SPECIMEN
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1.

Consideratio fractionum continuarum, quarum vsum vberrimum per totam Analysin iam aliquoties ostendi, deduxit me ad quantitates certo quodam modo ex indicibus formatas, quarum natura ita est comparata, vt singularem algorithmum requirat. Cum igitur summa Analyseos inuenta maximam partem algorithmo ad certas quasdam quantitates accommodato innitantur, non immerito suspicari licet, et hunc algorithmum singularem non exigui vsus in Analysi esse futurum, si quidem diligentius excolatur: etiamsi ei tantum non tribuendum censeam, vt cum receptis algorithmis comparari mereatur.

2. Sequenti autem modo ad eas quantitates, de quibus hic agere constitui, sum deductus: si habeatur fractio continua $a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d}}}$ cuius valor sit inuestigandus, ex numeris a, b, c, d, tanquam indicibus assumtis, sequenti modo fractiones formantur:

$$\begin{array}{ccccc} & a & b & c & d \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+1}{b}, & \frac{abc+c+a}{bc+1}, & \frac{abcd+cd+ad+ab+1}{bcd+d+b} \end{array}$$

Primum ſcilicet locum obtinet ſemper fractio $\frac{1}{0}$, ſecundum $\frac{a}{1}$, cuius numerator eſt primus indicum a, denominator vero vnitas. Sequentis cuiusque fractionis tam numerator, quam denominator, inuenitur, ſi praecedentium vltimus per indicem ſupra ſcriptum multiplicetur, et ad productum penultimus addatur.

3. Conſtat autem harum fractionum poſtremam ipſi fractioni continuae propoſitae eſſe aequalem, praecedentes autem tam prope ad hunc ipſum valorem accedere, vt nulla fractio numeris non maioribus contenta exhiberi queat, quae ad illum propius accedat. Atque ex hoc fonte problema illud a Walliſio olim tractatum facile reſoluitur, quo propoſita quacunque fractione ex ingentibus numeris conſtante, aliae quaeruntur fractiones ex minoribus numeris conſtantes, quae tam parum a propoſita diſcrepent, vt minus diſcrepantes exhiberi plane nequeant, niſi maiores numeros adhibere velimus.

4. Hoc autem aliisque vſibus, quos fractiones continuae ſuppeditant, praetermiſſis, hic inprimis obſeruo, in ſerie illa fractionum ex indicibus formatarum, tam numeratores, quam denominatores, eandem, progreſſionis legem ſequi, et ſeorſim efformari poſſe. In vtraque enim ſerie, ſiue numeratorum, ſiue denominatorum, quilibet terminus per indicem ſupra ſcriptum multiplicatus, et termino antecedente auctus, praebet terminum ſequentem. Vltimus autem numerus ſuperioris ſeriei componitur ex omnibus quatuor indicibus a, b, c, d, penultimus tantum ex tribus a, b, c, antepenultimus tantum

tantum ex duobus a, et b. Inferiores autem numeri primum indicem a plane non inuoluunt, ſed ex reliquis b, c, d aequali lege formantur.

5. Quoniam igitur ratio formationis ex indicibus, tam pro numeratoribus, quam pro denominatoribus, eſt eadem, ac datis indicibus numerus inde formatus innoteſcit, hos ipſos numeros, quatenus ex indicibus ſunt formati, hic ſum contemplaturus, eorumque algorithmum traditurus. Propoſitis autem indicibus quibuscunque et quotcunque a, b, c, d, numerum ex iis formatum hoc modo (a, b, c, d) denotabo, eritque ergo euolutione inſtituta:

$$(a, b, c, d) = abcd + cd + ad + ab + 1$$

ſimilique modo pro denominatoribus indicem primum a omittendo

$$(b, c, d) = bcd + d + b.$$

6. Haec ergo teneatur definitio ſignorum ( ), inter quae indices ordine a ſiniſtra ad dextram ſcribere conſtitui; atque indices hoc modo clauſulis incluſi inpoſterum denotabunt numerum ex iſtis indicibus formatum. Ita a ſimpliciſſimis caſibus inchoando, habebimus:

$$(a) = a$$
$$(a,b) = ab + 1$$
$$(a,b,c) = abc + c + a$$
$$(a,b,c,d) = abcd + cd + ad + ab + 1$$
$$(a,b,c,d,e) = abcde + cde + ade + abe + abc + e + c + a$$

etc.

ex

ex qua progreſſione patet, vnitatem tenere locum huius ſigni ( ) ſi ſcilicet nullus adſit index.

