



DE  
I N T E G R A T I O N E  
A E Q U A T I O N I S D I F F E R E N T I A L I S .

$$\frac{m dx}{\sqrt{(1-x^2)}} = \frac{n dy}{\sqrt{(1-y^2)}}$$

Auctore

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§. I.

Cum primum occasione inventionum Ill. Comitís Fagnani hanc aequationem esse contemplatus, eiusmodi quidem relationem algebraicam inter variables  $x$  et  $y$  elici, quae huic aequationi satisfaceret; sed ea relatio non pro aequatione integrali completa haberi poterat, propterea quod non completeretur quantitate constantem arbitrariam, cuiusmodi semper in calculum per integrationem introduci solet. Hinc enim, uti satis notum est, integralia incompleta et particularia distingui solent, quorum illa totam vim aequationum differentialium exhaustiunt, haec vero tantum ita satisfaciunt, ut aliae insuper expressiones aequae satisfacere queant. Criterium autem aequationis integralis completae in hoc consistit, quod ea quantitatem constantem involvere debeat, quae in aequatione differentiali non apparet.

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§. 2.

## § 8 DE INTEGRATIONE

§. 2. Quae quo clarius perspiciantur, sufficiet aequationem differentialem simplicissimam  $dx = dy$  considerasse, cui vixque satisfacit haec integralis  $x = y$ , in rem tamen haec integralis minus late patet, quam differentialis  $dx = dy$ , cum huic aeque satisfaciat haec integralis  $x = y + a$  multo latius patens, sumendo pro  $a$  quantitatem constantem quamcumque, atque haec datum integralis totam vim aequationis differentialis  $dx = dy$  exhaustivè censeatur, ex quo etiam aequatio integralis completa appellatur; propterea quod in ea inest quantitas constans  $a$ , quae in aequatione differentiali non occurrit. Quodsi vero loco istius constantis indefinitae  $a$  valores determinati substituuntur, ex integrali completo obtinentur integralia particularia, quae ob hanc ipsam rationem minus late patent, quam aequatio differentialis proposita.

§. 3. Saepe numero autem aequationis differentialis integrale particulare algebraicum exhiberi potest, cum tamen integrale completum sit transcendens; hoc scilicet evenit, si pars transcendens per constantem illam arbitrariam fuerit multiplicata, quae propterea, constans illa nihilo aequali posita, ex calculo evanescit, et integrale algebraicum particulare relinquit. Ita huic aequationi  $dy = dx + (y - x)dx$  manifestum est, satisfacere valorem  $y = x$ , quo tamen tantum integrale particulare continetur, cum completum sit  $y = x + ae^x$ , denotante  $e$  numerum, cuius logarithmus est  $= 1$ . Nisi igitur constans arbitraria  $a$  evanescens ponatur, integrale semper erit transcendens.

§. 4.

§. 4. Cum igitur euenire queat, vt aequatio differentialis integrale particulare algebraicum admittat, etiamfi integrale completum fit transcendens, ita etiam rationes dubitandi non defunt, quod integrale completum aequationis differentialis propositae  $\int \frac{m dx}{\sqrt{(1-x^2)}} = \int \frac{n dy}{\sqrt{(1-y^2)}}$  quantitates transcendentes inuoluat, etiamfi pro ea integrale particulare algebraicum exhibere licuerit. Cum enim integrale completum fit:

$$m \int \frac{dx}{\sqrt{(1-x^2)}} = n \int \frac{dy}{\sqrt{(1-y^2)}} + C$$

haec autem integralia nullo modo, neque circuli, neque hyperbolae, quadraturam in subsidium vocando, assignari queant, minime probabile videtur, istas formulas tantopere transcendentes in genere, ita vt constans C maneat indeterminata, ad relationem algebraicam inter  $x$  et  $y$  reuocari posse.

