Reliqua duo momenta Q et R praebet analogia ſine ulteriori calculo. Deinde vero eſt $\cos DZ = \cos \xi = \cos\zeta \cos l + \cos\eta \cos m + \cos\theta \cos n$, quae expreſſio uti unitatem nunquam ſuperare poteſt, ita unitati aequalis eſſe nequit, ſeu DZ $= 0$, niſi ſit $l = \zeta$, $m = \eta$ et $n = \theta$, ſcilicet has ternas determinationes ſimul ſuppeditat haec aequatio, $\cos\zeta \cos l + \cos\eta \cos m + \cos\theta \cos n = 1$. Cum enim praeterea ſit

$$\cos\zeta^2 + \cos\eta^2 + \cos\theta^2 = 1 \text{ et } \cos l^2 + \cos m^2 + \cos n^2 = 1,$$

ſi a ſumma harum illius duplum ſubtrahatur, prodit

$$(\cos\zeta - \cos l)^2 + (\cos\eta - \cos m)^2 + (\cos\theta - \cos n)^2 = 0,$$

trium autem quadratorum ſumma nihilo aequari nequit, niſi ſingula ſint nulla.

## SCHOLION.

896. Cum neque in his expreſſionibus pro momentis virium, P, Q, R inventis, neque in preſſione $\Pi = M\left(1 + \frac{f\,dd.\cos\xi}{2g\,dt^2}\right)$ radius ſphaerae $e$ baſin conſtituentis inſit, omnia quae ſupra de motu turbinis infra in cuſpidem deſinentis ſunt tradita, etiam valent de ejusmodi turbinibus, qui deſinunt in hemiſphaerium, ſeu aliam ſphaerae partem, dummodo punctum F, quod ante cuſpidem denotabat, hic in centro figurae ſphaericae G conſtituatur. Perinde ergo eſt, ſive turben gyretur ſuper cuſpide, ſive ſuper hemiſphaerio, dummodo $f$ ſit diſtantia centri inertiae I a centro baſis ſphaericae, quantuscunque enim fuerit radius hujus baſis $e$, is in computum non ingreditur, eo autem evaneſcente baſis turbinis abit in cuſpidem. Totum igitur caput praecedens hic inferi intelligatur, ita ut Theoria turbinum ſine ullo labore haud mediocriter amplificata ſit cenſenda. Baſin autem ſphaericam faciendo caſus occurrit ante excluſus, ſcilicet quo centrum inertiae I fundo propius eſt, quam centrum ſphaericitatis, hicque ſit quantitas $f$ negativa. Sive jam tale corpus ſit globus completus, ſive baſin habeat MFN portionem ſphaerae, centro G deſcriptae, qua plano horizontali incumbat, ejus motum, quatenus contactus in hanc baſin cadit, inveſtigemus. Hic autem cogimur corpori talem indolem tribuere, ut axis proprius AGIF, qui ſi fuerit verticalis, ſtatum quietus exhibet, ſimul ſit corporis axis principalis, reliqui vero bini axes principales habeant momenta inter ſe aequalia. Scilicet ſi reſpectu axis IA momentum inertiae ſit $= Maa$ reſpectu binorum reliquorum vero $Mbb$, et $Mcc$, ſtatuemus $bb = cc$. Hujusmodi ergo corpus quemcunque receperit motum impreſſum, quomodo motum ſit continuaturum, determinemus.

Fig. 115.

PRO-

[illegible]

P R O B L E [illegible]

S[illegible] ſphaerica MFN [illegible] ſit A[illegible] principalis, [illegible] Maa reſpectu [illegible] acceperit quemcunque, determinare motum [illegible]

## S O L U T I O.

Sit radius baſis ſphaericae FG $= e$, [illegible] infra centrum baſis G ad intervallum GI $=$ [illegible] ſivo, ſi centrum inertiae I haberet motum [illegible] rizontalem, eum conſervat in directione [illegible] motu verticali vicetur, is cognita preſſione [illegible]

quod ſi declinatio axis AF a ſitu verticali ponatur $=$ [illegible]

$\frac{f\,dd.\cos\xi}{2g\,dt^2}$, exiſtente M corporis maſſa [illegible] preſſio $\Pi$, quam corpus in planum horizontale [illegible] tu gyratorio cognoſci poteſt. Teneat ergo [illegible] ſphaerae fixae, in qua Z eſt punctum verticale [illegible] ejusmodi ſitum, ut ejus axes principales [illegible] nenturque arcus ZA $= l$, ZB $= m$, ZC $=$ [illegible] $= \lambda$, XZB $= \mu$, XZC $= \nu$, ita ut ſit $l = \zeta$ [illegible] circa polum O in ſenſum ABC celeritate angulari [illegible] cubus OA $= \alpha$, OB $= \beta$, OC $= \gamma$ ſit brevitatis [illegible] $u \cos\beta = y$, $u \cos\gamma = z$. Momenta autem virium [illegible] ſunt P $= 0$, Q $= -\Pi f \cos n$; R $= +\Pi f \cos m$, [illegible] quentes aequationes:

$$dx = 0;\ dy + \frac{aa - cc}{cc}\,xz\,dt = \frac{-2\Pi f g\,dt \ldots}{Mcc}; \ldots$$

$$xy\,dt = \frac{2\Pi f g\,dt \cos m}{Mcc}$$

$dl\, \sin l = dt\,(y \cos n - z \cos m)$; $d\lambda \sin l^2 = -dt$ [illegible]
$dm \sin m = dt\,(z \cos l - x \cos n)$; reliqui [illegible]
$dn \sin n = dt\,(x \cos m - y \cos l)$; [illegible]

Si porro ponamus $\cos l = p$, $\cos m = q$, $\cos n =$ [illegible] quationes congruunt cum iis, quas ſupra probl. 9[illegible] quod $f$ capiatur negative habebimus in finitis has aequationes [illegible]

Eee

I. $qy + rz = B - \frac{Aaap}{cc}$

II. $(qz - ry)^2 = \frac{(Ccc+4fgp)(1-pp) - cc(B - \frac{Aaap}{cc})^2}{cc+ff(1-pp)}$

III. $yy + zz = \frac{(Ccc+4fgp+ff(B-\frac{Aaap}{cc})^2}{cc+ff(1-pp)}$

IV. $\frac{\Pi}{M} = \frac{2gcc - Afaa(B-\frac{Aaap}{cc})}{2g(cc+ff(1-pp))} + \frac{fcep(Ccc+4fgp+ff(B-\frac{Aaap}{cc})^2)}{2g(cc+ff(1-pp))^2}$

V. $dt = \frac{dp\sqrt{(cc+ff(1-pp))}}{\sqrt{((Ccc+4fgp)(1-pp)-cc(B-\frac{Aaap}{cc})^2)}}$

VI. $d\lambda = \frac{-dt}{1-pp}\left(B - \frac{Aaap}{cc}\right)$

VII. $\varkappa\varkappa = AA + \frac{Ccc+4fgp+ff(B-\frac{Aaap}{cc})^2}{cc+ff(1-pp)}$

VIII. $\frac{ydz-zdy}{yy+zz} = \frac{A(aa-cc)dt}{cc} + \frac{2\Pi fg dt(B-\frac{Aaap}{cc})(cc+ff-ffpp)}{Mcc(Ccc+4fgp+ff(B-\frac{Aaap}{cc})^2)}$

ubi constantes A, B, C et reliquae per integrationem ingressurae ex statu corporis initiali debent definiri.

### COROLL. 1.

898. Si corpus initio quieverit, axisque principalis A fuerit in $a$ declinatione ejus existente $Za = \mathfrak{l}$ et $cos\,\mathfrak{l} = \mathfrak{p}$, initio erat $x = 0$, $y = 0$ et $z = 0$, ob $\varkappa = 0$; atque $p = \mathfrak{p}$. Fit ergo $A = 0$; $B = 0$, et $Ccc = -4fg\mathfrak{p}$: Hinc elapso tempore $t$ erit $x = 0$; $qy + rz = 0$;

$$qz - ry = \frac{2\sqrt{fg(p-\mathfrak{p})(1-pp)}}{\sqrt{(cc+ff(1-pp))}};\; yy+zz = \frac{4fg(p-\mathfrak{p})}{cc+ff(1-pp)}$$

$$= \varkappa\varkappa,\ \text{et}\ \frac{\Pi}{M} = \frac{cc}{cc+ff(1-pp)} + \frac{2ffcep(p-\mathfrak{p})}{(cc+ff(1-pp))^2}$$

CO-



### COROLL. 2.

899. Praeterea vero in eodem casu est $d\lambda = 0$, ideoque axis, qui initio in $a$ erat, per ipsum arcum $aZ$ movebitur, eritque $dt = \frac{dp\sqrt{(cc+ff(1-pp))}}{2\sqrt{fg(p-\mathfrak{p})(1-pp)}}$, unde quia $p > \mathfrak{p}$ seu $l < \mathfrak{l}$ axis ab $a$ recta ad Z progreditur. Denique ob $ydz - zdy = 0$, fit $z = \delta y$, et $y = \frac{2\sqrt{fg(p-\mathfrak{p})}}{\sqrt{(1+\delta\delta)(cc+ff(1-pp))}}$: atque $q(yy+zz) = \frac{2z\sqrt{fg(p-\mathfrak{p})(1-pp)}}{\sqrt{(cc+ff(1-pp))}}$, seu $q = \frac{\delta\sqrt{(1-pp)}}{\sqrt{(1+\delta\delta)}}$ et $r = \frac{-\sqrt{(1-pp)}}{\sqrt{(1+\delta\delta)}}$, unde fit $cos\,ZAB = \frac{q}{\sqrt{(1-pp)}}$ $= \frac{\delta}{\sqrt{(1+\delta\delta)}}$, qui ergo angulus manet constans.

### COROLL. 3.

900. Si ergo corpus initio quiescat, ejusque axis principalis IA tenuerit situm inclinatum I$a$, inde recta se eriget ex $a$ ad Z ascendens, gyrabitur autem circa punctum O; ut ob $x = \varkappa\,cos\,\alpha = 0$ arcus AO sit quadrans, et quia $cos\,ZO = cos\,\alpha\,cos\,l + cos\,\mathfrak{C}\,cos\,m + cos\,\gamma\,cos\,n = \frac{qy+rz}{\varkappa} = 0$, erit etiam ZO quadrans, sicque O erit polus circuli XZY. Et cum axis in Z pervenerit, erit celeritas angularis $\varkappa = \frac{2\sqrt{fg(1-\mathfrak{p})}}{c}$.

### SCHOLION. 1.

901. Si corpus initio non quieverit, sed motum quemcunque acceperit, continuatio motus ex iisdem formulis determinatur, dummodo constantes A, B, C statui initiali convenienter definiantur; ubi autem ad ejusmodi formulas integrandas devenitur, quae nonnisi concessis quadraturis superioris ordinis expediri possunt. Quin etiam casus hic simplicissimus, quo corpus initio in situ inclinato quievit, ab integratione formulae hujus $dt = \frac{dp\sqrt{(cc+ff(1-pp))}}{2\sqrt{fg(p-\mathfrak{p})(1-pp)}}$ pendet, quae neque per logarithmos neque arcus circulares absolvi potest. At si declinatio initialis Z$a$ fuerit quasi infinite parva, negotium ad arcus circulares perducitur: sit enim initio Z$a = \mathfrak{l}$, et elapso tempore $t$ declinatio ZA $= l$, ob $\mathfrak{l}$ et $l$ arcus minimos, erit $p = 1 - \frac{1}{2}ll$, $dp = -ldl$, et $\mathfrak{p} = 1 - \frac{1}{2}\mathfrak{ll}$

tum vero ſint corporis momenta inertiae $Maa$, $Mbb$, $Mcc$ reſpectu axium principalium IA, IB, IC. Nunc autem corpus gyretur circa axem IO in ſenſum ABC celeritate angulari $= u$, ſintque arcus AO $= \alpha$, BO $= \beta$, CO $= \gamma$: ac ponatur $u \cos \alpha = x$, $u \cos \beta = y$, $u \cos \gamma = z$. Quoniam igitur initio ubi $t = 0$, corpus ex quiete motum incipere aſſumitur, erat tum $x = 0$, $y = 0$, et $z = 0$. Tum vero quia motus corporis perpetuo manet tardiſſimus, quantitates $x$, $y$, $z$ ſemper manebunt minimae, ita ut binarum producta $xy$, $xz$, et $yz$ prae ſingulis pro evaneſcentibus haberi queant. Cum ergo momenta virium ſollicitantium P, Q, R, modo ſint definita ex §. 810. ſequentes adipiſcimur aequationes.

$$dx = \frac{2fg\,dt}{aa}(r \cos \eta - q \cos \theta)$$
$$dy = \frac{2fg\,dt}{bb}(p \cos \theta - r \cos \zeta)$$
$$dz = \frac{2fg\,dt}{cc}(q \cos \zeta - p \cos \eta).$$

Deinde quia eſt $\cos l = \cos \zeta + p$, $\cos m = \cos \eta + q$, et $\cos n = \cos \theta + r$ ob $\zeta$, $\eta$, $\theta$ conſtantes erit $dl \sin l = - dp$; $dm \sin m = - dq$ et $dn \sin n = - dr$, unde inſuper hae ternae aequationes accedunt

$$-dp = dt\,(y \cos \theta - z \cos \eta);$$
$$-dq = dt\,(z \cos \zeta - x \cos \theta);$$
$$-dr = dt\,(x \cos \eta - y \cos \zeta);$$

ubi producta $yr$, $zq$, $zp$, $xr$, $xq$, $yp$ ut minima prae terminis hic exhibitis omittimus.

Denique ſi arcus ZA a circulo quodam verticali fixo nunc declinare ſtatuatur angulo $\lambda$, ob $\sin l^2 = \sin \zeta^2 - 2p \cos \zeta$ habebimus hanc aequationem: $d\lambda = \frac{-dt\,(y \cos \eta + z \cos \theta)}{\sin \zeta^2 - 2p \cos \zeta}$. Quia autem in ſuperioribus aequationibus quantitates $x$, $y$, $z$ et $p$, $q$, $r$ ubique unam dimenſionem occupant, atque $x$, $y$, $z$ poſito $t = 0$ evaneſcere debent, manifeſtum eſt, tam huic conditioni, quam ſex illis aequationibus ſatisfieri poſſe ponendo:

$$x = A \sin \delta t,\ y = B \sin \delta t;\ z = C \sin \delta t;$$
$$p = D \cos \delta t;\ q = E \cos \delta t,\ r = F \cos \delta t,$$

tum enim ternae priores aequationes per $\cos \delta t$ diviſae, et ternae poſteriores per $\sin \delta t$ diviſae dabunt.

$$A\delta = \frac{2fg}{aa}(F \cos \eta - E \cos \theta);\ D\delta = B \cos \theta - C \cos \eta$$
$$B\delta = \frac{2fg}{bb}(D \cos \theta - F \cos \zeta);\ E\delta = C \cos \zeta - A \cos \theta$$



$$C\delta = \frac{2fg}{cc}(E \cos \zeta - D \cos \eta);\ F\delta = A \cos \eta - B \cos \zeta.$$

Ex poſterioribus ſubſtituantur valores coefficientium D, E, F in prioribus et obtinebimus:

$$\frac{A\delta\delta aa}{2fg} = A \cos \eta^2 - B \cos \zeta \cos \eta - C \cos \zeta \cos \theta + A \cos \theta^2$$
$$\frac{B\delta\delta bb}{2fg} = B \cos \theta^2 - C \cos \eta \cos \theta - A \cos \zeta \cos \eta + B \cos \zeta^2$$
$$\frac{C\delta\delta cc}{2fg} = C \cos \zeta^2 - A \cos \zeta \cos \theta - B \cos \eta \cos \theta + C \cos \eta^2.$$

Quodſi jam brevitatis gratia ponamus A $\cos \zeta$ + B $\cos \eta$ + C $\cos \theta$ = G, ob $\cos \zeta^2 + \cos \eta^2 + \cos \theta^2 = 1$ erit

$$A\left(1 - \frac{\delta\delta aa}{2fg}\right) = G \cos \zeta,\ B\left(1 - \frac{\delta\delta bb}{2fg}\right) = G \cos \eta$$
$$\text{et } C\left(1 - \frac{\delta\delta cc}{2fg}\right) = G \cos \theta.$$

Ponamus brevitatis cauſa $\frac{\delta\delta}{2fg} = u$; ut fiat

$$A = \frac{G \cos \zeta}{1 - aau};\ B = \frac{G \cos \eta}{1 - bbu};\ C = \frac{G \cos \theta}{1 - ccu}$$

Cum autem ſit A $\cos \zeta$ + B $\cos \eta$ + C $\cos \theta$ = G, erit

$$\frac{\cos \zeta^2}{1 - aau} + \frac{\cos \eta^2}{1 - bbu} + \frac{\cos \theta^2}{1 - ccu} = 1,$$

qua aequatione evoluta conſequimur per $u$ dividendo,

$$\left.\begin{array}{l} aabbcc\,uu - bbcc\,u \sin \zeta^2 + aa \cos \zeta^2 \\ \qquad - aacc\,u \sin \eta^2 + bb \cos \eta^2 \\ \qquad - aabb\,u \sin \theta^2 + cc \cos \theta^2 \end{array}\right\} = 0.$$

Statuantur quantitates cognitae

$$bbcc \sin \zeta^2 + aacc \sin \eta^2 + aabb \sin \theta^2 = K\,aabbcc$$
$$aa \cos \zeta^2 + bb \cos \eta^2 + cc \cos \theta^2 = L\,aabbcc$$

ut ſit $uu - Ku + L = 0$, hincque

$$u = \frac{\delta\delta}{2fg} = \tfrac{1}{2}K + \sqrt{(\tfrac{1}{4}KK - L)}$$

et quantitas G manet indefinita ex ſtatu initiali definienda; dum contra

tra quantitates K et L sunt ex natura corporis datae. Cum igitur hinc inventus sit valor ipsius $u$, inde habemus $\delta = \sqrt{2fgu}$, et

$$A = \frac{G\,cos\,\zeta}{1-aau};\ B = \frac{G\,cos\,\eta}{1-bbu};\ C = \frac{G\,cos\,\theta}{1-ccu};$$

$$D = \frac{Gu(bb-cc)\,cos\,\eta\,cos\,\theta}{\delta(1-bbu)(1-ccu)};\ E = \frac{Gu(cc-aa)\,cos\,\zeta\,cos\,\theta}{\delta(1-ccu)(1-aau)}$$

$$\text{et } F = \frac{Gu(aa-bb)\,cos\,\zeta\,cos\,\eta}{\delta(1-aau)(1-bbu)}.$$

Si jam initio fuer arcus $ZD = r$, qui nunc est $\varrho$, cum sit initio $p = D$: $q = E$; $r = F$, habebimus

$$DD + EE + FF = rr,$$

unde per r invenitur constans G. Denique pro angulo $\lambda$ inveniendo

prodit $d\lambda = \dfrac{-dt\,(B\,cos\,\eta + C\,cos\,\theta)\,fin\,\delta t}{fi\zeta^2}$, ideoque $d\lambda = \dfrac{(B\,cos\,\eta + C\,cos\,\theta)(cos\,\delta t - 1)}{\delta fi\zeta^2}$ si quidem arcus ZA initio fuerit in verticali fixo, indeque in sensum XOY moveri sumatur, quatenus ergo haec expressio pro $\lambda$ est negativa, in sensum contrarium axis IA circa Z gyrari est censendus.

Denique cum sit $pp + qq + rr = \varrho\varrho$, erit $\varrho = r\,cos\,\delta t$, ob $r = \sqrt{(DD + EE + FF)}$, unde patet axem ID in situm verticalem erigi elapso tempore $= \dfrac{\pi}{2\delta}$ et titubationes isochronas fore oscillationibus penduli, cujus longitudo est $= \dfrac{2g}{\delta\delta} = \dfrac{1}{fu} = \dfrac{K - \sqrt{(KK-4L)}}{2Lf}$.

### COROLL. 1.

904. Cum sit $DD + EE + FF = rr$, erit

$$\delta\delta\, rr = AA\,(cos\,\eta^2 + cos\,\theta^2) + BB\,(cos\,\zeta^2 + cos\,\theta^2) + CC\,(cos\,\zeta^2 + cos\,\eta^2) - 2BC\,cos\,\eta\,cos\,\theta - 2AC\,cos\,\zeta\,cos\,\theta - 2AB\,cos\,\zeta\,cos\,\eta$$

et quia $G = A\,cos\,\zeta + B\,cos\,\eta + C\,cos\,\theta$, hujus quadratum eo additum dabit

$$\delta\delta\, rr + GG = AA + BB + CC,$$

ubi



ubi si brevitatis gratia ponatur $\frac{1}{1-aau} = \mathfrak{P}$; $\frac{1}{1-bbu} = \mathfrak{Q}$; $\frac{1}{1-ccu} = \mathfrak{R}$, ob $\mathfrak{P}\,cos\,\zeta^2 + \mathfrak{Q}\,cos\,\eta^2 + \mathfrak{R}\,cos\,\theta^2 = 1$ et $A = G\mathfrak{P}\,cos\,\zeta$; $B = G\mathfrak{Q}\,cos\,\eta$ et $C = G\mathfrak{R}\,cos\,\theta$, fiet

$$\delta\delta\, rr = GG\,(\mathfrak{P}\mathfrak{P}\,cos\,\zeta^2 + \mathfrak{Q}\mathfrak{Q}\,cos\,\eta^2 + \mathfrak{R}\mathfrak{R}\,cos\,\theta^2 - 1)$$

ideoque ob $\mathfrak{P}\mathfrak{P} - \mathfrak{P} = \frac{aau}{(1-aau)^2}$ habebitur

$$\delta\delta\, rr = GGu\left(\frac{aa\,cos\,\zeta^2}{(1-aau)^2} + \frac{bb\,cos\,\eta^2}{(1-bbu)^2} + \frac{cc\,cos\,\theta^2}{(1-ccu)^2}\right).$$

### COROLL. 2.

905. Quia porro est $\delta\delta = 2fgu$, si in subsidium vocetur aequatio $uu - Ku + L = 0$, reperietur

$$GG = \frac{-2fgrr(1-aau)(1-bbu)(1-ccu)}{aa\,cos\,\zeta^2 + bb\,cos\,\eta^2 + cc\,cos\,\theta^2 - aabbccu}.$$

### EXPLICATIO.

906. Haec expressio pro GG satis concinna sequenti modo eruitur: Posito brevitatis gratia $\frac{1}{aa} = \mathfrak{a}$, $\frac{1}{bb} = \mathfrak{b}$, $\frac{1}{cc} = \mathfrak{c}$, habemus:

I. $K = \mathfrak{a} + \mathfrak{b} + \mathfrak{c} - \mathfrak{a}\,cos\,\zeta^2 - \mathfrak{b}\,cos\,\eta^2 - \mathfrak{c}\,cos\,\theta^2$

II. $L = \mathfrak{b}\mathfrak{c}\,cos\,\zeta^2 + \mathfrak{a}\mathfrak{c}\,cos\,\eta^2 + \mathfrak{a}\mathfrak{b}\,cos\,\theta^2$

III. $1 = cos\,\zeta^2 + cos\,\eta^2 + cos\,\theta^2$.

Hinc deducitur ob $uu - Ku + L = 0$

$$cos\,\zeta^2 = \frac{\mathfrak{a}K - L - \mathfrak{a}\mathfrak{a}}{(\mathfrak{a}-\mathfrak{b})(\mathfrak{c}-\mathfrak{a})},\ \text{et } u\,cos\,\zeta^2 = \frac{(\mathfrak{a}-u)(L-\mathfrak{a}u)}{(\mathfrak{a}-\mathfrak{b})(\mathfrak{c}-\mathfrak{a})}$$

ideoque

$$\frac{\delta\delta rr}{GG} = \frac{\mathfrak{a}(L-\mathfrak{a}u)}{(\mathfrak{a}-\mathfrak{b})(\mathfrak{c}-\mathfrak{a})(\mathfrak{a}-u)} + \frac{\mathfrak{b}(L-\mathfrak{c}u)}{(\mathfrak{b}-\mathfrak{c})(\mathfrak{a}-\mathfrak{b})(\mathfrak{b}-u)} + \frac{\mathfrak{c}(L-\mathfrak{c}u)}{(\mathfrak{c}-\mathfrak{a})(\mathfrak{b}-\mathfrak{c})(\mathfrak{c}-u)}$$

ex qua aequatione reducta illa expressio obtinetur.

### SCHOLION.

907. Quoniam haec ad titubationes omnium corporum, quorum basis est portio sphaerica, patent, quomodocunque ejus axes principales ratione axis naturalis DOIF fuerint dispositi, eorumque respectu momenta inertiae inaequalia, ne in tanta amplitudine confundamur,



con-

conveniet primo formulas nostras ad species corporum simpliciores accommodari, quo inde facilius ad species magis complicatas progredi liceat. Ac primo quidem casus, quo omnia momenta inertiae sunt inter se aequalia, seu $aa = bb = cc$, omnium est simplicissimus, quia tum etiam axis DF pro principali haberi potest, et titubationes eaedem prodire debent, quas jam ante definivimus. Tum vero duo saltem momenta inertiae aequalia statuamus, scilicet $bb = cc$.

