DE PROGRESSIONIBVS ARCVVM CIRCVLARIVM QVORVM TANGENTES SECVNDVM CERTAM LEGEM PROCEDVNT.

Auctore

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1.

Infinitas huiusmodi progreſſiones exhiberi poſſe vel ex his exemplis liquet, quae olim propoſui, ſcilicet denotante π arcum duos angulos rectos metientem inueni, eſſe

$\frac{\pi}{4} = A\,\text{tang}\,\frac{1}{2} + A\,\text{tang.}\,\frac{1}{8} + A\,\text{tang.}\,\frac{1}{18} + A\,\text{tang}\,\frac{1}{32} + A\,\text{tang.}\,\frac{1}{50} + \text{etc.}$

quae ſeries arcuum in infinitum progreditur, tangente cuiusque indefinite exiſtente $= \frac{1}{2xx}$, ſimili modo eſt

$\frac{\pi}{4} = A\,\text{tang}\,\frac{1}{3} + A\,\text{tang.}\,\frac{1}{7} + A\,\text{tang.}\,\frac{1}{13} + A\,\text{tang}\,\frac{1}{21} + A\,\text{tang}\,\frac{1}{31} + \text{etc.}$

hac arcuum ſerie pariter in infinitum continuata, cuius quisque terminus indefinite eſt $A\,\text{tang.}\,\frac{1}{xx+\gamma+1}$. Tales autem ſeries eo magis videntur omni attentione dignae, quod nulla adhuc conſtet methodus earum ſummam a priori inueniendi, atque etiam ipſi arcus omnes inter ſe ſint incommenſurabiles. Quin etiam ne expectare quidem licet methodum, cuius ope in genere huiusmodi ſerierum, quamcunque legem tangentes ſequantur, ſumma inueſtigari queat; ſed potius, niſi haec

haec lex certis conditionibus ſit adſtricta, nullo modo eae ad ſummam reuocari poſſe videntur, quae quidem arcu circulari exprimatur. Quam ob rem in hoc negotio alia via non patet, niſi vt à poſteriori huiusmodi ſeries inueſtigemus, quarum deinceps contemplatio fortaſſe viam quandam directam pateſaciet; hincque modum exponam facilem ad quotcunque huiusmodi ſeries perueniendi, qui cum, ſimpliciſſimis principiis innixus, ad tam ardua perducat, omnino mereri videtur, vt diligentius euoluatur.

2. Non ſolum autem hoc modo ad ſeries infinitas deducimur, ſed pro lubitu progreſſiones dato terminorum numero conſtantes conſequi poſſumus. Fundamentum enim totius inueſtigationis in eo conſiſtit, vt pro lubitu numeros quotcunque aſſumamus, qui ſint:

$$\alpha,\quad \beta,\quad \gamma,\quad \delta,\quad \varepsilon,$$

qui vt tangentes angulorum ſpectentur. Cum enim manifeſto ſit

$$\left.\begin{array}{l} +\text{A tang.}\,\alpha+\text{A tang.}\,\beta+\text{A tang.}\,\gamma+\text{A tang.}\,\delta \\ -\text{A tang.}\,\beta-\text{A tang.}\,\gamma-\text{A tang.}\,\delta-\text{A tang.}\,\varepsilon \end{array}\right\} = \text{A tang.}\,\alpha-\text{A tang.}\,\varepsilon$$

binis arcubus ſubſcriptis colligendis ob $\text{A tang.}\,p-\text{A tang.}\,q = \text{A tang.}\,\frac{p-q}{pq+1}$ habebimus

$$\text{A tang.}\,\alpha-\text{A tang.}\,\varepsilon = \text{A tang.}\,\frac{\alpha-\beta}{\alpha\beta+1} + \text{A tang.}\,\frac{\beta-\gamma}{\beta\gamma+1} + \text{A tang.}\,\frac{\gamma-\gamma}{\gamma\delta+1} + \text{A tang.}\,\frac{\delta-\varepsilon}{\delta\varepsilon+1} = \text{A tang.}\,\frac{\alpha-\varepsilon}{\alpha\varepsilon+1}.$$