§. 5. Notum quidem est, integrale completum huius aequationis differentialis  $\int \frac{m dx}{\sqrt{(1-x^2)}} = \int \frac{n dy}{\sqrt{(1-y^2)}}$  semper algebraice exhiberi posse, dummodo proportio coefficientium  $m$  et  $n$  fuerit rationalis; sed quia vtriusque formulae integrale arcum circuli indicat, ita vt integrale completum sit  $m A \text{ fin. } x = n A \text{ fin. } y + C$ , ratio autem sinuum, qui ad arcus proportionem rationalem inter se tenentes spectant, algebraice exprimi potest, mirum non est, aequationem integram completam his casibus quoque algebraice exhiberi posse. Cum autem huiusmodi comparatio in formulis transcendensibus  $\int \frac{dx}{\sqrt{(1-x^2)}}$  et  $\int \frac{dy}{\sqrt{(1-y^2)}}$  locum non habeat, seu saltem non consistet,

## 40 DE INTEGRATIONE

stet, inde reductio integralis ad quantitates algebraicas peti non poterit.

§. 6. Nihilominus tamen minus observari, si proposta fuerit huiusmodi aequatio differentialis

$$\frac{m dx}{\sqrt{(1-x^2)}} = \frac{n dy}{\sqrt{(1-y^2)}}$$

etiam integrale completum, quod scilicet quantitatem constantem arbitrariam involvat, semper algebraice exprimi posse, dummodo ratio  $m:n$  fuerit rationalis: quod mihi quidem eo magis notatu dignum videtur, quod nulla certa methodo ad hoc integrale sum perducitur, sed id potius tentando, vel diuinando, elicui. Vnde nullum est dubium, quin methodus directa, ad idem hoc integrale perducens, fines analyseos non mediocriter sit amplificatura; cuius propterea investigatio Analytici omni studio commendanda videtur.

§. 7. Completum autem integrale aequationis istius differentialis, quaecumque fuerit ratio rationalis coefficientium  $m$  et  $n$ , derivare mihi licuit ex integratione completa huius aequationis  $\frac{dx}{\sqrt{(1-x^2)}} = \frac{dy}{\sqrt{(1-y^2)}}$ : hac enim concessa methodum certam indicabo, ex ea quoque integrale completum huius aequationis multo latius patentis  $\frac{m dx}{\sqrt{(1-x^2)}} = \frac{n dy}{\sqrt{(1-y^2)}}$  concludendi. Quae methodus etiam in genere ad huiusmodi aequationum  $mX dx = nY dy$  integralia inveniendae adhiberi queat, si modo integrale completum huius  $X dx = Y dy$  fuerit erutum, atque  $Y$  talem significet functionem ipsius  $y$ , qualis  $X$  est ipsius  $x$ .

§. 8.

§. 8. Exordiar igitur ab hac aequatione

$$\sqrt{\frac{dx}{1-x^2}} = \sqrt{\frac{dy}{1-y^2}}$$

cui quidem primo intuitu satisfacere perspicuum est aequationem  $x=y$ , quae propterea eius est integrale particulare. Tum vero eidem aequationi quoque satisfacit iste valor algebraicus  $x = -\sqrt{\frac{1-y^2}{1+yy}}$ , cum enim sit  $dx = -\frac{2y dy}{(1+yy)\sqrt{1+yy}}$  et  $\sqrt{\frac{dx}{1-x^2}} = \frac{2y}{1+yy}$  erit  $\frac{dx}{\sqrt{1-x^2}} = \frac{2y dy}{\sqrt{1-y^2}}$ . Hinc iste etiam valor, seu aequatio  $xyy + xx + yy - 1 = 0$  est integralis particularis aequationis differentialis propositae. Unde integrale completum, quod constantem arbitriam involuat, ita comparatum sit necesse est, ut tribuendo huic constanti certum quandam valorem, prodeat  $x=y$ ; sin autem eidem constanti alius quidem valor tribuatur, ut prodeat  $x = -\sqrt{\frac{1-y^2}{1+yy}}$  seu  $xyy + xx + yy - 1 = 0$ .