## CASUS. I.

quo $aa = bb = cc$.

908. Hoc ergo casu habemus:

$$A = \frac{G\cos\zeta}{1-aau};\ B = \frac{G\cos\eta}{1-aau};\ C = \frac{G\cos\theta}{1-aau},$$

hincque $G = \dfrac{G\cos\zeta^2 + G\cos\eta^2 + G\cos\theta^2}{1-aau} = \dfrac{G}{1-aau}$, ita ut sit $u = 0$.

Verum iisdem quoque formulis satisfit ponendo $u = \frac{1}{aa}$ et $G = 0$, ut sit $A\cos\zeta + B\cos\eta + C\cos\theta = 0$, neque quicquam praeterea determinetur, sicque habebimus $\delta = \frac{\Gamma 2fg}{a}$; tum vero

$$D = \frac{B\cos\theta - C\cos\eta}{\delta};\ E = \frac{C\cos\zeta - A\cos\theta}{\delta};\ F = \frac{A\cos\eta - B\cos\zeta}{\delta}$$

atque $\delta\delta rr = AA + BB + CC$: ut sit $\varrho = \mathfrak{r}\cos\delta t$. Videamus jam circa quemnam polum O corpus sit gyraturum, ac primo habemus $\cos OD = \cos\alpha\cos\zeta + \cos\beta\cos\eta + \cos\gamma\cos\theta$, seu $z\cos OD = x\cos\zeta + y\cos\eta + z\cos\theta = 0$, sicque arcus OD quadrans. Deinde est $\cos OZ = \cos\alpha\cos l + \cos\beta\cos m + \cos\gamma\cos n$ seu

$$z\cos OZ = x\cos l + y\cos m + z\cos n = 0 + px + qy + rz = 0$$

ob $AD + BE + CF = 0$, eritque ergo etiam OZ quadrans. Ex quo perspicitur corpus circa punctum O, quod est polus circuli verticalis ZDX gyrari, sicque axem ex D recta in situm verticalem Z erigi, ita ut elapso tempore $t$ sit $\varrho = \mathfrak{r}\cos\frac{t\Gamma 2fg}{a}$. Quare hae titubationes isochronae erunt oscillationibus penduli, cujus longitudo est $= \frac{aa}{f}$.

CA-



## CASUS. II.

quo duo tantum momenta principalia sunt aequalia seu $bb = cc$.

909. Hoc ergo casu est $K = \dfrac{cc\sin\zeta^2 + aa\sin\eta^2 + aa\sin\theta^2}{aacc} = \dfrac{cc\sin\zeta^2 + aa + aa\cos\zeta^2}{aacc}$, et $L = \dfrac{aa\cos\zeta^2 + cc\sin\zeta^2}{aac^4}$: sive cum aequatio, unde $u$ definiri debet, sit $\dfrac{\cos\zeta^2}{1-aau} + \dfrac{\sin\zeta^2}{1-ccu} = 1$, erit

$$aa\cos\zeta^2 + cc\sin\zeta^2 = aacc\,u, \text{ ideoque } u = \frac{aa\cos\zeta^2 + cc\sin\zeta^2}{aacc}$$

qui valor etiam ex generali forma elicitur, nisi quod hoc modo radix inutilis $u = \frac{1}{cc}$ excluditur. Quamobrem habebimus

$$\delta = \frac{\Gamma 2fg(aa\cos\zeta^2 + cc\sin\zeta^2)}{ac},\ \text{tum vero}$$

$$A = \frac{Gcc}{(aa+cc)\cos\zeta};\ B = \frac{-Gaa\cos\eta}{(aa-cc)\sin\zeta^2};\ C = \frac{Gaa\cos\theta}{(aa-cc)\sin\zeta^2}.$$

Deinde pro G ex $\varrho$ inveniendo fit

$$\delta\delta rr + GG = \frac{GG(a^4\cos\zeta^2 + c^4\sin\zeta^2)}{(aa-cc)^2\sin\zeta^2\cos\zeta^2} \text{ sive}$$

$$\delta\delta rr = \frac{GG(aa\cos\zeta^2 + cc\sin\zeta^2)^2}{(aa-cc)^2\sin\zeta^2\cos\zeta^2},\ \text{et } G = \frac{(aa-cc)\delta\mathfrak{r}\sin\zeta\cos\zeta}{aa\cos\zeta^2 + cc\sin\zeta^2},$$

$$\text{vel } G = \frac{(aa-cc)\mathfrak{r}\sin\zeta\cos\zeta\,\Gamma 2fg}{ac\Gamma(aa\cos\zeta^2 + cc\sin\zeta^2)}.\ \text{Deinde vero obtinemus}$$

$$D = 0,\ E = \frac{\mathfrak{r}\cos\theta}{\sin\zeta};\ F = \frac{-\mathfrak{r}\cos\eta}{\sin\zeta};\ \text{atque}$$

$$A = \frac{-c\mathfrak{r}\sin\zeta\,\Gamma 2fg}{a\Gamma(aa\cos\zeta^2 + cc\sin\zeta^2)};\ B = \frac{a\mathfrak{r}\cos\zeta\cos\eta\,\Gamma 2fg}{c\sin\zeta\,\Gamma(aa\cos\zeta^2 + cc\sin\zeta^2)};$$

$$C = \frac{a\mathfrak{r}\cos\zeta\cos\theta\,\Gamma 2fg}{c\sin\zeta\,\Gamma(aa\cos\zeta^2 + cc\sin\zeta^2)}$$

ex quibus consequimur

$x = u \cos \alpha = A \sin \partial t$; $y = u \cos \beta = B \sin \partial t$; $z = u \cos \gamma = C \sin \partial t$
$p = \cos l - \cos \zeta = D \cos \partial t$; $q = \cos m - \cos \eta = E \cos \partial t$; $r = \cos n - \cos \theta = F \cos \partial t$;

atque $\lambda = \frac{-a r \sin \zeta \cos \zeta (1 - \cos \partial t) \sqrt{2fg}}{\partial c \sqrt{(aa \cos \zeta^2 + cc \sin \zeta^2)}} = \frac{-aar \sin \zeta \cos \zeta (1 - \cos \partial t)}{aa \cos \zeta^2 + cc \sin \zeta^2}$

estque $\lambda$ angulus VZA, existente ZV circulo verticali fixo, a quo declinationem poli A computamus. Deinde vero est $\varrho = r \cos \partial t$, et ut obtineamus angulum DZV, quaeramus angulum DZA, ex formula cos

$$DZA = \frac{\cos \zeta - \cos l \cos \varrho}{\varrho \sin l} = \frac{\cos \zeta - \cos \zeta \cos \varrho - p \cos \varrho}{\varrho \sin \zeta} = \tfrac{1}{2} \varrho \frac{\cos \zeta}{\sin \zeta},$$

ob $D = 0$ ideoque $p = 0$, ergo $\cos DZA = \frac{r \cos \zeta \cos \partial t}{2 \sin \zeta}$, qui cum sit infinite parvus, patet angulum DZA esse rectum proxime et angulo ZDA aequalem. Quare cum initio angulus ZDA fuerit non rectus, haec solutio, quippe quae manifesto tantum est particularis, eo non extenditur. Ceterum vero et hae titubationes erunt isochronae oscillationibus penduli, cujus longitudo est $= \frac{[illegible]acc}{f(aa \cos \zeta^2 + cc \sin \zeta^2)}$.

Denique cum sit $u = \sin \partial t . \sqrt{(AA + BB + CC)}$, prodibit

$$u = \frac{r \sqrt{2fg (a^4 \cos \zeta^2 + c^4 \sin \zeta^2)}}{ac \sqrt{(aa \cos \zeta^2 + cc \sin \zeta^2)}} . \sin \partial t.$$

Pro polo autem gyrationis O invenimus:

$u \cos OD = (A \cos \zeta + B \cos \eta + C \cos \theta) \sin \partial t = G \sin \partial t$ et
$u \cos OZ = (A \cos l + B \cos m + C \cos n) \sin \partial t = G \sin \partial t$.

ita ut sit $OD = OZ$, ob $Ap + Bq + Cr = 0$.

## SCHOLION.

910. Mirum non est, hanc solutionem non esse generalem, cum enim ex data indole corporis, quantitatibus $aa$, $bb$, $cc$ et angulis $\zeta$, $\eta$, $\theta$ comprehensa, et ex situ axis DF initiali ejusve declinatione $r$ a situ verticali omnes coefficientes A, B, C, D, E, F cum numero $\partial$ determinentur, ex iis angulus ADZ, quo arcus DA initio ab arcu DZ deviabat, sponte determinatur, neque amplius arbitrio nostro, uti natura rei postulat, relinquitur. Verum cum in genere pro quantitate $u$ geminum valorem elicuerimus, quorum neutrum prae altero rejicere fas est, si utrumque simul adhibeamus, solutionem ampliorem obtinebimus, unde simul effici potest, ut angulus ADZ initio fuerit dato angulo aequalis. Cum enim in aequationibus differentialibus quantitates $x$, $y$, $z$, et $p$, $q$, $r$ ubique unicam habeant dimensionem, si iis duplici modo satisfieri queat, pro qualibet quantitate summa binorum ejus valorum statui poterit, hincque solutionem generalem impetrabimus, quam hic exponamus.



## PROBLEMA. 109.

911. Si corpus basi sphaerica praeditum de situ aequilibrii quomodocunque infinite parum declinetur, subitoque dimittatur, definire motum titubatorium, quo agitabitur.

## SOLUTIO.

Retentis denominationibus superioris problematis, quoniam posito

$$\frac{\sin \zeta^2}{aa} + \frac{\sin \eta^2}{bb} + \frac{\sin \theta^2}{cc} = K \text{ et } \frac{\cos \zeta^2}{bbcc} + \frac{\cos \eta^2}{aacc} + \frac{\cos \theta^2}{aabb} = L$$

pro $u$ geminum invenimus valorem, sint ii

$u = \tfrac{1}{2} K + \sqrt{(\tfrac{1}{4} KK - L)}$ et $u' = \tfrac{1}{2} K - \sqrt{(\tfrac{1}{4} KK - L)}$

unde pro $\partial$ etiam binos adipiscimur valores, qui sint $\partial = \sqrt{2fgu}$ et $\partial' = \sqrt{2fgu'}$

atque hinc pro senis quantitatibus $x$, $y$, $z$ et $p$, $q$, $r$ sequentes impetrabimus valores

$$x = u \cos \alpha = \frac{G \cos \zeta \sin \partial t}{1 - aau} + \frac{H \cos \zeta \sin \partial' t}{1 - aau'}$$

$$y = u \cos \beta = \frac{G \cos \eta \sin \partial t}{1 - bbu} + \frac{H \cos \eta \sin \partial' t}{1 - bbu'}$$

$$z = u \cos \gamma = \frac{G \cos \theta \sin \partial t}{1 - ccu} + \frac{H \cos \theta \sin \partial' t}{1 - ccu'}$$

tum vero porro

$$p = \cos l - \cos \zeta = \frac{Gu(bb - cc) \cos \eta \cos \theta \cos \partial t}{\partial (1 - bbu)(1 - ccu)} + \frac{Hu'(bb - cc) \cos \eta \cos \theta \cos \partial' t}{\partial' (1 - bbu')(1 - ccu')}$$

 $q =$

$$q = \cos m - \cos\eta = \frac{Gu(cc-aa)\cos\zeta\cos\theta\cos\partial t}{\partial(1-ccu)(1-aau)} + \frac{Hu'(cc-aa)\cos\zeta\cos\theta\cos\partial' t}{\partial'(1-ccu')(1-aau')}$$

$$r = \cos n - \cos\theta = \frac{Gu(aa-bb)\cos\zeta\cos\eta\cos\partial t}{\partial(1-aau)(1-bbu)} + \frac{Hu'(aa-bb)\cos\zeta\cos\eta\cos\partial' t}{\partial'(1-aau')(1-bbu')}.$$

Hic jam habemus binas quantitates constantes arbitrarias G et H, atque hi valores ita satisfaciunt, ut facta substitutione in aequationibus differentialibus termini tam per G quam per H affecti seorsim se destruant. Verum si initio arcus ZD fuerit $= \mathfrak{r}$, posito $t = 0$, fieri debet $pp + qq + rr = \mathfrak{r}\mathfrak{r}$. Deinde vero si initio fuerit angulus ZDA $= \mathfrak{f}$, ob $\cos\mathfrak{f} = \frac{\cos l - \cos\zeta\cos\mathfrak{r}}{\sin\zeta\sin\mathfrak{r}} = \frac{\cos\zeta(1-\cos\mathfrak{r})+p}{\sin\zeta\sin\mathfrak{r}} = \frac{\mathfrak{r}\cos\zeta}{\sin\zeta} + \frac{p}{\mathfrak{r}\sin\zeta}$ et ob $\mathfrak{r}$ infinite parvum, erit $p = \mathfrak{r}\sin\zeta\cos\mathfrak{f}$. Si hic ergo pro $p$ ejus valor superior posito $t = 0$ substituatur, habebitur alia aequatio ex qua cum illa conjuncta binae constantes G et H determinabuntur. At posito angulo VZA $= \lambda$ erit $d\lambda = \frac{-dt(y\cos\eta + z\cos\theta)}{\sin\zeta^2}$, cujus integrale facile exhibetur. Simili autem modo positis angulis VZB $= \mu$ et VZC $= \nu$, erit

$$d\mu = \frac{-dt(z\cos\theta + x\cos\zeta)}{\sin\eta^2}; \text{ et } d\nu = \frac{-dt(x\cos\zeta + y\cos\eta)}{\sin\theta^2}.$$

Hic autem notari convenit, si sit $bb = cc$, fore binos valores $u = \frac{1}{cc}$ et $u' = \frac{\sin\zeta^2}{aa} + \frac{\cos\zeta^2}{cc}$, ideoque pro priore fractionum superiorum quasdam numeratores ac denominatores simul evanescere. Ad earum ergo valores investigandos ponatur $\frac{1}{bb} = \frac{1}{cc} + \omega$, existente $\omega$ quantitate evanescente, reperieturque

$$u = \frac{1}{cc} + \frac{\omega\cos\theta^2}{\sin\zeta^2} \text{ et } u' = \frac{\sin\zeta^2}{aa} + \frac{\cos\zeta^2}{cc}; \text{ hincque si}$$

$\frac{G}{\omega}$



$\frac{G}{\omega}$ ponatur $= I$, ut sit $G = I\omega = 0$, fiet

$$x = u\cos\alpha = \frac{H\cos\zeta\sin\partial' t}{1-aau'} = \frac{-Hcc\sin\partial' t}{(aa-cc)\cos\zeta}$$

$$y = u\cos\beta = \frac{I\sin\zeta^2\sin\partial t}{cc\cos\eta} + \frac{H\cos\eta\sin\partial' t}{1-ccu'} = \frac{I\sin\zeta^2\sin\partial t}{cc\cos\eta} + \frac{Haa\cos\eta\sin\partial' t}{(aa-cc)\sin\zeta^2}$$

$$z = u\cos\gamma = \frac{-I\sin\zeta^2\sin\partial t}{cc\cos\theta} + \frac{H\cos\theta\sin\partial' t}{1-ccu'} = \frac{-I\sin\zeta^2\sin\partial t}{cc\cos\theta} + \frac{Haa\cos\theta\sin\partial' t}{(aa-cc)\sin\zeta^2}$$

deinde vero

$$p = \cos l - \cos\zeta = \frac{I\sin\zeta^2\cos\partial t}{\partial cc\cos\eta\cos\theta}$$

$$q = \cos m - \cos\eta = \frac{-I\sin\zeta^2\cos\zeta\cos\partial t}{\partial cc\cos\theta} - \frac{Hu'(aa-cc)\cos\zeta\cos\theta\cos\partial' t}{\partial'(1-aau')(1-ccu')}$$

$$r = \cos n - \cos\theta = \frac{-I\sin\zeta^2\cos\zeta\cos\partial t}{\partial cc\cos\eta} + \frac{Hu'(aa-cc)\cos\zeta\cos\eta\cos\partial' t}{\partial'(1-aau')(1-ccu')}$$

ubi est $\frac{aa-cc}{(1-aau')(1-ccu')} = \frac{-aacc}{(aa-cc)\sin\zeta^2\cos\zeta^2}$.

Vel si ponamus $I = \frac{\mathfrak{G}\partial cc\cos\eta\cos\theta}{\sin\zeta^2}$ et $H = \frac{\mathfrak{H}\partial'(aa-cc)\sin\zeta^2\cos\zeta}{aacc}$ ob $u = \frac{1}{cc}$ et $u' = \frac{aa\cos\zeta^2 + cc\sin\zeta^2}{aacc}$, ideoque $\partial = \frac{\sqrt{2fg}}{c}$ et $\partial' = \frac{\sqrt{2fg(aa\cos\zeta^2 + cc\sin\zeta^2)}}{ac}$,

$$\text{erit } x = u\cos\alpha = \frac{-\mathfrak{H}\partial'\sin\zeta^2\sin\partial' t}{ac}$$

$y =$

$$y = u\cos\beta = \frac{\mathfrak{H}\,\partial'\cos\zeta\cos\eta\sin\partial' t}{cc} + \mathfrak{G}\,\partial\cos\theta\sin\partial t$$

$$z = u\cos\gamma = \frac{\mathfrak{H}\,\partial'\cos\zeta\cos\theta\sin\partial' t}{cc} - \mathfrak{G}\,\partial\cos\eta\sin\partial t$$

$$p = \cos l - \cos\zeta = \mathfrak{G}\sin\zeta^2\cos\partial t$$
$$q = \cos m - \cos\eta = -\mathfrak{G}\cos\zeta\cos\eta\cos\partial t + \mathfrak{H}u'\cos\theta\cos\partial' t$$
$$r = \cos n - \cos\eta = -\mathfrak{G}\cos\zeta\cos\theta\cos\partial t - \mathfrak{H}u'\cos\eta\cos\partial' t$$

quae formulae jam fine ulla difficultate ad omnes casus accommodari possunt.

### COROLL. 1.

912. Haec integralia adhuc latius extendi possunt, cum $x$, $y$, $z$ et $p$, $q$, $r$ partes constantes recipiant; ac forma litterarum G et H mutata, habebimus:

$$x = \cos\zeta\,(\mathfrak{E} + \mathfrak{G}(1 - bbu)(1 - ccu)\sin\partial t + \mathfrak{H}(1 - bbu')(1 - ccu')\sin\partial' t)$$
$$y = \cos\eta\,(\mathfrak{E} + \mathfrak{G}(1 - aau)(1 - ccu)\sin\partial t + \mathfrak{H}(1 - aau')(1 - ccu')\sin\partial' t)$$
$$z = \cos\theta\,(\mathfrak{E} + \mathfrak{G}(1 - aau)(1 - bbu)\sin\partial t + \mathfrak{H}(1 - aau')(1 - bbu')\sin\partial' t)$$

atque

$$p = \mathfrak{F}\cos\zeta + (bb - cc)\cos\eta\cos\theta\left(\frac{\mathfrak{G}u(1 - aau)\cos\partial t}{\partial} + \frac{\mathfrak{H}u'(1 - aau')\cos\partial' t}{\partial'}\right)$$

$$q = \mathfrak{F}\cos\eta + (cc - aa)\cos\zeta\cos\theta\left(\frac{\mathfrak{G}u(1 - bbu)\cos\partial t}{\partial} + \frac{\mathfrak{H}u'(1 - bbu')\cos\partial' t}{\partial'}\right)$$

$$r = \mathfrak{F}\cos\theta + (aa - bb)\cos\zeta\cos\eta\left(\frac{\mathfrak{G}u(1 - ccu)\cos\partial t}{\partial} + \frac{\mathfrak{H}u'(1 - ccu')\cos\partial' t}{\partial'}\right).$$





### COROLL. 2.

913. Angulorum etiam $\partial t$ et $\partial' t$ uterque quantitate constante augeri potest, ac si eorum loco scribamus $\partial t + \mathfrak{g}$ et $\partial' t + \mathfrak{h}$, integralia continebunt sex constantes arbitrarias $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, ideoque erunt integralia completa harum sex aequationum differentialium:

$$aadx = 2fgdt\,(r\cos\eta - q\cos\theta);\; dp = dt\,(z\cos\eta - y\cos\theta)$$
$$bbdy = 2fgdt\,(p\cos\theta - r\cos\zeta);\; dq = dt\,(x\cos\theta - z\cos\zeta)$$
$$ccdz = 2fgdt\,(q\cos\zeta - p\cos\eta);\; dr = dt\,(y\cos\zeta - x\cos\eta).$$

### COROLL. 3.

914. Si corpus initio quieverit, ut in problemate assumsimus, ita ut tum fuerit $x = 0$, $y = 0$, $z = 0$; poni debet $\mathfrak{E} = 0$, $\mathfrak{g} = 0$, et $\mathfrak{h} = 0$, reliquas autem constantes ex situ corporis initiali definiri oportet

### COROLL. 4.

915. Nempe si pro initio, quo $t = 0$, ponantur anguli ZDA $=$ l, ZDB $=$ m, et ZDC $=$ n; ut sit

$$\sin(l - m) = \frac{-\cos\theta}{\sin\zeta\sin\eta};\; \sin(m - n) = \frac{-\cos\zeta}{\sin\eta\sin\theta};\; \sin(n - l) = \frac{-\cos\eta}{\sin\zeta\sin\theta}$$

$$\cos(l - m) = \frac{-\cos\zeta\cos\eta}{\sin\zeta\sin\eta};\; \cos(m - n) = \frac{-\cos\eta\cos\theta}{\sin\eta\sin\theta};$$
$$\cos(n - l) = \frac{-\cos\zeta\cos\theta}{\sin\zeta\sin\theta}$$

pro initio $t = 0$, constantes ita definiri oportet, ut si tum fuerit ZD $=$ r, fiat

$$p = r\sin\zeta\cos l;\; q = r\sin\eta\cos m;\; r = r\sin\theta\cos n.$$

### EXPLICATIO.

916. Ad constantes $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ in genere ex statu initiali modo descripto definiendas, ponamus brevitatis gratia

$$aa\cos\zeta^2 + bb\cos\eta^2 + cc\cos\theta^2 = \mathfrak{A}$$
$$bbcc\cos\zeta^2 + aacc\cos\eta^2 + aabb\cos\theta^2 = \mathfrak{B}$$

 fitque

ſitque $\frac{\mathfrak{G} u \cos a}{\delta} = X$, et $\frac{\mathfrak{H} u' \cos \mathfrak{h}}{\delta'} = Y$, quo calculus facilius expediatur. Eo autem abſoluto reperietur

$$\mathfrak{F} = r \sin\zeta \cos\zeta \cos \mathfrak{l} + r \sin\eta \cos\eta \cos \mathfrak{m} + r \sin\theta \cos\theta \cos \mathfrak{n}$$

$$X + Y = \frac{\frac{+r\sin\zeta \cos\mathfrak{l}}{\cos\eta\cos\theta}(\mathfrak{B} - bbcc) + \frac{r\sin\eta\cos\mathfrak{m}}{\cos\zeta\cos\theta}(\mathfrak{B} - aacc) + \frac{r\sin\theta\cos\mathfrak{n}}{\cos\zeta\cos\eta}(\mathfrak{B} - aabb)}{(bb-cc)(cc-aa)(aa-bb)}$$

$$uX + u'Y = \frac{\frac{-r\sin\zeta\cos\mathfrak{l}}{\cos\eta\cos\theta}(\mathfrak{A} - aa) - \frac{r\sin\eta\cos\mathfrak{m}}{\cos\zeta\cos\theta}(\mathfrak{A} - bb) - \frac{r\sin\theta\cos\mathfrak{n}}{\cos\zeta\cos\theta}(\mathfrak{A} - cc)}{(bb-cc)(cc-aa)(aa-bb)}.$$

Ex his autem valoribus $\mathfrak{A}$ et $\mathfrak{B}$ eſt pro ſuperioribus

$$L = \frac{\mathfrak{A}}{aabbcc} \text{ et } K = \frac{aabb + aacc + bbcc - \mathfrak{B}}{aabbcc}$$

ex quibus fit

$$u = \tfrac{1}{2}K - \sqrt{(\tfrac{1}{4}KK - L)} \text{ et } u' = \tfrac{1}{2}K + \sqrt{(\tfrac{1}{4}KK - L)},$$

ita ut ſit $u + u' = K$ et $u' - u = \sqrt{(KK - 4L)}$.

Haec analyſis in genere valet, etiamſi corpori initio motus fuerit impreſſus, quoniam loco angulorum $\delta t$ et $\delta' t$ hic adhibuimus $\delta t + \mathfrak{g}$ et $\delta' t + \mathfrak{h}$. Simili modo, quo hic ex ſitu initiali conſtantes $\mathfrak{F}$, $\mathfrak{G}$ et $\mathfrak{H}$ definivimus, ex motu initio impreſſo, quantitates $x$, $y$, $z$ datos obtinebunt valores, quibus ſi formulae coroll. traditae et pro $\delta t$ et $\delta' t$ ſcribendo $\delta t + \mathfrak{g}$ et $\delta' t + \mathfrak{h}$ extenſae, poſito $t = 0$ aequentur, determinabuntur reliquae conſtantes $\mathfrak{E}$, $\mathfrak{g}$ et $\mathfrak{h}$: quae quidem, uti jam ante notavimus evaneſcunt, ſi motus a quiete incipiat.