7. Quemadmodum hae expreſſiones creſcente indicum numero vlterius ſint continuandae ex formationis lege, qua quilibet ex duobus antecedentibus componitur, ſponte liquet. Eſt ſcilicet:

$$(a, b) = b(a) + 1 = b(a) + (\,)$$
$$(a, b, c) = c(a, b) + (a)$$
$$(a, b, c, d) = d(a, b, c) + (a, b)$$
$$(a, b, c, d, e) = e(a, b, c, d) + (a, b, c)$$

In genere ergo habebitur:

$$(a, b, c \ldots p, q, r) = r(a, b, c \ldots p, q) + (a, b, c \ldots p)$$

quae connexio, tanquam corollarium definitionis numerorum, quos hic contemplamur, ſpectari debet.

8. In euolutione horum valorum, vti ante §. 6 ſunt exhibiti, difficulter ratio compoſitionis cernitur. Poſſunt autem ii quoque hoc modo repraeſentari:

$$(a) = a(1)$$
$$(a,b) = ab\left(1 + \tfrac{1}{ab}\right)$$
$$(a,b,c) = abc\left(1 + \tfrac{1}{ab} + \tfrac{1}{bc}\right)$$
$$(a,b,c,d) = abcd\left(1 + \tfrac{1}{ab} + \tfrac{1}{bc} + \tfrac{1}{cd} + \tfrac{1}{abcd}\right)$$
$$(a,b,c,d,e) = abcde\left(1 + \tfrac{1}{ab} + \tfrac{1}{bc} + \tfrac{1}{cd} + \tfrac{1}{de} + \tfrac{1}{abcd} + \tfrac{1}{abde} + \tfrac{1}{bcde}\right)$$

In his autem denominatoribus occurrunt primo facta ex binis indicibus contiguis, tum vero producta ex binis illorum factorum, qui nullum indicem communem inuoluunt, tum ſequentur producta ex ternis, quaternisnis etc. combinationibus, quae nullum implicant indicem communem; vnde ratio compositionis iam fit perspicua.

9. Ex hac euolutione iam manifestum est, si indices ordine retrogrado disponantur, eosdem plane prodire numeros inde formatos. Erit scilicet

$$(a, b) = (b, a)$$
$$(a, b, c) = (c, b, a)$$
$$(a, b, c, d) = (d, c, b, a)$$
$$(a, b, c, d, e) = (e, d, c, b, a).$$

Dummodo ergo ordo indicum detur, siue sit directus, siue retrogradus, perinde est; vtroque enim modo idem numerus inde formatus obtinetur.

10. Hinc ergo sequitur, fore formulas §. 7 hoc modo inuertendo:

$$(a, b) = a(b) + 1$$
$$(a, b, c) = a(b, c) + (c)$$
$$(a, b, c, d) = a(b, c, d) + (c, d)$$
$$(a, b, c, d, e) = a(b, c, d, e) + (c, d, e)$$

atque in genere erit pro quotcunque indicibus:

$$(a, b, c, d, \text{etc.}) = a(b, c, d, \text{etc.}) + (c, d, \text{etc.})$$

11. Si ergo ponatur:

$$(a, b, c, d, e, \text{etc.}) = A$$
$$(b, c, d, e, \text{etc.}) = B$$
$$(c, d, e, \text{etc.}) = C$$
$$(d, e, \text{etc.}) = D$$
$$(e, \text{etc.}) = E$$

 habe-

habebimus has aequalitates:

$A = aB + C$ seu $\frac{A}{B} = a + \frac{C}{B}$

$B = bC + D$ seu $\frac{B}{C} = b + \frac{D}{C}$

$C = cD + E$ seu $\frac{C}{D} = c + \frac{E}{D}$

etc. etc.