### Theoremâ.

§. 9. Dico igitur huius aequationis differentialis

$$\sqrt{\frac{dx}{1-x^2}} = \sqrt{\frac{dy}{1-y^2}}$$

aequationem integram completam esse:

$$xx + yy + ct + xxyy = ct + 2xy\sqrt{(1-t)}$$

### Demonstratio.

Posita enim hac aequatione, eius differentiale erit:

$$2xdx + ydy + cxdy + ydx = (xxy + yyy)\sqrt{(1-t)}$$

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42 DE INTEGRATIONE

vnde fit

$$dx(x+ccxyy-y\sqrt{(1-c^4)})+dy(y+ccxyy-x\sqrt{(1-c^4)})=0$$

Ex eadem vero aequatione resoluta colligitur:

$$y = \frac{x\sqrt{(1-c^4)} + c\sqrt{(1-c^4)}}{1+ccxyy}$$

$$\text{et } v = \frac{y\sqrt{(1-c^4)} - c\sqrt{(1-c^4)}}{1+ccxyy}$$

Si enim ibi radicali  $\sqrt{(1-x^4)}$  tribuitur signum +, hic radicali  $\sqrt{(1-y^4)}$  signum - tribui debet; vt posito  $x=0$ , vtrinque idem valor prodeat  $y=c$ : Erit ergo

$$x + cccxyy - y\sqrt{(1-c^4)} = -c\sqrt{(1-y^4)}$$

$$y + cccxyy - x\sqrt{(1-c^4)} = c\sqrt{(1-x^4)}$$

quibus valoribus in aequatione differentiali substitutis, prodit

$$-c dx\sqrt{(1-y^4)} + c dy\sqrt{(1-x^4)} = 0,$$

$$\text{sive } \frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}}$$

Huius ergo aequationis differentialis integrale est:

$$xx + yy + ccxyy = cc + 2xy\sqrt{(1-c^4)}$$

et quia constantem  $c$  ab arbitrio nostro pendentem continet, erit simul integrale completum. Q. E. D.

§. 10. Si igitur habeatur haec aequatio  $\frac{dx}{\sqrt{(1-x^4)}}$ :

$$= \frac{dy}{\sqrt{(1-y^4)}} \text{ valor integralis completus ipsius } x \text{ est:}$$

$$x = \frac{y\sqrt{(1-c^4)} + c\sqrt{(1-y^4)}}{1+ccxyy}$$

vnde si constantis arbitraria  $c$  evanescat fit  $x=y$ ; sin autem ponatur  $c=1$ , habemus  $x = \frac{1+\sqrt{(1-y^4)}}{1+yy}$  qui sunt ambo illi valores particulares iam supra exhibit.

abici. Hinc eruntur alii valores particulares prae caeteris simpliciores, sed qui ad imaginaria devolvuntur.

Ita

$$\text{posito } c = \infty \text{ fit } x = \frac{\sqrt{-1}}{y}; \text{ et}$$

$$\text{posito } cc = -1; \text{ fit } x = \sqrt{\frac{2y+1}{y-1}}$$

qui itidem aequationi propositae satisfaciunt.

Tab. I.  
Fig. 1. 2.

§. 11. Quo autem ratio huius integralis clarius perspiciatur, concepiatur curva AM cuius haec sit indoles, ut posita abscissa AP = u, sit arcus ei respondens AM =  $\int \frac{dx}{\sqrt{(1-x^4)}}$ . Deinde eadem curva denuo descripta, capiatur abscissa ap = x, erit arcus am =  $\int \frac{dx}{\sqrt{(1-x^4)}}$ . Sumto igitur

$$x = \frac{u \sqrt{(1-cc^4)} + c \sqrt{(1-u^4)}}{1+cc \cdot uu}$$

fiat  $\frac{dx}{\sqrt{(1-x^4)}} = \frac{du}{\sqrt{(1-u^4)}}$ ; ideoque arc. am = arc. AM + Const.

Pro constantis autem huius determinatione, posito u = 0 quo casu arcus AM evanescit, fit x = c. Quare si capiatur abscissa ab = c; cui arcus ad respondeat, erit arcus dm = arcui AM.