### SCHOLION.

917. Pro caſu ergo ejusmodi corporum pro quibus eſt $bb = cc$, erit $u = \frac{1}{cc}$, $u' = \frac{aa\cos\zeta^2 + cc\sin\zeta^2}{aacc}$, atque $\delta = \sqrt{2fgu}$, $\delta' = \sqrt{2fgu'}$, integralia in genere ita ſe habebunt:

$$x = \mathfrak{E}\cos\zeta \quad ; \quad \frac{\mathfrak{H}\delta\sin\zeta^2\sin(\delta' t + \mathfrak{h})}{aa}$$

$$y = \mathfrak{E}\cos\eta + \mathfrak{G}\delta\cos\theta\sin(\delta t + \mathfrak{g}) + \frac{\mathfrak{H}\delta'\cos\zeta\cos\eta\sin(\delta' t + \mathfrak{h})}{cc}$$

$z =$



$$z = \mathfrak{E}\cos\theta - \mathfrak{G}\delta\cos\eta\sin(\delta t + \mathfrak{g}) + \frac{\mathfrak{H}\delta'\cos\zeta\cos\theta\sin(\delta' t + \mathfrak{h})}{cc}$$

atque

$$p = \mathfrak{F}\cos\zeta + \mathfrak{G}\sin\zeta^2\cos(\delta t + \mathfrak{g})$$
$$q = \mathfrak{F}\cos\eta - \mathfrak{G}\cos\zeta\cos\eta\cos(\delta t + \mathfrak{g}) + \mathfrak{H}u'\cos\theta\cos(\delta' t + \mathfrak{h})$$
$$r = \mathfrak{F}\cos\theta - \mathfrak{G}\cos\zeta\cos\theta\cos(\delta t + \mathfrak{g}) - \mathfrak{H}u'\cos\eta\cos(\delta' t + \mathfrak{h}).$$

Quare ſi initio $t = 0$ fuerit

$$p = r\sin\zeta\cos\mathfrak{l};\ q = r\sin\eta\cos\mathfrak{m},\ r = r\sin\theta\cos\mathfrak{n}$$

reperitur

$$\mathfrak{H} = \frac{r\sin\eta\cos\theta\cos\mathfrak{m} - r\cos\eta\sin\theta\cos\mathfrak{n}}{u'\sin\zeta^2\cos\mathfrak{h}}$$

$$\mathfrak{G} = \frac{r\sin\zeta^2\cos\mathfrak{l} - r\sin\eta\cos\zeta\cos\eta\cos\mathfrak{m} - r\sin\theta\cos\zeta\cos\theta\cos\mathfrak{n}}{\sin\zeta^2\cos\mathfrak{g}}$$

$$\mathfrak{F} = r\sin\zeta\cos\zeta\cos\mathfrak{l} + r\sin\eta\cos\eta\cos\mathfrak{m} + r\sin\theta\cos\theta\cos\mathfrak{n}.$$

At datis angulis $\mathfrak{l}$, $\mathfrak{m}$, $\mathfrak{n}$, ſimul dantur $\zeta$, $\eta$, $\theta$

$$\cos\zeta^2 = \frac{\cos(\mathfrak{l}-\mathfrak{m})\cos(\mathfrak{n}-\mathfrak{l})}{\sin(\mathfrak{l}-\mathfrak{m})\sin(\mathfrak{n}-\mathfrak{l})};\ \cos\eta^2 = \frac{\cos(\mathfrak{m}-\mathfrak{n})\cos(\mathfrak{l}-\mathfrak{m})}{\sin(\mathfrak{m}-\mathfrak{n})\sin(\mathfrak{l}-\mathfrak{m})};$$

$$\cos\theta^2 = \frac{\cos(\mathfrak{n}-\mathfrak{l})\cos(\mathfrak{m}-\mathfrak{n})}{\sin(\mathfrak{n}-\mathfrak{l})\sin(\mathfrak{m}-\mathfrak{n})}.$$

$$\sin\zeta^2 = \frac{-\cos(\mathfrak{m}-\mathfrak{n})}{\sin(\mathfrak{l}-\mathfrak{m})\sin(\mathfrak{n}-\mathfrak{l})};\ \sin\eta^2 = \frac{-\cos(\mathfrak{n}-\mathfrak{l})}{\sin(\mathfrak{m}-\mathfrak{n})\sin(\mathfrak{l}-\mathfrak{m})},$$

$$\sin\theta^2 = \frac{-\cos(\mathfrak{l}-\mathfrak{m})}{\sin(\mathfrak{n}-\mathfrak{l})\sin(\mathfrak{m}-\mathfrak{n})}.$$

Ex his autem formulis colligitur, eſſe

$$\sin\zeta\cos\zeta\cos\mathfrak{l} + \sin\eta\cos\eta\cos\mathfrak{m} + \sin\theta\cos\theta\cos\mathfrak{n} = 0,$$

ita ut conſtans ſupra definita $\mathfrak{F}$ ſemper ſit $= 0$. Simili vero modo eſt

$$\frac{\sin\zeta\cos\mathfrak{l}}{\cos\eta\cos\theta} - \frac{\sin\eta\cos\mathfrak{m}}{\cos\zeta\cos\theta} + \frac{\sin\theta\cos\mathfrak{n}}{\cos\zeta\cos\eta} = 0$$

unde valores coëfficientium ſupra definiti multo ſimplicius determinantur, ita ut litterae $\mathfrak{A}$ et $\mathfrak{B}$ ex illis prorſus elidantur. Valent haec in genere etiamſi non ſit $bb = cc$.

### PROBLEMA. 110.

918. Si corpus baſi ſphaerica praeditum habeat duos axes principales pares, eique cum de ſitu quietis infinite parum fuerit declina-



(tum)

tum, motus minimus quicunque fuerit impressus, definire motus continuationem.

## SOLUTIO.

Fig. 116. Sit ID axis corporis aequilibrii per centrum inertiae I et centrum basis G transiens, sitque hoc illo altius situm existente intervallo GI $= f$. Sit porro IA axis corporis singularis principalis ejusque, respectu momentum inertiae $= Maa$, respectu axium omnium autem ad hunc normalium $= Mcc$, quos omnes cum aeque pro principalibus habere liceat, sumatur alter IB in arcu DA producto, eritque alter IC, ut quadrans AC sit ad AD normalis, ideoque DC etiam quadrans ad AD normalis. Posito ergo DA $= \zeta$, erit DB $= \eta = \zeta + 90°$ et DC $= \theta$ $= 90°$. Initio autem quo $t = 0$, fuerit arcus DZ $= \mathrm{r}$, et angulus ZDA $= \mathrm{l}$, erit ZDB $= \mathrm{m} = \mathrm{l}$ et ZDC $= \mathrm{n} = \mathrm{l} + 90°$. Ex formulis ergo praecedentibus habebimus

$$u = \frac{1}{cc};\ u' = \frac{aa\cos\zeta^2 + cc\sin\zeta^2}{aacc};\ \delta = \frac{\mathrm{r}\sqrt{2fg}}{c};\ \gamma = \frac{\mathrm{r}\sqrt{2fg}\,(aa\cos\zeta^2 + cc\sin\zeta^2)}{ac}$$

tum vero ex hoc situ initiali fiet primo $\mathfrak{F} = 0$, tum vero ergo

$$\mathrm{r}\sin\zeta\cos\mathrm{l} = \mathfrak{G}\sin\zeta^2\cos\mathfrak{g};$$

$$\mathrm{r}\cos\zeta\cos\mathrm{l} = \mathfrak{G}\sin\zeta\cos\zeta\cos\mathfrak{g};\quad \mathfrak{G} = \frac{\mathrm{r}\cos\mathrm{l}}{\sin\zeta\cos\mathfrak{g}}$$

$$-\mathrm{r}\sin\mathrm{l} = \mathfrak{H}\,u'\sin\zeta\cos\mathfrak{h};\quad \mathfrak{H} = \frac{-\mathrm{r}\sin\mathrm{l}}{u'\sin\zeta\cos\mathfrak{h}}.$$

Deinde initio corpori motus sit impressus circa axem IO celeritate angulari $= \varepsilon$ in sensum ABC, existente OA $= \mathfrak{a}$, OB $= \mathfrak{b}$, OC $= \mathfrak{c}$, fierique debet

$$\varepsilon\cos\mathfrak{a} = \mathfrak{E}\cos\zeta - \frac{\mathfrak{H}\delta\sin\zeta^2\sin\mathfrak{h}}{aa}$$

$$\varepsilon\cos\mathfrak{b} = -\mathfrak{E}\sin\zeta - \frac{\mathfrak{H}\delta'\sin\zeta\cos\zeta\sin\mathfrak{h}}{cc}.$$

$$\varepsilon\cos\mathfrak{c} = \mathfrak{G}\,\delta\sin\zeta\sin\mathfrak{g}$$

unde concludimus

$$\varepsilon\,(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\sin\zeta) = \mathfrak{E}\,(aa\cos\zeta^2 + cc\sin\zeta^2)$$

$$\text{et } \varepsilon\,(\cos\mathfrak{a}\sin\zeta + \cos\mathfrak{b}\cos\zeta) = -\mathfrak{H}\,\delta'\sin\zeta\sin\mathfrak{h}\left(\frac{\sin\zeta^2}{aa} + \frac{\cos\zeta^2}{cc}\right)$$

Ergo



Ergo $$\mathfrak{E} = \frac{\varepsilon\,(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\sin\zeta)}{aa\cos\zeta^2 + cc\sin\zeta^2}$$

$$\mathfrak{G} = \frac{\varepsilon\cos\mathfrak{c}}{\delta\sin\zeta\sin\mathfrak{g}}$$

$$\mathfrak{H} = \frac{-\varepsilon\,(\cos\mathfrak{a}\sin\zeta + \cos\mathfrak{b}\cos\zeta)}{\delta\,u'\sin\zeta\sin\mathfrak{h}}$$

hinc erit $\tan\mathfrak{g} = \dfrac{\varepsilon\cos\mathfrak{c}}{\delta\mathrm{r}\cos\mathrm{l}}$, et $\tan\mathfrak{h} = \dfrac{\varepsilon\,(\cos\mathfrak{a}\sin\zeta + \cos\mathfrak{b}\cos\zeta)}{\delta\,\mathrm{r}\sin\mathrm{l}}$ unde anguli $\mathfrak{g}$, et $\mathfrak{h}$ hincque numeri $\mathfrak{G}$ et $\mathfrak{H}$ innotescunt.

His definitis teneat corpus elapso tempore $t$ situm in figura repraesentatum, sitque ZD $= \varrho$, ZA $= l$, ZB $= m$, ZC $= n$: ac ponatur $\cos l = \cos\zeta + p$, $\cos m = \cos\eta + q$; $\cos n = \cos\theta + r$ seu $\cos m = -\sin\zeta + q$ et $\cos n = r$. Deinde gyretur nunc circa axem IO celeritate angulari $= \mathfrak{z}$ in sensum ABC existentibus arcubus OA $= \alpha$, OB $= \mathfrak{C}$, OC $= \gamma$, ac ponendo $\mathfrak{z}\cos\alpha = x$, $\mathfrak{z}\cos\mathfrak{C} = y$ et $\mathfrak{z}\cos\gamma = z$, habebimus ex §. 917.

$$\mathfrak{z}\cos\alpha = \frac{\varepsilon\cos\zeta\,(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\sin\zeta)}{aa\cos\zeta^2 + cc\sin\zeta^2} + \frac{\delta'\mathrm{r}\sin\zeta\sin\mathrm{l}\sin(\delta' t + \mathfrak{h})}{aa\,u'\cos\mathfrak{h}}$$

$$\mathfrak{z}\cos\mathfrak{C} = \frac{-\varepsilon\sin\zeta\,(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\sin\zeta)}{aa\cos\zeta^2 + cc\sin\zeta^2} + \frac{\delta'\mathrm{r}\cos\zeta\sin\mathrm{l}\sin(\delta' t + \mathfrak{h})}{aa\,u'\cos\mathfrak{h}}$$

$$\mathfrak{z}\cos\gamma = \frac{\delta\mathrm{r}\cos\mathrm{l}\sin(\delta t + \mathfrak{g})}{\cos\mathfrak{g}}.$$

tum vero praeterea:

$$p = \frac{\mathrm{r}\sin\zeta\cos\mathrm{l}\cos(\delta t + \mathfrak{g})}{\cos\mathfrak{g}}$$

$$q = \frac{\mathrm{r}\cos\zeta\cos\mathrm{l}\cos(\delta t + \mathfrak{g})}{\cos\mathfrak{g}}$$

$$r = \frac{-\mathrm{r}\sin\mathrm{l}\cos(\delta t + \mathfrak{h})}{\cos\mathfrak{h}}.$$

Ex his si ponatur arcus ZD $= \varrho$, erit

$$\varrho = \mathrm{r}\sqrt{\left(\frac{\cos\mathrm{l}^2\cos(\delta t + \mathfrak{g})^2}{\cos\mathfrak{g}^2} + \frac{\sin\mathrm{l}^2\cos(\delta t + \mathfrak{h})^2}{\cos\mathfrak{h}^2}\right)}.$$

Porro ex triangulo AZD est $\cos \mathrm{ADZ} = \frac{\cos l - \cos \zeta \cos \varrho}{\sin \zeta \sin \varrho} = \frac{p + \frac{1}{2}\varrho\varrho \cos \zeta}{\varrho \sin \zeta}$ $= \frac{p}{\varrho \sin \zeta}$ evanescente termino $\frac{\varrho \cos \zeta}{2 \sin \zeta}$; hinc ergo erit

$$\cos \mathrm{ADZ} = \frac{\cos l \cos(\delta t + g)}{\cos g} : r\left(\frac{\cos l^2 \cos(\delta t + g)^2}{\cos g^2} + \frac{\sin l^2 \cos(\delta' t + h)^2}{\cos h^2}\right)$$

ideoque $\mathrm{tang}\, \mathrm{ADZ} = \frac{\cos g\, \mathrm{tang}\, l \cos(\delta' t + h)}{\cos h \cos(\delta t + g)}$.

Praeter arcum autem $\mathrm{DZ} = \varrho$ et angulum ADZ nosse oportet angulum XZD a circulo verticali fixo ZX computatum; est vero $\mathrm{DZA} = 180°$ $-$ ADZ, seu $\mathrm{tang}\, \mathrm{DZA} = \frac{-\cos g \cos(\delta t + h)}{\cos h \cos(\delta t + g)}\, \mathrm{tang}\, l$, cum initio esset $\mathrm{DZA} = 180° - l$ et $\mathrm{tang}\, \mathrm{DZA} = -\mathrm{tang}\, l$. Deinde vero posito angulo $\mathrm{XZA} = \lambda$, est $d\lambda = \frac{-dt(y \cos \eta + z \cos \theta)}{\sin \zeta^2} = \frac{-y\, dt}{\sin \zeta}$, hincque $\lambda =$ Const. $- \frac{\varepsilon t(aa \cos a \cos \zeta - cc \cos b \sin \zeta)}{aa \cos \zeta^2 + cc \sin \zeta^2} - \frac{r \cos \zeta \sin l \cos(\delta' t + h)}{cc u' \sin \zeta \cos h}$. Quod si ponamus initio angulum XZD evanuisse, initio fuerat $\lambda = 180° - l$, sicque constans hic ingressa est:

$$\text{Const.} = 180° - l + \frac{r \cos \zeta \sin l}{cc u' \sin \zeta}$$

unde habebimus:

$$\lambda = 180° - l - \frac{\varepsilon t(aa \cos a \cos \zeta - cc \cos b \sin \zeta)}{aa \cos \zeta^2 + cc \sin \zeta^2} + \frac{r \cos \zeta \sin l}{cc u' \sin \zeta}\left(1 - \frac{\cos(\delta t + h)}{\cos h}\right)$$

hincque $\mathrm{XZD} = \lambda - \mathrm{DZA}$, ex quibus ad tempus $t$ situs corporis perfecte cognoscitur. In hacque determinatione simul motus continetur.

### COROLL. 1.

919. Si motus corpori initio impressus evanescat; est $g = 0$ et $h = 0$; hincque

$x = \dots$



$$x = z \cos a = \frac{\delta' r \sin \zeta \sin l \sin \delta' t}{aa u'}$$

$$y = z \cos b = \frac{\delta' r \cos \zeta \sin l \sin \delta' t}{cc u'}$$

$$z = z \cos \gamma = \delta r \cos l \sin \delta t$$

$p = r \sin \zeta \cos l \cos \delta t$; $q = r \cos \zeta \cos l \cos \delta t$; $r = -r \sin l \cos \delta' t$

$\mathrm{tang}\, \mathrm{ADZ} = \mathrm{tang}\,(180° - \mathrm{DZA}) = \frac{\cos \delta' t}{\cos \delta t}\, \mathrm{tang}\, l$, et $\lambda = 180° - l + \frac{r \cos \zeta \sin l}{cc u' \sin \zeta}(1 - \cos \delta' t)$.

### COROLL. 2.

920. Sin autem corpori initio motus fuerit impressus celeritate angulari $\varepsilon$, haec non multo major esse debet quam r. Si enim $\frac{\varepsilon}{r}$ esset numerus praemagnus, anguli g et h prodirent fere recti, eorumque cosinus fere evanescentes, sicque numeri $p$, $q$, $r$ nimis fierent magni, quam ut tanquam valde parvae, uti natura solutionis exigit, considerari possent. Namque arcus $\mathrm{ZD} = \varrho$ semper minimus esse debet.

### COROLL. 3.

921. Cum sit $u = r(xx + yy + zz)$, nisi ternae quantitates $x$, $y$, $z$ seorsim evanescant, fieri nequit, ut corpus unquam ad quietem redigatur. Atque etiamsi corpus a quiete moveri inceperit, tamen fieri potest, ut corpus deinceps nunquam ad quietem revertatur, hocque adeo semper eveniet, nisi fuerit vel $\sin l = 0$ vel $\cos l = 0$; quin etiam sua axis corporis naturalis DF nunquam in situm verticalem perveniet.

### COROLL. 4.

922. Cum sit r quantitas valde exigua, si corpus initio nullum motum acceperit, ut sit $\varepsilon = 0$, erit satis exacte $\lambda = 180 - l$; scilicet angulus XZA manebit constans, motusque axis IA ita comparatus, ut in arcu ZA modo ad punctum verticale Z propius accedat modo ab eo longius recedat, erit autem $\mathrm{AZ} = \zeta - r \cos l \cos \delta t$, et ang. $\mathrm{ZAD} = \frac{r \sin l \cos \delta' t}{\sin \zeta}$.

C O.

### COROLL. 5.

923. In genere autem quicunque motus corpori initio fuerit impressus, erit

$$XZA = \lambda = 180^\circ - \mathfrak{l} - \frac{\varepsilon t(aac\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\sin\zeta)}{aa\cos\zeta^2 + cc\sin\zeta^2}$$

sicque arcus ZA uniformiter circa punctum verticale Z circumferetur: deinde vero cum sit $\sin$ AD : $\sin$ DZA = ZD ($\zeta$) : ZAD, erit ZAD = $\frac{\mathfrak{r}\sin\mathfrak{l}\cos(\delta t+\mathfrak{h})}{\sin\zeta\cos\mathfrak{h}}$ atque arcus ZA $= \zeta - \frac{\mathfrak{r}\cos\mathfrak{l}\cos(\delta t+\mathfrak{g})}{\cos\mathfrak{g}}$. Seu pro $\mathfrak{g}$ et $\mathfrak{h}$ substituendis valoribus:

$$\text{angulus } ZAD = \frac{\mathfrak{r}\sin\mathfrak{l}\cos\delta t}{\sin\zeta} - \frac{\varepsilon(\cos\mathfrak{a}\sin\zeta + \cos\mathfrak{b}\cos\zeta)\sin\delta t}{\delta\sin\zeta}$$

$$\text{et arcus } ZA = \zeta - \mathfrak{r}\cos\mathfrak{l}\cos\delta t + \frac{\varepsilon\cos\mathfrak{c}\sin\delta t}{\delta}.$$

### SCHOLION. 1.

924. Hae tres postremae formulae, angulos XZA, ZAD cum arcu ZA exhibentes, totam problematis solutionem complectuntur. Quodsi enim has res ad quodvis tempus assignare possumus, situm corporis perfecte cognoscimus. Quare si pro $\mathfrak{r}$ et $\delta$ valores supra inventos substituamus, universa problematis solutio his formulis continebitur:

$$\text{ang. } XZA = 180^\circ - \mathfrak{l} - \frac{\varepsilon t(aa\cos\mathfrak{a}\cos\zeta - cc\cos\mathfrak{b}\sin\zeta)}{aa\cos\zeta^2 + cc\sin\zeta^2}$$

$$\text{arc. } ZA = \zeta - \mathfrak{r}\cos\mathfrak{l}\cos\frac{t\sqrt{2fg}}{c} + \frac{\varepsilon c\cos\mathfrak{c}}{\sqrt{2fg}}\sin\frac{t\sqrt{2fg}}{c}$$

$$\text{ang. } ZAD = \frac{\mathfrak{r}\sin\mathfrak{l}}{\sin\zeta}\cos\frac{t\sqrt{2fg(aa\cos\zeta^2+cc\sin\zeta^2)}}{ac} - \frac{\varepsilon ac(\cos\mathfrak{a}\sin\zeta + \cos\mathfrak{b}\cos\zeta)}{\sin\zeta\sqrt{2fg(aa\cos\zeta^2+cc\sin\zeta^2)}}$$
$$\sin\frac{t\sqrt{2fg(aa\cos\zeta^2+cc\sin\zeta^2)}}{ac}.$$

Quodsi ergo omnia momenta inertiae fuerint aequalia, scilicet $aa = cc$, erit

$$\text{ang. } XZA = 180^\circ - \mathfrak{l} - \varepsilon t(\cos\mathfrak{a}\cos\zeta - \cos\mathfrak{b}\sin\zeta)$$

$$\text{arc. } ZA = \zeta - \mathfrak{r}\cos\mathfrak{l}\cos\frac{t\sqrt{2fg}}{c} + \frac{\varepsilon c\cos\mathfrak{c}}{\sqrt{2fg}}\sin\frac{t\sqrt{2fg}}{c}$$

ang.



$$\text{ang. } ZAD = \frac{\mathfrak{r}\sin\mathfrak{l}}{\sin\zeta}\cos\frac{t\sqrt{2fg}}{c} - \frac{\varepsilon c(\cos\mathfrak{a}\sin\zeta + \cos\mathfrak{b}\cos\zeta)}{\sin\zeta\sqrt{2fg}}\sin\frac{t\sqrt{2fg}}{c}$$

quo casu positio puncti A est plane arbitraria.

### SCHOLION. 2.

925. Argumentum, quod in hoc capite potissimum evolvendum suscepimus, motum scilicet titubatorium corporum basi sphaerica praeditorum, perfecte pertractavimus, dummodo titubationes fuerint quam minimae, quae hypothesis etiam in doctrina oscillationum statui solet: formulae enim §. 912 et seqq. exhibitae perfectam continent hujus quaestionis solutionem, si quidem illi anguli $\delta\iota$ et $\delta t$ constantibus $\mathfrak{g}$ et $\mathfrak{h}$ augeantur. Constantes autem ex statu initiali definire docuimus in §. 916. quae operatio vehementer sublevatur annotatione sub finem §. 917. adjuncta; quare ad motum corporum cylindricorum explicandum progrediamur.

# CAPUT XIX.

## DE MOTU CORPORUM CYLINDRICORUM SUPER PLANO HORIZONTALI.

### THEOREMA. 11.

926. **D**um corpus cylindricum plano horizontali movetur, pressio, qua plano innititur, est verticalis et per centrum cujusdam sectionis cylindri ad longitudinem normaliter factae transit.

### DEMONSTRATIO.

Corpus cylindricum plano horizontali incumbit secundum lineam rectam axi cylindri parallelam, in qua vires existunt cylindrum sustentantes, fierique potest ut hae vires per totam illam rectam sint dispersae. Cum autem istae vires omnes sint ad planum horizontale normales, ideoque verticales, ac parallelae inter se, una dabitur vis iis omnibus aequivalens: cujus ergo directio pariter erit verticalis, certoque rectae contactus puncto insistet. Quodsi igitur in hoc puncto cylindrus ad longitudinem normaliter secetur, sectio erit circulus, et vis omnibus pressionibus aequivalens, quia est in hac sectione ad punctum con-



tactus

tactus verticalis per ejus centrum transibit. Nisi ergo haec sectio transeat per centrum inertiae corporis, directio media pressionis non in sectione per centrum inertiae ad longitudinem normaliter facta versabitur.

## EXPLICATIO.

927. Corpora igitus hic consideramus cylindrica, in quibus primo notetur eorum axis quasi geometricus, ad quem omnes sectiones normaliter factae, sint circuli aequales, ita ut corpus sit cylindrus rectus, cujus motus, dum plano horizontali perpetuo incumbit, sumus exploraturi.