12. Cum igitur sit

$$\frac{C}{B} = \frac{1}{b + \frac{D}{C}};\quad \frac{D}{C} = \frac{1}{c + \frac{E}{D}};\quad \frac{E}{D} = \frac{1}{d + \frac{F}{E}}\ \text{etc.}$$

erit his valoribus substituendis:

$$\frac{A}{B} = a + \frac{C}{B} = a + \cfrac{1}{b + \frac{D}{C}} = a + \cfrac{1}{b + \cfrac{1}{c + \frac{E}{D}}} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \frac{F}{E}}}}.$$

Vnde si e sit indicum vltimus, ita vt sit $E = e$ et $F = 1$, erit

$$\frac{A}{B} = \frac{(a, b, c, d, e)}{(b, c, d, e)} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e}}}}.$$

sicque patet, quemadmodum per huiusmodi numeros valores fractionum continuarum commode exprimi queant.

13. Si ergo indicum numerus fuerit infinitus, etiam fractio continua in infinitum excurret, eiusque valor erit $= \frac{(a, b, c, d, e, \text{etc.})}{(b, c, d, e\ \text{etc.})}$. Vicissim autem fractionum continuarum proprietates cognitae nobis insignes affectiones huiusmodi numerorum ex indicibus formatorum manifesta-

nifeſtabunt, quas diligentius euoluere operae erit pretium. Sit igitur fractio continua, ſiue in infinitum excurrens, ſiue ſecus propoſita: cuius valor indicetur littera V

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f\text{ etc.}}}}}}$$

et ſumendis omnibus indicibus a, b, c, d, e, f etc. erit, vti demonſtrauimus:

$$V=\frac{(a, b, c, d, e, f \text{ etc.})}{(b, c, d, e, f, \text{ etc.}}.$$

14. Si quispiam horum indicum fiat infinite magnus, is in hac expreſſione cum omnibus ſequentibus poterit omitti, et valor fractionis continuae tantum per indices, qui infinitum praecedunt, exprimetur.

Ita ſi ſit $b=\infty$ erit $V=\frac{(a)}{1}$

ſi ſit $c=\infty$ erit $V=\frac{(a, b)}{(b)}$

ſi ſit $d=\infty$ erit $V=\frac{(a, b, c)}{(b, c)}$

ſi ſit $e=\infty$ erit $V=\frac{(a, b, c, d)}{b, c, d}$.

Vti ergo his caſibus fractio continua abrumpitur, ita etiam valor V alios indices non implicat, niſi qui indicem infinitum antecedunt.

15. Sin autem nullus indicum in infinitum excreſcit, hi ipſi valores continuo propius ad verum va-

lorem V accedunt. Scilicet ſi fuerit $V = \frac{(a, b, c, d, e \text{ etc.})}{(b, c, d, e, \text{etc.})}$ fractiones in ſequenti ſerie expoſitae:

$$\frac{(a)}{1}; \frac{(a, b)}{(b)}, \frac{(a, b, c)}{(b, c)}; \frac{(a, b, c, d)}{(b, c, d)}; \frac{(a, b, c, d, e)}{(b, c, d, e)}; \text{ etc.}$$

continuo proprius ad valorem V accedunt, earumque vltima demum eius valorem verum exhibebit; ſiquidem indices a, b, c, d etc. fuerint numeri vnitate maiores. Prima quidem $\frac{a}{1}$ notabiliter ab V diſcrepare poterit, ſecunda autem propius accedet, tertia adhuc propius, et ita porro, donec tandem vltima verum valorem V ſit praebitura.

16. Neceſſe ergo eſt, vt differentiae inter binas huiusmodi fractiones contiguas continuo fiant minores; quod quo clarius perſpiciatur, has differentias inueſtigemus, quae erunt:

$$\frac{(a)}{1} - \frac{(a, b)}{(b)} = \frac{(a)(b) - 1(a, b)}{1(b)}$$

$$\frac{(a, b)}{(b)} - \frac{(a, b, c)}{(b, c)} = \frac{(a, b)(b, c) - (b)(a, b, c)}{(b)(b, c)}$$

$$\frac{(a, b, c)}{(b, c)} - \frac{(a, b, c, d)}{(b, c, d)} = \frac{(a, b, c)(b, c, d) - (b, c)(a, b, c, d)}{(b, c)(b, c, d)}$$

$$\frac{(a, b, c, d)}{(b, c, d)} - \frac{(a, b, c, d, e)}{(b, c, d, e)} = \frac{(a, b, c, d)(b, c, d, e) - (b, c, d)(a, b, c, d, e)}{(b, c, d)(b, c, d, e)}.$$