En ergo formam maxime generalem, vnde omnes huiusmodi ſeries arcuum originem ducunt, ſiue in in-

 finitum

finitum excurrant, ſiue finito terminorum numero conſtent:

$$\text{A tang}\,\frac{\alpha-\beta}{\alpha\beta+1}+\text{A tang}\,\frac{\beta-\gamma}{\beta\gamma+1}+\text{A tang}\,\frac{\gamma-\delta}{\gamma\delta+1}+\text{A tang}\,\frac{\delta-\varepsilon}{\delta\varepsilon+1}+\dots\dots\dots+\text{A tang}\,\frac{\psi-\omega}{\psi\omega+1}=\text{A tang}\,\frac{\alpha-\omega}{\alpha\omega+1}.$$

3. Caſui, quo numerus terminorum eſt finitus, hic non immorans, ſtatim in ſeries infinitas inquiram. Primo ergo pro α, β, γ etc. ſeriem harmonicam aſſumam in genere:

Hypotheſis I. $\frac{1}{a}$, $\frac{1}{a+b}$, $\frac{1}{a+2b}$, $\frac{1}{a+3b}$, $\frac{1}{a+4b}$, $\frac{1}{a+5b}$ etc.

vnde cum ſit $\omega=0$ habebimus:

$$\text{A tang.}\,\frac{1}{a}=\text{A tang.}\,\frac{b}{aa+ab+1}+\text{A tang.}\,\frac{b}{aa+3ab+2bb+1}+\text{A tang.}\,\frac{b}{aa+5ab+6bb+1}+\text{A tang.}\,\frac{b}{aa+7ab+12bb+1}+\text{etc.}$$

in infinitum.

Ac ſi ſingulis illis fractionibus communem tribuamus numeratorem c, erit ſimili modo

$$\text{A tang}\,\frac{c}{a}=\text{A tang.}\,\frac{bc}{a(a+b)+cc}+\text{A tang.}\,\frac{bc}{(a+b)(a+2b)+cc}+\text{A tang}\,\frac{bc}{(a+2b)(a+3b)+cc}+\text{A tang.}\,\frac{bc}{(a+3b)(a+4b)+cc}+\text{etc.}$$

in infinitum.

Hinc praecipue notari merentur caſus, quibus numerator horum tangentium ſit vnitas, quod euenit, ſi fuerit vel $bc=1$, vel ſi denominatores ſinguli per bc ſiant diuiſibiles.

4. Vtroque caſu capi oportet vel $c=1$, vel $c=2$; ac ſi pro priori ſumatur $b=1$, prodibit

$$\text{A tang.}\,\frac{1}{a}=\text{A tang.}\,\frac{1}{aa+a+1}+\text{A tang.}\,\frac{1}{aa+3a+3}+\text{A tang.}\,\frac{1}{aa+5a+7}+\text{A tang.}\,\frac{1}{aa+7a+13}$$

cuius

cuius terminus in genere est A tang. $\frac{1}{aa+(2x-1)a+xx-x+1}$ vnde pro simplicioribus valoribus ipsius a nascuntur hae series:

$$A\,tang.\,\tfrac{1}{0} = A\,tang.\,1 + A\,tang.\,\tfrac{1}{3} + A\,tang.\,\tfrac{1}{7} + A\,tang.\,\tfrac{1}{13} + A\,tang.\,\tfrac{1}{21} + A\,tang.\,\tfrac{1}{31} + etc.$$

$$A\,tang.\,1 = A\,tang.\,\tfrac{1}{3} + A\,tang.\,\tfrac{1}{7} + A\,tang.\,\tfrac{1}{13} + A\,tang.\,\tfrac{1}{21} + A\,tang.\,\tfrac{1}{31} + etc.$$