Fig. 117. Si centrum inertiae in ipso axe geometrico esset, in omni situ cylindrus aequilibrium teneret; sin autem secus, concipiatur sectio cylindri ad axem normalis per centrum inertiae I facta, cujus centrum sit in G, atque ad statum aequilibrii requiritur, ut recta GI sit verticalis, ex quo duplex datur aequilibrii situs, alter quo centrum inertiae I infra G alter quo supra G versatur, quorum ille *stabilis*, hic *labilis* vocetur. [illegible] autem cylindri [illegible] geometricus sit horizontalis, recta [illegible] verticaliter [illegible]. Deinde ternos axes mechanicos cylindri nosse [illegible], quia [illegible] materiae distributionem admittimus, [illegible] secundum longitudinem ducto differre possunt, a quorum positione motus determinatio potissimum pendet. Per centrum inertiae I [illegible] ducta concipiatur recta axi geometrico parallela, quae plerumque axis principalis esse solet, et semper manet horizontalis. [illegible] igitur notatis, quomodocunque cylindrus plano horizontali incumbat, in sectione LMFN per centrum inertiae I facta notetur punctum T, [illegible] circulus planum horizontale tangit, deinde etiam illa sectio [illegible] tetur, in qua media directio pressionis versatur, [illegible] erit parallela; ac tam quam quantitas pressionis, quam distantia sectionis, in qua versatur, a sectione per centrum inertiae facta, erit incognita, demum ex motu deinceps determinanda: siquidem hac vi effici debet, ut axis cylindri longitudinalis perpetuo maneat horizontalis, et cylindrus plano horizontali incumbat.

## SCHOLION.

928. In his motibus investigandis non opus est, ut totum corpus in figuram cylindri sit efformatum, sed sufficit, si in locis, quibus plano horizontali incumbit, talem habuerit formam. Huc ergo pertinent motus omnium eorum corporum, quae in terminos cylindricos desinunt,



desinunt, quibus tantum plano horizontali [illegible] incumbant. [illegible] eos motus corporum [illegible] evenit in curris, quae [illegible] tanquam circulos spectare [illegible] dus est motus [illegible], quae non [illegible] assumimus, sed materiam [illegible] tur, dum his binis cylindris plano horizontali [illegible] quae massa penduli dependet. [illegible] non sunt circuli, tamen binas illas sectiones [illegible] inertiae versetur, in easque vis pressionis [illegible] etare licet, quoniam figura externa [illegible] corpus plano horizontali incumbit. [illegible] corpora integras circumvolutiones [illegible] toti termini sint cylindrici, sed sufficit, si [illegible] contactus, talem habeat figuram, [illegible] tum corpus extensum probe notetur [illegible] ad plurimos casus extendi [illegible] inertiae a viribus [illegible] verticalis recipere [illegible] nisi extrinsecus [illegible] esset, prosecutionem [illegible] hic non attendimus.

## PROBLEMA [illegible].

929. Si corpus cylindricum super plano horizontali [illegible] detur que pressio [illegible], definire motum [illegible] quo centrum inertiae corporis [illegible].

## SOLUTIO.

Ad axem cylindri [illegible] normalis per centrum inertiae I, [illegible] sive corpus sit cylindrus [illegible] tantum terminis cylindricis [illegible] praeditum, spectetur tanquam circulus LMFN [illegible] centrum sit in G, centrum autem inertiae corporis in I, existente intervallo $GI = f$, ita ut in statu quietis recta LIGF teneat situm verticalem, in quo centrum inertiae I supra centrum circuli G in figura [illegible] tur, quod si fuerit profundius, intervallum $GI = f$ negative est [illegible] dum. Nunc autem contactus respondeat puncto T, [illegible] planum circuli in T normalis sit linea contactus in planum horizontale [illegible] *Hh* cadens. Ducta igitur per centrum circuli recta $HGT$ ad [illegible]

T, parallela erit directioni pressionis, qua corpus a plano horizontali repellitur, quae vis, in quacunque alia sectione versetur, ponatur $= \Pi$, quam tanquam cognitam spectamus. Sit porro pondus cylindri $= M$, radius circuli $GF = GT = e$, et angulus declinationis $\Pi GL = \varrho$; erit elevatio centri inertiae I supra horizontem $IP = e + f \cos \varrho$, quae ponatur $= v$. Quoniam igitur in motum progressivum inquirimus, et gravitas deorsum urget secundum IP vi $= M$, pressionis vis $\Pi$ ipsi centro inertiae I sursum secundum IV applicata concipiatur, ita ut jam tota vis deorsum sollicitans sit $= M - \Pi$, et massa movenda $= M$: unde ex principiis motus habetur $ddv = \frac{-2g(M-\Pi)}{M} dt^2$, hincque $\frac{\Pi}{M} = 1 + \frac{ddv}{2g\,dt^2}$ seu $\Pi = M\left(1 + \frac{f\,dd.\cos\varrho}{2g\,dt^2}\right)$, qua aequatione angulus $\Pi GL = \varrho$, ex eoque elevatio centri inertiae $IP = e + f \cos \varrho$ innotescit: ac nisi corpori motus horizontalis fuerit impressus, punctum I in recta PIV agitaretur, in ea vel ascendens vel descendens, ita ut punctum P maneret fixum, ex quo punctum contactus T definitur, quia est $PT = f \sin \varrho$. Sin autem corpus acceperit motum horizontalem, cum constanter servaret uniformem in directum, motusque puncti I ex hoc aequabili rectilineo horizontali et illo verticali foret compositus.

### SCHOLION.

930. Praeterea autem in hoc corpore motus gyratorius generari potest, ita tamen ut tam punctum G quam recta ad circulum LMFN in G normalis, quae est axis proprius cylindri, perpetuo maneat in eodem plano horizontali. Ad hunc motum gyratorium investigandum, seposito motu progressivo centrum inertiae I tanquam in quiete spectabimus, circa quod sphaera descripta, in ea circulus LMFN perpetuo erit verticalis, ad quem si recta normalis per I ducatur, erit ea axis cylindri longitudinalis. Qua conditione observata, omnes motus gyratorii, quorum cylindrus est capax, facile in figura repraesentari possunt. Hic autem ante omnia ad situm axium principalium probe attendi oportet, quorum respectu momenta ex vi pressionis nata sunt definienda.

### PROBLEMA. 112.

931. Si corpus cylindricum plano horizontali incumbens habeat situm quemcunque, deturque tam pressio $\Pi$ quam sectio cylindri transversi, in qua versatur, invenire ejus momenta respectu axium principalium.



### SOLUTIO.

Sectio cylindri per centrum inertiae I ad longitudinem normaliter facta cadat in planum tabulae, in qua recta IZ sit normalis, et recta LIG per centrum hujus sectionis G transeat, ita ut sit intervallum $IG = f$, et angulus $ZIL = \varrho$. Ex G erigatur ad planum tabulae normalis GH usque ad sectionem, in qua vis pressionis $\Pi$ versatur, sitque intervallum $GH = h$, ac supra vidimus esse $\Pi = M\left(1 + \frac{f\,dd.\cos\varrho}{2g\,dt^2}\right)$ cujus vis directio erit $H\Pi$ verticalis ideoque parallela ipsi IZ. Iam radio arbitrario $= 1$, circa centrum inertiae I sphaera concipiatur descripta, ad cujus superficiei puncta A, B, C axes corporis principales dirigantur, vocenturque arcus pro horum punctorum determinatione $LA = \zeta$, $LB = \eta$, $LC = \theta$; item $ZA = l$, $ZB = m$, $ZC = n$ existente $ZL = \varrho$; tum vero anguli $ZLA = \mathfrak{f}$, $ZLB = \mathfrak{g}$, $ZLC = \mathfrak{h}$, ut sit

$$\cos l = \cos\zeta \cos\varrho + \cos\mathfrak{f} \sin\zeta \sin\varrho; \quad \cos m = \cos\eta \cos\varrho + \cos\mathfrak{g} \sin\eta \sin\varrho;$$
$$\cos n = \cos\theta \cos\varrho + \cos\mathfrak{h} \sin\theta \sin\varrho.$$

Ac primo quidem vis $H\Pi = \Pi$ resolvatur secundum directiones axibus principalibus parallelas, quae resolutio perinde instituitur, ac si vis haec in centro I secundum directionem IZ esset applicata: inde autem nascitur

$$\text{vis sec. IA} = \Pi \cos l; \quad \text{vis sec. IB} = \Pi \cos m; \quad \text{vis sec. IC} = \Pi \cos n,$$

quae autem vires jam in puncto H applicatae sunt intelligendae. Ducatur recta IH quae erit $= \surd(ff + hh)$, secans sphaeram in F, erit $\tang GIH = \frac{h}{f}$, et arcus LF cum arcu ZL faciet angulum ZLF rectum. Ponatur arcus $LF = \epsilon$, erit $h = -f \tang \epsilon$ et $IH = \frac{-f}{\cos\epsilon}$, ita ut loco intervalli $GH = h$ commode arcum $LF = \epsilon$ in calculo retineamus. Nunc autem investigari oportet, quomodo recta IF ad axes principales inclinetur, quae inclinatio per arcus FA, FB et FC definitur. Reperitur autem

$$\cos AF = \cos\zeta \cos\epsilon + \sin\mathfrak{f} \sin\zeta \sin\epsilon$$
$$\cos BF = \cos\eta \cos\epsilon + \sin\mathfrak{g} \sin\eta \sin\epsilon$$
$$\cos CF = \cos\theta \cos\epsilon + \sin\mathfrak{h} \sin\theta \sin\epsilon.$$

Repraesentent jam rectae inter se normales IA, IB, IC axes principales Fig. 120

les corporis, inter quos existat recta $IH = \frac{-f}{\cos e}$ et utrique coordinatae pro puncto H axibus parallelae

$$IN = IH \cos AF;\ NM = IH \cos BF;\ MH = IH \cos CF$$

et vires in H applicatae axibusque parallelae erunt

vis $Ha = \Pi \cos l$; vis $Hb = \Pi \cos m$; vis $Hc = \Pi \cos n$,

unde respectu axium principalium nascuntur momenta:

Mom. respectu axis IA in sensum $BC = \Pi . IH (\cos n \cos BF - \cos m \cos CF)$

Mom. respectu axis IB in sensum $CA = \Pi . IH (\cos l \cos CF - \cos n \cos AF)$

Mom. respectu axis IC in sensum $AB = \Pi . IH (\cos m \cos AF - \cos l \cos BF)$.

Quae momenta cum supra litteris P, Q, R indicaverimus, si valores supra exhibitos substituamus, obtinebimus:

$$P = \frac{-\Pi f}{\cos e}\left(\begin{array}{l}\sin e \cos \varrho (\sin \mathfrak{g} \sin \eta \cos \theta - \sin \mathfrak{h} \cos \eta \sin \theta) + \cos e \sin \varrho (\cos \mathfrak{h} \cos \eta \sin \theta - \cos \mathfrak{g} \sin \eta \cos \theta) \\ + \sin \eta \sin \theta \sin e \sin \varrho (\sin \mathfrak{g} \cos \mathfrak{h} - \cos \mathfrak{g} \sin \mathfrak{h})\end{array}\right)$$

At est $\sin \mathfrak{g} \cos \mathfrak{h} - \cos \mathfrak{g} \sin \mathfrak{h} = \sin (\mathfrak{g} - \mathfrak{h}) = \frac{-\cos \zeta}{\sin \eta \sin \theta}$, tum vero

$$\sin \mathfrak{g} \sin \eta \cos \theta - \sin \mathfrak{h} \cos \eta \sin \theta = \cos \mathfrak{f} \sin \zeta$$

$$\cos \mathfrak{h} \cos \eta \sin \theta - \cos \mathfrak{g} \sin \eta \cos \theta = \sin \mathfrak{f} \sin \zeta$$

ita ut tam pro P quam pro Q et R analogice habeamus

$$P = \frac{-\Pi f}{\cos e}(\cos \mathfrak{f} \sin \zeta \sin e \cos \varrho + \sin \mathfrak{f} \sin \zeta \cos e \sin \varrho - \cos \zeta \sin e \sin \varrho)$$

$$Q = \frac{-\Pi f}{\cos e}(\cos \mathfrak{g} \sin \eta \sin e \cos \varrho + \sin \mathfrak{g} \sin \eta \cos e \sin \varrho - \cos \eta \sin e \sin \varrho)$$

$$R = \frac{-\Pi f}{\cos e}(\cos \mathfrak{h} \sin \theta \sin e \cos \varrho + \sin \mathfrak{h} \sin \theta \cos e \sin \varrho - \cos \theta \sin e \sin \varrho)$$

## COROLL. 1.

932. Cum sit $-f \tang e = b$, et $b$ denotet intervallum GH, quo sectio, in quam cadit pressio, antrorsum distat a sectione, in qua est centrum inertiae, erit

$$P = -\Pi f \sin \mathfrak{f} \sin \zeta \sin \varrho + \Pi b (\cos \mathfrak{f} \sin \zeta \cos \varrho - \cos \zeta \sin \varrho)$$

$$Q = -\Pi f \sin \mathfrak{g} \sin \eta \sin \varrho + \Pi b (\cos \mathfrak{g} \sin \eta \cos \varrho - \cos \eta \sin \varrho)$$

$$R = -\Pi f \sin \mathfrak{h} \sin \theta \sin \varrho + \Pi b (\cos \mathfrak{h} \sin \theta \cos \varrho - \cos \theta \sin \varrho).$$

CO-



## COROLL. 2.

933. Dum ergo motus corporis determinatur, non solum quantitatem pressionis $\Pi$ sed etiam intervallum $GH = b$ definiri oportet, ut habeatur locus, ubi media directio pressionum est applicata.

## EXPLICATIO.

934. Relatio inter arcus $\zeta$, $\eta$, $\theta$ et angulos $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ insignes suppeditat proprietates, inter quas substitutiones in solutione adhibitae continentur. Primo enim pro illorum angulorum differentiis invenimus

$$\cos(\mathfrak{f} - \mathfrak{g}) = \frac{-\cos \zeta \cos \eta}{\sin \zeta \sin \eta};\ \cos(\mathfrak{g} - \mathfrak{h}) = \frac{-\cos \eta \cos \theta}{\sin \eta \sin \theta};\ \cos(\mathfrak{h} - \mathfrak{f}) = \frac{-\cos \zeta \cos \theta}{\sin \zeta \sin \theta}$$

$$\sin(\mathfrak{f} - \mathfrak{g}) = \frac{-\cos \theta}{\sin \zeta \sin \eta};\ \sin(\mathfrak{g} - \mathfrak{h}) = \frac{-\cos \zeta}{\sin \eta \sin \theta};\ \sin(\mathfrak{h} - \mathfrak{f}) = \frac{-\cos \eta}{\sin \zeta \sin \theta}$$

Hinc jam anguli $\mathfrak{g}$ et $\mathfrak{h}$ ad angulum $\mathfrak{f}$ reduci possunt, ob $\mathfrak{g} = \mathfrak{f} - (\mathfrak{f} - \mathfrak{g})$ et $\mathfrak{h} = \mathfrak{f} + (\mathfrak{h} - \mathfrak{f})$, unde colligitur

$$\sin \mathfrak{g} = \frac{-\sin \mathfrak{f} \cos \zeta \cos \eta + \cos \mathfrak{f} \cos \theta}{\sin \zeta \sin \eta};\ \sin \mathfrak{h} = \frac{-\sin \mathfrak{f} \cos \zeta \cos \theta - \cos \mathfrak{f} \cos \eta}{\sin \zeta \sin \theta}$$

$$\cos \mathfrak{g} = \frac{-\cos \mathfrak{f} \cos \zeta \cos \eta - \sin \mathfrak{f} \cos \theta}{\sin \zeta \sin \eta};\ \cos \mathfrak{h} = \frac{-\cos \mathfrak{f} \cos \zeta \cos \theta + \sin \mathfrak{f} \cos \eta}{\sin \zeta \sin \theta}$$

Quodsi binis conjungendis vel $\cos \mathfrak{f}$ vel $\sin \mathfrak{f}$ elidatur, obtinentur sequentes formulae:

I. $\sin \mathfrak{f} \sin \zeta \cos \zeta + \sin \mathfrak{g} \sin \eta \cos \eta + \sin \mathfrak{h} \sin \theta \cos \theta = 0$
II. $\cos \mathfrak{f} \sin \zeta \cos \zeta + \cos \mathfrak{g} \sin \eta \cos \eta + \cos \mathfrak{h} \sin \theta \cos \theta = 0$
III. $\sin \mathfrak{f} \sin \zeta = -\cos \mathfrak{g} \sin \eta \cos \theta + \cos \mathfrak{h} \cos \eta \sin \theta$
IV. $\sin \mathfrak{g} \sin \eta = -\cos \mathfrak{h} \sin \theta \cos \zeta + \cos \mathfrak{f} \cos \theta \sin \zeta$
V. $\sin \mathfrak{h} \sin \theta = -\cos \mathfrak{f} \sin \zeta \cos \eta + \cos \mathfrak{g} \cos \zeta \sin \eta$
VI. $\cos \mathfrak{f} \sin \zeta = \sin \mathfrak{g} \sin \eta \cos \theta - \sin \mathfrak{h} \cos \eta \sin \theta$
VII. $\cos \mathfrak{g} \sin \eta = \sin \mathfrak{h} \sin \theta \cos \zeta - \sin \mathfrak{f} \cos \theta \sin \zeta$
VIII. $\cos \mathfrak{h} \sin \theta = \sin \mathfrak{f} \sin \zeta \cos \eta - \sin \mathfrak{g} \cos \zeta \sin \eta$
IX. $\sin \mathfrak{f} \cos \mathfrak{f} \sin \zeta^2 + \sin \mathfrak{g} \cos \mathfrak{g} \sin \eta^2 + \sin \mathfrak{h} \cos \mathfrak{h} \sin \theta^2 = 0$
X. $\sin \mathfrak{f}^2 \sin \zeta^2 + \sin \mathfrak{g}^2 \sin \eta^2 + \sin \mathfrak{h}^2 \sin \theta^2 = 1$; XI. $\cos \mathfrak{f}^2 \sin \zeta^2 + \cos \mathfrak{g}^2 \sin \eta^2 + \cos \mathfrak{h}^2 \sin \theta^2 = 1$

quae

quarum ope aequationes, ad quas motus determinatio perducitur, simpliciores reddi possunt.

### PROBLEMA. 113.

Fig. 121. 935. Si corpus cylindricum quodcunque super plano horizontali moveatur utcunque, aequationes exhibere, quibus ad quodvis tempus ejus situs et motus gyratorius determinetur.

### SOLUTIO.

Manentibus denominationibus in praecedente problemate factis, consideretur centrum inertiae I ut quiescens, circa quod descripta sit sphaera, cujus punctum verticale Z, et circulus verticalis fixus ZDX, in quo recta centralis IL initio situm ID tenuerit. Elapso autem tempore $t$ ea pervenerit in L, ac ponatur arcus ZL $= \varrho$ et angulus XZL $=$ $\varphi$, atque hinc situs axium principalium, quorum poli sint A, B, C ita definitur, ut sint arcus LA $= \zeta$, LB $= \eta$, LC $= \theta$, et anguli ZLA $= \mathfrak{f}$, ZLB $= \mathfrak{g}$, ZLC $= \mathfrak{h}$, qui sunt quantitates constantes, ex quibus cum arcu variabili ZL $= \varrho$, ita definiuntur arcus ZA $= l$, ZB $= m$, ZC $= n$ ut sit:

$$cos\, l = cos\,\zeta\, cos\,\varrho + cos\,\mathfrak{f}\, si\,\zeta\, si\,\varrho;\ cos\, m = cos\,\eta\, cos\,\varrho + cos\,\mathfrak{g}\, si\,\eta\, si\,\varrho;\ cos\, n = cos\,\theta\, cos\,\varrho + cos\,\mathfrak{h}\, si\,\theta\, si\,\varrho.$$

Quodsi jam momenta inertiae corporis respectu axium principalium IA, IB, IC sint Maa, Mbb, Mcc existente M massa corporis, $\Pi$ autem sit pressio, et sectio in qua ea versatur ab I antrorsum distet intervallo $= s$, quod quia est variabile, in superioribus formulis loco $b$ scribi debet $s$. Gyretur nunc corpus circa polum O celeritate angulari $= \varkappa$ in sensum ABC, positisque arcubus OA $= \alpha$, OB $= \beta$, OC $= \gamma$, sit $\varkappa\, cos$ $\alpha = x$, $\varkappa\, cos\,\beta = y$, $\varkappa\, cos\,\gamma = z$, ac primo habemus $\Pi = M(\mathfrak{f}$ + $\frac{f\,dd.\,cos\,\varrho}{2\,g\,dt^2})$, tum vero has tres aequationes

$$aadx + (cc - bb)\, yzdt = \frac{-2\Pi fg}{M}\, dt\, si\,\mathfrak{f}\, si\,\zeta\, si\,\varrho + \frac{2\Pi g s}{M}\, dt$$
$$(cos\,\mathfrak{f}\, si\,\zeta\, cos\,\varrho - cos\,\zeta\, si\,\varrho)$$

$$bbdy + (aa - cc)\, xzdt = \frac{-2\Pi fg}{M}\, dt\, si\,\mathfrak{g}\, si\,\eta\, si\,\varrho + \frac{2\Pi g s}{M}\, dt$$
$$(cos\,\mathfrak{g}\, si\,\eta\, cos\,\varrho - cos\,\eta\, si\,\varrho)$$

ccdz



$$ccdz + (bb - aa)\, xydt = \frac{-2\Pi fg}{M}\, dt\, si\,\mathfrak{h}\, si\,\theta\, si\,\varrho + \frac{2\Pi g s}{M}\, dt$$
$$(cos\,\mathfrak{h}\, si\,\theta\, cos\,\varrho - cos\,\theta\, si\,\varrho).$$

Praeterea habemus has tres aequationes

$$dl\, si\, l = dt\,(y\, cos\, n - z\, cos\, m) = d\varrho\,(cos\,\zeta\, si\,\varrho - cos\,\mathfrak{f}\, si\,\zeta\, cos\,\varrho)$$
$$dm\, si\, m = dt\,(z\, cos\, l - x\, cos\, n) = d\varrho\,(cos\,\eta\, si\,\varrho - cos\,\mathfrak{g}\, si\,\eta\, cos\,\varrho)$$
$$dn\, si\, n = dt\,(x\, cos\, m - y\, cos\, l) = d\varrho\,(cos\,\theta\, si\,\varrho - cos\,\mathfrak{h}\, si\,\theta\, cos\,\varrho)$$

quarum autem binas tantum sumsisse sufficit, ita ut superfint sex aequationes, ex quibus variabiles totidem $x$, $y$, $z$, $\Pi$, $r$ et $\varrho$ ad [illegible] tempus $t$ determinari oporteat. Denique vero positis angulis XZA $=$ $\lambda$, XZB $= \mu$, XZC $= \nu$, fit

$$d\lambda\, si\, l^2 = -dt\,(y\, cos\, m + z\, cos\, n)$$

quam unam resolvisse sufficit. At cum sit LZA $= \lambda - \Phi$, erit $cos$ $(\lambda - \varphi) = \frac{cos\,\zeta - cos\, l\, cos\,\varrho}{si\, l\, si\,\varrho}$ et $si\,(\lambda - \varphi) = \frac{si\,\mathfrak{f}\, si\,\zeta}{si\, l}$, unde $(d\lambda - d\varphi)$ $cos\,(\lambda - \varphi) = \frac{-dl\, si\,\mathfrak{f}\, si\,\zeta\, cos\, l}{si\, l^2} = \frac{(d\lambda - d\varphi)(cos\,\zeta - cos\, l\, cos\,\varrho)}{si\, l\, si\,\varrho}$, ideoque

$$d\varphi = \frac{-dt\,(y\, cos\, m + z\, cos\, n)}{si\, l^2} + \frac{dl\, si\,\mathfrak{f}\, si\,\zeta\, cos\, l\, si\,\varrho}{si\, l\,(cos\,\zeta - cos\, l\, cos\,\varrho)},$$

hincque etiam ad datum tempus angulus $\varphi$ definitur: ex quibus rebus motus corporis perfecte cognoscitur.

### COROLL. 1.

936. Cum sit $cos\,\zeta - cos\, l\, cos\,\varrho = si\,\varrho\,(cos\,\zeta\, si\,\varrho - cos\,\mathfrak{f}\, si\,\zeta\, cos\,\varrho) = \frac{dl\, si\, l\, si\,\varrho}{d\varrho}$ erit $d\varphi = \frac{-dt\,(y\, cos\, m + z\, cos\, n)}{si\, l^2} + \frac{d\varrho\, si\,\mathfrak{f}\, si\,\zeta\, cos\, l}{si\, l^2}$, hincque

$$d\varphi\, si\, l^2 = -dt\,(y\, cos\, m + z\, cos\, n) + d\varrho\, cos\,\varrho\, si\,\mathfrak{f}\, si\,\zeta\, cos\,\zeta$$
$$+ d\varrho\, si\,\varrho\, si\,\mathfrak{f}\, cos\,\mathfrak{f}\, si\,\zeta^2.$$

Similes autem expressiones pro $d\varphi\, si\, m^2$ et $d\varphi\, si\, n^2$ reperiuntur, quae in unam summam collectae, ob $si\, l^2 + si\, m^2 + si\, n^2 = 2$, dabunt

$$d\varphi = -dt\,(x\, cos\, l + y\, cos\, m + z\, cos\, n)$$ per n° 1 et IX. §. 934

ubi $x\, cos\, l + y\, cos\, m + z\, cos\, n$ denotat cosinum arcus ZO per $\varkappa$ multiplicatum.