17. De harum differentiarum denominatoribus, qui ex binis factoribus ſunt conflati, primum obſeruo, hos factores inter ſe eſſe numeros primos, quod quidem ex antecedentibus eſt ſatis manifeſtum. Cum enim pro denominatore $(b, c, d)(b, c, d, e)$ ſit $(b, c, d, e) = e(b, c, d) + (b, c)$ erit $\frac{(b, c, d, c)}{(b, c, d)} = e + \frac{(b, c)}{(b, c, d)}$, vnde factores (b, c, d) et (b, c, d, e) communem diuiſorem non habebunt, niſi qui ſimul ſit communis diuiſor numerorum (b, c) et (b, c, d); verum ob eandem rationem horum numerorum communis

nis diuifor non datur, nifi qui fimul fit communis diuifor horum (b) et (b,c), ac denique horum 1 et b; qui cum nullum habeant communem diuiforem, neque illi habebunt, eruntque propterea numeri primi. Hinc vero etiam intelligitur, numeros $(a,b,c,d$, etc.$)$ et $(b,c,d$, etc.$)$ effe inter fe primos.

18. Differentiae ergo illae minores effe nequeunt, quam fi numeratores in vnitatem, fiue affirmatiuam, fiue negatiuam, abeant, id quod re vera euenire exempla declarant. Conueniet ergo, idem ex natura iftorum numerorum per indices formatorum demonftrari. Pro primo quidem numeratore cum fit $(a,b) = b(a) + 1$ per §. 7. erit:

$$(a)(b) - 1(a,b) = ab - b(a) - 1 = -1.$$

Tum vero pro fecundo, ob $(b,c) = c(b) + 1$ et $(a,b,c) = c(a,b) + (a)$ erit

$$(a,b)(b,c) - (b)(a,b,c) = (a,b)c(b) + (a,b) - (b)c(a,b) - (b)(a)$$

quae propter terminos $(a,b)c(b) - (b)c(a,b)$ fe tollentes abit in

$(a,b) - (b)(a) = +1$, ita vt fit fecundus numerator

$$(a,b)(b,c) - (b)(a,b,c) = +1.$$

19. Quemadmodum hic numerator fecundus ad primum negatiue fumtum eft reductus, ita tertius oftendi poteft fecundo negatiue fumto effe aequalis. Nam quia $(b,c,d) = d(b,c) + (b)$ et

$(a,b,c,d) = d(a,b,c) + (a,b)$ erit

$$(a,b,c)(b,c,d) - (b,c)(a,b,c,d) = (a,b,c)d(b,c) + (a,b,c)(b)$$
$$- (b,c)d(a,b,c) - (b,c)(a,b)$$

 Haec

Haec ergo expreſſio tranſit in $-(a,b)(b,c)+(b)(a,b,c)=-1$, quia eſt denominator ſecundus negatiue ſumtus. Eodem autem modo numerator quartus aequabitur tertio negatiue ſumto, et in genere quilibet ſequens praecedenti negatiue ſumto.

20. Hinc ergo conſequimur ſequentes reductiones non parum notatu dignas:

$$(a)(b) \quad -1(a,b)=-1$$
$$(a,b)(b,c) \quad -(b)(a,b,c)=+1$$
$$(a,b,c)(b,c,d) \quad -(b,c)(a,b,c,d)=-1$$
$$(a,b,c,d)(b,c,d,e)-(b,c,d)(a,b,c,d,e)=+1$$

et in genere

$$(a,b,c,d\ldots m)(b,c,d,\ldots m,n)-(b,c,d\ldots m)(a,b,c,d\ldots m,n)=\pm 1$$

vbi $+1$ valet, ſi in primis vinculis numerus indicum fuerit par, contra vero -1.