§. 12. Ope huius ergo integrationis completa aequationis  $\frac{dx}{\sqrt{(1-x^4)}} = \frac{du}{\sqrt{(1-u^4)}}$ , in curva proposita arcui cuiusque AM, qui abscissae AP = u respondeat, arcus aequalis dm, qui a dato puncto d incipiat, abscindi poterit. Posita enim abscissa dato puncto d sponte ab = c; si capiatur abscissa ap =  $x = \frac{c\sqrt{(1-u^4)} + u\sqrt{(1-c^4)}}{1+cc \cdot uu}$  erit arcus dm arcui AM aequalis. Simili autem modo cum  $\sqrt{(1-c^4)}$  negativum statui liceat, si capiatur abscissa

$$ap = \frac{c \sqrt{(1-u^4)} - u \sqrt{(1-c^4)}}{1+cc \cdot uu}$$

erit

F 2

erit itidem arcus  $d\mu$  arcui  $AM$  aequalis: sicque in hac curva a dato quouis puncto  $d$  utrinque abscindi potest arcus  $d\mu$  et  $d\mu$ , qui arcui  $AM$  sint aequales.

§. 13. Hinc ergo patet, si arcus  $ad$  aequalis capiatur arcui  $AM$ , seu  $e = u$ , fore arcum  $am$  duplum arcui  $AM$ . Hinc si statuatur  $ap = x = \frac{2uv(1-u^4)}{1+u^4}$ , prodibit arcus  $am = 2 \text{ arc. } AM$ . Simili modo si capiatur arcus  $ad = 2 AM$ , seu  $e = \frac{2uv(1-u^4)}{1+u^4}$ , statuaturque  $x = \frac{c\sqrt{(1-u^4)+u\sqrt{(1-c^4)}}}{1+ccuu}$  obtinebitur arcus  $am = 3 \text{ arc. } AM$ . Ac si iste valor ipsius  $x$  demum pro  $c$  substituitur, ut sit  $ad = 3 AM$  iterumque statuatur  $x = \frac{c\sqrt{(1-u^4)+u\sqrt{(1-c^4)}}}{1+ccuu}$ , nalcetur arcus  $am$  quadruplus arcui  $AM$ , atque ita porro successive quaecumque multipla arcus  $AM$  geometricè assignari poterunt.

§. 14. Sit arcus  $ad = n. AM$  et  $ab = z$ ; ita ut sit  $\int \frac{dx}{\sqrt{(1-z^4)}} = n \int \frac{dx}{\sqrt{(1-u^4)}}$ ; atque ex his patet si capiatur  $x = \frac{z\sqrt{(1-u^4)+u\sqrt{(1-z^4)}}}{1+uuaz}$  fore  $\int \frac{dx}{\sqrt{(1-x^4)}} = (n+r) \int \frac{dx}{\sqrt{(1-u^4)}}$ ; sin autem ponatur  $x = \frac{z\sqrt{(1-u^4)-u\sqrt{(1-z^4)}}}{1+uuaz}$ , tum futurum esse  $\int \frac{dx}{\sqrt{(1-x^4)}} = (n-r) \int \frac{dx}{\sqrt{(1-u^4)}}$ . Si igitur haec aequatio  $\frac{dx}{\sqrt{(1-x^4)}} = \frac{ndu}{\sqrt{(1-u^4)}}$  fuerit integrata, debitusque valor pro  $z$  inde erutus, etiam integrari poterit haec aequatio  $\frac{dx}{\sqrt{(1-x^4)}} = \frac{(n+r)du}{\sqrt{(1-u^4)}}$ , quippe cuius integrale erit  $x = \frac{z\sqrt{(1-u^4)+u\sqrt{(1-z^4)}}}{1+uuaz}$ . Ac si pro  $z$  assumtus fuerit eius valor completus, qui scilicet constantem arbitriam involuat, etiam pro  $x$  prodibit eius valor completus.



§. 15. Hinc igitur perspicuum est, quomodo æquatio integralis completa inueniri debeat, quæ conueniat huic æquationi differentiali  $\sqrt[4]{(1-x^4)} = \sqrt[4]{(1-u^4)}$ ; quæ si fuerit numerus integer. Simili autem modo assignari poterit  $y$ , ut sit  $\sqrt[4]{(1-y^4)} = \sqrt[4]{(1-u^4)}$ , unde si eliminando  $u$ , æquatio inter  $x$  et  $y$  quaeratur, ea erit integralis huius æquationis  $\sqrt[4]{(1-x^4)} = \sqrt[4]{(1-y^4)}$ , quæcunque numeri rationales pro  $m$  et  $n$  substituuntur: atque ut hoc integrale prodeat completum, sufficit pro altera tantum variabilium  $x$  et  $y$  valorem completum per  $u$  determinasse, cum hinc iam noua constans arbitria in calculum introducat.

