nunc etiam motum penduli, terminis cylindricis simili modo suspensi, qui scilicet utrinque binis planis inclinatis incumbant:

et quia hic motus eſt reciprocus, iſta plana aequaliter ad horizontem inclinata ſtatui conveniet.

P R O B L E M A. 9.

1018. Si pendulum oſcilletur circa axem horizontalem fixum, cujus termini cylindrici utrinque binis planis aequaliter inclinatis incumbant, definire ejus motum ob frictionem perturbatum.

S O L U T I O.

Fig. 135. Sit AEBF baſis alterius termini cylindrici, qui incumbat planis ML et NL ad horizontem inclinatis angulo $=\zeta$ erunt puncta contactus in E et F, ut radii GE et GF cum verticali ABLH angulos conſtituant $=\zeta$, quae omnia ad alteram partem perinde ſe habeant, ut axis gyrationis ſit recta horizontalis GG. Sit porro penduli forma utrinque ſibi ſimilis, ac jam elapſo tempore t declinet penduli centrum inertiae I a ſitu verticali angulo HGI $=\phi$, unde ad ſitum verticalem accedat celeritate angulari $=u$, ita ut motus gyratorius fiat in ſenſum EBF. Sit maſſa tota idemque pondus penduli $=$ M, diſtantia GI $=b$, et momentum inertiae ejus reſpectu axis gyrationis GG $=$ Mkk. Quod ergo ad actionem gravitatis attinet, totum pondus M in puncto I collectum concipere licet.

Ponatur jam terminorum cylindricorum radius GE $=$ GF $=f$, ſintque vires, quibus ii a planis ſuſtentantur, ſecundum EG $=$ E et ſecundum GF $=$ F: unde frictiones erunt ſecundum EM $=\delta$E, et ſecundum FL $=\delta$F.

Ex his autem viribus ut ſupra §. 1005. ubi $\eta=\zeta$ naſcuntur primo vis verticalis ſurſum tendens $=(\mathrm{E}+\mathrm{F})\cos\zeta+\delta(\mathrm{E}-\mathrm{F})\sin\zeta$ et vis horizontalis dextrorſum directa $=(\mathrm{E}-\mathrm{F})\sin\zeta-\delta(\mathrm{E}+\mathrm{F})\cos\zeta$. Pondus autem praebet vim deorſum tendentem $=$ M.

Unde pro motu progreſſivo ſeu motu centri inertiae I habemus primo vim verticaliter deorſum directam:

$$\mathrm{M}-2(\mathrm{E}+\mathrm{F})\cos\zeta-2\delta(\mathrm{E}-\mathrm{F})\sin\zeta=\mathrm{P}$$

et vim dextrorſum tendentem horizontalem

$$2(\mathrm{E}-\mathrm{F})\sin\zeta-2\delta(\mathrm{E}+\mathrm{F})\cos\zeta=\mathrm{Q}.$$

Motus autem hujus, cum celeritas centri inertiae ſit $=bu$, celeritas verticalis deorſum tendens eſt $=bu\sin\phi$ et celeritas horizontalis dextrorſum directa $=bu\cos\phi$; unde colligimus:

$$\frac{b\,du\sin\phi+bu\,d\phi\cos\phi}{2g\,dt}=\frac{P}{M}\ \text{et}\ \frac{b\,du\cos\phi-bu\,d\phi\sin\phi}{2g\,dt}=\frac{Q}{M}$$

ubi

ubi eſt $u\,dt=-d\phi$. Deinde cum corpus circa axem fixum GG gyretur, cujus reſpectu eſt momentum virium ad accelerandum $=\mathrm{M}b\sin\phi-2\delta(\mathrm{E}+\mathrm{F})f$, erit

$$\frac{du}{2g\,dt}=\frac{Mb\sin\phi-2\delta(E+F)f}{Mkk}$$

Qui valor ſi in illis ſubſtituatur, habebimus

$$\frac{Mbb\sin\phi^2-2\delta(E+F)fb\sin\phi}{Mkk}-\frac{buu\cos\phi}{2g}=\frac{P}{M}$$

$$\frac{Mbb\sin\phi\cos\phi-2\delta(E+F)fb\cos\phi}{Mkk}+\frac{buu\sin\phi}{2g}=\frac{Q}{M}$$

hincque

$$\frac{Mbb\sin\phi-2\delta(E+F)fb}{Mkk}=\frac{P\sin\phi+Q\cos\phi}{M}\ \text{et}$$

$$\frac{buu}{2g}=\frac{Q\sin\phi-P\cos\phi}{M}$$

ex quibus quantitatibus preſſiones E et F definiri debent. Cum autem ſit $\mathrm{P}+\delta\mathrm{Q}=\mathrm{M}-2(1+\delta\delta)(\mathrm{E}+\mathrm{F})\cos\zeta$, erit

$$\mathrm{M}-2(1+\delta\delta)(\mathrm{E}+\mathrm{F})\cos\zeta=\frac{\mathrm{M}bb\sin\phi(\sin\phi+\delta\cos\phi)-2\delta(F+F)fb(\sin\phi+\delta\cos\phi)}{kk}$$

$$-\frac{\mathrm{M}buu(\cos\phi-\delta\sin\phi)}{2g}$$

hincque

$$2(\mathrm{E}+\mathrm{F})((1+\delta\delta)kk\cos\zeta-\delta fb(\sin\phi+\delta\cos\phi))=$$

$$\mathrm{M}kk-\mathrm{M}bb\sin\phi(\sin\phi+\delta\cos\phi)+\frac{\mathrm{M}bkkuu(\cos\phi-\delta\sin\phi)}{2g}$$

unde valor ipſius E + F ſubſtitutus praebet

$$\frac{du}{2g\,dt}=\frac{(1+\delta\delta)b\cos\zeta\sin\phi-\delta f-\dfrac{\delta fbuu(\cos\phi-\delta\sin\phi)}{2g}}{(1+\delta\delta)kk\cos\zeta-\delta fb(\sin\phi+\delta\cos\phi)}$$

ex qua aequatione motus penduli ope formulae $u\,dt=-d\phi$ determinari poteſt.

COROLL. 1.

1019. De pressione in E nullum est dubium, quin ea fiat positiva; sed pressio in F sequenti modo determinatur.

$$2F\left((1+\delta\delta)\sin 2\zeta + \frac{2\delta fb}{kk}\cos\zeta\cos\phi - \frac{2\delta\delta fb}{kk}\cos\zeta\sin\phi\right) =$$

$$M\left(\sin\zeta - \delta\cos\zeta + \frac{\delta fb}{kk}\cos\phi\right) - \frac{Mbb\sin\phi}{kk}\left(\cos(\zeta-\phi)+\delta\sin(\zeta-\phi)\right)$$

$$+ \frac{Mbuu}{2g}\left(\sin(\zeta-\phi) - \delta\cos(\zeta-\phi) + \frac{\delta fb}{kk}\cos 2\phi\right)$$

unde valor ipsius F positivus prodire debet, quod fit, dum fuerit $\tang\zeta > \delta$ existente ϕ angulo parvo.

COROLL. 2.

1020. Si frictio esset nulla seu $\delta = 0$, foret $\frac{du}{2g\,dt} = \frac{b\sin\phi}{kk}$, unde motus pendulorum supra definitus facile eruitur, pro pressionibus autem E et F haberemus has aequationes:

$$2(E+F)\,kk\cos\zeta = M\left(kk - bb\sin\phi^2 + \frac{bkkuu\cos\phi}{2g}\right)$$

$$\text{et } 2Fk^2\sin 2\zeta = M\left(kk\sin\zeta - bb\sin\phi\cos(\zeta-\phi) + \frac{bkkuu\sin(\zeta-\phi)}{2g}\right)$$

$$\text{et } 2Ek^2\sin 2\zeta = M\left(kk\sin\zeta + bb\sin\phi\cos(\zeta+\phi) + \frac{bkkuu\sin(\zeta+\phi)}{2g}\right)$$

quarum utraque ut sit positiva debet esse:

$$\tang\zeta > \frac{2gbb\sin\phi\cos\phi + bkkuu\sin\phi}{2gkk - 2gbb\sin\phi^2 + bkkuu\cos\phi}$$

ubi notandum est, esse $kk > bb$.

COROLL. 3.

1021. Aequatio differentialis inventa ob $dt = \frac{-d\phi}{u}$ abit in hanc formam

$$0 = u\,du\left((1+\delta\delta)kk\cos\zeta - \delta fb(\sin\phi + \delta\cos\phi)\right) - \delta fbuu\,d\phi\,(\cos\phi - \zeta\sin\phi)$$

$+ 2$

$$+ 2(1+\delta\delta)gb\,d\phi\cos\zeta\sin\phi - 2\delta fg\,d\phi$$

quae per $(1+\delta\delta)kk\cos\zeta - \delta fb(\sin\phi + \delta\cos\phi)$ multiplicata fit integrabilis, prodibitque

$$C = uu\left((1+\delta\delta)kk\cos\zeta - \delta fb(\sin\phi + \delta\cos\phi)\right)^2$$
$$+ 4g\int d\phi\left((1+\delta\delta)b\cos\zeta\sin\phi - \delta f\right)\left((1+\delta\delta)kk\cos\zeta - \delta fb(\sin\phi + \delta\cos\phi)\right).$$

SCHOLION.

1022. Si hoc integrale evolvamus, reperiemus

$$C = uu\left((1+\delta\delta)kk\cos\zeta - \delta fb(\sin\phi + \delta\cos\phi)\right)^2 - 4(1+\delta\delta)^2 gbkk\cos\zeta^2\cos\phi$$
$$- \delta(1+\delta\delta)fgbb\cos\zeta(2\phi - \sin 2\phi - \delta\cos 2\phi) - 4\delta(1+\delta\delta)fgkk\phi\cos\zeta$$
$$- 4\delta\delta ffgb(\cos\phi - \zeta\sin\phi).$$

Quare si sumamus angulum HGI initio fuisse $= \theta$, indeque pendulum a quiete descensum inchoasse, constans C ita definitur, ut sit

$$C = -4(1+\delta\delta)^2 gbkk\cos\zeta^2\cos\theta - \zeta(1+\delta\delta)fgbb\cos\zeta(2\theta - \sin 2\theta - \delta\cos 2\theta)$$
$$- 4\delta(1+\delta\delta)fgkk\theta\cos\zeta - 4\delta\delta ffgb(\cos\theta - \delta\sin\theta)$$

quo valore substituto pendulum ex altera parte eousque ascendet, donec iterum fiat $u = 0$. Verum hanc determinationem in genere suscipere haud licet, Neque vero ipsum problema in latissimo sensu resolvimus, ut ad omnia cujuscunque formae pendula pateret, sed primo assumsimus, binos terminos cylindricos utrinque a centro gravitatis aeque esse remotos: deinde etiam talem structuram statuimus, ut recta per centrum inertiae I axi gyrationis GG parallela ducta simul esset corporis axis principalis. Quae conditio nisi locum haberet, non licuisset momenta virium statim ad axem gyrationis GG transferre, sed etiam ratio habenda fuisset virium obliquarum, quae in terminis axis GG inaequales pressiones produxissent, ideoque formulae multo magis intricatae prodiissent. Ut igitur hinc quicquam ad usum concludamus, statuamus oscillationes esse minimas, et quomodo earum motus a frictione perturbetur, diligentius investigemus.

PROBLEMA. 10.

1023. Si pendulum eo modo suspensum, uti in problemate praecedente assumsimus, oscillationes peragat quam minimas, earum motum a frictione perturbatum determinare. Fig. 135.

So-

SOLUTIO.

Maneant omnia uti in problemate praecedente constituimus, ac si initio pendulum ad angulum $HGI = \theta$ fuerit declinatum, unde descensum ex quiete inchoaverit, elapso autem tempore t angulus HGI sit $= \varphi$, et celeritas angularis in sensum $IH = v$, in praesenti hypothesi anguli θ et φ erunt minimi, qui ergo loco sinuum et cosinuum ita introducantur, ut eorum potestates quadrato altiores rejiciantur. Hinc aequatio integralis §. praec. eruta induet hanc formam:

$$C = vv\,((1 + \delta\delta)\,kk\cos\zeta - \delta fb\,(\varphi + \delta - \tfrac{1}{2}\delta\varphi\varphi))^2 - 4(1 + \delta\delta)^2 gbkk\cos\zeta^2 (1 - \tfrac{1}{2}\varphi\varphi)$$
$$+ \delta\delta(1 + \delta\delta)\,fgbb\cos\zeta\,(1 - \varphi\varphi 2) - 4\delta(1 + \delta\delta)\,fgkk\,\varphi\cos\zeta$$
$$- 4\,\delta\delta\,ffgb\,(1 - \delta\varphi - \tfrac{1}{2}\varphi\varphi)$$

ubi constans $C = -4(1 + \delta\delta)^2 gbkk\cos\zeta^2 (1 - \tfrac{1}{2}\theta\theta) + \delta\delta(1 + \delta\delta)\,fgbb\cos\zeta\,(1 - 2\theta\theta)$
$$- 4\delta(1 + \delta\delta)\,fgkk\,\theta\cos\zeta - 4\delta\delta\,ffgb\,(1 - \zeta\theta - \tfrac{1}{2}\theta\theta).$$

Hac igitur aequatione evoluta obtinebimus:

$$vv\,((1 + \delta\delta)\,kk\cos\zeta - \delta\delta fb)^2 =$$
$$2gb\,((1 + \delta\delta)^2 kk\cos\zeta^2 - \delta\delta(1 + \delta\delta)\,fb\cos\zeta + \delta\delta ff)\,(\theta\theta - \varphi\varphi)$$
$$- 4\delta fg\,((1 + \delta\delta)\,kk\cos\zeta - \delta\delta fb)\,(\theta - \varphi)$$

ubi in coefficiente ipsius vv angulum φ neglexi, quia in evolutione perducturus esset ad altiores potestates. Ad hanc aequationem resolvendam statuamus brevitatis gratia:

$$(1 + \delta\delta)\,kk\cos\zeta - \delta\delta fb = A$$
$$(1 + \delta\delta)^2 kk\cos\zeta^2 - \delta\delta(1 + \delta\delta)\,fb\cos\zeta + \delta\delta ff = B$$

ut sit

$$AA\,vv = 2Bgb\,(\theta\theta - \varphi\varphi) - 4A\delta fg\,(\theta - \varphi)$$

unde ponendo $v = 0$ invenimus, quousque pendulum sit ascensurum, donec iterum ad quietem perducatur. Divisione autem per $2g\,(\theta - \varphi)$ instituta oritur

$$Bb\,(\theta + \varphi) - 2A\delta f = 0$$

hincque $\varphi = -\theta + \dfrac{2A\delta f}{Bb}$, seu ad alteram partem ultra H tantum per angulum $\theta - \dfrac{2A\delta f}{Bb}$ ascendet.