21. Ipſae ergo differentiae ſupra expoſitae erunt:

$$\frac{(a)}{1}-\frac{(a,b)}{(b)}=\frac{-1}{1(b)}$$
$$\frac{(a,b)}{(b)}-\frac{(a,b,c)}{(b,c)}=\frac{+1}{(b)(b,c)}$$
$$\frac{(a,b,c)}{(b,c)}-\frac{(a,b,c,d)}{(b,c,d)}=\frac{-1}{(b,c)(b,c,d)}$$
$$\frac{(a,b,c,d)}{(b,c,d)}-\frac{(a,b,c,d,e)}{(b,c,d,e)}=+\frac{1}{(b,c,d)(b,c,d,e)}$$
$$\frac{(a,b,c,d,e)}{(b,c,d,e)}-\frac{(a,b,c,d,e,f)}{(b,c,d,e,f)}=-\frac{1}{(b,c,d,e)(b,c,d,e,f)}$$

etc.

vnde, cum hae differentiae minores eſſe nequeant, ipſae fractiones tam prope ad ſe inuicem accedunt, quam fieri poteſt.

22,

22. Cum sit ex §. 7. $(b,c)-1=c(b)$; $(b,c,d)-(b)=d(b,c)$; $(b,c,d,e)-(b,c)=e(b,c,d)$ etc. erit binis illarum differentiarum addendis

$$\frac{(a)}{1}-\frac{(a,b,c)}{(b,c)}=-\frac{c}{1(b,c)}$$
$$\frac{(a,b)}{(b)}-\frac{(a,b,c,d)}{(b,c,d)}=+\frac{d}{(b)(b,c,d)}$$
$$\frac{(a,b,c)}{(b,c)}-\frac{(a,b,c,d,e)}{(b,c,d,e)}=-\frac{e}{(b,c)(b,c,d,e)}$$
$$\frac{(a,b,c,d)}{(b,c,d)}-\frac{(a,b,c,d,e,f)}{(b,c,d,e,f)}=+\frac{f}{(b,c,d)(b,c,d,e,f)}$$
etc.

eritque hic $\frac{(a)}{1}=a$, et $\frac{(a,b)}{(b)}=a+\frac{1}{b}$; vnde reliquae formulae concinne poterunt exhiberi.

23. Ex formulis ergo §. 21. habebimus sequentes fractionum continuarum valores:

$$\frac{(a)}{1}=a$$
$$\frac{(a,b)}{(b)}=a+\frac{1}{1(b)}$$
$$\frac{(a,b,c)}{(b,c)}=a+\frac{1}{1(b)}-\frac{1}{(b)(b,c)}$$
$$\frac{(a,b,c,d)}{(b,c,d)}=a+\frac{1}{1(b)}-\frac{1}{(b)(b,c)}+\frac{1}{(b,c)(b,c,d)}$$
etc.

vnde in genere erit, si etiam indices in infinitum excurrant,

$$\frac{(a,b,c,d,e,\text{etc.})}{(b,c,d,e,\text{etc.})}=a+\frac{1}{1(b)}-\frac{1}{(b)(b,c)}+\frac{1}{(b,c)(b,c,d)}-\frac{1}{(b,c,d)(b,c,d,e)}+\text{etc.}$$

24. Ex formulis autem §. 22. obtinebimus:

$$\frac{(a,b,c)}{(b,c)}=a+\frac{c}{1(b,c)}$$
$$\frac{(a,b,c,d,e)}{(b,c,d,e)}=a+\frac{c}{1(b,c)}+\frac{e}{(b,c)(b,c,d,e)}$$

vnde

vnde generaliter:

$$\frac{(a,b,c,d,e,\text{etc.})}{(b,c,d,e,\text{etc.})} = a + \frac{1}{1(b,c)} + \frac{e}{(b,c)(b,c,d,e)} + \frac{g}{(b,c,d,e)(b,c,d,e,f,g)} \text{ etc.}$$

Tum vero etiam:

$$\frac{a,b,c,d)}{(b,c,d)} = a + \frac{1}{b} - \frac{d}{(b)(b,c,d)}$$

$$\frac{(a,b,c,d,e,f)}{(b,c,d,e,f)} = a + \frac{1}{b} - \frac{d}{(b)(b,c,d)} - \frac{f}{(b,c,d)(b,c,d,e,f)}$$

ideoque generaliter:

$$\frac{(a,b,c,d,e,\text{etc.})}{(b,c,d,e\text{ etc.}} = a + \frac{1}{b} - \frac{d}{(b)(b,c,d)} - \frac{}{(b,c,d)(b,c,d,e,f)} - \frac{h}{(b,c,d,e,f)(b,c,d,e,f,g,h)} - \text{etc.}$$