$$A\,tang.\,\tfrac{1}{2} = A\,tang.\,\tfrac{1}{7} + A\,tang.\,\tfrac{1}{13} + A\,tang.\,\tfrac{1}{21} + A\,tang.\,\tfrac{1}{31} + A\,tang.\,\tfrac{1}{43} + etc.$$

$$A\,tang.\,\tfrac{1}{3} = A\,tang.\,\tfrac{1}{13} + A\,tang.\,\tfrac{1}{21} + A\,tang.\,\tfrac{1}{31} + A\,tang.\,\tfrac{1}{43} + A\,tang.\,\tfrac{1}{57} + etc.$$

$$A\,tang.\,\tfrac{1}{4} = A\,tang.\,\tfrac{1}{21} + A\,tang.\,\tfrac{1}{31} + A\,tang.\,\tfrac{1}{43} + A\,tang.\,\tfrac{1}{57} + A\,tang.\,\tfrac{1}{73} + etc.$$

etc.

5. Pro b alios numeros capere non licet, nisi qui sint diuisores ipsius $aa+1$, vnde si $b=2$, necesse est sit a numerus impar: hincque sequentes nascuntur series:

$$A\,tang.\,1 = A\,tang.\,\tfrac{1}{2} + A\,tang.\,\tfrac{1}{8} + A\,tang.\,\tfrac{1}{18} + A\,tang.\,\tfrac{1}{32} + A\,tang.\,\tfrac{1}{50} + A\,tang.\,\tfrac{1}{72}\ etc.$$

$$A\,tang.\,\tfrac{1}{3} = A\,tang.\,\tfrac{1}{8} + A\,tang.\,\tfrac{1}{18} + A\,tang.\,\tfrac{1}{32} + A\,tang.\,\tfrac{1}{50} + A\,tang.\,\tfrac{1}{72} + A\,tang.\,\tfrac{1}{98}\ etc.$$

$$A\,tang.\,\tfrac{1}{5} = A\,tang.\,\tfrac{1}{18} + A\,tang.\,\tfrac{1}{32} + A\,tang.\,\tfrac{1}{50} + A\,tang.\,\tfrac{1}{72} + A\,tang.\,\tfrac{1}{98}\ etc.$$

$$A\,tang.\,\tfrac{1}{7} = A\,tang.\,\tfrac{1}{32} + A\,tang.\,\tfrac{1}{50} + A\,tang.\,\tfrac{2}{72} + A\,tang.\,\tfrac{1}{98}\ etc.$$

etc.

6. Si

6. Si statuatur $b = 5$, sumi debet $u = 5n \pm 2$; hinc fit:

$$A\,tang.\,\tfrac{1}{2} = A\,tang.\,\tfrac{1}{3} + A\,tang.\,\tfrac{1}{17} + A\,tang.\,\tfrac{1}{41} + A\,tang.\,\tfrac{1}{75} + A\,tang.\,\tfrac{1}{119} + A\,tang.\,\tfrac{1}{174} + \text{etc.}$$

$$A\,tang.\,\tfrac{1}{3} = A\,tang.\,\tfrac{1}{5} + A\,tang.\,\tfrac{1}{21} + A\,tang.\,\tfrac{1}{47} + A\,tang.\,\tfrac{1}{83} + A\,tang.\,\tfrac{1}{129} + A\,tang.\,\tfrac{1}{185} + \text{etc.}$$

$$A\,tang.\,\tfrac{1}{7} = A\,tang.\,\tfrac{1}{17} + A\,tang.\,\tfrac{1}{41} + A\,tang.\,\tfrac{1}{75} + A\,tang.\,\tfrac{1}{119} + A\,tang.\,\tfrac{1}{174}\ \text{etc.}$$

$$A\,tang.\,\tfrac{1}{8} = A\,tang.\,\tfrac{1}{21} + A\,tang.\,\tfrac{1}{47} + A\,tang.\,\tfrac{1}{83} + A\,tang.\,\tfrac{1}{129} + A\,tang.\,\tfrac{1}{185}\ \text{etc.}$$

Primae terminus generalis est $A\,tang.\,\frac{1}{5xx - x - 1}$, secundae $A\,tang.\,\frac{1}{5xx + x - 1}$, sequentes autem ex prioribus facile deducuntur, quas ideo posthac omittemus.