Porro ad durationem hujus oscillationis investigandam, cum sit $v = \dfrac{\sqrt{(2Bgb(\theta\theta - \varphi\varphi) - 4A\delta fg(\theta - \varphi))}}{A} = \dfrac{-d\varphi}{dt}$, erit

$$dt = \frac{-A\,d\varphi}{\sqrt{(2Bgb(\theta\theta - \varphi\varphi) - 4A\delta fg(\theta - \varphi))}} \text{ seu}$$
$$dt = \frac{-A\,d\varphi}{\sqrt{2g(\theta - \varphi)(Bb(\theta + \varphi) - 2A\delta f)}}$$

unde integrando colligitur:

$$t = \frac{A}{\sqrt{2Bgb}}.\ \text{Arc. cos}\ \frac{Bb - \varphi A\delta f}{Bb\theta - A\delta f}.$$

Statuatur nunc $\varphi = -\theta + \dfrac{2A\delta f}{Bb}$ seu $Bb\varphi - A\delta f = -Bb\theta + A\delta f$, erit tempus oscillationis integrae $= \dfrac{\pi A}{\sqrt{2Bgb}}$; quod ergo non pendet ab amplitudine oscillationis, ita ut omnes oscillationes minimae maneant isochronae perinde ac si nulla frictio adesset. Sed non pari tempore absolventur. Quantum autem frictio tempus cujusque oscillationis turbet, quaeratur valor $\dfrac{A}{\sqrt{B}}$, ubi si crassitiem terminorum cylindricorum seu f ut minimam spectamus, est $\dfrac{1}{\sqrt{B}} = \dfrac{1}{(1+\delta\delta)k\cos\zeta} + \dfrac{\delta\delta fb}{2(1+\delta\delta)^2 k^3 \cos\zeta^2}$ ideoque $\dfrac{A}{\sqrt{B}} = k - \dfrac{\delta\delta fb}{2(1+\delta\delta)k\cos\zeta}$: quare tempus unius oscillationis $= \dfrac{\pi}{\sqrt{2gb}}\left(k - \dfrac{\delta\delta fb}{2(1+\delta\delta)k\cos\zeta}\right)$, unde patet, ob frictionem tempora oscillationum minui.

COROLL. 1.

1024. Si radius terminorum cylindricorum f sit valde exiguus prae quantitatibus b et k, erit proxime $B = A\,(1 + \delta\delta)\cos\zeta$. Hinc si primus arcus descensus sit $= \theta$, erit sequens arcus ascensus $= \theta - \dfrac{2\delta f}{(1+\delta\delta)b\cos\zeta}$, qui simul est arcus descensus in secunda oscillatione.

CO-

COROLL. 2.

1025. Oscillationes ergo successivae sequenti modo se habebunt:

In Oscillatione	arcus descensus	arcus ascensus	totus arcus
prima	θ	$\theta - \frac{2\delta f}{(1+\delta\delta)b\cos\zeta}$	$2\theta - \frac{2\delta f}{(1+\delta\delta)b\cos\zeta}$
secunda	$\theta - \frac{2\delta f}{(1+\delta\delta)b\cos\zeta}$	$\theta - \frac{4\delta f}{(1+\delta\delta)b\cos\zeta}$	$2\theta - \frac{6\delta f}{(1+\delta\delta)b\cos\zeta}$
tertia	$\theta - \frac{4\delta f}{(1+\delta\delta)b\cos\zeta}$	$\theta - \frac{6\delta f}{(1+\delta\delta)b\cos\zeta}$	$2\theta - \frac{10\delta f}{(1+\delta\delta)b\cos\zeta}$
quarta	$\theta - \frac{6\delta f}{(1+\delta\delta)b\cos\zeta}$	$\theta - \frac{8\delta f}{(1+\delta\delta)b\cos\zeta}$	$2\theta - \frac{14\delta f}{(1+\delta\delta)b\cos\zeta}$

COROLL. 3.

1026. Oscillationes tamdiu durabunt, quamdiu arcus ascensuum manent positivi. Statim enimque ac vel evanescunt, vel adeo negativi evadunt, motus omnis cessat. Atque ut motus oriatur, necesse est, ut sit $\theta > \frac{A\delta f}{Bb}$, si enim fuerit $\theta =$ vel $< \frac{A\delta f}{Bb}$ pendulum ob frictionem plane in quiete coercetur, etsi teneat situm inclinatum.

COROLL. 4.

1027. Ut ergo pendulum saltem unam oscillationem peragat, debet esse $\theta > \frac{A\delta f}{Bb}$ existente $\frac{A}{B} = \frac{1}{(1+\delta\delta)\cos\zeta}$: ut duas peragat oscillationes, debet esse $\theta > \frac{3A\delta f}{Bb}$: ut tres, debet esse $\theta > \frac{5A\delta f}{Bb}$, atque in genere, ut peragat n oscillationes, debet esse $\theta > \frac{(2n-1)A\delta f}{Bb}$. Verum hic numerum n majorem assumere non licet, quam ut angulus θ adhuc satis parvus maneat.

SCHOLION. 1.

1028. Quod ad diminutionem temporis oscillationum singularum attinet, notasse juvabit, significare $\frac{kk}{l}$ distantiam centri oscillationis ab axe

axe gyrationis, quae si ponatur $= l$, erit tempus unius oscillationis $= \frac{\pi\sqrt{l}}{\sqrt{2g}}\left(1 - \frac{\delta\delta f}{2(1+\delta\delta)l\cos\zeta}\right)$. Hic autem primum observetur, capi debere $tang\,\zeta > \delta$, ut axis GG in loco suo maneat immotus. Quare si fuerit $l = 3$ pedum, quo casu pendulum, nisi frictio obstaret, fere singulis minutis secundis oscillationes absolveret; axiculorum autem radius sit $f = \frac{1}{150}$ pedis, tum vero sumatur $\delta = \frac{1}{3}$ et $\zeta = 20°$, fiet tempus unius oscillationis $= \frac{\pi\sqrt{l}}{\sqrt{2g}}(1 - \frac{1}{28191})$, ita ut ob frictionem demum post 28191 oscillationes peractas seu post 8 fere horas error unius minuti secundi producatur. Hoc eodem casu ut pendulum n oscillationes peragere possit, antequam ad quietem redigatur, debet esse $\theta > \frac{2n-1}{4098}$. seu $\theta > 4, 3905(2n-1)$ min. sec. Quare si 100 oscillationes absolvere debeat, primum sumi debet $\theta > 874''$ seu $\theta > 14', 34''$. Quod si ergo θ capiatur $= 5°$, pendulum peraget oscillationes 2050, antequam ad quietem redigetur. Si f sit major vel minor quam $\frac{1}{150}$, effectus frictionis in eadem ratione major vel minor evadet.

SCHOLION. 2.

1029. Cum jam determinaverimus motum corporum circa axem fixum, ad alias motus species progrediamur, quibus corpus, dum movetur, ad superficiem quandam atteritur. Hic igitur praecipue figura corporis, quatenus successive aliae atque aliae partes superficiei applicantur, spectari debet: ubi quidem primo ejusmodi corpora occurrunt, quae unico tantum puncto eodemque perpetuo superficiem tangunt. Hic scilicet est casus turbinum in cuspidem desinentium, qua continuo superficiei insistunt, quorum motum, quantum ob frictionem cuspidis perturbetur, definiri conveniet. Deinde occurrunt corpora, quae unico quidem puncto perpetuo superficiem tangunt, quod autem jugiter varietur, quemadmodum fit, si globi aliave corpora sphaeroidica super quadam superficie moveantur, ac praeter motum progressivum motu gyratorio quocunque ferantur. His casibus ad effectum frictionis cognoscendum directio motus, quo punctum contactus superficiem terit, quovis momento est spectanda, quippe cui directio vis frictionis est contraria. Sequuntur casus, quibus corpus eadem quidem basi superficiem perpetuo tangit, uti fit in motu progressivo, sed ubi corpus simul gyratur circa axem ad basin normalem, ita ut ipsa basis super

super-

superficie in gyrum agatur. Porro progrediamur ad motus corporum cylindricorum super planis superficiebus, ubi contactus perpetuo fit secundum lineam rectam, ex cujus motu et appressione frictio est definienda. Quae autem corpora figuram ejusmodi habent angulosam, ut, dum moventur, aliae atque aliae hedrae superficiei applicentur, quoniam conflictus talem motum comitatur, dum nova hedra ad contactum pertingit, eorum motus hic nondum evolvere licet, sed prius ratio conflictus explicari debet. Secundum hanc ergo divisionem motum turbinum in cuspidem desinentium super plano horizontali determinare aggrediamur.

CAPUT IV.

DE MOTU TURBINUM IN CUSPIDEM DESINENTIUM SUPER PLANO HORIZONTALI FRICTIONIS HABITA RATIONE.

PROBLEMA II.

Fig. 136. 1030. Si turbo super plano horizontali moveatur utcunque, deturque singulis momentis ejus pressio in planum, definire frictionem motumque turbinis progressivum.

SOLUTIO.

Repraesentet tabula planum horizontale, super quo turbo incedit, cujus axis transiens per centrum inertiae et cuspidem nunc elapso tempore t situm teneat AIF, ut I sit centrum inertiae in sublimi situm, F vero cuspis, qua fit contactus in plano horizontali; voceturque intervallum IF $= f$, quod est constans. Ex I in planum demittatur perpendiculum IG, et sumta in plano recta directrice OV ad fixam mundi plagam spectante, ad eam ex G et F ducantur normales GX et FZ, itemque per G recta KL ipsi OV parallela. Ponatur angulus FIG $= \varrho$, qui exprimit declinationem axis turbinis AF a situ verticali: et angulus KGH $= \phi$, qui praebet declinationem plani verticalis, in quo jam axis turbinis versatur, a plano verticali super OV vel LK exstructo.

Erit ergo GI $= f\cos\varrho$ et GF $= f\sin\varrho$: tum vero GN $= f\sin\varrho\cos\phi$ et FN $= f\sin\varrho\sin\phi$. Praeterea vero sit OX $= x$, et XG $= y$; unde pro puncto F fit OZ $= x - f\sin\varrho\cos\phi$ et ZF $= y + f\sin\varrho\sin\phi$: ex quibus motus cuspidis F colligi potest, cujus celeritas secundum directionem OV vel NG est $= \frac{dx - fd.\sin\varrho\cos\phi}{dt}$, et celeritas secundum directionem NF $= \frac{dy + fd.\sin\varrho\sin\phi}{dt}$: quarum utraque nisi evanescat, cuspis F super plano movebitur frictionemque excitabit; ad cujus directionem inveniendam, sit Ff directio secundum quam cuspis progreditur, quae retro in L producta dabit directionem frictionis FL, pro qua ponatur angulus FLG $= \omega$ eritque $\tang\omega = \frac{dy + fd.\sin\varrho\sin\phi}{dx - fd.\sin\varrho\cos\phi}$. Sit jam pressio, quam cuspis in planum exerit $= \Pi$, pondere totius turbinis existente $= M$, atque ob frictionem turbo in F sollicitatur secundum directionem FL vi $= \delta\Pi$, quae resoluta dat vim sec. XO $= \delta\Pi\cos\omega$ et sec. FZ $= \delta\Pi\sin\omega$. Ad motum ergo progressivum centri inertiae I definiendum, praeter has vires frictionis, et applicatam concipiatur vis deorsum urgens secundum IG, $= M - \Pi$, ac principia motus suppeditabunt has ternas aequationes:

$$\frac{ddx}{2g\,dt^2} = \frac{-\delta\Pi\cos\omega}{M}; \quad \frac{ddy}{2g\,dt^2} = \frac{-\delta\Pi\sin\omega}{M}$$

$$\text{et } \frac{f\,dd.\cos\varrho}{2g\,dt^2} = -1 + \frac{\Pi}{M}$$

unde statim colligimus $ddx\sin\omega = ddy\cos\omega$. Ex his aequationibus, si anguli ϱ et ω ad tempus t ut cogniti spectentur, inde primo $\frac{\Pi}{M}$ tum vero differentialia dx et dy determinantur; ex iisque tandem angulus ϕ ex formula $\tang\omega = \frac{dy + fd.\sin\varrho\sin\phi}{dx - fd.\sin\varrho\cos\phi}$.

COROLL. 1.

1031. Si pro $\frac{\Pi}{M}$ valor inventus per ϱ substituatur, pro quantitatibus x et y determinandis habebimus has aequationes differentiales secundi gradus.

$$ddx = -2\delta g dt^2 \cos\omega - df\cos\omega\, dd.\cos\rho$$
$$ddy = -2\delta g dt^2 \sin\omega - df\sin\omega\, dd.\cos\rho.$$

COROLL. 2.

1032. Pro directione frictionis FL, ratione rectae FH, cum sit angulus LGF $= \phi$ et angulus FLG $= \omega$, erit angulus GFL $= 180° - \phi - \omega$; ipsa autem frictio est $= \delta\Pi$: nisi sit celeritas cuspidis F nulla, quo casu frictio subito evanescit: id quod evenit, si fuerit $dx = fd.\sin\rho\cos\phi$ et $dy = -fd.\sin\rho\sin\phi$.

SCHOLION.

1033. Ex his aequationibus nihil adhuc concludere licet, cum relatio variabilium ω et Π seu ρ ad tempus t nondum constet, quae demum ex motu gyratorio erui debet. His autem inventis, per formulas hic traditas variabiles x et y, sicque motus progressivus centri inertiae I definiri poterit. Quamobrem angulum ω in determinationem motus gyratorii introducamus, etiamsi ejus relatio ad angulos ρ, ϕ et tempus assignari possit. Cum enim sit

$$dy\cos\omega + f\cos\omega\, d.\sin\rho\sin\phi - dx\sin\omega + f\sin\omega\, d.\sin\rho\cos\phi = 0,$$

quo x et y facilius elidere queamus, ponamus

$$f\cos\omega\, d.\sin\rho\sin\phi + f\sin\omega\, d.\sin\rho\cos\phi = s\,dt,$$

ut sit $dy\cos\omega - dx\sin\omega + s\,dt = 0$: quae aequatio differentiata ob $ddy\cos\omega = ddx\sin\omega$ dat

$$-dy\sin\omega - dx\cos\omega + \frac{ds\,dt}{d\omega} = 0.$$

Differentietur porro, et ob $ddy\sin\omega + ddx\cos\omega = \frac{-2\delta g\Pi dt^2}{M}$ prodit

$$\frac{2\delta g\Pi dt^2}{M} - dy\,d\omega\cos\omega + dx\,d\omega\sin\omega + dt\,d.\frac{ds}{d\omega} = 0:$$

addatur prima per $d\omega$ multiplicata, fietque per dt dividendo

$$\frac{2\delta g\Pi dt}{M} + s\,d\omega + d.\frac{ds}{d\omega} = 0,$$

qua aequatione relatio inter s, ω, Π et t exprimitur, quae forte in sequentibus usum habebit. Involvit autem s angulos ρ, ϕ et ω, estque $\frac{\Pi}{M} = 1 + \frac{fdd.\cos\rho}{2g\,dt^2}$, ita ut hic adhuc insint quatuor variabiles ρ, ϕ, ω et t.

PRO-

PROBLEMA. 12.

1034. Dum turbo utcunque super plano horizontali movetur, et frictionem patitur, determinare virium, quibus sollicitatur, momenta respectu axium principalium turbinis.

SOLUTIO.