25. Sed miſſis his, quae ad ſeries ſpectant, quoniam ea iam fuſius ſum perſecutus, perpendamus ea, quae ad ſingularem harum quantitatum algorithmum pertinent. Et formulas quidem iis ſimiles, quae in §. 20. ſunt inuentae, ſuppeditabit nobis §. 22. ex quo patet eſſe:

$$(a)(b,c) \quad -1(a,b,c) = -c$$

$$(a,b)(b,c,d) \quad -(b)(a,b,c,d) = +d$$

$$(a,b,c)(b,c,d,e) \quad -(b,c)(a,b,c,d,e) = -e$$

$$(a,b,c,d)(b,c,d,e,f) - (b,c,d)(a,b,c,d,e,f) = +1$$

ideoque generaliter:

$$(a,b\ldots\ldots l)(b\ldots\ldots l,m,n) - (b\ldots\ldots l)(a,b\ldots\ldots l,m,n) = \pm n$$

vbi ſignum + valet, ſi in primo vinculo numerus indicum ſit par; contra ſignum −.

26 Per ſimiles autem reductiones intelligitur fore,

$$(a)(b,c,d) \quad -1(a,b,c,d) = -(c,d)$$

$$(a,b)(b,c,d,e) \quad -(b)(a,b,c,d,e) = +(d,e)$$

$$(a,b,c)(b,c,d,e,f) - (b,c)(a,b,c,d,e,f) = -(e,f)$$

et generaliter:

$$(a,b\ldots k)(b\ldots k,l,m,n)-(b\ldots k)(a,b\ldots k,l,m,n)=\pm(m,n)$$

vbi fignorum, vel fuperius, vel inferius, valet, prout in primo vinculo numerus indicum fuerit, vel par, vel impar.

27. Ratio autem huius formulae ex fupra repertis facile deriuatur. Si enim ponatur:

$$(a,b\ldots k,l,m)(b\ldots k,l,m,n)-(b\ldots k,l,m)(a,b\ldots k,l,m,n)=A$$
$$(a,b\ldots\ldots k,l)(b\ldots k,l,m,n)-(b\ldots k,l)(a,b\ldots k,l,m,n)=B$$
$$(a,b\ldots\ldots\ldots k)(b\ldots k,l,m,n)-(b\ldots k)(a,b\ldots k,l,m,n)=C$$

manifeftum eft, effe $A=mB+C$. At eft $A=\pm 1$; et $B=\mp n$; ideoque $C=\pm 1\pm mn=\pm(m,n)$, vbi de ambiguitate fignorum tenenda funt praecepta fuperiora.

28 Si ordo indicum in his formulis inuertatur, eae fient:

$$(a\ldots\ldots y)(a,b\ldots y,z)-(a,b\ldots y,z)(a,b\ldots y)=0$$
$$(a,b\ldots y)(b,c\ldots y,z)-(a,b\ldots y,z)(b,c\ldots y)=\pm 1$$
$$(a,b,c\ldots y)(c,d\ldots y,z)-(a,b\ldots y,z)(c,d\ldots y)=\pm(a)$$
$$(a,b,c,d\ldots y)(d,e\ldots y,z)-(a,b\ldots y,z)(d,e\ldots y)=\pm(a,b)$$
$$(a,b,c,d,e\ldots y)(e,f\ldots y,z)-(a,b\ldots y,z)(e,f\ldots y)=\pm(a,b,c)$$
$$(a,b\ldots y)(f,g\ldots y,z)-(a,b\ldots y,z)(f,g\ldots y)=\pm(a,b,c,d)$$

vbi figna valent fuperiora, fi numerus indicum in fecundo vinculo fuerit par, contra autem valent inferiora.