7. Si $b = 10$, sumi debet $a = 10n \pm 3$, vnde oriuntur hae duae series:

$$A\,tang.\,\tfrac{1}{3} = A\,tang.\,\tfrac{1}{4} + A\,tang.\,\tfrac{1}{30} + A\,tang.\,\tfrac{1}{76} + A\,tang.\,\tfrac{1}{142} + A\,tang.\,\tfrac{1}{228} + \text{etc.}$$

termino generali existente $A\,tang.\,\frac{1}{10xx - 4x - 2}$.

$$A\,tang.\,\tfrac{1}{7} = A\,tang.\,\tfrac{1}{12} + A\,tang.\,\tfrac{1}{46} + A\,tang.\,\tfrac{1}{100} + A\,tang.\,\tfrac{1}{174} + A\,tang.\,\tfrac{1}{268} + \text{etc.}$$

termino generali existente $A\,tang.\,\frac{1}{10xx + 4x - 2}$.

8. Simili modo posito $b = 13$, et $a = 13n \pm 5$, prodibunt

$$A\,tang.\,\tfrac{1}{5} = A\,tang.\,\tfrac{1}{7} + A\,tang.\,\tfrac{1}{43} + A\,tang.\,\tfrac{1}{105} + A\,tang.\,\tfrac{1}{193} + \text{etc.}$$

termi-

termino generali existente A tang. $\frac{1}{13xx - 3x - 3}$

A tang. $\frac{1}{8}$ = A tang. $\frac{1}{13}$ + A tang. $\frac{1}{55}$ + A tang. $\frac{1}{123}$ + A tang. $\frac{1}{217}$ + etc.

termino generali existente A tang. $\frac{1}{13xx + 3x - 3}$.

9. Sit $b = 17$, capiaturque $a = 17n \pm 4$, ac prodibunt

A tang. $\frac{1}{4}$ = A tang. $\frac{1}{5}$ + A tang. $\frac{1}{47}$ + A tang. $\frac{1}{123}$ + A tang. $\frac{1}{233}$ + A tang. $\frac{1}{377}$ etc.

termino generali existente A tang. $\frac{1}{17xx - 9x - 3}$

A tang. $\frac{1}{13}$ = A tang. $\frac{1}{23}$ + A tang. $\frac{1}{83}$ + A tang. $\frac{1}{177}$ + A tang. $\frac{1}{305}$ + A tang. $\frac{1}{467}$ etc.

termino generali existente A tang. $\frac{1}{17xx + 9x - 3}$.

10. Sit $b = 25$, captoque $a = 25n \pm 7$, erit

A tang. $\frac{1}{7}$ = A tang. $\frac{1}{9}$ + A tang. $\frac{1}{73}$ + A tang. $\frac{1}{187}$ + A tang. $\frac{1}{351}$ + A tang. $\frac{1}{565}$ + etc.

termino generali existente A tang. $\frac{1}{25xx - 11x - 5}$

A tang. $\frac{1}{18}$ = A tang. $\frac{1}{31}$ + A tang. $\frac{1}{117}$ + A tang. $\frac{1}{253}$ + A tang. $\frac{1}{439}$ + A tang. $\frac{1}{675}$ etc.

termino generali existente A tang. $\frac{1}{25xx + 11x - 5}$.