In sphaera centro inertiae turbinis I descripta repraesentet circulus GZH planum verticale, in quo jam axis turbinis per centrum inertiae I et cuspidem F ductus AIF versetur, qui simul sit axis principalis turbinis, ejusque respectu momentum inertiae $= Maa$, bini reliqui vero axes principales ex I ad sphaerae puncta B et C pertingant, quorum respectu sint momenta inertiae aequalia $= Mcc$, ita ut in formulis nostris generalibus sit $bb = cc$ quemadmodum jam supra assumsimus. Posito Z puncto sphaerae verticali, erit arcus ZA $= \rho$, ponamus autem ductis arcubus ZB et ZC, ut supra ZA $= l$, ZB $= m$, et ZC $= n$, ut sit $\rho = l$. His positis vires, quibus turbo sollicitatur, sunt primo ejus pondus $= M$, quae vis centro inertiae I applicata nulla praebet momenta; deinde adest pressio, qua planum horizontale in cuspidem F reagit, cujus directio est verticalis sursum directa FΠ, quae vis sit $= \Pi$, vidimusque esse $\frac{\Pi}{M} = 1 + \frac{fdd.\cos\rho}{2g\,dt^2}$. Denique sollicitatur turbo in F a frictione $= \delta\Pi$, nisi cuspis quiescat, cujus directio FL est horizontalis: ac pro ejus situ ducatur circulus maximus horizontalis GΛH, in quo capiatur secundum §. 1032. arcus H$\Lambda = 180° - \phi - \omega$ seu G$\Lambda = \phi + \omega$, ubi ϕ denotat declinationem plani GZH a plano quodam verticali fixo; angulus ω autem ex formulis in praecedente problemate traditis definiri debet: eritque directio FL radio IΛ parallela. Jam ad virium harum momenta respectu axium principalium investiganda; primum ipsae vires secundum directiones horum axium resolvantur, quem in finem ut in centro inertiae applicatae considerentur. Vis ergo F$\Pi = \Pi$, in directione IZ applicata praebet vim sec. IA $= \Pi\cos$ ZA $= \Pi\cos l$; vim sec. IB $= \Pi\cos$ ZB $= \Pi\cos m$ et vim sec. IC $= \Pi\cos$ ZC $= \Pi\cos n$. Deinde vis FL $= \delta\Pi$ in IΛ applicata resolvitur in vires 1° sec. IA $= \delta\Pi\cos$ AΛ, 2° sec. IB $= \delta\Pi\cos$ BΛ, 3° sec. IC $= \delta\Pi\cos$ CΛ. Ad has autem evolvendas sit ZX ille circulus verticalis fixus, ideoque angulus XZ$\Lambda = \phi$, ponamus autem ut supra angulos XZA $= \lambda$, XZB $= \mu$ et XZC $= \nu$, ut sit $\phi = \lambda$; et ob AZ$\Lambda = 180° - \lambda - \omega$, erit XZ$\Lambda = 180° - \omega$, hincque BZ$\Lambda = \mu + \omega - 180°$, et CZ$\Lambda = 180° - \nu - \omega$, unde ob Z$\Lambda$ quadrantem prodit $\cos$ A$\Lambda = -\sin l\cos(\lambda + \omega)$.

cos

$\cos BA = -\sin m \cos(\mu + \omega)$ et $\cos CA = -\sin n \cos(\nu + \omega)$. Quocirca habebimus

$$\text{vim sec. } IA = \Pi \cos l - \delta\Pi \sin l \cos(\lambda + \omega)$$
$$\text{vim sec. } IB = \Pi \cos m - \delta\Pi \sin m \cos(\mu + \omega)$$
$$\text{vim sec. } IC = \Pi \cos n - \delta\Pi \sin n \cos(\nu + \omega)$$

has autem vires nunc in puncto F applicatas concipi oportet, existente $IF = f$: unde momenta earum respectu axium principalium, quae supra litteris P, Q, R designavimus, concluduntur

$$P = 0,$$
$$Q = \Pi f \cos n - \delta f \Pi \sin n \cos(\nu + \omega)$$
$$R = -\Pi f \cos m + \delta\Pi f \sin m \cos(\mu + \omega).$$

PROBLEMA. 13.

1035. His virium momentis inventis exhibere aequationes, quibus motus turbinis super plano horizontali incedentis et a frictione perturbatus, continetur.

SOLUTIO.

Fig. 137. Primo pro motu gyratorio tempore elapso tempore t turbo situm in figura repraesentatum, ubi omnes denominationes modo factae maneant. Ac nunc gyretur turbo circa axem IO in sensum ABC celeritate angulari $= \varepsilon$, pro puncto O autem sint arcus $AO = \alpha$, $BO = \beta$, $CO = \gamma$, ponaturque

$$\varepsilon \cos\alpha = x, \ \varepsilon \cos\beta = y, \ \varepsilon \cos\gamma = z,$$

quae quantitates per momenta modo inventa ita determinantur, ut primo sit $dx = 0$, ideoque $x =$ const. Ponatur ergo $x = b$, et pro y et z has habebimus aequationes

$$dy + \frac{(aa - cc)}{cc} bzdt = \frac{2\Pi f g dt}{Mcc}(\cos n - \delta \sin n \cos(\nu + \omega))$$

$$dz - \frac{(aa - cc)}{cc} bydt = \frac{-2\Pi f g dt}{Mcc}(\cos m - \delta \sin m \cos(\mu + \omega)).$$

Tum vero pro arcubus l, m, n itemque angulis λ, μ, ν ostendimus esse:

$$dl \sin l = dt(y \cos n - z \cos m); \ d\lambda \sin l^2 = -dt(y \cos m + z \cos n)$$

$$dm \sin m = dt(z \cos l - b \cos n); \ d\mu \sin m^2 = -dt(z \cos n + b \cos l)$$

$$dn \sin n = dt(b \cos m - y \cos l); \ d\nu \sin n^2 = -dt(b \cos l + y \cos m)$$

ubi praeterea hae relationes sunt notandae:

$$\cos(\mu - \lambda) = \frac{-\cos l \cos m}{\sin l \sin m}; \ \cos(\nu - \lambda) = \frac{-\cos l \cos n}{\sin l \sin n}$$

$$\sin(\mu - \lambda) = \frac{-\cos n}{\sin l \sin m}; \ \sin(\nu - \lambda) = \frac{+\cos m}{\sin l \sin n}$$

unde anguli μ et ν per λ ita definiuntur:

$$\cos\mu = \frac{-\cos\lambda \cos l \cos m + \sin\lambda \cos n}{\sin l \sin m}; \ \cos\nu = \frac{-\cos\lambda \cos l \cos n - \sin\lambda \cos m}{\sin l \sin n}$$

$$\sin\mu = \frac{-\sin\lambda \cos l \cos m - \cos\lambda \cos n}{\sin l \sin m}; \ \sin\nu = \frac{-\sin\lambda \cos l \cos n + \cos\lambda \cos m}{\sin l \sin n}.$$

Hicque est $\frac{\Pi}{M} = 1 + \frac{f dd.\cos l}{2g dt^2}$. At angulus ω ex motu progressivo est ingressus, pro quo si in fig. 136. ad situm centri inertiae I definiendum distinctionis causa vocemus coordinatas $OX = X$ et $XG = Y$, existente $GI = f \cos l$, ad superiores aequationes insuper has adjungere debemus:

$$\frac{ddX}{2g dt^2} = \frac{-\delta\Pi}{M}\cos\omega; \ \frac{ddY}{2g dt^2} = \frac{-\delta\Pi}{M}\sin\omega$$

et $dY \cos\omega - dX \sin\omega + f \cos\omega\, d.\sin l \sin\lambda + f \sin\omega\, d.\sin l \cos\lambda = 0$.

Atque in his aequationibus omnia, quae tam ad motum progressivum quam gyratorium spectant, determinantur. Si primo quantitates X et Y e calculo excludere velimus, loco harum trium postremarum aequationum sequentem unicam adhibuisse sufficit: pro qua si ponatur:

$$s dt = f \cos\omega\, d.\sin l \sin\lambda + f \sin\omega\, d.\sin l \cos\lambda$$

seu sumtis his differentialibus locoque dl et $d\lambda$ valoribus superioribus substitutis erit

$$s = -f y \sin n \sin(\omega + \nu) + f z \sin m \sin(\omega + \mu)$$

et aequatio loco illarum trium usurpanda supra inventa est

$$\frac{2\delta g \Pi dt}{M} + s d\omega + d.\frac{ds}{d\omega} = 0.$$

SCHOLION. 1.

1036. Multitudo harum aequationum, praecipue autem angulus ω primas aequationes ingrediens in causa est, quod earum resolutionem nullo modo suscipere liceat. Unde patet motum turbinum ob frictionem maxime fore perturbatum, ita ut ex his aequationibus nihil omnino.

omnino. unde hic motus cognosci posset, concludere valeamus. Quod si vero hujus motus causas obiter tantum contemplemur, evidens est centrum inertiae I non in recta tantum verticali, ut remota frictione eveniebat, ascendere vel descendere, sed etiam motum horizontalem adipisci, qui oritur a vi frictionis, cujus directio cum sit contraria motui cuspidis, motus centri inertiae secundum eandem directionem incitatur, unde neque uniformis neque rectilineus erit, et quatenus incurvatur, ejus convexitas in eam regionem spectabit, in quam cuspis progreditur. Simili modo etiam motus gyratorius tam ratione celeritatis quam ratione axis gyrationis maxime perturbabitur, de quo vix quicquam ex consideratione frictionis affirmare licet.

SCHOLION 2.

1037. Verum haec tanta motus perturbatio tamdiu duntaxat durat, donec frictio cessat, hoc autem tandem evenire debere per se est evidens, quandoquidem motus ob frictionem continuo retardatur. At frictio cessare nequit, nisi cuspis turbinis in eodem loco persistat, ex quo necesse est motum ita temperari debere, ut cuspis tandem in eodem plani puncto sit perseveraturus, dummodo hoc eveniat, antequam turbo procumbat. Si enim turbini primo motus gyratorius nimis tardus fuerit impressus, nullum est dubium, quin procumbat, antequam illud phaenomenon oriatur: ex quo vicissim concludere licet, si motus satis fuerit celer, fore, ut antequam turbo procumbat, cuspis a frictione ad idem plani horizontalis punctum redigatur. Quod cum evenerit, atque in turbine adhuc motus insit gyratorius, ex superioribus patet, axem turbinis verticalem esse debere; si enim esset inclinatus, nullo modo ita gyrari posset, ut cuspis eidem puncto insisteret. Ex his igitur conjunctis hanc conclusionem deducimus: turbinem, si modo ei satis celer motus gyratorius fuerit impressus, ob frictionem se tandem in situm verticalem erigere, et tum circa axem verticalem motum gyratorium esse continuaturum. Quod phaenomenon eo magis est notatu dignum, quod soli frictioni debeatur; ita ut ope frictionis linea verticalis, ideoque etiam planum horizontale obtineri queat: id quod in navigatione magnum usum habere potest, ad quem etiam in Anglia olim fuit commendatum.

CAPUT

CAPUT V.

DE MOTU GLOBORUM, CENTRUM INERTIAE IN IPSORUM CENTRO SITUM HABENTIUM, SUPER PLANO HORIZONTALI.

PROBLEMA 14.

1038. Si globus super plano horizontali utcunque tam motu progressivo quam gyratorio moveatur, determinare celeritatem et directionem, qua punctum contactus radit superficiem horizontalem.

SOLUTIO.

Sit I centrum simulque centrum inertiae globi, ejusque radius $= f$, et contactus fiet in puncto imo T. Motus autem globi ita sit comparatus, ut centrum inertiae moveatur secundum directionem PIR celeritate $= v$; simul vero gyretur circa axem quemcunque IO celeritate angulari $= u$, in eum sensum, ut punctum T circa O incedat per arculum Tt, ac pro positione puncti O statuamus angulum PTO $= \theta$ et arcum TO $= s$, ubi quidem arcus ita sumo, quasi radius globi esset $= 1$. Ducatur TV ipsi PIR parallela, ac si motus gyratorius abesset, punctum contactus T rasurum esset planum horizontale celeritate $= v$ in directione TV. Deinde si globus solo motu gyratorio ferretur, quia punctum T per Tt moveretur celeritate $= fu\, \sin$ TO $= fu \sin s$, cujus directio cum sit horizontalis, in plano per rectam TΘ referatur, ita ut sit angulus STΘ = PTt $= \theta - 90°$ ob OTt rectum. Fit ergo VT$\Theta = 270° - \theta$. Capiantur rectae TV $= v$ et T$\Theta = fu \sin s$, et quia punctum T his duobus motibus conjunctim movetur, ejus verus motus fiet secundum rectam TF diagonalem parallelogrammi TVFΘ. Ex F ad TV ducta normali FH, erit VH $= fu \sin s \sin \theta$ et FH $= -fu \sin s \cos \theta$, unde fit TH $= v - fu \sin s \sin \theta$ atque celeritas radens TF $= \sqrt{(vv - 2fuv \sin s \sin \theta + ffuu \sin s^2)}$

$$\text{et } tang\ \text{VTF} = \frac{-fu \sin s \cos \theta}{v - fu \sin s \sin \theta}$$

Ducatur ex centro I ipsi TF parallela IQ, erit arcus TQ quadrans, et ang-

Fig. 138.

angulus RTQ = VTF. Quare ſi IQ ſit directioni, ſecundum quam punctum T radit, parallela, erit

$$\mathrm{tang}\, PTQ = \frac{fu \sin s \cos\theta}{v - fu \sin s \sin\theta},$$

ac poſita celeritate radente $\sqrt{(vv - 2fuv \sin s \sin\theta + ffuu \sin s^2)} = u$ erit

$$\sin PTQ = \frac{-fu \sin s \cos\theta}{u} \text{ et } \cos PTQ = \frac{fu \sin s \sin\theta - v}{u}.$$

COROLL. 1.

1039. Fieri ergo poteſt, ut celeritas radens ideoque et attritus evaneſcat, quo caſu hae duae conditiones locum habere debent, altera $u \sin s \cos\theta = 0$, altera $v = fu \sin s \sin\theta$. Unde ſtatim patet, ſi nullus adſit motus progreſſivus, ſeu $v = 0$, nullum attritum affore, ſi $\sin s = 0$, hoc eſt ſi globus circa axem verticalem ZT gyretur.

COROLL. 2.

1040. Deinde motus globi ab attritu erit liber, ſi fuerit primo $\cos\theta = 0$, ſeu angulus PTO rectus: deinde celeritas progreſſiva v ad angularem u hanc relationem tenere debet, ut ſit $v = fu \sin s$, ſeu TV = TΘ, et angulus STΘ = 0.

COROLL. 3.

1041. Quando ergo globus hujusmodi motum eſt conſecutus, quia ſublato omni attritu etiam nulla adeſt frictio, globus eundem motum conſtanter conſervabit, ſiquidem axis gyrationis IO habeat proprietatem axis principalis.

SCHOLION. 1.

1042. Quemadmodum hic frictionem conſtituimus, ea non obſtat, quominus globus ſuper plano horizontali motum ſuum intemeratum conſervare poſſit, quod tamen minime fieri obſervamus, cum globus ſuper tabula tali motu latus mox omnem motum amittat, cujus rei cauſa reſiſtentiae aeris tribui nequit. Verum hic primum animadverto, experimenta nunquam Theoriae perfectiſſime congruere: veluti dum caſu hic tractato aſſumſimus, contactum unico fieri puncto, id ſemper in praxi ſecus evenit. Interim tamen ſi arcus TO eſt quadrans et PTO = 90°, exiſtente $v = fu$, tametſi contactus non fiat in uno

uno puncto, tamen attritus evaneſcit, ideoque hic motus extinctio neutiquam frictioni adſcribi poteſt. Ex quo concludere debemus, praeter frictionem, uti hic eam definivimus, aliud adhuc dari motus impedimentum, dum corpora ſuper ſuperficiebus incedunt, a frictione probe diſtinguendum, cujus ratio utcunque fuerit comparata, ejus effectus potius ſeorſim inveſtigari convenit, quam frictionis indolem hic ſtabilitam immutari. Et quemadmodum hic a reſiſtentia aeris mentem abſtrahimus, ita etiam licebit hoc obſtaculum frictionem concomitans a praeſenti argumento ſejungere.

SCHOLION. 2.