### COROLL. 2.

937. Ex aequationibus pro $dl\, si\, l$, $dm\, si\, m$, $dn\, si\, n$ inventis colligimus,



dl

$$dl \sin l \cos \zeta + dm \sin m \cos \eta + dn \sin n \cos \theta = d\rho \sin \rho$$

ac valoribus per $dt$ ſubſtitutis, impetrabimus,

$$d\rho = -dt\,(x \sin \mathfrak{f} \sin \zeta + y \sin \mathfrak{g} \sin \eta + z \sin \mathfrak{h} \sin \theta)$$

ope reductionum ſupra traditarum.

### COROLL. 3.

938. Ex tribus autem prioribus aequationibus deducimus, ob $x\,dl \sin l + y\,dm \sin m + z\,dn \sin n = 0$, hanc aequationem

$$aax\,dx + bby\,dy + ccz\,dz = \frac{-2\Pi fg}{M}\,dt \sin \rho\,(x \sin \mathfrak{f} \sin \zeta + y \sin \mathfrak{g} \sin \eta + z \sin \mathfrak{h} \sin \theta)$$

$$= \frac{2\Pi fg}{M}\,d\rho \sin \rho = -2fg\,d.\cos \rho\,\left(1 + \frac{f\,dd.\cos \rho}{2g\,dt^2}\right),$$ cujus ergo integrale eſt

$$aaxx + bbyy + cczz = C - 4fg \cos \rho - \frac{ff\,d\rho^2 \sin \rho^2}{dt^2}.$$

### SCHOLION.

939. Si in ſphaera noſtra dicatur circulus maximus horizontalis YMX, in eo perpetuo axis cylindri longitudinalis reperiatur neceſſe eſt. Pertingat ejus terminus anterior in M, et quia tam ML quam MZ ſunt quadrantes, erunt anguli MZL et MLZ recti, ideoque angulus ZML $= \rho$ et arcus XM $=$ angulo XZM $= 90° + \phi$. Tum vero quia punctum M aliter niſi in circulo XY moveri nequit, polus gyrationis O neceſſario in quadrante ZM ſitus ſit neceſſe eſt. Hinc ſi arcus OM ponatur $= \omega$, ob celeritatem angularem $= u$ in ſenſum ABC tendentem, punctum M tempusculo $dt$ regreditur verſus X per arculum $= u\,dt \sin \omega$: eſt vero $\sin \omega = \cos$ OL $= \cos \alpha \cos l + \cos \beta \cos m + \cos \gamma \cos n$, ideoque $u \sin \omega = x \cos l + y \cos m + z \cos n$, ita ut ſit $-d\phi = dt\,(x \cos l +$
Fig. 122. $y \cos m + z \cos n)$ uti in coroll. 1. invenimus. Deinde cum in triangulo OZL ſit ZO $= 90° - \omega$, ZL $= \rho$ et OZL $= 90°$, erit arcus $\cos$ OL $= \sin \omega \cos \rho$, $\sin$ OLZ $= \frac{\cos \omega}{\sin OL}$ et $\cos$ OLZ $= \frac{\sin \omega \sin \rho}{\sin OL}$ ob $\cot$ OLZ $= \frac{\sin \rho \sin \omega}{\cos \omega}$. Quare ſi tempusculo $dt$ punctum L circa O gyretur in $l$, erit L$l$ $= u\,dt \sin$ OL, et angulus OL$l$ rectus; hinc ducto circulo $l\lambda$ ad Z$l$ perpendiculari fiet L$\lambda$ $=$ L$l$ $\cos l$L$l$ $=$ L$l \sin$ OLZ $= u\,dt \cos \omega$, at eſt L$\lambda = -d\rho$ ideoque $d\rho = -u\,dt \cos \omega$. Quae formula comparata cum ea, quam §. 937. invenimus, dat





$$u \cos \omega = x \sin \mathfrak{f} \sin \zeta + y \sin \mathfrak{g} \sin \eta + z \sin \mathfrak{h} \sin \theta$$

at eſt $xx + yy + zz = uu = (x \cos l + y \cos m + z \cos n)^2 + (x \sin \mathfrak{f} \sin \zeta + y \sin \mathfrak{g} \sin \eta + z \sin \mathfrak{h} \sin \theta)^2$ quae aequalitas per aequationem $x\,dl \sin l + y\,dm \sin m + z\,dn \sin n = 0$ confirmatur. Verum ne multitudine litterarum obruamur, evolvamus caſum, quo axis cylindri longitudinalis ſimul eſt axis principalis.

### PROBLEMA. 114.

940. Si corporis cylindrici axis longitudinalis per ejus centrum inertiae ductus ſimul fuerit axis principalis, idque ſuper plano horizontali utcunque moveatur, definire ejus motum. Fig. 12.

### SOLUTIO.

Cum puncta A et M in unum incidant, bini reliqui poli principales B et C in circulo verticali ZL exiſtent, eritque propterea: LA $= \zeta = 90°$; LB $= \eta$; LC $= \theta = 90° - \eta$: ZLA $= \mathfrak{f} = 90°$, ZLB $= \mathfrak{g} = 180°$; ZLC $= \mathfrak{h} = 0$; hincque ZA $= l = 90$; ZB $= m = \eta + \rho$ et ZC $= n = \rho - \theta = \eta + \rho - 90°$. Quibus valoribus ſubſtitutis, habebimus iſtas aequationes:

$$\frac{\Pi}{M} = 1 + \frac{f\,dd.\cos \rho}{2g\,dt^2}$$

$$aadx + (cc - bb)\,yz\,dt = \frac{-2\Pi fg}{M}\,dt \sin \rho$$

$$bbdy + (aa - cc)\,xz\,dt = \frac{-2\Pi gs}{M}\,dt \sin (\eta + \rho)$$

$$ccdz + (bb - aa)\,xy\,dt = \frac{2\Pi gs}{M}\,dt \cos (\eta + \rho)$$

$$y \sin (\eta + \rho) - z \cos (\eta + \rho) = 0$$
$$-x\,dt \sin (\eta + \rho) = d\rho \sin (\eta + \rho) \text{ ſeu } d\rho = -x\,dt$$
$$\text{et } d\phi = -dt\,(y \cos (\eta + \rho) + z \sin (\eta + \rho)).$$

Ponatur $y = u \cos (\eta + \rho)$ et $z = u \sin (\eta + \rho)$, ac pro $dt$ ſcribatur $-\frac{d\rho}{x}$, ſeu $x = \frac{-d\rho}{dt}$, quo facto noſtrae aequationes erunt

I. $\frac{\Pi}{M} = 1 + \frac{f\,dd.\cos \rho}{2g\,dt^2}$

II. $-aa\,dd\rho + \frac{1}{2}(cc - bb)\,uu\,dt^2 \sin 2(\eta + \rho) + \frac{2\Pi}{M} fg\,dt^2 \sin \rho = 0$

$$\text{III. } bbdu \cos(\eta+\varrho) - (aa + bb - cc)\, ud\varrho \sin(\eta+\varrho) = \frac{-2\Pi}{M} gsdt \sin(\eta+\varrho)$$

$$\text{IV. } ccdu \sin(\eta+\varrho) + (aa - bb + cc)\, ud\varrho \cos(\eta+\varrho) = \frac{2\Pi}{M} gsdt \cos(\eta+\varrho)$$

et V. $d\varphi = -udt$.

Ex tertia et quarta eliminando $s$ nanciscimur,

$$bbdu \cos(\eta+\varrho)^2 + ccdu \sin(\eta+\varrho)^2 - 2(bb-cc)\, ud\varrho \sin(\eta+\varrho)\cos(\eta+\varrho) = 0$$

cujus integrale est

$$u = \frac{C}{bb+cc+(bb-cc)\cos(2\eta+2\varrho)}$$

qui valor in II substitutus praebet

$$-2aadd\varrho + \frac{CC(cc-bb)\, dt^2 \sin 2(\eta+\varrho)}{(bb+cc+(bb-cc)\cos 2(\eta+\varrho))^2} + dt^2 \sin\varrho \left(4fg + \frac{2ffdd.\cos\varrho}{dt^2}\right) = 0$$

quae aequatio per $d\varrho$ multiplicata et integrata dat,

$$-aad\varrho^2 - \frac{\frac{1}{2}CCdt^2}{bb+cc+(bb-cc)\cos 2(\eta+\varrho)} - 4fgdt^2 \cos\varrho - ffd\varrho^2 \sin\varrho^2 + Ddt^2 = 0$$

seu $d\varrho^2(aa+ff\sin\varrho^2) = dt^2\left(D - 4fg\cos\varrho - \frac{\frac{1}{2}CC}{bb+cc+(bb-cc)\cos 2(\eta+\varrho)}\right)$

unde fit

$$dt = \frac{d\varrho \sqrt{(aa+ff\sin\varrho^2)(bb+cc+(bb-cc)\cos 2(\eta+\varrho))}}{\sqrt{((D-4fg\cos\varrho)(bb+cc+(bb-cc)\cos 2(\eta+\varrho))-\frac{1}{2}CC)}}.$$

Nunc dato tempore $t$ per $\varrho$, pariter ac $u$, inde colligimus pressionem $\pi$ porroque intervallum $s$ ex hac aequatione

$$\frac{2\Pi}{M} gsdt = (cc-bb)\, du \sin(\eta+\varrho)\cos(\eta+\varrho) + aaud\varrho + (cc-bb)\, ud\varrho \cos 2(\eta+\varrho).$$

Tum vero obtinemus $x = \frac{-d\varrho}{dt}$; $y = u\cos(\eta+\varrho)$ et $z = u\sin(\eta+\varrho)$ ac denique $\varphi = -\int udt$.

CO-



## COROLL. 1.

941. Si initio punctum L fuerit in D ut sit ZD = r, ibique quieverit, posito $t = 0$ erat $u = 0$, $\frac{d\varrho}{dt} = 0$, ob $x = 0$; ideoque constantes ita definiri oportet, ut sit $C = 0$; et $D = 4fg\cos r$: unde fit $dt = d\varrho \sqrt{\frac{aa+ff\sin\varrho^2}{4fg(\cos r - \cos\varrho)}}$, sicque $\varrho > r$. Porro est $u = 0$, hinc $\varphi = 0$, et $s = 0$; pressio autem $\Pi$ hinc facile innotescit, et cum $\varrho$ ad 90° augeri possit, corpus quasi procumbet. Hic ergo motus neque a positione axium principalium IB et IC, neque a radio basium cylindri $e$ pendet.

## COROLL. 2.

942. Si initio recta IL fuerit verticalis seu $\varrho = 0$, et corpus circa eam gyrari coeperit celeritate angulari $\varepsilon$ in sensum AB, ut fuerit O in L, ideoque $\alpha = 90°$, $\beta = \eta$ et $\gamma = 90° - \eta$: initio erat $x = \frac{-d\varrho}{dt} = 0$, $y = \varepsilon\cos\eta$ et $z = \varepsilon\sin\eta$. Hinc fiunt constantes $C = \varepsilon(bb+cc+(bb-cc)\cos 2\eta)$ et $D = 4fg + \frac{1}{2}\varepsilon\varepsilon(bb+cc+(bb-cc)\cos 2\eta)$: unde colligitur $u = \frac{\varepsilon(bb+cc+(bb-cc)\cos 2\eta)}{bb+cc+(bb-cc)\cos 2(\eta+\varrho)}$ atque $\frac{d\varrho^2(aa+ff\sin\varrho^2)}{dt^2}$

$$= 4fg(1-\cos\varrho) + \frac{\frac{1}{2}\varepsilon\varepsilon(bb-cc)(bb+cc+(bb-cc)\cos 2\eta)(\cos 2(\eta+\varrho)-\cos 2\eta)}{bb+cc+(bb-cc)\cos 2(\eta+\varrho)}.$$

## COROLL. 3.

943. Si esset $bb = cc$, fieret $dt = d\varrho \sqrt{\frac{aa+ff\sin\varrho^2}{4fg(1-\cos\varrho)}}$, et recta IL perpetuo maneret verticalis: corpusque circa eam uniformiter gyrari pergeret: cum enim denominator contineat $\sqrt{(1-\cos\varrho)} = \sin\frac{1}{2}\varrho\sqrt{2}$ nonnisi tempore elapso infinito arcus $\varrho$ finitus evaderet: quod idem evenit, si fuerit vel $\eta = 0$, vel $\theta = 0$, hoc est si recta IL fuerit axis principalis.

## SCHOLION.

944. Nisi axis longitudinalis simul sit axis principalis corporis, ob multitudinem literarum vix patet, quomodo formulae [illegible]

 erutae

ritae generaliter evolvi queant, quod tamen inferius suscipiemus. Verum si hujusmodi corporum cylindricorum tantum motus quasi infinite parvos consideremus, ad quod necesse est, ut in recta centrali LF (fig 118.) centrum inertiae I infra centrum circuli G cadat, corpusque infinite parum de statu quietis deturbetur, oscillationes vel vacillationes minimae orientur, quarum indolem ex formulis nostris generalibus determinare licebit. Hic non opus est, ut totum corpus sit cylindricum, sed sufficit, si ejus termini circa M et N sint cylindrici, quibus super planis horizontalibus firmis P et Q sustentetur, quin etiam sufficit, si tantum circa contactum utriusque termini figura fuerit cylindrica, siquidem motus tantum admittimus infinite parvos. Deinde inter sustentacula P et Q annexum esse potest corpus pendulum quodcunque FMHn, ut oriatur pendulum non circa axem fixum linearem, sed circa terminos cylindricos planis horizontalibus incumbentes mobile, cujus motum oscillatorium definiri oporteat. In tali ergo pendulo primo notetur ejus centrum inertiae I, per quod ducatur recta $mn$ axi geometrico cylindri MN parallela, quae est axis longitudinalis jugiter manens horizontalis. Ducatur porro ex I ad MN recta perpendicularis IGL, quae si fuerit verticalis, corpus in quiete versabitur: ac si intervallum GI ponamus $= f$, in superioribus formulis litteram $f$ negative sumere debemus. Tum pro figura cylindrica terminorum sit radius basis $= e$, qui autem, ut vidimus, prorsus non in computum ingreditur, ita ut perinde sit sive termini sint crassiores sive graciliores. Quodsi recta IG $= f$ minor fuerit, quam GF $= e$, totumque corpus supra sustentacula P et Q versetur, motus prodit similis ei, quo cunae agitari solent. Quicquid autem sit centrum inertiae I, perpetuo in eadem recta verticali manebit, unde tota investigatio ad motum gyratorium definiendum perducitur, in quo centrum inertiae I ut quiescens consideramus.

Fig. 124.

### PROBLEMA 115.

945. Si corpus, quod basibus cylindricis super planis horizontalibus incumbit, infinite parum de situ quietis deturbetur, eique forte simul motus infinite parvus imprimatur, determinare motum vacillatorium, quo agitabitur.

### SOLUTIO.

Fig. 121. In formulis nostris generalibus primo intervallum GI $= f$ negativum statuatur: deinde arcus ZL $= \varrho$, quo recta centralis LGI a situ verticali declinat, ut infinite parvus spectari debet, perinde atque celeritas



ritas angularis $\varepsilon$: unde quantitates $x = \varepsilon \cos\alpha$, $y = \varepsilon \cos\beta$, $z = \varepsilon \cos\gamma$, ut evanescentes tractari debent. Quomodocunque ergo axes principales IA, IB, IC respectu rectae centralis GI et axis longitudinalis $mn$ fuerint dispositi, quorum situs cum arcubus LA $= \zeta$, LB $= \eta$, LC $= \theta$, tum angulis ZLA $= \mathfrak{f}$, ZLB $= \mathfrak{g}$, ZLC $= \mathfrak{h}$ definitur, primo habebimus $\sin\varrho = \varrho$, $\cos\varrho = 1$, deinde producta $xy$, $xz$ et $yz$ omitti poterunt; unde fiet $\cos l = \cos\zeta$, $\cos m = \cos\eta$ et $\cos n = \cos\theta$; et aequationes solutionem continentes ex probl. 113. ob $\frac{\Pi}{M} = 1 - \frac{fdd.\cos\varrho}{2g\,dt^2}$ $= 1$ erunt:

$$\begin{aligned}
&\text{I. } aadx = -2fg\varrho\,dt \sin\mathfrak{f} \sin\zeta + 2gs\,dt \cos\mathfrak{f} \sin\zeta\\
&\text{II. } bbdy = 2fg\varrho\,dt \sin\mathfrak{g} \sin\eta + 2gs\,dt \cos\mathfrak{g} \sin\eta\\
&\text{III. } ccdz = 2fg\varrho\,dt \sin\mathfrak{h} \sin\theta + 2gs\,dt \cos\mathfrak{h} \sin\theta
\end{aligned}$$

unde ex §. 938. haec integralis est derivata

$$aaxx + bbyy + cczz = C - 2fg\varrho\varrho - \frac{ff\varrho\varrho d\varrho^2}{dt^2}$$

ob $\cos\varrho = 1 - \frac{1}{2}\varrho\varrho$, quia hic infinite parvum $\varrho\varrho$ negligere non licet; Deinde habemus:

$$\text{IV. } y\cos\theta - z\cos\eta = -\frac{d\varrho}{dt}\cos\mathfrak{f}\sin\zeta$$

$$\text{V. } z\cos\zeta - x\cos\theta = -\frac{d\varrho}{dt}\cos\mathfrak{g}\sin\eta$$

$$\text{VI. } x\cos\eta - y\cos\zeta = -\frac{d\varrho}{dt}\cos\mathfrak{h}\sin\theta$$

atque ex §. §. 936. et 937.

$$\begin{aligned}
d\varrho &= -dt\,(x\sin\mathfrak{f}\sin\zeta + y\sin\mathfrak{g}\sin\eta + z\sin\mathfrak{h}\sin\theta)\\
d\phi &= -dt\,(x\cos\zeta + y\cos\eta + z\cos\theta).
\end{aligned}$$

Cum nunc sit IV. $x$ + V. $y$ + VI. $z = 0$, erit

$$x\cos\mathfrak{f}\sin\zeta + y\cos\mathfrak{g}\sin\eta + z\cos\mathfrak{h}\sin\theta = 0.$$

Deinde ex n°. I. II. III. in subsidium vocando formulas n°. I. et II. ex §. 934. colligimus

$$aax\cos\zeta + bby\cos\eta + ccz\cos\theta = A.$$

et pro intervallo $s$ determinando

$$aadx\cos\mathfrak{f}\sin\zeta + bbdy\cos\mathfrak{g}\sin\eta + ccdz\cos\mathfrak{h}\sin\theta = 2gs\,dt.$$

Statuamus $d\varrho = -u\,dt$ et $d\phi = -v\,dt$, atque ob $x\cos\mathfrak{f}\sin\zeta + y\cos\mathfrak{g}\sin\eta + z\cos\mathfrak{h}\sin\theta = 0$ consequimur

$x = u$

$$x = u \sin \mathfrak{f} \sin \zeta + v \cos \zeta; \quad y = u \sin \mathfrak{g} \sin \eta + v \cos \eta; \quad z = u \sin \mathfrak{h} \sin \theta + v \cos \theta$$ hincque
$$A = u\,(aa \sin \mathfrak{f} \sin \zeta \cos \zeta + bb \sin \mathfrak{g} \sin \eta \cos \eta + cc \sin \mathfrak{h} \sin \theta \cos \theta) + v\,(aa \cos \zeta^2 + bb \cos \eta^2 + cc \cos \theta^2).$$

Ponamus ad abbreviandum

$$\left.\begin{array}{l} bb \cos \mathfrak{h} \cos \eta \sin \theta - cc \cos \mathfrak{g} \sin \eta \cos \theta = \mathfrak{A} \\ cc \cos \mathfrak{f} \cos \theta \sin \zeta - aa \cos \mathfrak{h} \sin \theta \cos \zeta = \mathfrak{B} \\ aa \cos \mathfrak{g} \cos \zeta \sin \eta - bb \cos \mathfrak{f} \sin \zeta \cos \eta = \mathfrak{C} \end{array}\right\} \text{eritque } \mathfrak{A} \cos \mathfrak{f} \sin \zeta + \mathfrak{B} \cos \mathfrak{g} \sin \eta + \mathfrak{C} \cos \mathfrak{h} \sin \theta = 0$$

$$aa \cos \zeta^2 + bb \cos \eta^2 + cc \cos \theta^2 = \mathfrak{D}: \quad aa \sin \mathfrak{f} \sin \zeta \cos \zeta + bb \sin \mathfrak{g} \sin \eta \cos \eta + cc \sin \mathfrak{h} \sin \theta \cos \theta = \mathfrak{F}$$

et habebimus

$$v = \frac{A - \mathfrak{F} u}{\mathfrak{D}}$$

$$x = \frac{A \cos \zeta + \mathfrak{A} u}{\mathfrak{D}}; \quad y = \frac{A \cos \eta + \mathfrak{B} u}{\mathfrak{D}}; \quad z = \frac{A \cos \theta + \mathfrak{C} u}{\mathfrak{D}}$$

qui valores, in aequatione integrali vim vivam complectente substituti, ob $\frac{d\rho}{dt} = -u$ dabunt:

$$\frac{AA\mathfrak{D} + 2Au(\mathfrak{A}a^2 \cos \zeta + \mathfrak{B}b^2 \cos \eta + \mathfrak{C}c^2 \cos \theta) + uu(\mathfrak{A}^2 a^2 + \mathfrak{B}^2 b^2 + \mathfrak{C}^2 c^2)}{\mathfrak{D}\mathfrak{D}} = C - 2fg\rho\rho - ff\rho\rho uu$$

quae aequatio ob $\mathfrak{A}aa \cos \zeta + \mathfrak{B}bb \cos \eta + \mathfrak{C}cc \cos \theta = 0$ abit in hanc

$$AA\mathfrak{D} + (\mathfrak{A}^2 a^2 + \mathfrak{B}^2 b^2 + \mathfrak{C}^2 c^2)\,uu = C\mathfrak{D}\mathfrak{D} - 2\mathfrak{D}\mathfrak{D}fg\rho\rho - \mathfrak{D}\mathfrak{D}ff\rho\rho uu$$

ubi si loco $C\mathfrak{D}\mathfrak{D} - AA\mathfrak{D}$ ponatur $B\mathfrak{D}\mathfrak{D}$, fiet

$$u = \frac{\mathfrak{D}\sqrt{(B - 2fg\rho\rho)}}{\sqrt{(\mathfrak{A}^2 a^2 + \mathfrak{B}^2 b^2 + \mathfrak{C}^2 c^2 + \mathfrak{D}\mathfrak{D}ff\rho\rho)}}.$$

Statuamus porro $\mathfrak{A}^2 a^2 + \mathfrak{B}^2 b^2 + \mathfrak{C}^2 c^2 = \mathfrak{D}\mathfrak{D}\mathfrak{H}\mathfrak{H}$, et rejecto termino infinite parvo $\mathfrak{D}\mathfrak{D}ff\rho\rho$ habebimus

$$u = \frac{\sqrt{(B - 2fg\rho\rho)}}{\mathfrak{H}} \text{ et } dt = \frac{-\mathfrak{H}\,d\rho}{\sqrt{(B - 2fg\rho\rho)}},$$

unde colligimus $t = \text{Const.} + \frac{\mathfrak{H}}{\sqrt{2fg}} \text{Arc.} \cos \frac{\rho\sqrt{2fg}}{\sqrt{B}}$, seu

$$\rho = \frac{\sqrt{B}}{\sqrt{2fg}} \cos \frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}}, \text{ et } u = \frac{\sqrt{B}}{\mathfrak{H}} \sin \frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}}$$

tum



tum vero $v = \frac{A}{\mathfrak{D}} - \frac{\mathfrak{F}\sqrt{B}}{\mathfrak{D}\mathfrak{H}} \sin \frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}}$; hincque

$$\varphi = D - \frac{At}{\mathfrak{D}} - \frac{\mathfrak{F}\sqrt{B}}{\mathfrak{D}\sqrt{2fg}} \cos \frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}} = D - \frac{At}{\mathfrak{D}} - \frac{\mathfrak{F}\rho}{\mathfrak{D}}.$$

Deinde reperiemus:

$$s = \frac{-\sqrt{B}f}{\mathfrak{D}\mathfrak{H}\mathfrak{H}\sqrt{2g}}\,(aa\,bb \sin \mathfrak{h} \cos \mathfrak{h} \sin \theta^2 + aa\,cc \sin \mathfrak{g} \cos \mathfrak{g} \sin \eta^2 + bb\,cc \sin \mathfrak{f} \cos \mathfrak{f} \sin \zeta^2) \cos \frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}}.$$

Denique vero erit

$$ss = xx + yy + zz = \frac{AA - 2A\mathfrak{F}u + (\mathfrak{A}\mathfrak{A} + \mathfrak{B}\mathfrak{B} + \mathfrak{C}\mathfrak{C})\,uu}{\mathfrak{D}\mathfrak{D}}$$

sicque omnia ad datum tempus sunt definita. Ceterum hic notasse juvat, esse $\mathfrak{A}\mathfrak{A} + \mathfrak{B}\mathfrak{B} + \mathfrak{C}\mathfrak{C} = \mathfrak{D}\mathfrak{D} + \mathfrak{F}\mathfrak{F}$, ita ut sit $ss = vv + uu$.