11. Hinc iam in genere colligere poterimus, si fuerit a numerus quicunque, sitque $aa + 1 = mn$, fore

A tang. $\frac{1}{a}$ = A tang. $\frac{1}{a + m}$ + A tang. $\frac{1}{3a + m + 2n}$ + A tang. $\frac{1}{5a + m + 6n}$ + A tang. $\frac{1}{7a + m + 12n}$ etc.

 termi-

termino generali existente $A\,\text{tang.}\frac{1}{nxx+(2a-n)x-a+m}$

seu hoc modo: $A\,\text{tang.}\frac{1}{nx(x-1)+a(2x-1)+m}$

Si ergo summam huius seriei infinitae per $\int A\,\text{tang.}\frac{1}{nxx+(2a-n)x-a+m}$ indicemus, consequemur hoc Theorema:

$$\int A\,\text{tang.}\frac{1}{nxx+(2a-n)x-a+m} = A\,\text{tang.}\frac{1}{a}, \text{ existente } aa+1=mn.$$

12. Videamus ergo, quibus casibus series, cuius terminus generalis est $A\,\text{tang.}\frac{1}{Lxx+Mx+N}$, summari queat; et comparatione instituta deprehendemus, hoc fieri posse, quoties fuerit $MM+4=LL+4LN$; ideoque

$$\text{vel } L=-2N+\surd(MM+4NN+4), \text{ vel } M=\surd(LL+4LN-4)$$
$$\text{vel } N=\frac{MM-LL+4}{4L}.$$

Atque si haec relatio inter coefficientes L; M N locum habuerit, erit $\int A\,\text{tang.}\frac{1}{Lxx+Mx+N}=A\,\text{tang.}\frac{2}{L+M}$ siue erit

$$A\,\text{tang.}\frac{2}{L+M}=A\,\text{tang.}\frac{1}{L+M+N}+A\,\text{tang.}\frac{1}{4L+2M+N}$$
$$+A\,\text{tang.}\frac{1}{9L+3M+N}+A\,\text{tang.}\frac{1}{16L+4M+N}+\text{etc.}$$

Hypothesis II. 13. Cum haec ex progressione harmonica sequantur, pro α, β, γ, δ etc. sumamus hanc seriem:

$$\frac{c}{a};\ \frac{c+d}{a+b};\ \frac{c+2d}{a+2b};\ \frac{c+3d}{a+3b};\ \frac{c+4d}{a+4b};\ \text{etc.}$$

vnde cum sit $\alpha=\frac{c}{a}$ et $\omega=\frac{d}{b}$, hanc adipiscemur summationem:

$$A\,\text{tang.}\frac{bc-ad}{ab+cd}=A\,\text{tang.}\frac{bc-ad}{a(a+b)+c(c+d)}+A\,\text{tang.}\frac{bc-ad}{(a+b)(a+2b)+(c+d)(c+2d)}+\text{etc.}$$

cuius

cuius terminus generalis eſt A tang. $\frac{bc-ad}{(a+b(x-1))(a+bx)+(c+d(x-1))(c+dx)}$.

14. Vt iam numerator huius tangentis vnitati fiat aequalis, vel eſſe debet $bc-ad=1$, vel denominator per numeratorem $bc-ad$ diuiſibilis, quod poſterius euenit ſumendo:

$a=pr+qs$; $c=ps-qr$; $b=pt+qu$; $d=pu-qt$ dum ſit $st-ru=1$.

tum enim fiet:

A tang. $\frac{1}{rt+su}=\int$ A tang. $\frac{1}{rr+ss+(rt+su)(2x-1)+(tt+uu)x(x-1)}$.

Verum haec formula cum praecedente ita conuenit, vt hinc nullae nouae ſeries eliciantur.