1043. Corpora hic ſphaerica conſidero, in quorum centro ſitum ſit ipſorum centrum inertiae I, quod ergo ipſum etiam in plano horizontali moveatur, et puncto contactus T perpetuo verticaliter immineat. Ex quo patet, preſſionem in contactu ſemper fore ponderi corporis M aequalem. Si ergo corpus ex materia uniformi conſtaret, omnes ejus diametri proprietate axium principalium gauderent, ſed concipiamus materiae diſtributionem utcunque inaequabilem, ita tamen ut centrum inertiae cadat in centrum figurae. Quare neceſſe erit in globo ternos axes principales conſiderari, qui ex centro I pertingant in puncta A, B, C, quadrantibus a ſe invicem diſtantia, quorum reſpectu ſit, ut ſupra poſuimus, momenta inertiae Maa, Mbb, Mcc. Quanquam autem deinceps bina vel omnia haec momenta inter ſe aequalia ſtatuemus, tamen conveniet hujusmodi tria puncta in ſuperficie fixa notaſſe, quo ex eorum relatione ad ſpatium abſolutum, facilius motus globi definiri poſſit. Conſtitutis autem in globo his tribus punctis A, B, C, quoniam motus gyrationis circa O, quem in plagam Tt dirigi aſſumſimus, ſenſum habet CBA contrarium ei, quem ſupra ſtatuimus, in applicatione formularum generalium ad hunc caſum celeritatem angularem δ ut negativam ſpectare debemus.

PROBLEMA. 15.

1044. Si globus ſuper plano horizontali utcunque moveatur, definire vires, quibus ſollicitatur, earumque momenta reſpectu ternorum axium principalium globi.

SOLUTIO.

Incluſus concipiatur globus ſphaerae vel fixae vel cum eo parem motum progreſſivum habenti, in qua Z ſit punctum verticale ejusque Fig. 139.

 oppo-

oppoſitum T punctum contactus, DE vero ſit diameter horizontalis ad certam mund. plagam tendens, et DPQE circulus maximus horizontalis. Nunc autem elapſo tempore t, moveatur globus motu progreſſivo ſecundum directionem PI celeritate $= v$, ponaturque arcus DP ſeu angulus $DZP = \varphi$; axes autem principales nunc ſint in A, B, C. Tum vero globus jam gyretur circa axem IQ, celeritate angulari $= u$ in ſenſum ACB: ſitque pro ſitu puncti O angulus PTO ſeu $PZO = \theta$, et arcus $ZO = s$. Etſi enim ante arcum TO poſuimus $= s$, quia tantum ejus ſinus in computum intrat, perinde eſt. Erit ergo angulus $DZO = \theta + \varphi$, et $EZO = 180° - \theta - \varphi$. Deinde ſi a punctis A, B, C tam ad O quam ad Z arcus circulorum magnorum ducti concipiantur, ſint ut hactenus arcus $AO = \alpha$, $BO = \zeta$, $CO = \gamma$, $ZA = l$, $ZB = m$, $ZC = n$, et anguli $EZA = \lambda$, $EZB = \mu$, $EZC = \nu$. In praecedente autem problemate oſtendimus, punctum contactus T planum ſubjectum radere ſecundum directionem radio IQ parallelam celeritate $= \sqrt{}(vv - 2fu\,v\,si\,s\,si\,\theta + ff\,si\,s^2)$, foreque

$$tang\,PTQ = tang\,PZQ = \frac{fu\,si\,s\,cos\,\theta}{v - fu\,si\,s\,si\,\theta}$$

denotante f radium globi. Cum igitur preſſio in T ſit $= M$, frictio erit $= \delta M$, quae puncto T eſt applicata ſecundum directionem ipſi QI parallelam. Hac ergo vi reſoluta ſecundum directiones axium principalium IA, IB, IC, prodit vis ſec. $IA = -\delta M\,cos\,AQ$; vis ſec. $IB = -\delta M\,cos\,BQ$, et vis ſec. $IC = -\delta M\,cos\,CQ$, quae ternae vires in puncto T applicatae ſunt concipiendae, unde colliguntur momenta

Reſp. axis IA in ſenſum $BC = -\delta M\,f\,cos\,CQ\,cos\,BT + \delta M f\,cos\,BQ\,cos\,CT = P$

Reſp. axis IB in ſenſum $CA = -\delta M\,f\,cos\,AQ\,cos\,CT + \delta M f\,cos\,CQ\,cos\,AT = Q$

Reſp. axis IC in ſenſum $AB = -\delta M\,f\,cos\,BQ\,cos\,AT + \delta M f\,cos\,AQ\,cos\,BT = R.$

Erunt ergo haec tria momenta:

$$P = \delta M f\,(cos\,m\,cos\,CQ - cos\,n\,cos\,BQ)$$
$$Q = \delta M f\,(cos\,n\,cos\,AQ - cos\,l\,cos\,CQ)$$
$$R = \delta M f\,(cos\,l\,cos\,BQ - cos\,m\,cos\,AQ).$$

Pro puncto autem Q ponamus angulum $PZQ = \xi$ ut ſit $tang\,\xi = \frac{fu\,si\,s\,cos\,\theta}{v - fu\,si\,s\,si\,\theta}$ et poſita celeritate radente $\sqrt{}(vv - 2fuv\,si\,s\,si\,\theta + ffuu\,si\,s^2) = u$,

erit $si\,\xi = \frac{-fu\,si\,s\,cos\,\theta}{u}$ et $cos\,\xi = \frac{fu\,si\,s\,si\,\theta - v}{u}$.

Fit ergo $DZQ = \varphi + \xi$ et $EZQ = 180° - \xi - \varphi$; hincque

$$AZQ = 180° - \xi - \varphi - \lambda;\; BZQ = \mu + \xi + \varphi - 180°;$$
$$CZQ = 180° - \xi - \varphi - \nu$$

ergo

$$cos\,AQ = -cos\,(\xi + \varphi + \lambda)\,si\,l$$
$$cos\,BQ = -cos\,(\xi + \varphi + \mu)\,si\,m$$
$$cos\,CQ = -cos\,(\xi + \varphi + \nu)\,si\,n.$$

Ex relatione igitur, quae inter angulos λ, μ et ν intercedit, concludemus momenta virium:

$$P = \delta M f\,si\,l\,si\,(\lambda + \varphi + \xi)$$
$$Q = \delta M f\,si\,m\,si\,(\mu + \varphi + \xi)$$
$$R = \delta M f\,si\,n\,si\,(\nu + \varphi + \xi).$$

PROBLEMA 16.

1045. Si motum gyratorium ad quodvis tempus ut datum ſpectemus, definire motum progreſſivum globi.

SOLUTIO.

Quia centrum globi in plano horizontali movetur, deſcripſerit id tempore t lineam GI, quae referatur ad directricem GX ſuperiori directioni fixae DE parallelam, ductaque IX ad GX normali, ſint coordinatae $GX = X$, $XI = Y$. Per I ducatur recta DE ipſi GX parallela, quae erit ipſa diameter DE in fig. 139. Ducatur IP, ut ſit $DIP = EIR = \varphi$ et centrum I per hypotheſin progreditur in directione IR celeritate $= v$, ita ut ſit celeritas ſecundum $GX = v\,cos\,\varphi$ et celeritas ſecundum $XI = v\,si\,\varphi$, ideoque $dX = vdt\,cos\,\varphi$ et $dY = vdt\,si\,\varphi$. Ducatur recta QIS, ut IQ ſit directioni, qua punctum contactus radit parallela, erit angulus $EIQ = DIS = 180° - \xi - \varphi$, eſt enim aequalis angulo EZQ in praec. figura, unde globus ſollicitari cenſendus eſt vi $= \delta M$ in directione IS. Hinc ergo oritur vis ſecundum $ID = -\delta M\,cos\,(\xi + \varphi)$ et vis ſecundum $XI = \delta M\,si\,(\xi + \varphi)$. Ex quibus colligitur Fig. 14

$$\frac{d.v\,cos\,\varphi}{2g\,dt} = \frac{dv\,cos\,\varphi - v\,d\varphi\,si\,\varphi}{2g\,dt} = \delta\,cos\,(\xi + \varphi)$$

$$\frac{d\,v\,si\,\varphi}{2g\,dt} = \frac{dv\,si\,\varphi + v\,d\varphi\,cos\,\varphi}{2g\,dt} = \delta\,si\,(\xi + \varphi)$$

hincque porro

$$\frac{dv}{2g\,dt} = \delta \cos \xi \quad \text{et} \quad \frac{v\,d\phi}{2g\,dt} = \delta \sin \xi$$

ita ut sit $\frac{v\,d\phi}{dv} = \tan \xi = \frac{f\varepsilon \sin s \cos\theta}{v - f\varepsilon \sin s \sin\theta}$.

PROBLEMA 17.

1046. Definito motu progressivo globi, determinare ejus motum gyratorium.

SOLUTIO.

Fig. 139. Spectetur nunc centrum globi I ut quiescens, et maneant omnes denominationes in probl. 15. adhibitae, sintque Maa, Mbb, Mcc momenta inertiae respectu axium principalium IA, IB, IC, quae primo ut inaequalia consideremus. Quoniam vero hic celeritatem angularem ε ut negativam spectare debemus, quia tendit in sensum ACB, si ponamus $\varepsilon \cos \alpha = x$, $\varepsilon \cos \beta = y$, et $\varepsilon \cos \gamma = z$, in formulis generalibus has literas x, y, z negative sumi oportet, ex §. 810. habebimus has aequationes motum determinantes,

$$dx + \frac{bb-cc}{aa}\,yzdt + \frac{2\delta fg}{aa}\,dt \sin l \sin(\lambda+\phi+\xi) = 0$$

$$dy + \frac{cc-aa}{bb}\,xzdt + \frac{2\delta fg}{bb}\,dt \sin m \sin(\mu+\phi+\xi) = 0.$$

$$dz + \frac{aa-bb}{cc}\,xydt + \frac{2\delta fg}{cc}\,dt \sin n \sin(\nu+\phi+\xi) = 0$$

$$dl \sin l = dt\,(z \cos m - y \cos n)\,; \quad d\lambda \sin l^2 = dt\,(y \cos m + z \cos n)$$

$$dm \sin m = dt\,(x \cos n - z \cos l)\,; \quad d\mu \sin m^2 = dt\,(z \cos n + x \cos l)$$

$$dn \sin n = dt\,(y \cos l - x \cos m)\,; \quad d\nu \sin n^2 = dt\,(x \cos l + y \cos m).$$

Tum vero ex motu progressivo habemus:

$$dv = 2\delta g dt \cos \xi\,; \quad v d\phi = 2\delta g dt \sin \xi$$

et $\tan \xi = \frac{f\varepsilon \sin s \cos\theta}{v - f\varepsilon \sin s \sin\theta}$.

Ubi est $EZO = \theta$ et $ZO = s$. Cum ergo sit $EZO = 180° - \theta - \phi$: erit $AZO = 180° - \lambda - \theta - \phi$: hincque

cos

$$\cos \alpha = \cos l \cos s - \sin l \sin s \cos(\lambda + \theta + \phi)$$

$$\cos \beta = \cos m \cos s - \sin m \sin s \cos(\mu + \theta + \phi)$$

$$\cos \gamma = \cos n \cos s - \sin n \sin s \cos(\nu + \theta + \phi)$$

existente $\cos s = \cos l \cos \alpha + \cos m \cos \beta + \cos n \cos \gamma$

unde sequitur fore

$$\sin l \cos l \cos(\lambda+\theta+\phi) + \sin m \cos m \cos(\mu+\theta+\phi) + \sin n \cos n \cos(\nu+\theta+\phi) = 0.$$

Ponamus $\varepsilon \cos s = p$ et $\varepsilon \sin s = q$, ut sit $\tan \xi = \frac{fq\cos\theta}{v - fq\sin\theta} = \frac{v\,d\phi}{dv}$; eritque

$$x = p \cos l - q \sin l \cos(\lambda + \theta + \phi)$$

$$y = p \cos m - q \sin m \cos(\mu + \theta + \phi)$$

$$z = p \cos n - q \sin n \cos(\nu + \theta + \phi)$$

ex quibus valoribus fit

$$dl = qdt \sin(\lambda+\theta+\phi)\,; \quad d\lambda = pdt + qdt \cot l \cos(\lambda+\theta+\phi)$$

$$dm = qdt \sin(\mu+\theta+\phi)\,; \quad d\mu = pdt + qdt \cot m \cos(\mu+\theta+\phi)$$

$$dn = qdt \sin(\nu+\theta+\phi)\,; \quad d\nu = pdt + qdt \cot n \cos(\nu+\theta+\phi)$$

indeque porro

$$dx = dp \cos l - dq \sin l \cos(\lambda+\theta+\phi) + q\,(d\theta + d\phi) \sin l \sin(\lambda+\theta+\phi)$$

$$dy = dp \cos m - dq \sin m \cos(\mu+\theta+\phi) + q\,(d\theta + d\phi) \sin m \sin(\mu+\theta+\phi)$$

$$dz = dp \cos n - dq \sin n \cos(\nu+\theta+\phi) + q\,(d\theta + d\phi) \sin n \sin(\nu+\theta+\phi).$$

At sine subsidio harum substitutionum ex aequationibus terminis prioribus cum in genere sit $\sin l \cos l \sin(\lambda + A) + \sin m \cos m \sin(\mu + A) + \sin n \cos n \sin(\nu + A) = 0$, elicimus hanc aequationem

$$aadx \cos l + bbdy \cos m + ccdz \cos n - aaxdl \sin l - bbydm \sin m - cczdn \sin n = 0$$

cujus integrale est

$$aax \cos l + bby \cos m + ccz \cos n = C$$

quae adhibitis substitutionibus abit in hanc formam:

$$p\,(aa \cos l^2 + bb \cos m^2 + cc \cos n^2) - q\,(aa \sin l \cos l \cos(\lambda+\theta+\phi) + bb \sin m \cos m \cos(\mu+\theta+\phi) + cc \sin n \cos n \cos(\nu+\theta+\phi)) = \text{Const.}$$

Deinde

Deinde etiam per reductiones §. 934. traditas pro vi viva colligitur haec aequatio differentialis

$$aaxdx + bbydy + cczdz = 2\delta fgqdt\, si\,(\xi - \theta).$$

SCHOLION.

1047. Ad reductiones hic factas intelligendas ex formulis supra traditis, ubi angulos μ et ν per λ, l, m, n expressimus, notari convenit fieri:

$$\cos(\mu+\theta+\varphi) = \frac{-\cos l \cos m \cos(\lambda+\theta+\varphi) + \cos n\, si(\lambda+\theta+\varphi)}{si\, l\, si\, m}$$

$$\cos(\nu+\theta+\varphi) = \frac{-\cos l \cos n \cos(\lambda+\theta+\varphi) - \cos m\, si(\lambda+\theta+\varphi)}{si\, l\, si\, n}$$

$$si(\mu+\theta+\varphi) = \frac{-\cos l \cos m\, si(\lambda+\theta+\varphi) - \cos n \cos(\lambda+\theta+\varphi)}{si\, l\, si\, m}$$

$$si(\nu+\theta+\varphi) = \frac{-\cos l \cos n\, si(\lambda+\theta+\varphi) + \cos m \cos(\lambda+\theta+\varphi)}{si\, l\, si\, n}$$

Ac simili modo anguli $\mu+\varphi+\xi$ et $\nu+\varphi+\xi$ ad angulum $\lambda+\varphi+\xi$ revocari possunt. Deinde etiam pro sequentibus reductionibus haec forma imprimis est notanda

$$si(\mu+B)\cos(\nu+C) = si(\nu+B)\cos(\mu+C)$$

quae ob $si\, M \cos N = \frac{1}{2} si(M+N) + \frac{1}{2} si(M-N)$ reducitur ad $si(\mu-\nu)\cos(B-C)$; hocque modo reductionem pro aliis formulis instituendo, reperimus:

$$si(\mu+B)\cos(\nu+C) - si(\nu+B)\cos(\mu+C) = si(\mu-\nu)\cos(B-C)$$

$$si(\mu+B)\, si(\nu+B) - si(\nu+B)\, si(\mu+C) = -si(\mu-\nu)\, si(B-C)$$

$$\cos(\mu+B)\cos(\nu+C) - \cos(\nu+B)\cos(\mu+C) = -si(\mu-\nu)\, si(B-C)$$

ubi $si(\mu-\nu)$ per formulas usurpatas datur, est enim $si(\mu-\nu) = \frac{\cos l}{si\, m\, si\, n}$.