§. 16. Methodus, quæ hic in Theorematis demonstratione sum usus, etsi non ex rei natura est perita, sed indirecte ad id, quod propositum erat, perduxit, tamen multo lauius patet: simili enim modo colligitur, huius æquationis differentialis

$$\sqrt[4]{(1+mx^4+nx^4)} = \sqrt[4]{(1+my^4+ny^4)}$$

integrale completum esse:

$$0 = cc - xx - yy + nccxxyy + xxy\sqrt{(1+mc^4+nc^4)}$$

Vndè idem, quod ante, ratiocinium adhibendo, integrale quoque completum obtinebitur huius æquationis

$$\sqrt[4]{(1+mx^4+nx^4)} = \sqrt[4]{(1+my^4+ny^4)}$$

siquidem litteris  $\mu$  et  $\nu$  numeri integri designentur.

§. 17. Inuestigatio autem huius integrationis ita se habet: Pingatur primo pro arbitrio relatio inter variabiles  $x$  et  $y$  hac æquatione contenta:

(1)  $e$

## DE INTEGRATIONE

$$(1) \alpha xx + \alpha yy = 2\beta xy + \gamma xxy + \delta$$

quae differentiata dat:

$$\alpha x dx + \alpha y dy = \beta y dx + \gamma xy dx + \gamma xxy dy$$

unde conficitur

$$(2) dx(\alpha x - \beta y - \gamma xy) + dy(\alpha y - \beta x - \gamma xxy) = 0$$

Deinde ex aequatione (1) eliciantur valores utriusque variabilis:

$$x = \frac{\beta y + \sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^2)}}{\alpha - \gamma yy}$$

$$y = \frac{\beta x - \sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^2)}}{\alpha - \gamma xx}$$

Atque hinc obtineimus:

$$(3) \alpha x - \beta y - \gamma xy = \sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^2)}$$

$$(4) \alpha y - \beta x - \gamma xy = -\sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^2)}$$

qui valores in aequatione (2) substituti praebebunt

$$(5) \frac{\sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^2)}}{dx} = \frac{\sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^2)}}{dy}$$

cuius ergo aequationis integrale est aequatio (1).

§. 18. Quo istas formas simpliciores reddamus, ponamus  $\alpha\delta = A$ ;  $\beta\beta - \alpha\alpha - \gamma\delta = C$ ;  $\alpha\gamma = E$

eritque  $\delta = \frac{A}{x}$ ;  $\gamma = \frac{E}{x}$  et  $\beta = \sqrt{C + \alpha\alpha + \frac{AE}{\alpha\alpha}}$

Quare huius aequationis differentialis

$$(6) \frac{\sqrt{(A + Cxx + Exx)}}{dx} = \frac{\sqrt{(A + Cyy + Eyy)}}{dy}$$

aequatio integralis est haec;

$$(7) \alpha(xx + yy) = \frac{A}{\alpha} + \frac{E}{\alpha} xxyy + 2xy\sqrt{C + \alpha\alpha + \frac{AE}{\alpha\alpha}}$$

quae simul est integralis completa;

§. 19. Vel ponamus  $A = f\alpha\alpha$ ;  $C = g\alpha\alpha$  et  $E$

$= h\alpha\alpha$ , ut habeamus hanc aequationem differentialem

$$\frac{\sqrt{(f + gxx + hxx)}}{dx} = \frac{\sqrt{(f + gyy + hyy)}}{dy}$$

cuius

eius propterea aequatio integralis completa erit:

$$xx + yy = f + bxy + 2xy\sqrt{f + g + fb}$$

quae etiam novam constantem involvere non videtur, tamen est completa, cum in differentiali tantum ratio quantiatum  $f, g$ , et  $h$  spectatur, ita ut pro  $f, g$ , et  $h$  scribere liceat  $fcc$ ,  $gee$  et  $bcc$ , unde aequatio integralis manifesto completa prodit:

$$xx + yy = fcc + bccxy + 2xy\sqrt{f + g + fbc}$$

$$\text{vel } f(xx + yy) = fee + beexxy + 2xy\sqrt{f(f + gee + be)}$$

positio  $cc = \frac{ee}{f}$ .