### COROLL. 1.

946. Cum sit $\rho = \frac{\sqrt{B}}{\sqrt{2fg}} \cos \frac{(t+\delta)\sqrt{2fg}}{\mathfrak{H}}$; patet arcum $ZL = \rho$ seu declinationem rectae LI a situ verticali ad similitudinem penduli variari, hujusque lineae LI vacillationes isochronas fore oscillationibus penduli, cujus longitudo est $= \frac{\mathfrak{H}\mathfrak{H}}{f}$, quae longitudo est $= \frac{\mathfrak{A}^2 a^2 + \mathfrak{B}^2 b^2 + \mathfrak{C}^2 c^2}{\mathfrak{D}\mathfrak{D}f}$.

### COROLL. 2.

947. Deinde cum sit $\varphi = D - \frac{At}{\mathfrak{D}} - \frac{\mathfrak{F}\rho}{\mathfrak{D}}$, punctum L motu medio revolvitur circa verticem Z celeritate angulari $= \frac{A}{\mathfrak{D}}$; verum locus medius corrigi debet particula $\frac{\mathfrak{F}\rho}{\mathfrak{D}}$. Sin autem sit constans $A = 0$, angulus DZL parumper mutatur, nisi sit $\mathfrak{F} = 0$.

### COROLL. 3.

948. Si ergo revolutiones corporis circa axem verticalem IZ excludantur, ut sit $A = 0$, atque initio fuerit $\varphi = 0$; $\xi = t$



et

et celeritas angularis $x = \epsilon$, constantes ita definientur, ut fit

$$D = \frac{\mathfrak{F}r}{\mathfrak{D}};\; r = \frac{\sqrt{B}}{\sqrt{2fg}} \cos\frac{\delta\sqrt{2fg}}{\mathfrak{H}} \text{ et } \epsilon\epsilon = \frac{(\mathfrak{D}\mathfrak{D}+\mathfrak{F}\mathfrak{F})B}{\mathfrak{D}\mathfrak{D}\mathfrak{H}\mathfrak{H}}\left(\sin\frac{\delta\sqrt{2fg}}{\mathfrak{H}}\right)^2$$

$$\text{Ergo } \sqrt{B} = \frac{r\sqrt{2fg}}{\cos\frac{\delta\sqrt{2fg}}{\mathfrak{H}}} = \frac{\epsilon\mathfrak{D}\mathfrak{H}}{\sin\frac{\delta\sqrt{2fg}}{\mathfrak{H}}\sqrt{(\mathfrak{D}\mathfrak{D}+\mathfrak{F}\mathfrak{F})}}, \text{ ideoque}$$

$$\tan\frac{\delta\sqrt{2fg}}{\mathfrak{H}} = \frac{\epsilon\mathfrak{D}\mathfrak{H}}{r\sqrt{2fg}\sqrt{(\mathfrak{D}\mathfrak{D}+\mathfrak{F}\mathfrak{F})}},$$ unde et constans B innotescit.

Sin autem fuerit $\epsilon = 0$; prodit $\sqrt{B} = r\sqrt{2fg}$, et $\delta = 0$.

*EXEMPLUM.*

Fig. 123. 949. Ponamus rectam IM, quae per centrum inertiae I axi geometrico cylindri (MN fig. 124.) parallela ducitur, simul esse corporis axem principalem, et habebimus uti §. 940. $\mathfrak{f} = 90°$, $\mathfrak{g} = 180°$, $\mathfrak{h} = 0$ et $\zeta = 90°$, atque $\theta = 90° - \eta$. Hinc autem colligimus:

$$\mathfrak{A} = bb\cos\eta^2 + cc\sin\eta^2;\; \mathfrak{B} = 0;\; \mathfrak{C} = 0;\; \mathfrak{D} = \mathfrak{A} \text{ et } \mathfrak{F} = 0,$$

ergo $aa = \mathfrak{H}\mathfrak{H}$: unde longitudo penduli simplicis isochroni fit $= \frac{aa}{f}$. Tum vero axis IA horizontalis manebit immotus. Ac si initio ubi $\varrho = r$, corpus motum a quiete inceperit, erit $\delta = 0$, et $\sqrt{B} = r\sqrt{2fg}$: ex quibus reliquae quantitates variabiles colliguntur,

$$\varrho = r\cos\frac{t\sqrt{2fg}}{a};\; u = \frac{r\sqrt{2fg}}{a}\sin\frac{t\sqrt{2fg}}{a};\; v = 0;\text{ ob } A = 0$$

$$\text{et } x = u = \frac{r\sqrt{2fg}}{a}\sin\frac{t\sqrt{2fg}}{a};\; y = 0 \text{ et } z = 0, \text{ atque } x = x.$$

Fig. 124. Revera autem adjuncto motu progressivo centrum inertiae I in recta verticali alternatim ascendet ac descendet, cylindro superiore MN hunc motum sequente, dum super planis P et Q liberrime incedere potest, neque a frictione impediri assumitur.

*SCHOLION.*

950. Quia magnitudo cylindri MN in computum non ingreditur, eadem solutio valebit, si eius crassities evanescat, corpusque annexum ab axe lineari esset suspensum. Ex quo hic motus convenire debere videtur cum motu oscillatorio supra definito, quod tamen longe aliter usu venit; quoniam pro motu oscillatorio vero longitudo penduli simplicis isochroni prodiit $= f + \frac{aa}{f} = \frac{aa+ff}{f}$, cum hic tantum fit $= \frac{aa}{f}$.

Causa



Causa hujus discriminis in eo est sita, quod supra in doctrina oscillationum axem MN fixum assumsimus, dum hic liberrime mobilis statuitur. Hinc patet, ob libertatem axis, etsi plano horizontali incumbat, oscillationes multo promtiores fieri, quam si axis in eodem loco firmiter detineretur. Atque hoc etiam Theoriae omnino est conforme, si enim (fig. 118.) circulus ΠMTN planum semper in eodem puncto T tangere debeat, praeter pressionem Π vis quaedam horizontalis in calculum introduci debet, quae si ponatur $= \Theta$ secundum TH urgens, ut punctum T maneat constans, ob TP $= f\sin\varrho$ esse oportet $\frac{f\,dd.\,f\sin\varrho^2}{dt^2} = \frac{-2\Theta g}{M}$. Ex hac autem vi quoque nascitur momentum respectu axium principalium, qua propterea motus gyratorius afficitur, ut talis prodeat, qualem supra in motus oscillatorii investigatione determinavimus. Ceterum hic probe notasse juvabit, si axiculi penduli planis horizontalibus politissimis incumbant, motum oscillatorium plurimum discrepare posse ab eo, qui orietur, si firmiter detinerentur, et multo quidem promtiorem esse futurum. Minima autem frictio hoc discrimen tollere, motumque ad oscillationum legem reducere valebit. Hujus autem problematis solutio nos ad solutionem problematis generalis n°. 113. manuducet.

*PROBLEMA 116.*

951. Si corpus cylindricum quodcunque super plano horizontali moveatur utcunque, aequationes supra inventas, quibus ejus motus definitur, resolvere atque ad integrationem perducere.

*SOLUTIO.*

Maneant hic omnia, uti supra in problemate 113. sunt constituta, atque in recta centrali LIGF sumamus ut ibi centrum inertiae I a puncto F magis remotum, quam centrum sectionis cylindri G, ponendo intervallum GI $= f$. Ex aequationibus igitur differentialibus ibi exhibitis jam unam aequationem integralem eruimus, quae est: Fig. 121.

$$aaxx + bbyy + cczz = C - 4fg\cos\varrho - \frac{ff\,d\varrho^2\sin\varrho^2}{dt^2}.$$

Praeterea vero ternae priores aequationes cum ternarum posteriorum iis postremis terminis applicatarum abeunt in has formas



I. $aax$

$$\text{I. } aadx + (cc-bb)\,yzdt = \frac{-2\Pi fg}{M}.\,dt \sin\mathfrak{f}\sin\zeta\sin\varrho - \frac{2\Pi gs}{M}.\frac{dt\,dl\sin l}{d\varrho}$$

$$\text{II. } bbdy + (aa-cc)\,xzdt = \frac{-2\Pi fg}{M}.\,dt \sin\mathfrak{g}\sin\eta\sin\varrho - \frac{2\Pi gs}{M}.\frac{dt\,dm\sin m}{d\varrho}$$

$$\text{III. } ccdz + (bb-aa)\,xydt = \frac{-2\Pi fg}{M}.\,dt \sin\mathfrak{h}\sin\theta\sin\varrho - \frac{2\Pi gs}{M}.\frac{dt\,dn\sin n}{d\varrho}.$$

Hinc jam colligatur forma I. $\cos l$ + II. $\cos m$ + III. $\cos n$, et quia $dl\sin l\cos l + dm\sin m\cos m + dn\sin n\cos n = 0$, termini ultimi intervallum $s$ involventes se destruent: tum vero etiam per relationes §. 934. traditas reperitur

$$\sin\mathfrak{f}\sin\zeta\cos l + \sin\mathfrak{g}\sin\eta\cos m + \sin\mathfrak{h}\sin\theta\cos n = 0$$

ita ut quoque penultimi tollantur. Quocirca perveniemus ad hanc aequationem.

$$aadx\cos l + bbdy\cos m + ccdz\cos n + aaxzdt\cos m + bbxydt\cos n + ccyzdt\cos l = 0$$
$$-aaxydt\cos n - bbyzdt\cos l - ccxzdt\cos m$$

at ex terminis posterioribus, est

$$z\cos m - y\cos n = \frac{-dl\sin l}{dt};\ x\cos n - z\cos l = \frac{-dm\sin m}{dt};\ y\cos l - x\cos m = \frac{-dn\sin n}{dt}$$

quibus valoribus substitutis obtinemus

$$aadx\cos l + bbdy\cos m + ccdz\cos n - aax\,dl\sin l - bby\,dm\sin m - ccz\,dn\sin n = 0$$

cujus integralis est

$$aax\cos l + bby\cos m + ccz\cos n = \mathrm{D}.$$

Deinde loco $x$, $y$ et $z$ introducamus novas variabiles hinc definiendas

$$x\cos\zeta + y\cos\eta + z\cos\theta = p$$
$$x\cos\mathfrak{f}\sin\zeta + y\cos\mathfrak{g}\sin\eta + z\cos\mathfrak{h}\sin\theta = q$$
$$x\sin\mathfrak{f}\sin\zeta + y\sin\mathfrak{g}\sin\eta + z\sin\mathfrak{h}\sin\theta = r$$

eritque primo $d\varrho = -rdt$; porro ob $x\cos l + y\cos m + z\cos n = p\cos\varrho + q\sin\varrho$; erit $d\Phi = -dt\,(p\cos\varrho + q\sin\varrho)$. Praeterea ob $x\,dl\sin l + ydm\sin m + zdn\sin n = 0$, fit, $p\sin\varrho - q\cos\varrho = 0$. Quam ob rem ponamus

$$p = u\cos\varrho \text{ et } q = u\sin\varrho \text{ eritque } d\Phi = -udt \text{ et } d\varrho = -rdt,$$

at ex illis aequationibus assumtis elicimus

$$x = r\sin\mathfrak{f}\sin\zeta + u\cos l;\ y = r\sin\mathfrak{g}\sin\eta + u\cos m;\ z = r\sin\mathfrak{h}\sin\theta + u\cos n$$

hincque $xx + yy + zz = rr + uu = ss$.

Nunc aequatio integralis modo ante inventa praebet

D = r



$$\mathrm{D} = r\,(aa\sin\mathfrak{f}\sin\zeta\cos l + bb\sin\mathfrak{g}\sin\eta\cos m + cc\sin\mathfrak{h}\sin\theta\cos n)$$
$$+ u\,(aa\cos l^2 + bb\cos m^2 + cc\cos n^2)$$

qua $u$ determinatur per $r$ et $\varrho$; ideoque et $x$, $y$, $z$. Denique aequatio integralis primum inventa

$$aaxx + bbyy + cczz = \mathrm{C} - 4fg\cos\varrho - ffrr\sin\varrho^2$$

quia tantum $r$ et $\varrho$ continet, determinabit $r$ per $\varrho$, indeque aequatio $dt = \frac{-d\varrho}{r}$ pro dato tempore $t$ omnes quantitates motum continentes manifestabit.

Quodsi ad abbreviandam ponantur constantes:

$$aa\cos\zeta^2 + bb\cos\eta^2 + cc\cos\theta^2 = \mathfrak{A}$$
$$aa\cos\mathfrak{f}\sin\zeta\cos\zeta + bb\cos\mathfrak{g}\sin\eta\cos\eta + cc\cos\mathfrak{h}\sin\theta\cos\theta = \mathfrak{B}$$
$$aa\cos\mathfrak{f}^2\sin\zeta^2 + bb\cos\mathfrak{g}^2\sin\eta^2 + cc\cos\mathfrak{h}^2\sin\theta^2 = \mathfrak{C}$$
$$aa\sin\mathfrak{f}\sin\zeta\cos\zeta + bb\sin\mathfrak{g}\sin\eta\cos\eta + cc\sin\mathfrak{h}\sin\theta\cos\theta = \mathfrak{D}$$
$$aa\sin\mathfrak{f}\cos\mathfrak{f}\sin\zeta^2 + bb\sin\mathfrak{g}\cos\mathfrak{g}\sin\eta^2 + cc\sin\mathfrak{h}\cos\mathfrak{h}\sin\theta^2 = \mathfrak{E}$$
$$aa\sin\mathfrak{f}^2\sin\zeta^2 + bb\sin\mathfrak{g}^2\sin\eta^2 + cc\sin\mathfrak{h}^2\sin\theta^2 = \mathfrak{F}$$

nostrae aequationes integrales erunt

$$\mathrm{D} = r\,(\mathfrak{D}\cos\varrho + \mathfrak{E}\sin\varrho) + u\,(\mathfrak{A}\cos\varrho^2 + 2\mathfrak{B}\sin\varrho\cos\varrho + \mathfrak{C}\sin\varrho^2)$$
$$\mathrm{C} - 4fg\cos\varrho - ffrr\sin\varrho^2 = \mathfrak{F}rr + 2ru\,(\mathfrak{D}\cos\varrho + \mathfrak{E}\sin\varrho)$$
$$+ uu\,(\mathfrak{A}\cos\varrho^2 + 2\mathfrak{B}\sin\varrho\cos\varrho + \mathfrak{C}\sin\varrho^2)$$

ex quibus concluditur

$$rr = \frac{DD - (\mathfrak{A}\cos\varrho^2 + 2\mathfrak{B}\sin\varrho\cos\varrho + \mathfrak{C}\sin\varrho^2)(C - 4fg\cos\varrho)}{(\mathfrak{D}\cos\varrho + \mathfrak{E}\sin\varrho)^2 - (\mathfrak{A}\cos\varrho^2 + 2\mathfrak{B}\sin\varrho\cos\varrho + \mathfrak{C}\sin\varrho^2)(\mathfrak{F} + ff\sin\varrho^2)}$$

Hinc pro tempore adipiscimur $t = \int\frac{-d\varrho}{r}$, et cum sit $u = \frac{D - r(\mathfrak{D}\cos\varrho + \mathfrak{E}\sin\varrho)}{\mathfrak{A}\cos\varrho^2 + 2\mathfrak{B}\sin\varrho\cos\varrho + \mathfrak{C}\sin\varrho^2}$, erit angulus $\Phi = -\int udt = \int\frac{ud\varrho}{r}$.

Cum autem ad quodvis tempus $t$ tam arcum $\varrho$ quam angulum $\Phi$ determinaverimus, totus motus erit perfecte cognitus.

### COROLL. 1.

952. Quantitates ergo $\mathfrak{A}$, $\mathfrak{C}$, et $\mathfrak{F}$ necessario sunt positivae, et $\mathfrak{B}$ ad $\mathfrak{A}$ et $\mathfrak{B}$ ad $\mathfrak{A}$ et $\mathfrak{C}$ ita referuntur, ut sit

$$\mathfrak{A}\mathfrak{C} - \mathfrak{B}\mathfrak{B} = aabb\sin\mathfrak{h}^2\sin\theta^2 + aacc\sin\mathfrak{g}^2\sin\eta^2 + bbcc\sin\mathfrak{f}^2\sin\zeta^2$$

unde patet, formam $\mathfrak{A}\cos\varrho^2 + 2\mathfrak{B}\sin\varrho\cos\varrho + \mathfrak{C}\sin\varrho^2$ in duos factores simplices resolvi non posse,

### COROLL. 2.

953. Ex hac solutione generali casus in praecedente problemate evolutus facile deducitur, sumendo $f$ negative, et arcum $\varrho$ infinite parvum, unde fit

$$rr = \frac{DD - \mathfrak{A}C - 4\mathfrak{A}fg\cos\varrho}{\mathfrak{D}\mathfrak{D} - \mathfrak{A}\mathfrak{F}} = \frac{Const. + 4\mathfrak{A}fg\cos\varrho}{\mathfrak{A}f - \mathfrak{D}\mathfrak{D}}.$$

Reperitur autem valoribus evolutis

$$\mathfrak{A}\mathfrak{F} - \mathfrak{D}\mathfrak{D} = aabb\cos\mathfrak{h}^2 \sin\theta^2 + aacc\cos\mathfrak{g}^2 \sin\eta^2 + bbcc\cos\mathfrak{f}^2 \sin\zeta^2$$

unde longitudo penduli simplicis isochroni simplicius quam supra ita exhibetur, ut sit $= \dfrac{aabb\cos\mathfrak{h}^2 \sin\theta^2 + aacc\cos\mathfrak{g}^2 \sin\eta^2 + bbcc\cos\mathfrak{f}^2 \sin\zeta^2}{f(aa\cos\zeta^2 + bb\cos\eta^2 + cc\cos\theta^2)}$.

### SCHOLION.

954. His de motu corporum cylindricorum super plano horizontali expeditis, institueram pauca de motu super plano inclinato adjungere: verum si motus fuerit simplex, res nullam habet difficultatem, sin autem sit complicatus, in calculos incommodos incideremus. Quare cum in praxi frictionem ab his motibus separare haud liceat, motus saltem simpliciores super plano inclinato ita persequabimus, ut simul frictionis rationem habeamus, ex quo peculiarem tractatum de motu corporum rigidorum a frictione perturbato adjungi conveniet.



# SUPPLEMENTUM
## DE
# MOTU CORPORUM RIGIDORUM A FRICTIONE PERTURBATO.

# CAPUT I.
## DE FRICTIONE IN GENERE.

### *DEFINITIO.*

955. *Frictio* eſt reſiſtentia, quam corpus ſuper ſuperficie aſpera incedens eamque radens, in motu ſuo patitur.

Eſt ergo frictio vis motus directioni contraria, et baſi corporis, qua ſuperficiem tangit, applicata

### *COROLL.* 1.

956. Quamdiu corpus quieſcit, frictio nullam plane vim exerit, ſtatim autem atque corpus movetur, ſubito ejus vis exiſtit motui ſemper contraria eamque propterea retardans.

### *COROLL.* 2.

957. Si corpus a vi quapiam ſollicitetur, etiamſi quieſcat, frictio ſe illi vi opponit, quoniam in prima motus generatione ſtatim exiſtit, ac niſi vis ſollicitans frictionem ſuperet, corpus movere non valebit.

### *COROLL.* 3.

958. Quia directio frictionis motus directioni jugiter eſt contraria, mutata motus directione ſimul frictionis directio mutatur. Statim autem atque corpus ad quietem redigitur, uti motus directio tollitur, ita ſubito frictio evaneſcit.

### *EXPLICATIO.*

959. Ad haec, quae ad frictionem pertinent, dilucidanda, ad omnes circumſtantias, quae ad frictionem quicquam conferre poſſe videantur, attendi conveniet, etſi adhuc minime pateat, quid quisque efficere valeat. Primo igitur ſuperficies, ſuper qua fit inceſſus, conſiderari debet, quae ſive ſit plana ſive ſecus parum refert, quoniam

Fig. 125. quovis tempore ad contactum eſt reſpiciendum. Sit igitur EF ſuperficies, quam tanquam planam contemplemur, ſiquidem hinc facile ad ſuperficies convexas et concavas judicium extendere licebit: hujus ergo aſperitas praecipuum locum inter cauſas frictionis tenet, quoniam ſi ſuperficies perfecte eſſet polita et laevigata; frictioni nullus locus relinqueretur: ex quo colligitur, quo magis ſuperficies fuerit aſpera, eo majorem frictionem fieri oportere. Deinde baſis corporis AB, qua fit contactus, in computum eſt ducenda, cujus magnitudo et figura an quicquam ad frictionem conferat, nondum liquet, aſperitas vero certe cum aſperitate ſuperficiei conjuncta, ubi imprimis motui eſt obſtaculo, ita frictionem generare eſt putanda. Circa ipſum denique corpus ABCD praeter ejus maſſam reliquasque proprietates, ejus preſſio ad ſuperficiem ſine dubio maximi eſt momenti, quoniam ſi nulla vi ad eam apprimeretur, nulla certe frictio adeſſet, corpusque perinde moveretur, ac ſi ſuperficies abeſſet. Cum tandem frictio non niſi in motu cernatur, celeritas quoque tanquam inſigne frictionis momentum videri poſſet, ſed praeter expectationem videbimus, celeritatem nullo modo ad frictionem determinandam concurrere, quod eo magis eſt mirandum, cum ſublata celeritate omnis frictio certe ceſſet. Quod ſi ergo corpus ſecundum directionem BF ſuper ſuperficie promoveatur, vis aderit, qua id ſecundum directionem oppoſitam AE ſollicitatur, haecque vis frictio vocatur.

### SCHOLION.

960. Frictionem hic primo tanquam phaenomenon conſiderabo, cujus quantitas et indoles nobis experientia innotuerit, deinceps in ejus cauſas, quantum fieri licet, inquiſiturus. Cum enim hic phyſicae corporum qualitates, cujusmodi ſunt aſperitates ſuperficierum, et ratio, qua duae ſuperficies invicem appreſſae ſibi mutuo cedant, et minimis particulis quasdam impreſſiones inducant, totum quaſi negotium conficiant; ob defectum talis cognitionis corporum contenti eſſe debemus phaenomena frictionis ita accipere, prouti ea nobis ab experientia ſuppeditantur, quemadmodum etiam aliarum virium, quarum effectus in Mechanica evolvimus, origo minime eſt perſpecta. Quae ergo per experientiam nobis circa frictionis indolem innotuerunt, breviter recenſeamus.

### PHAENOMENON. 1.

961. *Si cetera ſint paria, frictio non pendet a corporis celeritate, ſed ſive id celerius incedat ſive tardius, eandem exerit vim, cujus directio ſemper eſt contraria motus directioni.*

CO-



### COROLL. 1.

962. Frictio ergo non tanquam functio quaedam celeritatis ſpectari poteſt, cum perpetuo eandem quantitatem ſervet, ſive motus ſit celerrimus, ſive tardiſſimus. Interim tamen, motu penitus ceſſante, ſubito evaneſcit.

### COROLL. 2.

963. Etſi autem frictio a motus celeritate neutiquam pendet, tamen ejus directio per motus directionem unice determinatur, quippe cui eſt contraria et in ipſo contactu applicata.

### SCHOLION.

964. De motu corporis abſoluto haec ſunt intelligenda, ſi ſuperficies, in qua corpus incedit, abſolute quieſcat, ſin autem haec ſuperficies ipſa moveatur, ex motu corporis reſpectivo ad ſuperficiem relato judicium eſt petendum. Scilicet ſi corpus reſpectu ſuperficiei quieſcat, etiamſi utcumque moveatur abſolute, frictio eſt nulla, ſin autem reſpectu ſuperficiei moveatur, frictio eam impetrat quantitatem, quam reliquae circumſtantiae exigunt, neque quantitas motus huc quicquam confert. Directio autem frictionis per directionem reſpectivam corporis reſpectu ſuperficiei conſtanter determinatur: neque igitur hic motum ſecundum duas treſve directiones reſolvere licet, et pro quolibet, quaſi ſolus adeſſet, frictionem definire, indeque frictionem totam colligere: ſed uti quantitas frictionis non a motus quantitate pendet, ita directio ſemper ex directione, ſecundum quam corpus ſuper ſuperficie incedit, definiri debet. Ceterum hoc phaenomenon non ita accurate per experimenta indicatur, ut nullis plane dubiis ſit ſubjectum: quin potius motus celerrimi ab hac regula aliquantillum recedere videntur. Quodſi forte veritati fuerit conſentaneum, id potius alii cauſae tribuamus, quam ſtabilitam frictionis notionem immutemus: et cum aberratio ſit valde parva, eam eo magis negligamus, cum alias nonnullas exiguas vires, quae ex eodem fonte atque frictio originem trahere videntur, negligere cogamur. Hic ſcilicet in eos tantum effectus, qui a frictione prouti vulgo concipi ſolet, inquirere conſtitui, de aliis motus obſtaculis minime ſollicitus.