15. Fractiones autem continuae admodum idoneos praebent valores pro numeris α, β, γ, δ, etc. aſſumendos. Si enim fuerit: *Hypotheſis III.*

$$z=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\cfrac{1}{g+\text{etc.}}}}}}}$$

hinc ſequens ſeries fractionum conſtituitur, pro $\alpha, \beta, \gamma, \delta$ etc. capiendarum:

$$\overset{\alpha}{\frac{1}{0}};\ \overset{\beta}{\frac{a}{1}};\ \overset{\gamma}{\frac{ab+1}{b}};\ \overset{\delta}{\frac{abc+c+a}{bc+1}};\ \overset{\varepsilon}{\frac{abcd+cd+ad+ab+1}{bcd+d+b}};\ \text{etc.}$$

qua-

quarum vltima ipsi valori ipsius z est aequalis.

16. Quodsi hic eam notandi formam, quam in Algorithmi singularis specimine tradidi, introducamus, hae fractiones ita exprimentur:

$$\overset{\alpha}{\frac{1}{0}};\quad \overset{\beta}{\frac{(a)}{1}};\quad \overset{\gamma}{\frac{(a,b)}{(b)}};\quad \overset{\delta}{\frac{(a,b,c)}{(b,c)}};\quad \overset{\varepsilon}{\frac{(a,b,c,d)}{(b,c,d)}};\quad \overset{\zeta}{\frac{(a,b,c,d,e)}{(b,c,d,e)}}\ \text{etc.}$$

vti notari oportet, esse

$$(a,b)=a(b)+1=b(a)+1$$
$$(a,b,c)=a(b,c)+(c)=c(a,b)+(a)$$
$$(a,b,c,d)=a(b,c,d)+(c,d)=d(a,b,c)+(a,b)$$
$$(a,b,c,d,e)=a(b,c,d,e)+(c,d,e)=e(a,b,c,d)+(a,b,c)$$

etc.

17. Cum igitur sit $\alpha=\frac{1}{0}$, et $\omega=z$, erit

$$\text{A tang.}\frac{1}{z}=\text{A tang.}\frac{1}{\beta}+\text{A tang.}\frac{\beta-\gamma}{\beta\gamma+1}+\text{A tang.}\frac{\gamma-\delta}{\gamma\delta+1}$$
$$+\text{A tang.}\frac{\delta-\varepsilon}{\delta\varepsilon+1}+\text{etc.}$$

vbi est $\beta=a$; et $\frac{\beta-\gamma}{\beta\gamma+1}=\frac{-1}{(a)(a,b)+(b)}$; tum vero

$$\frac{\gamma-\delta}{\gamma\delta+1}=\frac{+1}{(a,b)(a,b,c)+(b)(b,c)}$$
$$\frac{\delta-\varepsilon}{\delta\varepsilon+1}=\frac{-1}{(a,b,c)(a,b,c,d)+(b,c)(b,c,d)}$$
$$\frac{\varepsilon-\zeta}{\varepsilon\zeta+1}=\frac{-1}{(a,b,c,d)(c,b,c,d,e)+(b,c,d)(b,c,d,e)}$$

etc.

ita vt omnes numeratores iam sint, vel $+1$, vel -1.

18. Quodsi breuitatis gratia loco illius serici scribamus

$$\overset{a}{\frac{1}{0}};\quad \overset{b}{\frac{a}{1}};\quad \overset{c}{\frac{B}{\mathfrak{B}}};\quad \overset{d}{\frac{C}{\mathfrak{C}}};\quad \overset{e}{\frac{D}{\mathfrak{D}}};\quad \overset{f}{\frac{E}{\mathfrak{E}}};\quad \overset{g}{\frac{F}{\mathfrak{F}}};\ \text{etc}$$

vt

vt fit:

$B = ab + 1$	$\mathfrak{B} = b$
$C = cB + a$	$\mathfrak{C} = c\mathfrak{B} + 1$
$D = dC + B$	$\mathfrak{D} = d\mathfrak{C} + \mathfrak{B}$
$E = eD + C$	$\mathfrak{E} = e\mathfrak{D} + \mathfrak{C}$
$F = fE + D$	$\mathfrak{F} = f\mathfrak{E} + \mathfrak{D}$
etc.	etc.