29. Si haec indicum series in fine duobus truncetur, orietur simili modo:

$$(a \ldots x)(a \ldots z) - (a \ldots z)(a \ldots x) = 0$$
$$(a \ldots x)(b \ldots z) - (a \ldots z)(b \ldots x) = \pm (z)$$
$$(a \ldots x)(c \ldots z) - (a \ldots z)(c \ldots x) = \pm (a)(z)$$
$$(a \ldots x)(d \ldots z) - (a \ldots z)(d \ldots x) = \pm (a, b)(z)$$
$$(a \ldots x)(e \ldots z) - (a \ldots z)(e \ldots x) = \pm (a, b, c)(z)$$

atque hinc tandem colligitur fore generaliter:

$$(a \ldots\ldots l, m, n \ldots\ldots p)\ (n \ldots\ldots p, q, r \ldots\ldots z)$$
$$- (a \ldots l, m, n \ldots\ldots p, q, r \ldots\ldots z)\ (n \ldots\ldots p)$$
$$= \pm (a \ldots\ldots l)\ (r \ldots\ldots z).$$

30. Quo ratio ambiguitatis signorum pateat, notandum est, si sit $m = a$, fore $(a \ldots l) = 1$, et si sit $q = z$, fore $(r \ldots z) = 1$, vnde casus speciales, in quibus ratio signorum est cognita, erunt

$$(a)(b) - (a, b)\ 1 = -1$$
$$(a)(b, c) - (a, b, c)\ 1 = -(c)$$
$$(a, b)(c) - (a, b, c)\ 1 = -(a)$$
$$(a, b)(b, c, d) - (a, b, c, d)(b) = +(d)$$
$$(a)(b, c, d) - (a, b, c, d)\ (1) = -(c, d)$$

vnde concluditur, valorem fore affirmatiuum, si in extremo quaternorum vinculorum numerus indicum sit impar, sin autem fuerit par, valor erit negatiuus. Ita in exemplis subiunctis erit

$$(a, b, c, d)(e, f, g, h) - (a, b, c, d, e, f, g, h)\ 1 = -(a, b, c)(f, g, h)$$
$$(a, b, c, d, e)\ (c, d, e, f, g, h) - (a, b, c, d, e, f, g, h)\ (c, d, e)$$
$$= +(a)\ (g, h).$$

31. Hu-

31. Huiusmodi autem formulae, quot lubuerit, facile ſequenti modo exhiberi poſſunt; ſumatur tertium vinculum, quod eſt completum, et omnes indices continet, abſcindantur ab initio ſuperne ii indices, qui primum vinculum conſtituant, tum inferne a fine ii, qui vinculum ſecundum conſtituant; ita tamen, vt in duobus primis vinculis omnes indices occurrant. Tum qui locis abſciſſis vtrinque ſunt vicini puncto notentur, indeque facile huiusmodi formulae exhibentur:

vt $\overline{a,\ b,\ \underline{c,\ d,}}\ \underline{e,\ f,}$ dabit

$$(a,b,c,d)(c,d,e,f)-(a,b,c,d,e,f)(c,d)=-(a)(f)$$

vt $\overline{a,\ b,\ c,}\ \underline{d,\ e,\ f,}$ dat

$$(a,b,c)(d,e,f)-(a,b,c,d,e,f)\,1=-(a,b)(e,f)$$

vt $\overline{a,\ b,\ c,\ \underline{d,}}\ \underline{e,\ f}$ dat

$$(a,b,c,d)(d,e,f)-(a,b,c,d,e,f)(d)=+(a,b)(f).$$

32. Quodſi ergo in duobus vinculis nulla littera bis occurrat, quartum vinculum erit vnitas, vnde ſequentes formulae naſcuntur:

$$\begin{cases}(a,b,c)=(a,b)(c)+(a)\\(a,b,c)=(a)(b,c)+(c).\end{cases}$$

$$\begin{cases}(a,b,c,d)=(a,b,c)(d)+(a,b)\\(a,b,c,d)=(a,b)(c,d)+(a)(d)\\(a,b,c,d)=(a)(b,c,d)+(c,d)\end{cases}$$

$$\begin{cases}(a,b,c,d,e)=(a,b,c,d)(e)+(a,b,c)\\(a,b,c,d,e)=(a,b,c)(d,e)+(a,b)(e)\\(a,b,c,d,e)=(a,b)(c,d,e)+(a)(d,e)\\(a,b,c,d,e)=(a)(b,c,d,e)+(c,d,e)\end{cases}$$

 $(a,b,c,$

$$\begin{cases}(a,b,c,d,e,f)=(a,b,c,d,e)(f)+(a,b,c,d)\\(a,b,c,d,e,f)=(a,b,c,d)(e,f)+(a,b,c)(f)\\(a,b,c,d,e,f)=(a,b,c)(d,e,f)+(a,b)(e,f)\\(a,b,c,d,e,f)=(a,b)(c,d,e,f)+(a)(d,e,f)\\(a,b,c,d,e,f)=(a)(b,c,d,e,f)+(c,d,e,f)\end{cases}$$

etc.