PROBLEMA. 18.

1048. Si globus ex materia uniformi constet: vel saltem ita fuerit comparatus, ut omnia momenta inertiae sint inter se aequalia, eique initio impressus fuerit motus quicunque, determinare eius continuationem.

SO-

SOLUTIO.

Cum hic sit $aa = bb = cc$, seu momentum inertiae respectu omnium diametrorum $= Maa$, prima aequatio integrata praebet [illegible] Const, unde p erit quantitas constans. Statuatur ergo $p = b$; et ternae aequationes differentiales priores induent has formas:

$$\text{I.}\ -dq\cos(\lambda+\theta+\varphi) + q(d\theta+d\varphi)\, si(\lambda+\theta+\varphi) + \frac{2\delta fg}{aa}dt\, si(\lambda+\varphi+\xi) = 0$$

$$\text{II.}\ -dq\cos(\mu+\theta+\varphi) + q(d\theta+d\varphi)\, si(\mu+\theta+\varphi) + \frac{2\delta fg}{aa}dt\, si(\mu+\varphi+\xi) = 0$$

$$\text{III.}\ -dq\cos(\nu+\theta+\varphi) + q(d\theta+d\varphi)\, si(\nu+\theta+\varphi) + \frac{2\delta fg}{aa}dt\, si(\nu+\varphi+\xi) = 0$$

quarum autem sufficit binas considerasse, quia inde jam nata est conclusio $p = b$. Iam per superiores reductiones binae posteriores ita combinentur:

II. $\cos(\nu+\theta+\varphi)$ − III. $\cos(\mu+\theta+\varphi)$ praebet

$$q(d\theta+d\varphi)\, si(\mu-\nu) + \frac{2\delta fg}{aa}dt\, si(\mu-\nu)\cos(\xi-\theta) = 0$$

$$\text{seu } q(d\theta+d\varphi) + \frac{2\delta fg}{aa}dt\cos(\xi-\theta) = 0$$

Deinde II. $si(\nu+\theta+\varphi)$ − III. $si(\mu+\theta+\varphi)$ dat

$$dq\, si(\mu-\nu) - \frac{2\delta fg}{aa}dt\, si(\mu-\nu)\, si(\xi-\theta) = 0$$

$$\text{seu } dq = \frac{2\delta fg}{aa}dt\, si(\xi-\theta)$$

qui valor in ultima aequatione pro viribus vivis substitutus praebet $xdx + ydy + zdz = qdq$

hincque $xx + yy + zz = uu =$ Const. $+ qq =$ Const. $+ uu\, si\, s^2$ ita ut sit $uu \cos s^2 =$ const. ut jam invenimus ob $u \cos s = p = b$. Hinc istas habemus aequationes a litteris l, m, n, λ, μ, ν immunes:

$$\text{I. } q(d\theta + d\varphi) + \frac{2\delta f g}{aa}\, dt \cos(\xi - \theta) = 0; \quad \text{II. } dq = \frac{2\delta f g}{aa}\, dt \sin(\xi - \theta)$$

$$\text{III. } dv = 2\delta g\, dt \cos\xi; \quad \text{IV. } v\, d\varphi = 2\delta g\, dt \sin\xi$$

quibus adjungatur haec finita $\mathrm{tang}\,\xi = \frac{fq\cos\theta}{v - fq\sin\theta}$, quae transformata in hanc $v \sin\xi - fq\cos(\xi - \theta) = 0$, differentietur

$$dv \sin\xi + v\,d\xi \cos\xi - f\,dq \cos(\xi - \theta) + fq\,d\xi \sin(\xi - \theta) - fq\,d\theta \sin(\xi - \theta) = 0.$$

Iam I. $\sin(\xi - \theta)$ + II. $\cos(\xi - \theta)$ dat

$$q(d\theta + d\varphi)\sin(\xi - \theta) + dq\cos(\xi - \theta) = 0$$

quae per f multiplicata illi addatur

$$dv \sin\xi + v\,d\xi\cos\xi + fq(d\xi + d\varphi)\sin(\xi - \theta) = 0.$$

Porro ob $\frac{dv}{v\,d\varphi} = \frac{\cos\xi}{\sin\xi}$, erit

$$v(d\varphi + d\xi)\cos\xi + fq(d\xi + d\varphi)\sin(\xi - \theta) = 0$$

seu $(d\varphi + d\xi)(v\cos\xi + fq\sin(\xi - \theta)) = 0$
quorum factorum finitus $v\cos\xi + fq\sin(\xi - \theta)$ evanescere nequit, ob $v\sin\xi - fq\cos(\xi - \theta) = 0$, sequeretur enim inde $v\cos\theta = 0$, et $fq\cos\theta = 0$; quod nonnisi casu $\theta = 90°$ locum habet. Relinquitur ergo ut sit $d\varphi + d\xi = 0$ ideoque $\varphi + \xi = \mathrm{Const.}$

Hoc impetrato reliqua non difficulter expedientur; ad integrationes autem determinandas pro statu initiali $t = 0$, ponamus fuisse celeritatem progressivam $v = e$, ang. $\varphi = 0$; ang. $PZO = \theta = \mathfrak{h}$, arcum $ZO = s = \mathfrak{s}$; et celeritatem angularem $u = \varepsilon$ in sensum ACB; hincque $p = b = u\cos s = \varepsilon\cos\mathfrak{s}$, et $q = \varepsilon\sin\mathfrak{s}$; porro $\mathrm{tang}\,\xi = \frac{\varepsilon f \sin\mathfrak{s}\cos\mathfrak{h}}{e - \varepsilon f\sin\mathfrak{s}\sin\mathfrak{h}}$. Statuatur $\frac{\varepsilon f\sin\mathfrak{s}\cos\mathfrak{h}}{e - \varepsilon f\sin\mathfrak{s}\sin\mathfrak{h}} = \mathrm{tang}\,\zeta$ ut fuerit initio $\xi = \zeta$, ac perpetuo erit $\xi + \varphi = \zeta$, ita ut angulus $DZQ = \zeta$ maneat constans. Quare cum sit $\xi = \zeta - \varphi$: erit $v\sin(\zeta - \varphi) = fq\cos(\zeta - \theta - \varphi)$. Supra autem invenimus:

$$\frac{d.v\cos\varphi}{2g\,dt} = \delta\cos(\xi + \varphi) = \delta\cos\zeta; \text{ et } \frac{d.v\sin\varphi}{2g\,dt} = \delta\sin(\xi + \varphi) = \delta\sin\zeta,$$

unde

unde integrando colligimus

$$v\cos\varphi = e + 2\delta gt\cos\zeta \text{ et } v\sin\varphi = 2\delta gt\sin\zeta$$

hincque $v = \sqrt{(ee + 4\delta egt\cos\zeta + 4\delta\delta ggtt)}$ et $\mathrm{tang}\,\varphi = \frac{2\delta gt\sin\zeta}{e + 2\delta gt\cos\zeta}$

atque $\mathrm{tang}(\zeta - \varphi) = \frac{e\sin\zeta}{e\cos\zeta + 2\delta gt} = \frac{fq\cos\theta}{v - fq\sin\theta} = \mathrm{tang}\,\xi.$

Deinde ob $d\varphi = -d\xi$ binae priores aequationes abeunt in

$$\text{I. } q(d\xi - d\theta) = \frac{2\delta fg}{aa}\, dt\cos(\xi - \theta); \quad \text{II. } dq = \frac{2\delta fg}{aa}\, dt\sin(\xi - \theta),$$

quarum haec per illam divisa dat:

$\frac{dq}{q(d\xi - d\theta)} = \frac{\sin(\xi - \theta)}{\cos(\xi - \theta)}$, quae integrata dat $q\cos(\xi - \theta) = \mathrm{Const.}$ ideoque $q\cos(\xi - \theta) = \varepsilon\sin\mathfrak{s}\cos(\zeta - \mathfrak{h})$, unde valor ipsius q in prima substitutus praebet:

$$\frac{\varepsilon(d\xi - d\theta)\sin\mathfrak{s}\cos(\zeta - \mathfrak{h})}{\cos(\xi - \theta)^2} = \frac{2\delta fg}{aa}\, dt, \text{ et integrando}$$

$$\varepsilon\sin\mathfrak{s}\cos(\zeta - \mathfrak{h})\,\mathrm{tang}(\xi - \theta) = C + \frac{2\delta fg}{aa}\, t,$$

ubi $C = \varepsilon\sin\mathfrak{s}\sin(\zeta - \mathfrak{h})$. At $\mathrm{tang}(\xi - \theta) = \mathrm{tang}(\zeta - \varphi - \theta)$ $= \frac{\mathrm{tang}(\zeta - \varphi) - \mathrm{tang}\,\theta}{1 + \mathrm{tang}\,\theta\,\mathrm{tang}(\zeta - \varphi)}$, et $\mathrm{tang}\,\theta = \frac{\mathrm{tang}\,\xi - \mathrm{tang}(\xi - \theta)}{1 + \mathrm{tang}\,\xi\,\mathrm{tang}(\xi - \theta)}$.

Sed per hypothesin est $\varepsilon\sin\mathfrak{s} = \frac{e\sin\zeta}{f\cos(\zeta - \mathfrak{h})}$, unde fit

$\mathrm{tang}(\xi - \theta) = \mathrm{tang}(\zeta - \mathfrak{h}) + \frac{2\delta ffgt}{eaa\sin\zeta}$, at $\mathrm{tang}\,\xi = \frac{e\sin\zeta}{e\cos\zeta + 2\delta gt}$

Hincque angulus θ facile determinatur; indeque $q = \frac{e\sin\zeta}{f\cos(\xi - \theta)}$.

Verum hic notari oportet, cum sit $\mathrm{tang}\,\zeta = \frac{\varepsilon f\sin\mathfrak{s}\cos\mathfrak{h}}{e - \varepsilon f\sin\mathfrak{s}\sin\mathfrak{h}}$, esse ut sup. de angulo ξ ostendimus,

$$\sin\zeta = \frac{-\varepsilon f \sin\mathfrak{f} \cos\mathfrak{h}}{\surd(ee - 2\varepsilon ef \sin\mathfrak{f} \sin\mathfrak{h} + \varepsilon\varepsilon ff \sin\mathfrak{f}^2)} \text{ et } \cos\zeta = \frac{-e + \varepsilon f \sin\mathfrak{f} \sin\mathfrak{h}}{\surd(ee - 2\varepsilon ef \sin\mathfrak{f} \sin\mathfrak{h} + \varepsilon\varepsilon ff \sin\mathfrak{f}^2)}$$

unde $\cos(\zeta - \mathfrak{h}) = \dfrac{-e \cos\mathfrak{h}}{\surd(ee - 2\varepsilon ef \sin\mathfrak{f} \sin\mathfrak{h} + \varepsilon\varepsilon ff \sin\mathfrak{f}^2)}$.

His inventis cum sit $u \cos s = \varepsilon \cos\mathfrak{f}$ et $u \sin s = q$, erit $u = \surd(qq + \varepsilon\varepsilon \cos\mathfrak{f}^2)$ et $\tang s = \dfrac{q}{\varepsilon \cos\mathfrak{f}}$. Sicque tam motus progressivus, quam ad quodvis tempus axis gyrationis IO cum celeritate angulari u potest assignari, id quod ad motus cognitionem sufficit. Determinatio autem situs punctorum A, B, C ad quodvis tempus nimis est ardua, quam ut eam perficere liceat.

COROLL. 1.

1049. Cum sit celeritas angularis $u = \dfrac{\varepsilon \cos\mathfrak{f}}{\cos s}$, seu cosinui arcus SO reciproce proportionalis; sequitur, si polus gyrationis O initio fuerit in superiori hemisphaerio DZE, eum nunquam in inferius pervenire posse: in transitu enim per circulum horizontalem DE prodiret celeritas angularis u infinita.

COROLL. 2.

1050. Ob eandem rationem, si polus gyrationis O initio fuerit in hemisphaerio inferiori DTE, is nunquam in superius ascendet. Sin autem initio fuerit in ipso circulo horizontali DE, perpetuo in eodem manebit. Scilicet si initio axis gyrationis fuerit horizontalis, perpetuo horizontalis manebit.

COROLL. 3.

1051. Si fuerit initio angulus DZO $= \mathfrak{h}$ rectus, fiet $\sin\zeta = 0$ et ob $\tang(\zeta - \mathfrak{h}) = \dfrac{\varepsilon f \sin\mathfrak{f} - e \sin\mathfrak{h}}{e \cos\mathfrak{h}}$, sit etiam $\xi - \theta$ rectus. Sed ob $\tang \xi = \dfrac{e \sin\zeta}{e \cos\zeta + 2\delta g t}$, angulus ξ evanescit, unde angulus $\theta =$ PZO prodit rectus. Simulque igitur angulus PZO factus fuerit rectus, perpetuo rectus manebit.

COROLL. 4.

1052. Memorabilis est etiam proprietas, quod angulum $\xi + \zeta$ ... DZQ et in fig. 140. angulus DIQ fit constans. Recta enim QIS ... perpetuo manebit parallela, et quia globus in motu progressivo sollicitatur vi constante δM secundum eandem directionem IS, curva ab eo descripta GI parabola sit necesse est.

SCHOLION. 1.

1053. Hic autem motus globi, uti nostris formulis est definitus, diutius non durat, quam revera frictio adest, seu planum horizontale in puncto contactus T raditur. Si enim eveniat, ut ratio cesset, seu celeritas radens in T evanescat, subito frictio evanescit, formulaeque inventae non amplius locum habent. Tum igitur globus motu tam progressivo, quam gyratorio uniformiter in directum progredietur, neque axis gyrationis ullam amplius mutationem patietur. Ac si statim initio motus globo impressus ita fuerit comparatus, ut frictio fuerit nulla, quod evenit, si tam $\varepsilon f \sin\mathfrak{f} \cos\mathfrak{h} = 0$ quam $e = \varepsilon f \sin\mathfrak{f} \sin\mathfrak{h}$, deinceps etiam globus nullam frictionem sentiet, et statim ab initio motum progressivum uniformiter in directum prosequetur, simulque uniformiter circa eundem axem gyrabitur. Verum si corpori ab initio alius motus quicunque fuerit impressus, semper aliquo tempore elapso eo reducetur, ut frictio evanescat, indeque motum suum uniformiter prosequetur; quod memorabile temporis punctum in sequenti problemate investigabimus.

SCHOLION. 2.

1054. Quae in solutione problematis eliciuimus, huc redeunt: Ex motu primum impresso habemus celeritatem motus progressivi $= e$ secundum directionem DI: ac si gyretur circa axem IO celeritate angulari ε in sensum ACB seu ZETD, qui sensus *antrorsum tendens* dici solet, sueritque arcus ZO $= \mathfrak{f}$ et angulus DZO $= \mathfrak{h}$: tum vero radius globi sit $= f$ ejusque momentum inertiae $= Maa$ respectu omnium diametrorum, existente M ejus massa: ex his datis colligitur celeritas radens in puncto contactus $= \surd(ee - 2\varepsilon ef \sin\mathfrak{f} \sin\mathfrak{h} + \varepsilon\varepsilon ff \sin\mathfrak{f}^2)$ Fig. 139.
quae si ponatur $= k$, quaeratur angulus ζ, ut sit $\sin\zeta = \dfrac{-\varepsilon f \sin\mathfrak{f} \cos\mathfrak{h}}{k}$ et $\cos\zeta = \dfrac{\varepsilon f \sin\mathfrak{f} \sin\mathfrak{h} - e}{k}$, qui sit DZQ $= \zeta$, eritque IQ directio motus radentis

Tum.