erit

$$\text{Atang.}\tfrac{1}{z} = \text{Atang.}\tfrac{1}{a} - \text{Atang.}\frac{1}{a+b(aa+1)} + \text{Atang.}\frac{1}{a+b(aa+1)+c(BB+\mathfrak{B}\mathfrak{B})}$$

$$- \text{A tang.}\frac{1}{a+b(aa+1)+c(BB+\mathfrak{B}\mathfrak{B})+d(CC+\mathfrak{C}\mathfrak{C})}$$

$$+ \text{A tang.}\frac{1}{a+b(aa+1)+c(BB+\mathfrak{B}\mathfrak{B})+d(CC+\mathfrak{C}\mathfrak{C})+e(DD+\mathfrak{D}\mathfrak{D})}$$

etc.

19. Consideremus fractionem continuam definitam hanc:

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 \text{ etc.}}}}}$$ cuius valor est $= \sqrt{2}$

vnde fractiones, quarum vltima est $= \sqrt{2}$, sunt

$$\overset{1}{\tfrac{1}{0}};\ \overset{1}{\tfrac{1}{1}};\ \overset{2}{\tfrac{3}{2}};\ \overset{2}{\tfrac{7}{5}};\ \overset{2}{\tfrac{17}{12}};\ \overset{2}{\tfrac{41}{29}};\ \overset{2}{\tfrac{99}{70}};\ \overset{2}{\tfrac{239}{169}} \text{ etc.}$$

Cum iam sit $\alpha = \infty$, et $\omega = \sqrt{2}$, erit

$$\text{A tang.}\tfrac{1}{\sqrt{2}} = \text{A tang.}\, 1 - \text{A tang.}\tfrac{1}{5} + \text{A tang.}\tfrac{1}{31} - \text{A tang.}\tfrac{1}{179} + \text{A tang.}\tfrac{1}{1045} - \text{etc.}$$

 cuius

cuius feriei lex non fatis eft perfpicua, propterea quod in fractione continua ordo indicum eft interruptus, qui fi obferuatur, vt fit

$$2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\text{etc.}}}} = 1+\sqrt{2}$$

vnde oriuntur hae fractiones

$$\begin{array}{cccccccc} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ \frac{1}{0}, & \frac{2}{1}, & \frac{5}{2}, & \frac{12}{5}, & \frac{29}{12}, & \frac{70}{29}, & \frac{169}{70}, & \frac{408}{169} \text{ etc.} \end{array}$$

$$\text{A tang.} \frac{1}{1+\sqrt{2}} = \text{A tang.} \frac{1}{2} - \text{A tang.} \frac{1}{12} + \text{A tang} \frac{1}{70} - \text{A tang.} \frac{1}{408} + \text{A tang.} \frac{1}{2378} - \text{etc.}$$

vbi eft $70 = 6.12 - 2$; $408 = 6.70 - 12$; $2378 = 6.408 - 70$ etc.

et $\text{A tang.} \frac{1}{1+\sqrt{2}} = \frac{\pi}{8}$.

20. Hoc modo quaecunque alia fractio continua tractari poteft; veluti cum fit

$$\sqrt{3} = 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2\ \text{etc.}}}}}}}$$

hinc oriuntur fequentes ex indicibus fractiones:

$$\begin{array}{ccccccccc} 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ \frac{1}{0}, & \frac{1}{1}, & \frac{2}{1}, & \frac{5}{3}, & \frac{7}{4}, & \frac{19}{11}, & \frac{26}{15}, & \frac{71}{41}, & \frac{97}{56} \text{ etc.} \end{array}$$