33. Si ordo indicum inuertatur, ſequentes formulae hinc facile elicientur:

$$(\alpha)(a,b,c,d\ldots)=(\alpha,a,b,c,d\ldots)-(b,c,d\ldots)$$
$$(\alpha,\beta)(a,b,c,d\ldots)=(\alpha,\beta,a,b,c,d\ldots)-(\alpha)(b,c,d\ldots)$$
$$(\alpha,\beta,\gamma)(a,b,c,d\ldots)=(\alpha,\beta,\gamma,a,b,c,d\ldots)-(\alpha,\beta)(b,c,d\ldots)$$

etc.

vnde producto ex duobus huius generis numeris ad eiusmodi numeros ſimplices reuocari poterunt:

$$(\alpha)(a,b,c,d\ldots)=(\alpha,a,b,c,d\ldots)-(b,c,d\ldots)$$
$$(\alpha,\beta)(a,b,c,d\ldots)=(\alpha,\beta,a,b,c,d\ldots)-(\alpha,b,c,d\ldots)+(c,d\ldots)$$
$$(\alpha,\beta,\gamma)(a,b,c,d\ldots)=\begin{cases}+(\alpha,\beta,\gamma,a,b,c,d\ldots)\\-(\alpha,\beta,b,c,d\ldots)\\+(\alpha,c,d\ldots)\\-(d\ldots)\end{cases}$$
$$(\alpha,\beta,\gamma,\delta)(a,b,c,d,e\ldots)=\begin{cases}+(\alpha,\beta,\gamma,\delta,a,b,c,d,e\ldots)\\-(\alpha,\beta,\gamma,b,c,d,e\ldots)\\+(\alpha,\beta,c,d,e\ldots)\\-(\alpha,d,e\ldots)\\+(e\ldots)\end{cases}$$

etc.

quia ergo in vtroque factore ordo indicum inuerti poteſt, hae formae pluribus modis variari poterunt.

34. Re-

34 Reuertamur autem ad fractiones continuas, vnde haec sunt nata, sitque valor huius $a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \text{etc.}}}}}} = S$

atque supra iam inuenimus hos valores

$$A = \frac{(a)}{1};\ B = \frac{(a,b)}{(b)};\ C = \frac{(a,b,c)}{(b,c)};\ D = \frac{(a,b,c,d)}{(b,c,d)};$$

$$E = \frac{(a,b,c,d,e)}{(b,c,d,e)}\ \text{etc.}$$

continuo propius ad valorem S accedere. Horum terminorum igitur singulas differentias perpendamus;

$A-B = -\frac{1}{1(b)}$	$B-C = +\frac{1}{(b)(b,c)}$	$C-D = -\frac{1}{(b,c)(b,c,d)}$
$A-C = -\frac{(c)}{1(b,c)}$	$B-D = +\frac{(d)}{(b)(b,c,d)}$	$C-E = -\frac{(e)}{(b,c)(b,c,d,e)}$
$A-D = -\frac{(c,d)}{1(b,c,d)}$	$B-E = +\frac{(d,e)}{(b)(b,c,d,e)}$	$C-F = -\frac{(e,f)}{(b,c)(b,c,d,e,f)}$
$A-E = -\frac{(c,d,e)}{1(b,c,d,e)}$	$B-F = +\frac{(d,e,f)}{(b)(b,c,d,e,f)}$	$C-G = -\frac{(e,f,g)}{(b,c)(b,c,d,e,f,g)}$

35. Quoniam igitur in doctrina de fractionibus continuis, cuius iam aliquot specimina edidi, huius generis numeri per indices formati totum negotium conficiunt: algorithmi eorum species, quam hic exposui, nec non insignes comparationes inuentae, non exiguum praestabunt vsum in hoc argumento vberius excolendo, vnde has animaduersiones vsu non carituras esse confido.