Tum si elapso tempore t globi centrum proferatur celeritate v secundum directionem PI, et gyretur celeritate angulari $= \varepsilon$ in sensum ZETD circa polum O, ponaturque DZP $= \phi$, PZO $= \theta$, et ZO $= s$: invenimus primo: $\operatorname{tang}\phi = \frac{2\delta g t \sin\zeta}{e + 2\delta g t \cos\zeta}$ et celeritatem centri D $= \sqrt{(ee + 4\delta e g t \cos\zeta + 4\delta\delta g g t t)}$, at celeritas radens etiamnum fiet in directione IQ, existente DZQ $= \zeta$: unde posito PZQ $= \xi$ erit $\operatorname{tang}\xi = \frac{e\sin\zeta}{e\cos\zeta + 2\delta g t}$. Porro est $\operatorname{tang}(\xi - \theta) = \operatorname{tang}(\zeta - \eta) + \frac{2\delta f f g t}{a a e \sin\zeta}$ existente $\operatorname{tang}(\zeta - \eta) = \frac{\varepsilon f \sin\mathfrak{f} - e\sin\eta}{e\cos\eta}$, unde angulus θ innotescit, hincque ob DZO $= \phi + \theta = \zeta - \xi + \theta$ concluditur tang DZO $= \operatorname{tang}(\phi + \theta) = \frac{\varepsilon a a k \sin\mathfrak{f}\sin\eta + 2\delta f g t (e - \varepsilon f \sin\mathfrak{f}\sin\eta)}{\varepsilon a a k \sin\mathfrak{f}\cos\eta - 2\delta\varepsilon f f g t \sin\mathfrak{f}\cos\eta}$. Atque ex his tandem nacti sumus:

$$u\cos s = \varepsilon\cos\mathfrak{f} \text{ et } u\sin s = \frac{e\sin\zeta}{f\cos(\xi-\theta)}.$$

Denique pro celeritate radente secundum IQ ea est $= \sqrt{(vv - 2ufv\sin s\sin\theta + uuff\sin s^2)}$; quae si vocetur $= w$, supra ostendimus esse

$$\sin\xi = \frac{-uf\sin s\cos\theta}{w} \text{ et } \cos\xi = \frac{uf\sin s\sin\theta - v}{w},$$

unde u et s definiuntur. Sed pro situ punctorum A, B, C in globo fixorum ad quodvis tempus determinando formulae adeo fiunt intricatae, ut nihil inde concludi queat. Interim si pro puncto A vocetur ZA $= l$ et EZA $= \lambda$, ad has binas aequationes totum negotium reducitur:

$$\text{I. } dl = dt\left(\varepsilon\sin\mathfrak{f}\sin(\eta + \lambda) - \frac{2\delta f g t}{aa}\cos(\zeta + \lambda)\right).$$

$$\text{II. } d\lambda\sin l = \varepsilon dt\cos\mathfrak{f}\sin l + dt\cos l\left(\varepsilon\sin\mathfrak{f}\cos(\eta + \lambda) + \frac{2\delta f g t}{aa}\sin(\zeta + \lambda)\right)$$

quarum resolutio vereor ne frustra suscipiatur. Cum autem ad quodvis tempus axem gyrationis cum celeritate angulari assignare valeamus, quod ad motus cognitionem, qualis vulgo desideratur, sufficere potest, eo magis mirum videtur, quod motus singulorum globi punctorum quasi vires analyseos superet. Multo minus igitur de motu globorum,

rum, in quibus momenta inertiae non sunt aequalia, quicquam definire licebit.

PROBLEMA 19.

1051. Si globo, cujus omnia momenta inertiae sunt inter se aequalia, motus quicunque fuerit impressus, assignare temporis punctum ubi celeritas radens ideoque frictio evanescit, indeque globus motu uniformi progredi pergat.

SOLUTIO.

Supra §. 1039. vidimus, ut attritus evanescat, has duas requiri conditiones, alteram $u\sin s\cos\theta = 0$ alteram $v = fu\sin s\sin\theta$, seu in expressione $\operatorname{tang}\xi = \frac{fu\sin s\cos\theta}{v - fu\sin s\sin\theta}$ tam numeratorem quam denominatorem simul evanescere debere. Cum autem invenerimus $\operatorname{tang}\xi = \frac{e\sin\zeta}{e\cos\zeta + 2\delta g t}$, ubi numerator $e\sin\zeta$ est constans, si in illa formula numerator evanescit, necesse est denominator simul evanescat, quia alioquin aequalitas inter has duas fractiones subsistere nequit. Unde posito $\cos\theta = 0$ tempus quaesitum declarabit. Verum idem luculentius determinabimus, si ad quodvis tempus elapsum t celeritatem radentem w investigemus. Cum igitur ex formula $\sin\xi = \frac{-uf\sin s\cos\theta}{w}$ fit $w = \frac{-uf\sin s\cos\theta}{\sin\xi}$, quae expressio, ob $u\sin s = \frac{e\sin\zeta}{f\cos(\xi - \theta)}$, abit in $w = \frac{-e\sin\zeta\cos\theta}{\sin\xi\cos(\xi - \theta)}$: atque ob $\theta = \xi - (\xi - \theta)$ in hanc $w = -e\sin\zeta(\cot\xi + \operatorname{tang}(\xi - \theta))$, si hic pro $\operatorname{tang}\xi$ et $\operatorname{tang}(\xi - \theta)$ valores supra inventos substituamus, reperiemus:

$$w = -\left(e\cos\zeta + 2\delta g t + e\sin\zeta\operatorname{tang}(\zeta - \eta) + \frac{2\delta f f g t}{aa}\right).$$

At $\cos\zeta + \sin\zeta\operatorname{tang}(\zeta - \eta) = \frac{\cos\eta}{\cos(\zeta - \eta)}$, et $\cos(\zeta - \eta) = \frac{-e\cos\eta}{k}$, unde $e\cos\zeta + e\sin\zeta\operatorname{tang}(\zeta - \eta) = -k$, ubi k denotat celeritatem radentem initialem. Quamobrem elapso tempore t habebimus celeritatem radentem $w = k - 2\delta g\left(1 + \frac{ff}{aa}\right)t$, ita ut ea labente tempore uniformiter

formiter decrescat. Tandem ergo certo evanescet, idque fiet elapso tempore $t = \frac{aak}{2\delta g(aa+ff)}$; eritque tum $\mathrm{cof}\,\theta = 0$ et $\theta = 90° =$ PZO. Quod ergo cum evenerit, videamus quomodo reliquae motus determinationes se sint habiturae: et quoniam $2\delta gt = \frac{aak}{aa+ff}$, erit tang $\varphi = \frac{aak\,\mathrm{fi}\,\zeta}{e(aa+ff)+aak\,\mathrm{cof}\,\zeta}$ et $\mathrm{tang}\,\xi = \frac{e(aa+ff)\,\mathrm{tang}\,\zeta}{e(aa+ff)+aak}$. Hinc fit $u\,\mathrm{fi}\,s = \frac{e\,\mathrm{fi}\,\zeta}{\mathrm{fi}\,\xi}$. Cum autem fit $v = \surd\left(ee + \frac{2aaek\,\mathrm{cof}\,\zeta}{aa+ff} + \frac{a^4kk}{(aa+ff)^2}\right)$; erit $\mathrm{fi}\,\varphi = \frac{aak\,\mathrm{fi}\,\zeta}{(aa+ff)v}$ et $\mathrm{cof}\,\varphi = \frac{e(aa+ff)+aak\,\mathrm{cof}\,\zeta}{(aa+ff)v}$, atque $\mathrm{fi}\,\xi = \frac{e\,\mathrm{fi}\,\zeta}{v}$, ideoque $u\,\mathrm{fi}\,s = \frac{v}{f}$. Porro quia $u\,\mathrm{cof}\,s = \varepsilon\,\mathrm{cof}\,\mathfrak{f}$, erit $\mathrm{tang}\,s = \frac{v}{\varepsilon f\,\mathrm{cof}\,\mathfrak{f}}$ et $u = \surd\left(\frac{vv}{ff} + \varepsilon\varepsilon\,\mathrm{cof}\,\mathfrak{f}^2\right)$ seu $u =$

$$\frac{\surd(eeff + 2\varepsilon eaaf\,\mathrm{fi}\,\mathfrak{f}\,\mathrm{fi}\,\mathfrak{h} + \varepsilon\varepsilon a^4\,\mathrm{fi}\,\mathfrak{f}^2 + \varepsilon\varepsilon(aa+ff)^2\,\mathrm{cof}\,\mathfrak{f}^2)}{aa+ff}$$

ob $kk = ee - 2\varepsilon ef\,\mathrm{fi}\,\mathfrak{f}\,\mathrm{fi}\,\mathfrak{h} + \varepsilon\varepsilon ff\,\mathrm{fi}\,\mathfrak{f}^2$.

COROLL. 1.

1056. Quo major ergo initio fuerit celeritas radens k, eo diutius motus durat, antequam celeritas frictione ad uniformitatem redigatur. Ac si globus constet ex materia homogenea, fit $aa = \frac{2}{5}ff$, ideoque motus uniformitas incipit elapso tempore $t = \frac{k}{7\delta g}$ min. sec. hinc in hypothesi $\delta = \frac{1}{4}$ fit $t = \frac{3k}{7g}$, existente $g = 15\frac{5}{8}$ ped. Rhen.

COROLL. 2.

1057. Ut centrum globi eodem tempore ad quietem redigatur, statu initialis ita comparatus esse debet, ut fit $\mathrm{cof}\,\zeta = -1$ et $e = \frac{aak}{aa+ff}$. Fit ergo $k = e - \varepsilon f\,\mathrm{fi}\,\mathfrak{f}\,\mathrm{fi}\,\mathfrak{h}$, et $\mathrm{fi}\,\mathfrak{h} = 1$; seu $\mathfrak{h} = 90°$; et $k = e - \varepsilon f\,\mathrm{fi}\,\mathfrak{f}$; hincque

hincque $\varepsilon\,\mathrm{fi}\,\mathfrak{f} = \frac{-ef}{aa}$. Porro ob $v = 0$ fit $s = 0$; et $u = \varepsilon\,\mathrm{cof}\,\mathfrak{f}$, qua celeritate angulari jam globus circa axem verticalem quiescentem gyrabitur: elapso ab initio tempore $t = \frac{e}{2\delta g}$ min. sec.

COROLL. 3.

1058. Hoc autem casu, quo initio est $\mathfrak{h} = 90°$, et $\varepsilon = \frac{-ef}{aa\,\mathrm{fi}\,\mathfrak{f}}$, fit $\zeta = 180$; $\varphi = 0$; $\xi = 180$; $\theta = 90°$; $v = e - 2\delta gt$; tum vero $u\,\mathrm{cof}\,s = \frac{-ef\,\mathrm{cof}\,\mathfrak{f}}{aa\,\mathrm{fi}\,\mathfrak{f}}$; $u\,\mathrm{fi}\,s = \frac{-ef}{aa}\left(1 - \frac{2\delta gt}{e}\right)$, hincque $\mathrm{tang}\,s = \left(1 - \frac{2\delta gt}{e}\right)\mathrm{tang}\,\mathfrak{f}$ et $u = \frac{-ef}{aa\,\mathrm{fi}\,\mathfrak{f}}\surd\left(1 - \frac{4\delta gt}{e}\,\mathrm{fi}\,\mathfrak{f}^2 + \frac{4\delta\delta ggtt}{ee}\,\mathrm{fi}\,\mathfrak{f}^2\right)$. At initio erat celeritas radens $k = e\left(1 + \frac{ff}{aa}\right)$, elapso autem tempore t ea est $w = \left(1 + \frac{ff}{aa}\right)(e - 2\delta gt)$; sicque posito $t = \frac{e}{2\delta g}$ simul fit $w = 0$, $v = 0$ et $s = 0$, ut ante.

COROLL. 4.

1059. Ne valor $u\,\mathrm{fi}\,s = \frac{e\,\mathrm{fi}\,\zeta}{f\,\mathrm{cof}(\xi - \theta)}$ indefinitus videatur, quod fit si numerator ac denominator evanescant, seu $\zeta = 0$, conveniet loco $\mathrm{fi}\,\zeta$ et $\mathrm{cof}(\xi - \theta)$ valores ex superioribus substitui, atque hinc reperietur:

$$u\,\mathrm{fi}\,s = \surd\left(\varepsilon\varepsilon\,\mathrm{fi}\,\mathfrak{f}^2 - \frac{4\delta\varepsilon fgt\,\mathrm{fi}\,\mathfrak{f}(\varepsilon f\,\mathrm{fi}\,\mathfrak{f} - e\,\mathrm{fi}\,\mathfrak{h})}{aak} + \frac{4\delta\delta ffggtt}{a^4}\right)$$

unde ob $u\,\mathrm{cof}\,s = \varepsilon\,\mathrm{cof}\,\mathfrak{f}$ prodit:

$$uu = \varepsilon\varepsilon - \frac{4\delta\varepsilon fgt\,\mathrm{fi}\,\mathfrak{f}(\varepsilon f\,\mathrm{fi}\,\mathfrak{f} - e\,\mathrm{fi}\,\mathfrak{h})}{aak} + \frac{4\delta\delta ffggtt}{a^4}.$$

COROLL. 5.

1060. Cum fit vis viva globi $= M(vv + aa\,uu)$, erat ea initio $= M(ee + \varepsilon\varepsilon\,aa)$, elapso autem tempore t ea erit $=$

$$M\left(ee + \varepsilon\varepsilon aa - 4\delta gkt + 4\left(1 + \frac{ff}{aa}\right)\delta\delta gg tt\right).$$

At elapſo tempore $t = \frac{aak}{2\delta g(aa+ff)}$, vis viva fiet $=$
$\frac{M(eeff + 2\varepsilon eaaf\,\mathfrak{f}\,\sin\mathfrak{h} + \varepsilon\varepsilon aa(aa+ff\cos\mathfrak{f}^2))}{aa+ff}$, cujus defectus ab initiali eſt $= \frac{Maa(ee - 2\varepsilon ef\sin\mathfrak{f}\sin\mathfrak{h} + \varepsilon\varepsilon ff\sin\mathfrak{f}^2)}{aa+ff} = \frac{Maakk}{aa+ff}$, ita ut iſta vis viva fit $= M\left(ee + \varepsilon\varepsilon aa - \frac{aakk}{aa+ff}\right)$.

SCHOLION.

1061. Ex his ergo formulis totus motus globi aſſignari poteſt, quicunque motus ei initio fuerit impreſſus: interim tamen hae formulae non parum ſunt complexae, unde ad clariorem explicationem haud abs re erit, caſus quosdam magis notabiles evolvi. Cujusmodi ſunt, uti jam ſupra innuimus, duo potiſſimum, alter quo arcus ZO initio erat quadrans, alter vero quo angulus DZO $= \mathfrak{h}$ erat rectus: utrumque igitur ſeorſim explicemus.

PROBLEMA. 20.

1062. Si globo, in quo omnia momenta inertiae ſunt aequalia, initio motus gyratorius circa axem horizontalem fuerit impreſſus praeter motum progreſſivum, definire continuationem motus.

SOLUTIO.