vbi ob $\alpha = \infty$, et $\omega = \sqrt{3}$, erit

$$\text{A tang.} \frac{1}{\sqrt{3}} = \text{A tang.}\, 1 - \text{A tang.} \frac{1}{3} + \text{A tang.} \frac{1}{13} - \text{A tang.} \frac{1}{47} + \text{A tang.} \frac{1}{177} - \text{A tang.} \frac{1}{659} + \text{etc.}$$

fin

fin autem tantum fractiones alternae fumantur

$$\frac{1}{0},\ \frac{2}{1},\ \frac{7}{4},\ \frac{26}{15},\ \frac{97}{56},\ \frac{362}{209}\ \text{etc.}$$

obtinebimus:

$$\text{A tang.}\frac{1}{\sqrt{3}} = \text{A tang.}\frac{1}{2} + \text{A tang.}\frac{1}{18} + \text{A tang.}\frac{1}{242} + \text{A tang.}\frac{1}{3362} + \text{etc.}$$

cuius denominatores omnes funt duplicata quadrata; fcilicet

$$\text{A tang.}\frac{1}{\sqrt{3}} = \text{A tang.}\frac{1}{2.1^2} + \text{A tang.}\frac{1}{2.3^2} + \text{A tang.}\frac{1}{2.11^2}$$
$$+ \text{A tang.}\frac{1}{2.41^2} + \text{A tang.}\frac{1}{2.153^2} + \text{etc.}$$

Sumtis autem alteris alternis, prodit ob $\text{A tang.}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$ et $\text{A tang.}\,1 = \frac{\pi}{4}$, $\frac{\pi}{12} = \text{A tang.}\frac{1}{4} + \text{A tang.}\frac{1}{64} + \text{A tang.}\frac{1}{900}$ $+ \text{A tang.}\frac{1}{12544} + \text{etc.}$

qui denominatores funt quadrata, quorum radices hanc progreffionem conftituunt:

1	2	3	4	5		x
2,	8,	30,	112,	418,	. . . . .	$\frac{(2+\sqrt{3})^x - (2-\sqrt{3})^x}{\sqrt{3}}$

21. Cum autem fractiones continuae ad huiusmodi feries arcuum deduxerint, viciffim fumma talis feriei ope fractionis continuae exhiberi poterit, quod commodiffime fequenti modo praeftabitur.

Sit $\text{A tang.}\frac{1}{z} = \text{A tang.}\frac{1}{a} - \text{A tang.}\frac{1}{b} + \text{A tang.}\frac{1}{c} - \text{A tang}\frac{1}{d}$
$+ \text{A tang.}\frac{1}{e} - \text{A tang.}\frac{1}{f}$ etc.

ac ponatur

$$\text{A tang.}\frac{1}{z} = \text{A tang.}\frac{1}{a} - \text{A tang.}\frac{1}{B} \text{ erit } z = \frac{aB+1}{B-a} = a + \frac{aa+1}{-a+B}$$
$$\text{A tang.}\frac{1}{B} = \text{A tang.}\frac{1}{b} - \text{A tang.}\frac{1}{C} \text{ erit } B = \frac{bC+1}{C-b} = b + \frac{bb+1}{-b+C}$$
$$\text{A tang.}\frac{1}{C} = \text{A tang.}\frac{1}{c} - \text{A tang.}\frac{1}{D} \text{ erit } C = \frac{cD+1}{D-c} = c + \frac{cc+1}{-c+D}$$
$$\text{A tang.}\frac{1}{D} = \text{A tang.}\frac{1}{d} - \text{A tang.}\frac{1}{E} \text{ erit } D = \frac{dE+1}{E-d} = d + \frac{dd+1}{-d+E}$$

etc. etc.

 hinc

hinc ergo colligendo habebitur per fractionem continuam:

$$z = a + \cfrac{aa+1}{-a+b+\cfrac{bb+1}{-b+c+\cfrac{cc+1}{-c+d+\cfrac{dd+1}{-d+e+\cfrac{ee+1}{-e+f+\text{etc.}}}}}}$$

vnde valor ipsius z definitur.

SPECI.