### PHAENOMENON. 2.

965. *Si cetera ſint paria, quantitas frictionis etiam neque a figura neque magnitudine baſis, qua corpus ſuperficiem contingit, pendet; ſed ſive*

*ve ea fuerit major sive minor, et cujuscunque figurae, frictio eandem semper vim exerit.*

### COROLL. 1.

Fig. 125. 966. Quodsi ergo basis, qua corpus superficiem contingit, AB ponatur $= bb$; haec quantitas non in expressionem frictionis ingreditur, aeque parum ac velocitas corporis.

### COROLL. 2.

967. Neque etiam frictio mutatur, licet contactus in unico fiat puncto, quemadmodum evenit, si corpus sit globus seu corpus basi convexa praeditum: dummodo corpus superficiem radat.

### SCHOLION.

968. Hoc phaenomenon, etsi certissimis experimentis confirmatum, exceptionem tamen patitur, si corpus in acutissimam desinat cuspidem; qua superficiei infigi queat, quo casu sine dubio penitus coerceretur. Excipiendi scilicet hinc sunt casus, quibus superficies ab incedente corpore damnum patitur, de quibus etiam hic non tractabimus. Ceterum maxime paradoxum videbitur, quod a contactu in unico puncto facto tanta frictio nasci queat, quanta a basi sata vasta, cum frictio ab asperitate ambarum superficierum, quae se mutuo terunt producatur, in ampliori autem contactu plus asperitatis superari debeat. Verum hoc dubium mox evanescet, cum ostendemus, quomodo frictio se ratione pressionis habere debeat.

### PHAENOMENON. 3.

969. *Si cetera sint paria, frictio proportionalis est pressioni, qua corpus ad superficiem apprimitur: eoque majori pressionis parti aequatur, quo major fuerit asperitas superficierum se mutuo atterentium.*

### COROLL. 1.

970. Quodsi corpus nulla plane vi ad superficiem, super qua incedit, apprimatur, nullam etiam patietur frictionem; quae autem eo major evadet, quo magis appressio augetur.

### COROLL. 2.

971. Si ergo asperitas fuerit eadem, frictio, quam corpora super superficiebus incedentia patiuntur, certae cuidam parti pressionis aequatur, qua parte cognita, frictionis quantitas perfecte determinatur.

CO-



### COROLL. 3.

972. Quodsi ergo corpus ABCD vi $= P$ ad superficiem apprimatur, ac super ea incedat in directione BF, frictio erit $= \delta P$ (denotante $\delta$ partem illam memoratam) qua corpus secundum directionem oppositam AE retrahitur. Fig. 125.

### SCHOLION. 1.

973. Haec manifesta sunt, quando corpus motu progressivo incedit super superficie, quo casu frictio motus directioni est contraria. Verum si corpus insuper habeat motum quempiam gyratorium, videndum est, in quanam directione basis superficiem terat, huicque erit contraria frictionis directio, cujus quantitas cum ex pressione constet, effectus frictionis in motu corporis perturbando ex principiis supra stabilitis definiri poterit. Ceterum quemadmodum frictio a solo attritu corporis et superficiei oritur, patet si corpus ita volvendo promoveatur, ut nullus attritus existat, cujusmodi motus provolutio perfecta vocatur, nulla etiam frictio locum habebit: simulatque autem motus volutorius tantillo fuerit celerior vel tardior, quam illa conditio postulat, sicque attritus sese admisceat, etiamsi sit minimus, tamen statim subito plena frictio $\delta P$ effectum suum exerit. Quare phaenomena hinc orta ingentem saltum implicare debent, cum pro certa motus specie omnis frictio subito tollatur, dum autem motus tantillum inde discrepat, pleno effectu adsit.

### SCHOLION. 2.

974. Insigne calculi compendium hinc consequimur, quod frictio tam simpliciter exprimitur, et a sola pressione P cum fractione $\delta$, quam asperitas definit, pendet; si enim insuper tam a celeritate corporis quam ab ejus basi penderet, facile in calculos inextricabiles illaberemur. Ac si calculum ad praxin accomodare velimus, totum negotium ad valorem fractionis $\delta$ reducitur, quem unico experimento pro singulis corporum generibus assignasse sufficit. Pro corporibus autem ligneis experimenta ostendunt litterae $\delta$ valorem circiter $\frac{1}{3}$ tribui debere, si quidem eorum superficies mediocriter fuerit dolata, sin autem magis sit rudis et aspera, majorem valorem sortitur, quemadmodum e contrario corpora metallica probe polita pro littera $\delta$ fractionem $\frac{1}{4}$ adeoque minorem exigunt. Verum ex sequentibus patebit, quomodo quovis casu per experimenta conveniens fractionis $\delta$ quantitas facile explorari queat. Experientia autem didicimus, nullam superficiem ne-



que

que corpus tam perfecte poliri posse, ut frictio plane evanescat, quin potius semper satis notabili adhuc parti frictionis aequari deprehenditur. Quare quae supra de motu corporum super plano politissimo, quod nullam gignat frictionem, sunt allata, in praxi neutiquam locum inveniunt.

### PROBLEMA. 1.

975. Si corpus superficiei cuicunque incumbens quiescat, simulque a viribus quibuscunque sollicitetur, distinguere casus, quibus id vel ad motum impellatur vel in quiete perseveret.

### SOLUTIO.

Fig. 125. Omnes vires, quibus corpus ABCD sollicitatur, resolvantur in binas, quarum altera sit ad superficiem normalis, altera eidem parallela. Sit P summa omnium ad superficiem perpendicularium, quatenus corpus ab iis ad superficiem apprimitur, erit P pressio, foretque $\delta P$ frictio, si corpus moveretur. Quod ad alteras vires attinet, consideremus hic tantum casum, quo ab iis corpori motus progressivus induceretur, si nulla esset frictio; quoniam motus gyratorius ampliorem postulat evolutionem infra suscipiendam. Cum igitur corpus alium motum nisi secundum directionem superficiei recipere nequeat, vires huic parallelae quasi uni puncto applicatae spectentur, earumque quaeratur aequivalens, quae sit $= V$ secundum directionem BF urgens, atque manifestum est, quamdiu fuerit $V < \delta P$ corpus in quiete esse perseveraturum, neque id commoveri posse, nisi vis sollicitans V major fuerit, quam $\delta P$. Habemus ergo pro vi sollicitante V terminum $\delta P$, quo si vis fuerit minor, nullus motus sit consecuturus, sin autem fuerit major, tum demum motus producatur.

### COROLL. 1.

976. Cum corpus in quiete persistere pergat, quamdiu fuerit $V < \delta P$, frictio censenda est vim exercere ipsi vi V aequalem et contrariam: si enim fortius urgeret, corpus in plagam oppositam AE moveri deberet, quod esset absurdum, cum in plagam BF incitetur.

### COROLL. 2.

977. Dum ergo corpus quiescit, frictio non determinatam exerit vim, sed quovis casu tantam, quanta opus est ad corpus in quiete conservandum, nisi opus fuerit vi majori quam $\delta P$. Unde si corpus a nulla vi sollicitetur ad motum, etiam frictio nullam vim exercet.



### COROLL. 3.

978. Quamdiu ergo motus a vi, quae non superet $\delta P$, impediri potest, eam vim frictio suppeditat, et quidem secundum eam directionem, qua opus est ad motum impediendum. Sin autem quietis conservatio majorem postulet vim, quoniam frictio tantum praestare nequit, motus generabitur.

### SCHOLION. 1.

979. Cum supra dixerimus, in quiete corporum nullam dari frictionem, id de vera quiete tantum, in qua corpus esset perseveraturum, etiamsi nulla adesset frictio, est intelligendum. Statim enim atque corpus a viribus sollicitatur, quibus ad motum incitaretur, si nulla esset frictio, huic etiam motus productioni frictio reluctatur, etiamsi corpus adhuc sit in quiete. Ita igitur frictio tam ratione motus quam quietis est definienda, ut dum corpus movetur, vim exerat perpetuo ipsi $\delta P$ aequalem et secundum directionem motui contrariam: dum autem corpus quiescit, eadem vim non per se definitam, sed tantam duntaxat exerceat, quanta motui impediendo sufficit, nisi forte ad hoc majori vi opus sit quam $\delta P$: tum enim hac tantum vi $\delta P$ motus productioni resistit, quae cum motum coercere non valeat, motus revera generabitur. Vis scilicet $\delta P$ est maximus conatus, quo frictio anniti potest, quo revera semper ipsi motui resistit, et quo etiam motus generationi reluctatur, si opus est. Sin autem minor vis sufficiat, etiam minorem tantum exerit: seu quoties vis ad motus productionem cohibendam necessaria non fuerit major quam $\delta P$, ea vis a frictione suppeditatur. Haec autem tantum de motu progressivo sunt tenenda, si enim motus gyratorius accedat, praecipue si axis gyrationis fuerit ad superficiem inclinatus, res est altioris indaginis, et quia hoc casu non omnia basis elementa secundum eandem directionem moventur, superficiemque terunt, frictio singulorum elementorum considerari debet, ex quo etiam basis figura et magnitudo in computum ingredietur. Atque ad hanc circumstantiam supra, ubi basis figuram a determinatione frictionis removimus, non respeximus.

### SCHOLION. 2.

980. Difficile sane est frictionis, quemadmodum hic eam experientiae consentaneam statuimus, causam assignare, facile autem causas, quae

quae forte menti occurrant, refellere. Perspicuum enim est, neque ab abrasione quadam particularum, neque a depressione filamentorum, dum corpus super superficie incedit, frictionem oriri posse, quia tum necessario baseos magnitudo in computum intraret. Quod ad frictionem, quatenus motus generationi resistit, attendamus, ea sequenti modo haud inepte explicari posse videtur. Dum nempe corpus ABCD superficiei EF incumbit, contactus non secundum planum AB, ut sensui ostendit, fieri est concipiendus, sed ob minimas utrinque prominentias et cavitates secundum superficiem sinuosam et quasi undulatam, *ab ab ab*, dum ob pressionem prominentiae alterius in cavitates alterius se insinuant. Hoc admisso corpus moveri nequit, quin simul supra superficiem AB aliquantillum elevetur; seu prima motus impressio non secundum directionem OV ipsi AB parallelam, sed secundum quandam directionem OS inclinatam fieri debet, quae scilicet parallela sit maximae quasi declivitati in contactu illo sinuoso: atque haec declivitas seu obliquitas respondet asperitati utriusque superficiei in contactu data, ut pro majore minoreve asperitate angulus VOS major minorve sit concipiendus. Statuatur ergo iste angulus $VOS = \zeta$, corpusque superficiei apprimatur vi $OP = P$, ac jam videamus, quanta vi, secundum directionem OV agente opus sit, ut corpus de situ suo dimovere valeat. Agat ergo vis $OV = V$, a qua corpus secundum directionem OS sollicitabitur vi $= V \cos\zeta$: at vis pressionis $OP = P$ huic actioni resistit vi $= P \sin\zeta$. Quare nisi fuerit $V \cos\zeta > P \sin\zeta$ seu $V > P \tang\zeta$, corpus de quiete non deturbabitur; vel quamdiu vis sollicitans $OV = V$ minor fuerit quam $P \tang\zeta$, corpus in quiete perseverabit. Id quod egregie cum supra traditis convenit, cum loco fractionis illius $\delta$ hic habeamus tangentem cujuspiam anguli $\zeta$. Verum fateri cogor, hinc non intelligi, cur dum corpus movetur, frictionis vis motui contraria etiam ipsi $P \tang\zeta$ aequalis esse debeat: cum enim basis corporis alternatim se ex illis sinuositatibus expediat, iterumque se eo insinuet, minus patet quantum detrimentum hinc motus sit passurus. Quoniam tamen hypothesis stabilita hinc non evertitur, ei inhaereamus, causamque hic assignatam tanquam a vero non abhorrentem spectemus.

Fig. 126.

CAPUT



# CAPUT II.

## DE MOTU PROGRESSIVO CORPORUM GRAVIUM A FRICTIONE IMPEDITO.

### PROBLEMA. 2.

981. Si corpus grave super plano horizontali motu progressivo incedat, determinare motus retardationem a frictione oriundam.

### SOLUTIO.

Sit M corporis massa idemque ejus pondus, quod planum horizontale EF tangat basi sua AB, quam pariter planam esse oportet. Consideretur corporis centrum inertiae O, in quo ejus pondus M collectum concipiatur, ita ut corpus deorsum sollicitetur vi $OP = M$, quae cum ad planum EF sit normalis, tanta quoque vi ad planum apprimitur: ubi primum observo, nisi recta OP intra corporis basin AB cadat, motum progressivum esse non posse. Verum ne hoc quidem sufficit, cum enim progrediente corpore secundum directionem BF id secundum directionem contrariam BE ob frictionem retrahatur vi $= \delta M$, denotante $1 : \delta$ rationem pressionis ad frictionem, haec vis conatur corpori motum gyratorium circa horizontalem axem per O transeuntem inducere, cujus momentum est $= \delta M . OP$. Cui vi si corpus obsequatur, primo instanti basis punctum A elevari incipiet, ita ut jam totum corpus extremitati basis B innitatur, quo etiam pressio transferetur. In hoc ergo statu ad gyrandum proclivi corpus in B sursum urgeri censendum est vi $BM = M$, unde momentum gyrationi resistens nascitur $= M . BP$: quod nisi superet illud $\delta M . OP$, corpus revera gyrari incipiet. Quare cum hic tantum motum progressivum contemplari statuerimus, haec conditio insuper requiritur, ut sit $BP > \delta . OP$, quam ergo hic locum habere assumamus. Fuerit ergo initio corporis celeritas secundum directionem $EF = c$, et elapso tempore $t$ confecerit spatium $= s$, habeatque celeritatem $= v$. Atque ob vim $\delta M$ motui contrariam erit $\frac{dv}{2g\,dt} = \frac{-\delta M}{M} = -\delta$, ideoque $v = c - 2g\delta t$. Porro quia est $ds = v\,dt$, fiet $s = ct - \delta g t t$. Motus autem tamdiu tantum durabit, quoad corpus ad quietem fuerit reductum, frictione $\delta M$ tum fi-

Fig. 127.



bi-

bito cessante: corpus ergo ad quietem redigetur elapso tempore $t = \frac{c}{2\delta g}$ et percurso spatio $= \frac{cc}{A\delta g}$.

### COROLL. 1.

982. Ut ergo corpus grave super plano horizontali motu progressivo incedere possit, perpendiculum OP ex centro inertiae corporis O in planum demissum non solum intra basin AB cadere, sed etiam a termino basis anteriori B tanto intervallo BP remotum esse debet, ut sit BP $> \delta$. OP.

### COROLL. 2.

983. Ductis igitur ex centro inertiae O cum perpendiculo OP, tum ad anteriorem basis terminum B recta OB, angulus BOP [illegible] esse oportet angulo, cujus tangens est $= \delta$. Unde si fuerit [illegible] angulus BOP major esse debet quam 18°, 26'. Sin autem fuerit minor, corpus progrediendo simul provolvetur.

### [illegible]

984. At si corpus motu progressivo puro promoveatur, ejus motus erit uniformiter retardatus, et similis ei, quo corpus celeritate $c$ sursum projectum ascenderet, deorsum sollicitatum vi, quae sit ad ejus massam ut $\delta$ ad 1. Hoc tantum discrimine, quod hic corpus ad quietem redactum perpetuo in quiete sit permansurum.

### SCHOLION. 1.

985. Ut tali corpori quieto motus imprimatur, necesse est ut secundum directionem horizontalem impellatur vi, quae major sit quam $\delta M$: quamdiu autem sollicitatur vi minore, in quiete perseverabit, nisi forte ad provolutionem incitetur, quod quando evenire debeat, accuratius evolvamus. Sollicitetur ergo primo corpus secundum directionem horizontalem OS, quae per ejus centrum inertiae O transeat, vi OS $=$ S, ut sit S $< \delta M$, et frictio pari vi S secundum BA renitetur. An autem circa extremitatem B provolvatur? judicium petetur ex momento frictionis S. OP et momento pressionis M in B translatae, quod est $=$ M. BP: hinc si fuerit S. OP $>$ M. BP, corpus provolvetur, sin minus, in quiete persistet: quia enim vis sollicitans OS $=$ S ipsi centro inertiae est applicata, ea nihil huc confert. Sit nunc vis S infra centrum



trum inertiae in R applicata, et quia hinc momentum provolutioni contrarium nascitur $=$ S. OR, ne corpus provolvatur, esse oportet S. OR $+$ M. BP $>$ S. OP, seu S. PR $<$ M. BP; unde simul patet, si vis horizontalis S sublimius in $r$ esset applicata, corpus provolutioni non fore obnoxium, si fuerit S. P$r$ $<$ M. BP, ubi quidem assumimus esse S $<$ $\delta$M. Idem etiam hinc magis fit perspicuum, si punctum B ut axem fixum, corpusque circa eum mobile spectemus, tum enim vis $r v$ $=$ S momentum in sensum DC est $=$ S. P$r$. ex pondere autem corporis M in O collecto oritur momentum in sensum contrarium M. BP: ideoque corpus provolvetur si S. P$r$ $>$ M. BP, quiescet vero si S. P$r$ $<$ M. BP.

### SCHOLION. 2.

986. Sin autem vis $rv$ $=$ S major fuerit quam $\delta$M, motus corpori progressivus inducetur ab excessu S $-$ $\delta$M, quia frictio jam tantum vi $=$ $\delta$M secundum directionem BE reluctatur. Utrum autem simul corpus sit motum gyratorium adepturum, nec ne? hoc modo cognoscetur. Seposito nimirum motu progressivo, assumo corpori alium motum gyratorium imprimi non posse, nisi circa axem horizontalem per centrum inertiae O transeuntem et ad motus directionem OS normalem, ad quem investigandum, cum basis punctum B maneat in plano horizontali, simul ac punctum A elevari incipit, tota pressio in puncto B exercetur, ita ut tum in B habeatur vis sursum urgens BM $=$ M. Nunc igitur ex viribus $rv$ $=$ S, BE $=$ $\delta$M, OP $=$ M et BM $=$ M colligitur momentum provolutionem producens $=$ S. O$r$ $+$ $\delta$M. PO $-$ M. BP; quare ut corpus solo motu progressivo feratur, haec conditio requiritur, ut sit S. O$r$ $+$ $\delta$M. PO $<$ M. BP, ubi per hypothesin est S $>$ $\delta$M. Si vis horizontalis S infra centrum inertiae in R esset applicata, corpus provolutioni non erit obnoxium, si fuerit $\delta$M. PO $<$ M. BP $+$ S. OR seu S. OR $+$ M. BP $>$ $\delta$M. PO. Hinc igitur clare intelligimus, quantum cum amplitudo basis, seu distantia perpendiculi ex centro inertiae demissi OP ab ejus terminis, tum elevatio centri inertiae supra planum horizontale, tum altitudo in qua vis horizontalis applicatur, atque ipsa frictio conferant, ut nulla provolutio sit metuenda.

### PROBLEMA. 3.

987. Si corpus grave ABCD plano inclinato EF imponatur, definire conditiones, sub quibus id ob frictionem in quiete sit permansurum. Fig. 128.

### SOLUTIO.

Sit angulus, quem planum inclinatum EF cum horizonte GF conſtituit, $GFE = \zeta$, corporis autem ei impoſiti maſſa $= M$, et centrum inertiae O, baſi autem AB plano inclinato incumbat. Ducatur recta verticalis OQR, ſecundum quam corpus ob gravitatem ſollicitari cenſendum eſt vi $= M$, quae reſolvatur ſecundum directiones OP et OC, quarum illa in planum EF ſit normalis, haec vero eidem parallela, et ob angulum $POQ = GFE = \zeta$, erit vis $OP = M \, cof \, \zeta$ et vis $OC = M \, fin \, \zeta$. Illa autem vi OP corpus ad planum EF apprimitur, unde ſi moveretur, frictio foret $= \delta M \, cof \, \zeta$: hac vero vi $OC = M \, fi \, \zeta$ ad motum ſecundum plani inclinati EF directionem ſollicitatur. Niſi ergo haec vis $M \, fi \, \zeta$ major ſit, quam $\delta M \, cof \, \zeta$, corpus nullum motum progreſſivum adipiſcetur; quare ut corpus quieſcat, neceſſe eſt, ſit $M \, fi \, \zeta < \delta M \, cof \, \zeta$ ſeu $tang \, \zeta < \delta$. Prima ergo conditio ad conſervationem quietis neceſſaria exigit, ut anguli inclinationis $F = \zeta$ tangens minor ſit quam fractio $\delta$ qua frictio determinatur. Deinde manifeſto requiritur, ut recta verticalis OQ intra baſin AB cadat. Nam ne corpus circa baſis extremitatem B provolvatur, neceſſe eſt, ut vis $OQ = M$ momentum reſpectu puncti B, quod eſt $M.\, BQ \, cof \, \zeta$ ſit poſitivum, ideoque BQ poſitivum, ſeu punctum Q intra baſin AB cadere debet. Quod etiam ex motu gyratorio circa O generando ita oſtendi poteſt. Fingamus enim corpus jam talem motum gyratorium incipere, et dum punctum A elevatur, tota preſſio $M \, cof \, \zeta$ in B transferetur, ut nunc corpus in B ſollicitetur primo vi $BM = M \, cof \, \zeta$; ob frictionem autem vi $BA = M \, fi \, \zeta$, ex quibus momentum generans motum gyratorium erit $= M \, fi \, \zeta. \, OP - M \, cof \, \zeta \, BP$. Quare ne talis motus oriatur, debet eſſe $BP \, cof \, \zeta > OP. \, fi \, \zeta$ ſeu $BP > OP \, tang \, \zeta$, at $OP \, tang \, \zeta = PQ$, ergo ob $BP > PQ$ intervallum BQ poſitivum eſſe oportet. Conſequenter ut corpus ABCD plano inclinato EF impoſitum quieſcat, primo requiritur, ut verticalis OQ intra baſin AB cadat, deinde ut tangens anguli inclinationis F minor ſit quam $\delta$.

### COROLL. 1.

988. Hinc igitur facillimum modum nanciſcimur, explorandi frictionem ſeu fractionem $\delta$: planum enim EF eousque elevetur, quoad corpus ſuper eo deſcendere incipiat, et tangens anguli maximi F, quo corpus etiamnum in quiete perſiſtit, dabit valorem fractionis $\delta$.

CO-



### COROLL. 2.

989. Quodſi fuerit $\delta = \frac{1}{3}$, corpus tamdiu in quiete permanebit, quamdiu angulus elevationis GFE non ſuperat $18^\circ, 26'$. Sin autem ſit $\delta = \frac{1}{4}$, hunc angulum minorem eſſe oportet, quam $14^\circ, 2'$, ſicque viciſſim ex hoc angulo valor ipſius $\delta$ innoteſcit.

### COROLL. 3.

990. Ut autem corpus ſuper plano inclinato quieſcat, non ſufficit ut ſit $tang \, GFE < \delta$, ſed etiam baſis corporis tam ampla eſſe debet, ut ſit $BP > OP \, tang \, GFE$; ſeu ut angulus BOP major ſit quam angulus GFE.

### SCHOLION.

991. In figura repraeſentatur ſectio corporis verticalis per ejus centrum inertiae O facta, quae ſimul ad planum inclinatum ſit normalis; in qua propterea recta OP ad id eſt perpendicularis, et OC ſit directio motus progreſſivi, quem gravitas corpori imprimere conatur. Ex dictis autem manifeſtum eſt, motum progreſſivum coerceri, ſi fuerit $tang \, F < \delta$. Verum ad judicium expediendum, num corpus motum gyratorium ſit accepturum, non ſufficit ad ſolam ſectionem ABCD ejusque baſin AB ſpectare, cum fieri poſſet, ut in hac ſectione corpus plano nusquam incumberet, ſed contactus in extremitatibus corporis tantum exiſteret. Tum igitur univerſus contactus conſiderari ac diſpici debet, quomodo et circa quinnam lineam provolutio oriri poſſit, quae utique ex figura baſis eſt dijudicanda. Quodſi ergo corpora tam irregularia adhibeantur, ut hoc judicium nimis difficile evadat, experientiam conſulere conveniet, an corpus ad provolutionem ſit proclive? prior vero concluſio de angulo F manet, et ab hac irregularitate neutiquam pendet.

### PROBLEMA. 4.

992. Si elevatio plani inclinati EF major fuerit, quam ut grave ei incumbens ABCD in quiete perſiſtere poſſit, definire conditiones, quibus id ſolo motu progreſſivo ſuper plano inclinato EF ſit deſcenſurum.