Fig. 139. Cum initio axis gyrationis fuerit horizontalis, erit $\mathfrak{f} = ZO = 90°$. Denotante ergo e celeritatem progreſſivam ſecundum directionem DIE, et ε celeritatem angularem circa axem IO in ſenſum ZETD, ſit pro puncto O angulus DZO $= \mathfrak{h}$: manente f = radio globi et Maa = momento inertiae. Ex his erat initio celeritas radens $k = \sqrt{(ee - 2\varepsilon ef\sin\mathfrak{h} + \varepsilon\varepsilon ff)}$ et pro ejus directione IQ angulus DZQ $= \zeta$ ut ſit $\sin\zeta$

Fig. 140. $= \frac{-\varepsilon f\cos\mathfrak{h}}{k}$ et $\cos\zeta = \frac{\varepsilon f\sin\mathfrak{h} - e}{k}$. His pro ſtatu initiali conſtitutis elapſo tempore t centrum globi deſcripſerit viam GI, ut jam ſit in I ubi ejus celeritas ſecundum IR erit $= v = \sqrt{(ee + \frac{4\delta\varepsilon gt(\varepsilon f\sin\mathfrak{h} - e)}{k}} +$ $4\delta\delta$

$gg tt)$: unde poſitis coordinatis GX $=$ X et XI $=$ Y ob tang IIX $= \text{tang}\,\varphi = \frac{-2\delta\varepsilon fgt\cos\mathfrak{h}}{ek + 2\delta gt(\varepsilon f\sin\mathfrak{h} - e)}$ erit $dX = edt + \frac{2\delta gtdt}{k}$ $(\varepsilon f\sin\mathfrak{h} - e)$ et $dY = \frac{-2\delta\varepsilon fgt\,dt\cos\mathfrak{h}}{k}$, ideoque GX $=$ X $= et +$ $\frac{\delta gtt}{k}(\varepsilon f\sin\mathfrak{h} - e)$ et XI $=$ Y $= \frac{-\delta\varepsilon fgtt}{k}\cos\mathfrak{h}$. Tum vero pro motu gyratorio, qui jam fiat in ſenſum ZETD celeritate angulari $= \varepsilon$ circa polum O exiſtente ZO $= s$, PZO $= \theta$, et DZQ $= \varphi + \xi$, ubi IQ refert directionem celeritatis radentis, quia conſtanter eſt $\varphi + \xi = \zeta$, ſeu directio IQ conſtans, erit $\text{tang}\,\xi = \frac{-\varepsilon f\cos\mathfrak{h}}{\varepsilon ef\sin\mathfrak{h} - ee + 2\delta gkt}$, et $\text{tang}(\xi - \theta) = \frac{\varepsilon f - e\sin\mathfrak{h}}{e\cos\mathfrak{h}} - \frac{2\delta fgkt}{\varepsilon eaa\cos\mathfrak{h}}$, unde ambo anguli ξ et θ definiuntur. Vel erit $\text{tang}(\varphi + \theta) = \frac{\varepsilon aak\sin\mathfrak{h} + 2\delta fgt(e - \varepsilon f\sin\mathfrak{h})}{\varepsilon aak\cos\mathfrak{h} - 2\delta\varepsilon ffgt\cos\mathfrak{h}}$.

Fig. 140.

Celeritas autem radens ſecundum directionem IQ eſt $w = k - 2\delta g(1 + \frac{ff}{aa})t$. Tum vero ob $\varepsilon\cos s = 0$, erit arcus ZO $= s$ quadrans, et

$$\varepsilon = \sqrt{\left(\varepsilon\varepsilon - \frac{4\delta\varepsilon fgt(\varepsilon f - e\sin\mathfrak{h})}{aak} + \frac{4\delta\delta ffggtt}{a^4}\right)}.$$

Hic motus inaequabilis autem tantum durabit per tempus $t = \frac{aak}{2\delta g(aa+ff)}$, quo elapſo eſt $\text{tang}\,\varphi = \frac{-\varepsilon aaf\cos\mathfrak{h}}{e(aa+ff) + aa(\varepsilon f\sin\mathfrak{h} - e)}$ $= \frac{-\varepsilon aa\cos\mathfrak{h}}{ef + \varepsilon aa\sin\mathfrak{h}} = \sqrt{\left(ee + \frac{2aae(\varepsilon f\sin\mathfrak{h} - e)}{aa+ff} + \frac{a^4kk}{(aa+ff)^2}\right)}$, $= 90°$, et $\varepsilon = \frac{v}{f} = \frac{\sqrt{(eeff + 2\varepsilon eaaf\sin\mathfrak{h} + \varepsilon\varepsilon a^4)}}{aa+ff}$, ſubſtituto pro kk valore. Tum autem fit angulus $\theta = 90°$, et $\sin\xi = \frac{e\sin\zeta}{v}$.

COROLL. 1.

1063. Si initio fuerit angulus DZO $= \mathfrak{f} = 0$, erit $k = \sqrt{(ee + \varepsilon\varepsilon ff)}$

pro

pro angulo DZQ $= \zeta$ fit $\sin\zeta = \frac{-\epsilon f}{k}$; $\cos\zeta = \frac{-o}{k}$; tum post tempus t prodit $v = \surd\left(\epsilon\epsilon - \frac{4\delta\epsilon\epsilon g t}{\epsilon k} + 4\delta\delta g g t t\right)$; $\tan\phi = \frac{-2\delta\epsilon f g t}{\epsilon(k - 2\delta g t)}$; $X = et\left(1 - \frac{\delta g t}{k}\right)$; $Y = \frac{-\delta\epsilon f g t t}{k}$. Porro $\tan\xi = \frac{\epsilon e f}{ee - 2\delta g k t}$;

$$\tan(\xi - \theta) = \frac{\epsilon f}{e} - \frac{2\delta f g k t}{\epsilon e a a}; \quad \tan(\phi + \theta) = \frac{2\delta e f g t}{\epsilon a a k - 2\delta\epsilon f f g t};$$

$$u = \surd\left(\epsilon\epsilon - \frac{4\delta\epsilon\epsilon f f g t}{a a k} + \frac{4\delta\delta f f g g t t}{a^4}\right) \text{ et } w = k - 2\delta g\left(1 + \frac{ff}{aa}\right)t.$$

Elapso autem tempore $t = \frac{aak}{2\delta g(aa+ff)}$ erit $\tan\phi = \frac{-\epsilon a a}{e f f}$; $v = \frac{f\surd(e e f f + \epsilon\epsilon a^4)}{aa+ff} = fx$; $\theta = 90°$ et $\tan\xi = \frac{\epsilon e f(aa+ff)}{ee(aa+ff) - aakk}$
$$= \frac{\epsilon e(aa+ff)}{f(ee - \epsilon\epsilon a a)}.$$

COROLL. 2.

1064. Si angulus DZO $= \mathfrak{f}$ esset $= 180°$, eaedem formulae motum indicabunt, sumta celeritate angulari ϵ negativa seu motu gyratorio in contrarium verso. At si sit $\epsilon = 0$, seu globo solus motus progressivus fuerit impressus, fit $k = e$, $\zeta = 180°$; $v = e - 2\delta g t$; $\phi = 0$; $X = t(e - \delta g t)$, $Y = 0$; $\xi = 180°$; $\theta = 90°$, $u = \frac{2\delta f e t}{aa}$; et elapso tempore $t = \frac{aae}{2\delta g(aa+ff)}$, fit $v = \frac{eff}{aa+ff}$, $u = \frac{ef}{aa+ff}$ et $X = \frac{et(aa+2ff)}{2(aa+ff)} = \frac{aaee(aa+2ff)}{4\delta g(aa+ff)^2}$.

SCHOLION.

1065. Casus hic, quo globus initio nullum motum gyratorium est adeptus, in genere valet, neque ad ullam hypothesin angulorum $\mathfrak{f}$ et $\mathfrak{h}$ est adstrictus. Tum igitur globus in directum progreditur motu progressivo

gressivo retardato, motumque paulatim gyratorium accipiet, donec elapso tempore $t = \frac{aae}{2\delta g(aa+ff)}$ motum uniformem acquirat, quo deinceps continuo progreditur. Hinc deducimur ad casum, quo globus initio motum tantum gyratorium acceperit sine ullo motu progressivo, cujus evolutio est facilis. Posito enim $e = 0$, erit $k = \epsilon f \sin\mathfrak{f}$, hincque fit $\sin\zeta = -\cos\mathfrak{h}$ et $\cos\zeta = \sin\mathfrak{h}$, ergo $\zeta = \mathfrak{h} - 90°$: ubi pro axe gyrationis initio impressae IO est ZO $= \mathfrak{f}$ et DZO $= \mathfrak{h}$, existente celeritate angulari in sensum ZETD $= \epsilon$. Elapso ergo tempore t fit $\phi = \zeta$, scilicet sublato ab angulo DZO $= \mathfrak{h}$ angulo recto PZO, erit PI directio motus progressivi, quem globus acquiret, cujus celeritas erit $v = 2\delta g t$, ideoque tempori proportionalis. Tum vero erit $\tan\xi = 0$ et $\tan(\xi - \theta) = \infty$, ergo ob $\phi + \xi = \zeta = \mathfrak{h} - 90°$, erit $\xi = 0$, et $\theta = 90°$, hinc DZO $= \zeta + 90 = \mathfrak{h}$, ita ut polus gyrationis O in eodem perpetuo circulo verticali reperiatur. Denique ex §. 1059. est $u \sin r$
$$= \surd\left(\epsilon\epsilon \sin\mathfrak{f}^2 - \frac{4\delta\epsilon f g t \sin\mathfrak{f}}{aa} + \frac{4\delta\delta f f g g t t}{a^4}\right) = \epsilon \sin\mathfrak{f} - \frac{2\delta f g t}{aa} \text{ et } u$$
$\cos r = \epsilon\cos\mathfrak{f}$, unde fit $\tan r = \tan\mathfrak{f} - \frac{2\delta f g t}{\epsilon a a \cos\mathfrak{f}}$, ita ut arcus ZO diminuatur, nisi fuerit quadrans vel eo major, et $u = \surd\left(\epsilon\epsilon - \frac{4\delta f\epsilon t\sin\mathfrak{f}}{aa} + \frac{4\delta\delta f f g g t t}{a^4}\right)$. Motus autem ad uniformitatem reducetur elapso tempore $t = \frac{\epsilon a a f \sin\mathfrak{f}}{2\delta g(aa+ff)}$; fitque tum $u = \frac{\epsilon\surd(a^4\sin\mathfrak{f}^2 + (aa+ff)^2\cos\mathfrak{f}^2)}{aa+ff}$; $v = \frac{\epsilon a a f \sin\mathfrak{f}}{aa+ff}$ et $\tan r = \frac{aa\tan\mathfrak{f}}{aa+ff}$. Si ergo fuisset $\mathfrak{f} = 0$, seu globo motus gyratorius circa axem verticalem impressus esset sine ullo motu progressivo, eundem motum sine ulla mutatione esset conservaturus.

PROBLEMA. 21.

1066. Si globo, in quo omnia momenta inertiae sunt aequalia, motus gyratorius fuerit impressus circa axem ad motus progressivi directionem normalem; definire continuationem motus.

 SO-

SOLUTIO.

Fig. 139. Cum motus progressivi initio impressi directio sit recta DIE, et celeritas $= c$, angulus DZO $= \eta$ est rectus, et sumto ZO $= \mathfrak{f}$ erat O polus circa quem initio globus accepit celeritatem angularem $= \varepsilon$ in sensum ZETD. Habemus ergo $k = \pm (c - \varepsilon f \sin \mathfrak{f})$ ubi valorem positivum pro k sumi oportet: ita ut hic prodeant duo casus seorsim tractandi.

Casus I. Sit $c > \varepsilon f \sin \mathfrak{f}$, erit $k = c - \varepsilon f \sin \mathfrak{f}$, quae est celeritas radens initio, ejusque directio IQ, ut sit $\sin DQ = 0$ et $\cos DQ = -1$ ideoque DQ $= \zeta = 180°$, et Q cadit in E: globusque a frictione δM secundum ID constanter retrahatur: unde statim colligitur globi centrum I in eadem recta DE esse incessurum. Elapso tempore t ergo ob $\cos \zeta = -1$ fit celeritas centri $v = c - 2\delta g t$, et celeritas radens $w = c - \varepsilon f \sin \mathfrak{f} - 2\delta g \left(1 + \frac{ff}{aa}\right) t$: tum vero $\varphi = 0$ et $\xi = 180°$, atque $\theta = 90°$. Quare pro axe gyrationis praesente IO, est DIO $= 90°$, et posito arcu ZO $= s$ et celeritate angulari $= x$ habemus $x \cos s = \varepsilon \cos \mathfrak{f}$ et ex §. 1059. $x \sin s = \varepsilon \sin \mathfrak{f} + \frac{2\delta f g t}{aa}$, unde colligitur $\tang s = \tang \mathfrak{f} + \frac{2\delta f g t}{\varepsilon aa \cos \mathfrak{f}}$, et $x = \sqrt{\left(\varepsilon\varepsilon + \frac{4\delta \varepsilon f g t \sin \mathfrak{f}}{aa} + \frac{4\delta\delta ff gg tt}{a^4}\right)}$. Hocque tempore t percurrit centrum I lineam rectam GX $= X = t(c - \delta g t)$. Hic autem motus inaequabilis durabit per tempus $t = \frac{aa(c - \varepsilon f \sin \mathfrak{f})}{2\delta g(aa + ff)}$ quo elapso erit spatium $X = \frac{aa(c - \varepsilon f \sin \mathfrak{f})(c(aa + 2ff) + \varepsilon aaf \sin \mathfrak{f})}{2\delta g(aa + ff)^2}$, et celeritas $v = \frac{f(cf + \varepsilon aa \sin \mathfrak{f})}{aa + ff}$. At pro motu gyratorio $\tang s = \tang$ ZO $= \tang \mathfrak{f} + \frac{f(c - \varepsilon f \sin \mathfrak{f})}{\varepsilon(aa + ff)\cos \mathfrak{f}} = \frac{cf + \varepsilon aa \sin \mathfrak{f}}{\varepsilon(aa + ff)\cos \mathfrak{f}}$, existente perpetuo DIO $= 90°$ et celeritas angularis

$$x = \frac{\sqrt{(ccff + 2\varepsilon c aaf \sin \mathfrak{f} + \varepsilon\varepsilon a^4 \sin \mathfrak{f}^2 + \varepsilon\varepsilon(aa + ff)^2 \cos \mathfrak{f}^2)}}{aa + ff}.$$

Casus II. Sit $c < \varepsilon f \sin \mathfrak{f}$ seu $k = \varepsilon f \sin \mathfrak{f} - c$, quae est celeritas radens initio, ejusque directio IQ talis ut sit $\sin DQ = 0$ et $\cos DQ = 1$, ergo DQ $= \zeta = 0$, et Q in D cadat. Globus ergo a frictione δM secundum directionem IE constanter acceleratur, ejusque centrum I in eadem recta IE progreditur; atque elapso tempore t erit ejus celeritas $v = c + 2\delta g t$, et celeritas radens $w = \varepsilon f \sin \mathfrak{f} - c - 2\delta g \left(1 + \frac{ff}{aa}\right) t$. Tum vero fit $\varphi = 0$ et $\xi = 0$, atque $\theta = 90°$. Quare pro axe gyrationis praesente IO est DIO $= 90°$, et posito arcu ZO $= s$ et celeritate angulari $= x$ habemus $x \cos s = \varepsilon \cos \mathfrak{f}$ et $x \sin s = \varepsilon \sin \mathfrak{f} - \frac{2\delta f g t}{aa}$, unde fit $\tang s = \tang \mathfrak{f} - \frac{2\delta f g t}{\varepsilon aa \cos \mathfrak{f}}$ et $x = \sqrt{\left(\varepsilon\varepsilon - \frac{4\delta \varepsilon f g t \sin \mathfrak{f}}{aa} + \frac{4\delta\delta ff gg tt}{a^4}\right)}$ hocque tempore t centrum globi percurrit lineam rectam GX $= X = t(c + \delta g t)$. Hic autem motus inaequabilis durabit tantum tempore $t = \frac{aa(\varepsilon f \sin \mathfrak{f} - c)}{2\delta g(aa + ff)}$, quo elapso erit celeritas $v = \frac{f(cf - \varepsilon aa \sin \mathfrak{f})}{aa + ff}$, et spatium $X = \frac{aa(\varepsilon f \sin \mathfrak{f} - c)(c(aa + 2ff) + \varepsilon aaf \sin \mathfrak{f})}{2\delta g(aa + ff)^2}$. At pro motu gyratorio reperitur $\tang s = \tang$ ZO $= \frac{cf + \varepsilon aaf \sin \mathfrak{f}}{\varepsilon(aa + ff)\cos \mathfrak{f}}$ existente perpetuo DIO $= 90°$, et celeritas angularis

$$x = \frac{\sqrt{(ccff + 2\varepsilon c aaf \sin \mathfrak{f} + \varepsilon\varepsilon a^4 \sin \mathfrak{f}^2 + \varepsilon\varepsilon(aa + ff)^2 \cos \mathfrak{f}^2)}}{(aa + ff)}.$$

COROLL. 1.

1067. Si fuerit $c = \varepsilon f \sin \mathfrak{f}$, globus statim ab initio motum prosequetur uniformem, tam progressivum quam gyratorium, qui casus limitem constituit inter binos tractatos.

COROLL. 2.

1068. Ad priorem casum quo $c > \varepsilon f \sin \mathfrak{f}$ referendi sunt ii, quibus ε habet valorem negativum, seu globo impressus fuerit initio motus gyratorius in sensum ZDTE. Posito autem $-\varepsilon$ loco ε, fieri potest, ut globus

globus revertatur, antequam ad uniformitatem pervenerit: scilicet si fuerit $\varepsilon > \frac{cf}{aa \sin \zeta}$.

COROLL. 3.

1069. Casu hoc, quo ε negative capitur, habebimus ad tempus t: $\phi = 0$, $\theta = 0$, $\zeta = 180$, $v = c - 2\delta gt$; $w = c + \varepsilon f \sin \zeta - 2\delta g \left(1 + \frac{ff}{aa}\right) t$; $\tang s = \tang \zeta - \frac{2\delta fgt}{\varepsilon aa \cos \zeta}$; et $\varepsilon = \surd \left(\varepsilon\varepsilon - \frac{4\delta\varepsilon fgt \sin \zeta}{aa} + \frac{4\delta\delta ffggtt}{a^4}\right)$. At post tempus $t = \frac{aa(c + \varepsilon f \sin \zeta)}{2\delta g(aa+ff)}$ percurso spatio $X = \frac{aa(c+\varepsilon f \sin \zeta)(c(aa+2ff) - \varepsilon aaf \sin \zeta)}{2\delta g(aa+ff)^2}$, uniformitatem attinget, eritque tum $v = \frac{f(cf - \varepsilon aa \sin \zeta)}{aa+ff}$; $\tang s = \frac{\varepsilon aa \sin \zeta - cf}{\varepsilon (aa+ff) \cos \zeta}$ et

$$\varepsilon = \frac{\surd (ccff - 2c\varepsilon aaf \sin \zeta + \varepsilon\varepsilon a^4 \sin \zeta^2 + \varepsilon\varepsilon (aa+ff)^2 \cos \zeta^2)}{aa+ff}.$$

SCHOLION.

1070. Casus hic praecipue est memorabilis, quo globo ejusmodi motus imprimi potest, ut primo recedat, mox autem iterum revertatur, quod experimento ostendi solet, dum digito ad globum circa D applicato et deorsum presso duplex motus globo imprimitur, alter progressivus in directione DIE, alter gyratorius in sensum ZDTE. Sed ut phaenomenon succedat, necesse est, ut celeritas angularis prae progressiva certum quendam limitem excedat; quem quo facilius agnoscamus, calculum ad istum casum accommodemus, quo motus gyratorius globo circa axem horizontalem et ad directionem motus progressivi normalem imprimitur. Quod si ergo c denotet celeritatem progressivam secundum directionem DIE, et ε celeritatem angularem retro gyrantem in sensum ZDTE, existente f radio globi et Maa ejus momento inertiae, frictioneque $= \delta M$; primo globus in directione DIE procedet, et elapso tempore t ejus celeritas secundum eandem directionem erit $v = c - 2\delta gt$, confecto spatio $X = t(c - \delta gt)$: tum vero etiamnum circa eundem axem retro volvetur celeritate angulari $\varepsilon = \varepsilon - \frac{2\delta fgt}{aa}$.

Motus

Motus autem aequabilis evadit elapso tempore $t = \frac{aa(c + \varepsilon f)}{2\delta g(aa+ff)}$, eritque tum celeritas progressiva $v = \frac{f(cf - \varepsilon aa)}{aa+ff}$; et angularis $u = \frac{\varepsilon aa - cf}{aa+ff}$. Quare si fuerit $\varepsilon > \frac{cf}{aa}$, globus nunc retro movetur, gyratorio adhuc retro vergente: sin autem fuerit $\varepsilon < \frac{cf}{aa}$, globus adhuc procedit, et gyratio in sensum contrarium est versa. Illo casu globus regredi coepit elapso tempore $t = \frac{c}{2\delta g}$ et percurso spatio $X = \frac{cc}{4\delta g}$.

Si globus sit homogeneus, erit $aa = \frac{2}{5} ff$, et εf exprimit celeritatem gyrationis in puncto contactus, quae si vocetur $= b$, erit post tempus t celeritas progressiva $v = c - 2\delta gt$, et gyratoria in puncto contactus, quae sit $u = b - 5\delta gt$, et spatium percursum $= t(c - \delta gt)$: motus vero aequabilis evadet elapso tempore $t = \frac{c+b}{7\delta g}$, et confecto spatio $= \frac{(6c-b)(c+b)}{49\delta g}$; ubi erit $v = \frac{5c-2b}{7}$, et $u = \frac{2b-5c}{7}$. Ut ergo phaenomenum memoratum succedat, debet esse initio $b > \frac{5}{2} c$. Sin autem esset $b = \frac{5}{2} c$ uterque motus simul extingueretur elapso tempore $= \frac{c}{2\delta g}$ min. sec. et confecto spatio $= \frac{cc}{4\delta g}$.

Ttt

SUP-

SUPPLEMENTUM

ad Problema 80. §. 761. de motu quocunque libero corporis solidi a nullis viribus sollicitati.

Posito $x = u \cos \alpha$; $y = u \cos \mathfrak{C}$; et $z = u \cos \gamma$, aequationes resolvendae erunt novem sequentes:

I. $dx = \frac{bb-cc}{aa} yzdt$; II. $dy = \frac{cc-aa}{bb} xzdt$; III. $dz = \frac{aa-bb}{cc} xydt$

IV. $dl\, \mathrm{si}\, l = dt\,(y \cos n - z \cos m)$; VII. $d\lambda\, \mathrm{si}\, l^2 = -dt\,(y \cos m + z \cos n)$

V. $dm\, \mathrm{si}\, m = dt\,(z \cos l - x \cos n)$; VIII. $d\mu\, \mathrm{si}\, m^2 = -dt\,(z \cos n + x \cos l)$

VI. $dn\, \mathrm{si}\, n = dt\,(x \cos m - y \cos l)$; IX. $d\nu\, \mathrm{si}\, n^2 = -dt\,(x \cos l + y \cos m)$

unde novem quantitates x, y, z, l, m, n, λ, μ, ν definiri oportet. Trium priorum quidem solutio jam in antecedentibus problematibus est tradita; ad usum autem sequentium statuatur

$$\frac{bb-cc}{aa} = A;\quad \frac{cc-aa}{bb} = B;\quad \frac{aa-bb}{cc} = C \text{ et } xyzdt = du$$

eritque $xdx = Adu$; $ydy = Bdu$; $zdz = Cdu$

unde integrando elicitur:

$$xx = 2Au + \mathfrak{A};\ yy = 2Bu + \mathfrak{B};\ zz = 2Cu + \mathfrak{C}$$

ideoque $dt = \frac{du}{\sqrt{(2Au+\mathfrak{A})(2Bu+\mathfrak{B})(2Cu+\mathfrak{C})}}$

Ratione autem quantitatum A, B, C eae ita inter se sunt comparatae, ut sit: $Aaa + Bbb + Ccc = 0$ et $Aa^4 + Bb^4 + Cc^4 = 0$; Quare fiet $aaxx + bbyy + cczz = \mathfrak{A}aa + \mathfrak{B}bb + \mathfrak{C}cc =$ quantitati constanti. Restitutis autem pro x, y, z valoribus assumtis fit

$$aaxx + bbyy + cczz = uu\,(aa \cos \alpha^2 + bb \cos \mathfrak{C}^2 + cc \cos \gamma^2) = \text{Const.}$$

At posita massa corporis $=$ M expressio $M\,(aa \cos \alpha^2 + bb \cos \mathfrak{C}^2 + cc \cos \gamma^2)$ denotat momentum inertiae corporis respectu axis IO, circa quem corpus nunc gyratur, quod momentum ergo si dicatur $= Mrr$, erit $Mrr\,uu$ vis viva corporis, quae ergo manet constans.

Deinde cum sit $\cos \alpha^2 + \cos \mathfrak{C}^2 + \cos \gamma^2 = 1$ erit

$$u = \sqrt{(xx + yy + zz)} = \sqrt{(2\,(A + B + C)\,u + \mathfrak{A} + \mathfrak{B} + \mathfrak{C})}$$

et ex cognitis x, y, z per u, etiam anguli α, $\mathfrak{C}$, γ, per u definiuntur. Atque hucusque quidem in problematibus antecedentibus pertingere licuit;

nunc igitur videamus, quomodo solutio propria probl. 80. expediri queat. Omnem autem difficultatem in aequationibus IV, V, VI sitam esse patet, ad quam superandam, statuamus

$$\cos l = px,\quad \cos m = qy,\quad \text{et } \cos n = rz,$$

ut prodeant hae aequationes:

$$\text{IV. } 0 = pdx + xdp + dt\,(ryz - qyz) \text{ at est } yzdt = \frac{dx}{A}$$

$$\text{V. } 0 = qdy + ydq + dt\,(pxz - rxz) \qquad xzdt = \frac{dy}{B}$$

$$\text{VI. } 0 = rdz + zdr + dt\,(qxy - pxy) \qquad xydt = \frac{dz}{C}$$

unde hae aequationes in sequentes formas mutantur

$$\text{IV. } 0 = pdx + xdp + \frac{(r-q)\,dx}{A};\ \text{seu } \frac{dx}{x} = \frac{A\,dp}{q-r-Ap} = \frac{Adu}{2Au+\mathfrak{A}}$$

$$\text{V. } 0 = qdy + ydq + \frac{(p-r)\,dy}{B};\ \text{seu } \frac{dy}{y} = \frac{A\,dp}{r-q-Bq} = \frac{Bdu}{2Bu+\mathfrak{B}}$$

$$\text{VI. } 0 = rdz + zdr + \frac{(q-p)\,dz}{C};\ \text{seu } \frac{dz}{z} = \frac{C\,dr}{p-q-Cr} = \frac{Cdu}{2Cu+\mathfrak{C}}.$$

Multiplicetur IV. per aax; V. per bby et VI. per ccz, ut habeatur

$$\text{IV. } aapxdx + aaxxdp = a^2\,\frac{(q-r)\,xdx}{A} = a^2\,(q-r)\,du$$

$$\text{V. } bbqydy + bbyydq = \frac{bb\,(r-p)\,ydy}{B} = bb\,(r-p)\,du$$

$$\text{VI. } ccrzdz + cczzdr = \frac{cc\,(p-q)\,zdz}{C} = cc\,(p-q)\,du.$$

Ex ternis autem primis colligitur

I. $aapxdx = Aaapdu = (bb - cc)\,pdu$

II. $bbqydy = Bbbqdu = (cc - aa)\,qdu$

III. $ccrzdz = Cccrdu = (aa - bb)\,rdu.$

Hi

His sex aequationibus in unam summam conjectis, partes posteriores se mutuo destruunt, proditque aequatio integrabilis:

$$2aapxdx + aaxxdp + 2bbqydy + bbyydq + 2ccrzdz + cczzdr = 0$$

cujus integrale est

$$aapxx + bbqyy + ccrzz = \text{Const.}$$

in quo maxima vis inest ad integrationem desideratam absolvendam, si conjungatur cum aequatione $\cos l^2 + \cos m^2 + \cos n^2 = 1$, quae abit in $ppxx + qqyy + rrzz = 1$. Cum enim x, y, z dentur per u ex his duabus aequationibus quantitates p et q per u et r definiri poterunt, qui in aequatione $\frac{dr}{p-q-Cr} = \frac{du}{2Cu+\mathfrak{C}}$ substituti perducent ad aequationem binas tantum variabiles u et r involventem, ex qua etiam r per u determinare licebit.

Primum autem observo, aequationibus nostris satis fieri posse, tribuendo litteris p, q, et r valores constantes: ad hoc enim necesse est fiat

$$q - r - Ap = 0;\ r - p - Bq = 0;\ p - q - Cr = 0;$$

unde fit $p = n(1 - B)$; $q = n(1 + A)$ et $r = n(1 + AB)$ si modo sit $A + B + C + ABC = 0$, quod autem revera evenit. Erit ergo pro A, B, C valores assumtos substituendo

$$p = \frac{n(aa+bb-cc)}{bb};\ q = \frac{n(aa+bb-cc)}{aa} \text{ et } r = \frac{ncc(aa+bb-cc)}{aabb}$$

quare sumto $n = \frac{maabb}{aa+bb-cc}$, colligitur

$$p = maa;\ q = mbb;\ \text{et } r = mcc,$$

ubi coefficiens m ita debet esse comparatus, ut fiat $ppxx + qqyy + rrzz = 1$ seu $mm(a^4(2Au + \mathfrak{A}) + b^4(2Bu + \mathfrak{B}) + c^4(2Cu + \mathfrak{C})) = 1$;

quare cum sit $Aa^4 + Bb^4 + Cc^4 = 0$, erit $m = \frac{1}{\sqrt{(\mathfrak{A}a^4 + \mathfrak{B}b^4 + \mathfrak{C}c^4)}}$, simulque fit

$$aapxx + bbqyy + ccrzz = m(a^4(2Au + \mathfrak{A}) + b^4(2Bu + \mathfrak{B}) + c^4(2Cu + \mathfrak{C}))$$

cujus ergo expressionis valor constans est $= +(\mathfrak{A}a^4 \sqrt{} \mathfrak{B}b^4 + \mathfrak{C}c^4)$. Observo autem, hanc integrationem non esse pro incompleta habendam, propterea quod vertex sphaerae immobilis Z pro lubitu assumi potest. Eum ergo semper ita accipere licebit, ut quantitates p, q, r fiant constantes. Posito itaque brevitatis gratia $\sqrt{(\mathfrak{A}a^4 + \mathfrak{B}b^4 + \mathfrak{C}c^4)} = n$ omnia per u sequenti modo definientur: ut sit

$$x = \sqrt{(2Au + \mathfrak{A})};\ p = \frac{aa}{n};\ \cos l = \frac{aa}{n}\sqrt{(2Au + \mathfrak{A})}$$

$$y = \sqrt{(2Bu + \mathfrak{B})};\ q = \frac{bb}{n};\ \cos m = \frac{bb}{n}\sqrt{(2Bu + \mathfrak{B})}$$

$$z = \sqrt{(2Cu + \mathfrak{C})};\ r = \frac{cc}{n};\ \cos n = \frac{cc}{n}\sqrt{(2Cu + \mathfrak{C})}.$$

Pro ternis postremis aequationibus ob $dt = \frac{du}{xyz}$ fiet

$$d\lambda = \frac{-n\,dt\,\sqrt{(\mathfrak{B}bb + \mathfrak{C}cc - 2Aaau)}}{\mathfrak{B}b^4 + \mathfrak{C}c^4 - 2Ac^4u},$$

sufficit autem unicum ternorum angulorum λ, μ, ν determinasse, cum bini reliqui ex eo per se constent.