### SOLUTIO.

Sit maſſa, idemque pondus corporis $= M$, et ejus centrum inertiae O ut ante, atque $\delta$ exponens frictionis. Vocato ergo angulo elevationis $GFE = \zeta$, erit per hypotheſin $tang \, \zeta > \delta$. Jam ex vi gra- Fig. 128.



vitatis

vitatis OQR $= M$ colligimus pressionem in planum inclinatum, seu vim OP $= M \cos\zeta$, et vim ad descensum sollicitantem OC $= M \sin\zeta$. Cum igitur frictio ei renitatur vi $= \delta M \cos\zeta$, corpus revera ad descensum incitabitur excessu virium $M \sin\zeta - \delta M \cos\zeta = M(\sin\zeta - \delta \cos\zeta)$, a qua motus progressivus producetur, dummodo praeterea in corpore nullus motus gyratorius generetur. Videamus ergo, sub quibusnam conditionibus corpori motus gyratorius circa axem horizontalem et ad planum COP normalem per centrum inertiae O ductum generari possit; statim autem ac talis motus incipit, tota pressio $M \cos\zeta$ in B transfertur, ita ut nunc corpus sollicitetur a vi BM $= M \cos\zeta$, et ob frictionem a vi BA $= \delta M \cos\zeta$, unde momentum gyrationem in sensum BADC generans est $= \delta M \cos\zeta \cdot OP - M \cos\zeta \cdot BP$. Quare ne corpus provolutioni sit obnoxium, oportet hanc quantitatem esse negativam, ideoque BP $> \delta$ OP. seu $tang$ BOP $> \delta$.

*COROLL. 1.*

993. Quia conditio inventa $tang$ BOP $> \delta$ non pendet ab inclinatione plani EF, si corpus in minori inclinatione provolutioni non fuerit obnoxium, etiam in majori elevatione nulla provolutio erit metuenda.

*COROLL. 2.*

994. Quodsi ergo fuerit $\delta = \frac{1}{3}$, dummodo angulus BOP major sit quam $18^\circ, 26'$, corpus nullum motum volutorium accipiet, sed super plano inclinato vel quiescet, vel solo motu progressivo descendet.

*SCHOLION.*

995. In hoc autem judicio pro puncto B non tam extremitas in ipsa sectione ABCD per centrum inertiae O facta est sumenda, sed in tota basi, qua fit contactus, linea per terminos a puncto P maxime remotos ducta est intelligenda, cujus a P distantia pro intervallo PB accipi debet.

*PROBLEMA 5.*

996. Si corpus ita fuerit comparatum, ut nulla provolutio sit metuenda, eius motum descensus super plano inclinato EF determinare.

*SOLUTIO.*

Posita corporis massa eodemque pondere $= M$, et elevatione plani supra horizontem seu angulo GFE $= \zeta$, ut sit $tang\,\zeta > \delta$, quia alioquin corpus in quiete perseveraret. Confecerit jam corpus tempore $= t$ super plano inclinato spatium $= s$, motu scilicet a quiete inchoato, et quia vis accelerans est $= M \sin\zeta$, a gravitate oriunda, retardans autem $= \delta M \cos\zeta$ a frictione profecta, hinc nanciscimur istam aequationem:

$$\frac{dds}{2g\,dt^2} = \frac{M\sin\zeta - \delta M \cos\zeta}{M} = \sin\zeta - \delta\cos\zeta,$$

hincque integrando $\frac{ds}{dt} = 2gt(\sin\zeta - \delta\cos\zeta)$, quae est celeritas corporis hoc tempore $t$ acquisita, ipsum autem spatium interea confectum fit $s = g\,tt(\sin\zeta - \delta\cos\zeta)$.

*COROLL. 1.*

997. Frictio ergo non impedit, quo minus corpus super plano inclinato descendat motu uniformiter accelerato, cum celeritates in ratione temporum crescant: verum in multo minore ratione crescunt; sublata enim frictione foret $s = gtt \sin\zeta$.

*COROLL. 2.*

998. Si observetur tempus $t$ quo datum spatium $s$ fuerit confectum, simulque elevatio plani seu angulus $\zeta$ fuerit exploratus, inde exponens frictionis $\delta$ colligi poterit: erit enim $\delta = tang\,\zeta - \frac{s}{gtt\cos\zeta}$.

*SCHOLION.*

999. Hoc modo explorari poterit, utrum pro quiete idem valor exponentis $\delta$ reperiatur, ac pro motu eoque sive celeriore sive tardiore: sed hujusmodi experimenta sunt lubrica, quia exiguus error in observatione temporis $t$ commissus multum turbat. Tum vero etiam resistentiae aeris ratio est habenda, quae praesertim in motibus velocioribus insigne momentum afferre potest. Quare nonnisi plurimis hujusmodi experimentis summa cura institutis quicquam certi in hoc negotio concludi poterit. Ne autem resistentia aeris moram facessat, planum non multum ultra statum quietis elevari convenit, quia in motibus tardioribus ejus effectus est minimus. Tum vero corpus quantum fieri potest, ponderosum efficiatur, frustum plumbi intra ejus volumen includendo, ut tamen basis ex ea constet materia, cujus frictionem explorare lubet.

EXEM-

*EXEMPLUM.*

1000. Ponamus tabulae EF longitudinem esse 6 ped. Rehn. tempusque $t$ observari, quo corpus descendendo totam hanc longitudinem conficiat, ac videamus, quantum discrimen in tempore $t$ frictione $\delta$ parumper mutata oriri debeat. Cum igitur sit $g = 15\frac{5}{8}$ ped. Rhen. erit tempus descensus $t = \sqrt{\frac{48}{125(\sin\zeta - \delta\cos\zeta)}}$.

Ponamus $\delta = \frac{1}{3}$, et angulum $\zeta = 20^\circ$, quia debet esse $\tan\zeta > \frac{1}{3}$, ac reperietur tempus descensus $t = 3$, [illegible] min. sec. seu $t =$ [illegible] sec. proxime.

Sit jam $\delta$ aliquantulum majus, nempe $\delta = \frac{1}{3} + \frac{1}{100}$, manente $\zeta = 20^\circ$, et prodit tempus $t = 4, 45 = 4\frac{9}{20}$ sec.

At si esset $\delta = \frac{1}{3} - \frac{1}{100}$ manente $\zeta = 20^\circ$, invenitur tempus $t = 3, 171 = 3\frac{1}{6}$ sec.

Pars igitur centesima unitatis in valore ipsius $\delta$ gignit temporis discrimen illo casu $\frac{4}{3}$ sec. hoc vero tantum $\frac{1}{5}$ sec. unde in observationibus temporis valde attentum esse oportet. Si plano minor tribuatur elevatio, ut motus multo lentior oriatur, dubium est, an observationibus multum confidere queamus. Levissima enim inaequalitas in superficie descensum vehementer perturbare valebit, ita ut si experimentum idem aliquoties repetatur, phaenomena multum discrepare possint. Atque hanc ob causam, etsi hic calculum hypothesi de frictione stabilitae superstruo, tamen si conclusiones inde deductas cum experientia conferre velimus, minime perfectum consensum expectare debemus.

# CAPUT. III.

## DE MOTU GYRATORIO CORPORUM GRAVIUM CIRCA AXEM FIXUM A FRICTIONE RETARDATO.

*PROBLEMA.* 6.

1001. **E**fficere ut corpus circa axem fixum per ejus centrum inertiae transeuntem gyrari possit.

SO-



*SOLUTIO.*

Si corpus debeat gyrari circa axem GG, necesse est, ut utrinque instructum sit terminis cylindricis CEFD, quos axis GG medium trajiciat, ita ut utriusque cylindri axis existat: atque hic quidem assumo, hunc axem GG per corporis centrum inertiae I transire, quanquam eadem structura est observanda, si recta GG non per corporis centrum gravitatis transire debeat. Ut jam durante motu gyratorio haec recta GG fixa maneat, id pluribus modis obtineri potest. Primo hi termini cylindrici annulis fixis ejusdem amplitudinis inseri possunt, intra quas libere, frictione saltem excepta, converti queant: verum si amplitudo annulorum non excedat amplitudinem cylindricorum CEFD, verendum est, ne ob nimis arctam insertionem ingens resistentia oriatur, ac si termini illi cylindrici vel minimum intumescant, motus omnis coerceatur. Fig. 129.

Deinde termini cylindrici utrinque canali MLN in figuram quadrati excavato imponi possunt, ut contactus tantum in tribus punctis E, H, F fiat; dum enim corpus intra has cavitates circumvolvitur, axis GG manet immotus. Ne autem motus nimis impediatur, non opus est, ut ambo parietes verticales M et N cylindrum tangant, sed majore intervallo a se invicem distare possunt. Statim enim atque corpus gyratur, cylindrici termini se alterutri parieti applicabunt, perindeque est, ac si alter abesset; qui tantum ideo adjicitur, ut corpus si forte in sensum contrarium gyretur, se ei pari modo applicare possit. Fig. 130.

Tertio termini cylindrici etiam utrinque cavitati MLN, ex duobus planis inclinatis ML et NL efformatae, in poni possunt; hoc modo contactus perpetuo fiet in duobus punctis E et F, axisque GG manebit in quiete; dummodo inclinatio illorum planorum tanta sit, ut termini cylindrici super illis non ascendant, quam conditionem deinceps investigabimus. Fig. 131

Quarto imponi etiam possunt ambo termini cylindrici fulcris in figuram circularem MLN excavatis, quibus quidem corpus dum quiescit ita incumbit, ut contactus fiat in imo puncto H. Quando autem gyratur, contactus fiet in alio puncto elevato, quod cum perpetuo maneat idem, uti ostendemus, axis GG, quamdiu motus gyratorius in eundem sensum durat, manebit immotus. Hic sufficit radium circuli MLM majorem fuisse radio termini cylindrici, sed tanta profunditas huic cavitati tribui debet, ut non sit verendum, ne corpus supra ejus oras M et N transiliat. Fig. 132.

*COROLL.* 1.

1002. Dum corpus hoc modo utrinque talibus cavitatibus incumbit, ob pondus ſuum eas premet: ac ſi centrum inertiae I in medio verſetur, utrinque preſſio aequalis exeretur: ſin autem id non fuerit in medio, preſſiones erunt diſtantiis reciproce aequales, ita ut ſumma ſit toti ponderi aequalis.

*COROLL.* 2.

1003. Quodſi autem corpus gyretur, preſſio non amplius a ſolo pondere corporis pendet, ſed ob ipſam frictionem immutabitur, ideoque ex frictionis ratione determinari debet, unde etiam ultimo caſu punctum contactus eſt definiendum.

*SCHOLION.*

1004. Preſſio etiam, ideoque et frictio, plurimum perturbatur viribus, quibus corpus dum gyratur, praeter gravitatem ſollicitatur. Quare quo hoc argumentum dilucide pertractemus, primo mentem ab hujusmodi viribus abſtrahamus, corpusque tantum grave ſpectemus, cui initio motus gyratorius fuerit impreſſus; et quantum is ob frictionem retardari debeat, indagemus. Tum vero etiam aſſumamus, axem gyrationis GG per centrum inertiae corporis I tranſire, ab eoque ambos terminos aeque eſſe remotos, ita ut corpus utrinque ſibi ſit ſimile. Quin etiam ne vires obliquae calculum turbent, ſtatuamus, rectam GG ſimul eſſe axem principalem corporis. Minime enim conſultum videtur, corpori figuram nimis irregularem tribuendo, inveſtigationes noſtras difficilibus calculis implicare, cum ipſa principia hactenus ſtabilita etiam his caſibus evolvendis ſufficiant, ſi quis laborem ſuſcipere voluerit. Caſus autem fig. 130. repraeſentatus in fig. 131. continetur, dum alterum planum ſit verticale et alterum horizontale; deinde vero etiam caſum fig. 132. ex eo dijudicari poſſe videbimus.

*PROBLEMA.* 7.

Fig. 133. 1005. Si corporis (in fig. 129. repraeſentati) termini cylindrici utrinque inter duo plana utcunque inclinata ML et NL ſuſtententur, corpusque in gyrum agatur celeritate quacunque, definire frictionem ejusque effectum in motu corporis retardando.

SO-



*SOLUTIO.*

Quia centrum inertiae I in medio axis GG ſitum aſſumimus, reſpectu terminorum cylindricorum utrinque omnia erunt paria. Sit igitur pro altero termino radius baſis circularis GE = GF = $f$, et puncta contactus in E et F. Ducta verticali GH ponantur anguli EGH = $\zeta$ et FGH = $\eta$, quibus poſitio planorum ML et NL determinantur: tum vero corpus jam elapſo tempore $t$ gyretur in ſenſum EF celeritate angulari = $\varepsilon$, quae initio fuerit = $\epsilon$. Quia ergo ex hac parte corpus in punctis E et F ſuſtinetur, ſint E et F preſſiones, quibus corpus planis innititur, ac viciſſim ſecundum directiones eo normales EG et FG urgetur. Frictio porro in punctis E et F, ubi fit attritus, ita ſe exeret, ut in E corpus ſollicitetur vi ſec. EM = $\delta$E et in F vi ſec. FL = $\delta$F, ita ut ex hac parte quatuor habeantur vires.

vis EG = E; vis EM = $\delta$E; vis FG = F; vis FL = $\delta$F,

totidemque pares ex altera parte. Poſita ergo maſſa eodemque pondere corporis = M, quia omnis motus progreſſivus excluditur, hae vires centro inertiae applicatae ſe mutuo deſtruere debent: Colligitur autem ex iſtis quaternis viribus vis verticaliter ſurſum tendens

$$E\,\cos\zeta + F\,\cos\eta + \delta E\,\sin\zeta - \delta F\,\sin\eta$$

et vis horizontalis dextorſum directa

$$E\,\sin\zeta - F\,\sin\eta - \delta E\,\cos\zeta - \delta F\,\cos\eta,$$

ubi haec debet evaneſcere, illa autem dimidio ponderi corporis aequari. Hinc nanciſcimur:

$$E\,\sin\zeta - F\,\sin\eta = \delta(E\,\cos\zeta + F\,\cos\eta) \text{ et}$$
$$(1+\delta\delta)(E\,\cos\zeta + F\,\cos\eta) = \tfrac{1}{2}M, \text{ ideoque}$$

$$E\,\cos\zeta + F\,\cos\eta = \frac{M}{2(1+\delta\delta)} \text{ et}$$

$$E\,\sin\zeta - F\,\sin\eta = \frac{M\delta}{2(1+\delta\delta)}$$

ex quibus elicitur

$$E = \frac{M(\sin\eta + \delta\cos\eta)}{2(1+\delta\delta)\sin(\zeta+\eta)};\quad F = \frac{M(\sin\zeta - \delta\cos\zeta)}{2(1+\delta\delta)\sin(\zeta+\eta)}$$

ubi ſtatim eſt obſervandum, cum vires E et F negative eſſe nequeant, neceſſario eſſe oportere $\sin\zeta > \delta\cos\zeta$ ſeu $\tang\zeta > \delta$.

Nunc denique colligantur momenta ex frictione nata, quae erunt

$$\delta(E+F)f = \frac{M\delta f(\sin\zeta + \sin\eta - \delta\cos\zeta + \delta\cos\eta)}{2(1+\delta\delta)\sin(\zeta+\eta)}$$ cujus duplum motui



opponitur.

opponitur. Quare si momentum inertiae corporis respectu axis GG fuerit $= Maa$, habebimus hanc aequationem:

$$\frac{dz}{2g\,dt} = \frac{-\delta f(fi\,\zeta + fi\,\eta - \delta\cos\zeta + \delta\cos\eta)}{(1+\delta\delta)aa\,fi(\zeta+\eta)},$$ et integrando

$$z = \varepsilon - \frac{2\delta f g t(fi\,\zeta + fi\,\eta - \delta\cos\zeta + \delta\cos\eta)}{(1+\delta\delta)aa\,fi(\zeta+\eta)}.$$

### COROLL. 1.

1006. Quo minor ergo est $f$, seu quo graciliores termini cylindrici, eo minor est effectus frictionis. Sed hos terminos non pro lubitu diminuere licet, quia eos satis fortes esse oportet ad onus gestandum, atque quantitas $f$ fere rationem subduplicatam ponderis M sequi debet.

### COROLL. 2.

1007. Si sit $\zeta = 90$ et $\eta = 0$, qui est casus fig. 130. momentum frictionis est $= \frac{M\delta(1+\delta)f}{1+\delta\delta}$; Sin autem sit $\eta = \zeta$, seu plana ML et NL aequaliter ad horizontem inclinata, erit momentum frictionis $=$ $\frac{2M\delta f\,fi\,\zeta}{(1+\delta\delta)fi\,2\zeta} = \frac{M\delta f}{(1+\delta\delta)\cos\zeta}$: ubi debet esse $tang\,\zeta > \delta$.

### COROLL. 3.

1008. Minimum autem fit momentum frictionis sumendo $tang\,\zeta = \delta$, tum enim ob F $= 0$, erit E $= \frac{M}{2(1+\delta\delta)\cos\zeta} = \frac{M}{2\sqrt{(1+\delta\delta)}}$, ideoque momentum frictionis $= \frac{M\delta f}{\sqrt{(1+\delta\delta)}}$. Hoc ergo casu corpus soli plano ML innititur, et alterum NL plane non in computum venit.

### COROLL. 4.

1009. Hinc casus fig. 132. quaecunque sit cavitatis MLN figura facile evolvitur. Termini enim cylindrici puncto O applicabuntur, ubi tangens cum horizonte facit angulum, cujus tangens est $= \delta$, eritque momentum frictionis $= \frac{M\delta f}{\sqrt{(1+\delta\delta)}}$.

SCHO-



### SCHOLION.

1010. Terminos ergo cylindricos ita sustentari convenit, ut contactus utrinque in unico fiat puncto, quia tum momentum frictionis minimum redditur: quem in finem eos cavitatibus MLN (fig. 132.) imponi expediet, quae in formam semicirculi crassitiem non multum superantis sint excavatae; ne situs, quem in motu obtinent, multum discrepet a situ quietis. Tum vero hos terminos cylindricos quam maxime tenues effici oportet, quantum quidem eorum firmitas ratione ponderis gestandi permittit. Praeterea etiam hi termini oleo aliave materia lubrica inungi solent, quo magis attritus diminuatur, fractionique $\delta$ minor valor concilietur. Interim tamen casu, quem sumus contemplati, motus mox extinguetur, quod fiet elapso tempore

$$t = \frac{\varepsilon(1+\delta\delta)aa\,fi(\zeta+\eta)}{2\delta f g(fi\,\zeta + fi\,\eta - \delta\cos\zeta + \delta\cos\eta)}.$$

Quando autem vires adhibentur, ad motum conservandum, ex iisdem principiis earum quantitas definiri potest, ut motus maneat uniformis. Quin etiam hujusmodi machinae, dum in gyrum aguntur, ad onera elevanda instrui solent, quae operatio ut motu uniformi perficiatur, tantis viribus opus est, quae non solum oneris resistentiam, sed etiam frictionem superare valeant; quem casum, cum in vita communi frequentissime occurrat, hic evolvamus.

### PROBLEMA 8.

1011. Si cylindrus (fig. 129.) adhibeatur ad onus quodpiam elevandum, determinare vires ei applicandas, ut habita frictionis ratione motus servetur uniformis.

### SOLUTIO.

Incumbat alter terminus cylindricus, cujus radius GE $=$ GF $= f$ binis planis inclinatis ML et NL, quae cum horizonte angulos faciant $\zeta$ et $\eta$, quibus aequales erunt anguli, quos radii GE et GF ad puncta contactus E et F ducti cum recta verticali GH faciunt. Dum autem corpus in sensum EF gyratur, ope cordae in medio circumvolutae elevet onus $=$ Q, quod pondere suo $=$ Q vecti horizontali GS $= s$ secundum directionem verticalem SQ motui reluctetur. Tum vero radio GR $= r$, a verticali GA declinanti angulo AGR $= \theta$ jugiter applicata sit vis RP $=$ P ad eum normalis, cujus quantitas quaeritur, ut motus maneat uniformis existente celeritate angulari circa axem GG $= \varepsilon$. Quod- Fig. 134.



si

unde $dt = \frac{-c\,dl}{\sqrt{2fg(1-ll)}}$, et $t = \frac{c}{\sqrt{2fg}}\, A\cos\frac{l}{1}$ seu $l = 1\cos\frac{t\sqrt{2fg}}{c}$. Quare axis IA fiet verticalis elapſo tempore $= \frac{\pi c}{2\sqrt{2fg}}$, et corpus titubationes iſochronas conficiet, uti pendulum ſimplex longitudinis $= \frac{cc}{f}$.

## SCHOLION. 2.

902. Niſi corpori ejusmodi indolem tribuiſſemus, ut ejus axis naturalis FD, qui in ſtatu quietis ſit verticalis, ſimul eſſet ejus axis principalis, binique reliqui haberent momenta inertiae aequalia, formulas quidem differentiales motum ejus continentes aſſignare, nullo autem modo ob analyſeos defectum ipſum motum definire potuiſſemus. Interim tamen, quemadmodum in caſu tractato, ubi corpori infinite parvam declinationem tribuimus, uſu venit, ut motus fieret ſatis ſimplex motuique penduli conformis, id adeo in genere locum habet, quomodocunque axes principales reſpectu axis naturalis fuerint diſpoſiti. In ſitu ſcilicet aequilibrii, ubi axis naturalis DF ſitum tenet verticalem, aſſumo centrum inertiae I infra centrum baſis ſphaericae G ad intervallum GI $= f$ cadere: tum vero hoc corpus infinite parum de ſitu ſuo quietis declinari ponamus, ut arcus ZD $= \varrho$ ſit infinite parvus, atque evidens eſt, corpus ſe reſtituendo oſcillationes ſeu titubationes eſſe peracturum, donec tandem motu ob reſiſtentiam extincto in ſtatu aequilibrii acquieſcat. Quoniam declinatio corporis hic perpetuo eſt minima, non opus eſt, ut tota corporis figura ſit ſphaerica, ſed ſufficit, ſi infima ejus portio eaque minima, qua plano horizontali applicatur, ſit pars ſuperficiei ſphaericae, cujus centrum eſt in G. Hunc igitur motum titubatorium inveſtigaturi primo diſpiciamus, quomodo formulae ſupra in genere erutae pro hoc caſu, quo axis corporis naturalis DF quam minime a ſitu verticali declinat, contrahi, indeque momenta virium P, Q, R ita commode definiri queant, ut deinceps ex iis motum aſſignare valeamus.

Fig. 114.

## PROBLEMA. 107.

902. Si corpus baſi ſphaerica inſtructum infinite parum a ſitu aequilibrii declinet, definire momenta virium reſpectu ternorum ejus axium principalium.

Fig. 114.

## SOLUTIO.

SO-



## SOLUTIO.

Circa corporis centrum inertiae I deſcripta ſphaera, in qua Z ſit punctum verticale, teneat axis corporis naturalis ID ſitum a verticali minime declinans, ut ſit arcus ZD $= \varrho$ minimus: axes autem corporis principales reſpondeant punctis A, B, C, quorum ſitus ratione puncti D ita ſe habeat, ut ſint arcus DA $= \zeta$, DB $= \eta$, DC $= \theta$, qui ſunt conſtantes. Nunc autem reſpectu puncti verticalis Z ſint arcus ZA $= l$, ZB $= m$, et ZC $= n$, qui ob arcum ZD $= \varrho$ minimum vix diſcrepabunt ab illis $\zeta$, $\eta$, $\theta$: quare ſi ponamus:

$$\cos l = \cos\zeta + p;\ \cos m = \cos\eta + q;\ \cos n = \cos\theta + r$$

quantitates $p$, $q$, $r$ erunt minimae. Quia vero eſt tam $\cos\zeta^2 + \cos\eta^2 + \cos\theta^2 = 1$; quam $\cos l^2 + \cos m^2 + \cos n^2 = 1$, fiet

$$2p\cos\zeta + 2q\cos\eta + 2r\cos\theta + pp + qq + rr = 0.$$

Deinde autem cum ſit $\cos\varrho = \cos\zeta\cos l + \cos\eta\cos m + \cos\theta\cos n$, erit $\cos\varrho = 1 + p\cos\zeta + q\cos\eta + r\cos\theta$, ideoque

$$p\cos\zeta + q\cos\eta + r\cos\theta = -\tfrac{1}{2}\varrho\varrho \text{ et } pp + qq + rr = \varrho\varrho.$$

Nunc igitur, poſita preſſione corporis in planum horizontale $= \Pi$, ex §. 894. tribuendo ipſi $f$ valorem negativum, obtinebimus momenta virium reſpectu axium principalium:

$$P = \Pi f(r\cos\eta - q\cos\theta);\ Q = \Pi f(p\cos\theta - r\cos\zeta)$$
$$\text{et } R = \Pi f(q\cos\zeta - p\cos\eta),$$

tum vero vidimus eſſe $\Pi = M\left(1 - \frac{f\,dd.\cos\varrho}{2g\,dt^2}\right)$; quia autem $\cos\varrho$ proxime eſt $= 1$, et minimas variationes ſubit, erit ſatis exacte $\Pi = M$, ita ut corpus toto ſuo pondere planum horizontale premere ſit cenſendum: ſicque habebimus

$$P = Mf(r\cos\eta - q\cos\theta)$$
$$Q = Mf(p\cos\theta - r\cos\zeta)$$
$$R = Mf(q\cos\zeta - p\cos\eta).$$

## PROBLEMA. 108.

903. Si corpus baſi ſphaerica praeditum de ſitu quietis, in quo axis DI eſt verticalis, parumper declinetur, iterumque dimittatur, ut ex quiete verſus ſtatum aequilibrii revertatur, determinare ejus motum.

## SOLUTIO.

Elapſo tempore $t$ teneat corpus ſitum in fig. 114. repraeſentatum, maneantque omnes denominationes in probl. praecedente ſtabilitae: Fig. 114.



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