§. 20. Quodsi ergo proposita sit haec aequatio differentialis

$$\sqrt{f + gxx + bxx^2} = \frac{dy}{\sqrt{f + gyy + by^2}}$$

valor ipsius  $y$  per functionem algebraicam ipsius  $x$  ex primi poterit, ita ut sit:

$$y = \frac{x\sqrt{f + gxx + fbc} \pm c\sqrt{f + gxx + bxx^2}}{1 - bccxx}$$

$$\text{vel } y = \frac{x\sqrt{f(f + gee + be)} \pm e\sqrt{f(f + gxx + bxx^2)}}{f - beexx}$$

Quodsi ergo sit  $g = 0$ , ut habeatur haec aequatio differentialis

$$\sqrt{f + bxx} = \frac{dy}{\sqrt{f + by^2}}$$

valor integralis completus ipsius  $y$  erit

$$y = \frac{x\sqrt{f(f + be)} \pm e\sqrt{f(f + bxx)}}{f - beexx}$$

unde constantem  $e$  pro lubita determinando, insinueri valores particulares pro  $y$  deduci possunt.

## 48 DE INTEGRATIONE.

§. 21. Methodi autem, qua supra visus sum, beneficio etiam huius aequationis

$$\sqrt[m]{\frac{U+gxx+bx^2}{d}} = \frac{\pi dy}{\sqrt{f+gyy+hy^2}}$$

si modo  $m$  et  $n$  sint numeri rationales, integrale completum, atque id quidem algebraice, exhiberi poterit.

§. 22. Quemadmodum in aequatione supra assumta, variables  $x$  et  $y$  inter se permutabiles sunt constitutae, ut ambae formulae inter se similes euaderent, ita omissa hac limitatione ad formularum differentialium disparium comparationem perueniemus. Ponamus ergo:

$$(1) \alpha xx + \beta yy = 2 \gamma xy + \delta xxx + \epsilon$$

unde fit

$$x = \frac{\gamma y + \sqrt{(\alpha\epsilon + (\gamma\gamma - \delta\epsilon - \alpha\beta)yy + \beta\delta y^2)}}{\alpha - \delta yy}$$

$$\text{et } y = \frac{\gamma x - \sqrt{(\beta\epsilon + (\gamma\gamma - \delta\epsilon - \alpha\beta)xx + \alpha\delta x^2)}}{\beta - \delta xx}$$

hincque

$$(2) \alpha x - \gamma y - \delta xy = \sqrt{(\alpha\epsilon + (\gamma\gamma - \delta\epsilon - \alpha\beta)yy + \beta\delta y^2)}$$

$$(3) \beta y - \gamma x - \delta xy = -\sqrt{(\beta\epsilon + (\gamma\gamma - \delta\epsilon - \alpha\beta)xx + \alpha\delta x^2)}$$

at aequatio (1) differentiata dat:

$$dx(\alpha x - \gamma y - \delta xy) + dy(\beta y - \gamma x - \delta xy) = 0$$

unde conficitur haec aequatio differentialis:

$$\frac{dx}{\sqrt{(\beta\epsilon + (\gamma\gamma - \delta\epsilon - \alpha\beta)xx + \alpha\delta x^2)}} = \frac{dy}{\sqrt{(\alpha\epsilon + (\gamma\gamma - \delta\epsilon - \alpha\beta)yy + \beta\delta y^2)}}$$

cuius propterea integralis est aequatio assumta.

§. 23. Verum haec disparitas facile tollitur, loco  $y$  ponendo  $z\sqrt{\frac{\alpha}{\beta}}$ , cuius rei ratio statim ex aequatione assumta potuisset esse manifesta. Sed alia patet via ad formulas disparis perueniendi, cuius hic exemplum tradidisse sufficiat. Assumatur aequatio:  $x^2 + 2\alpha xyy + 2bxx$

= 6,

$= c$ , cuius differentiale est  $dx(x^2 + axyy + bx)$   
 $+ axxydy = 0$ , seu

$$\frac{dx}{xy} = \frac{-axy}{xx + ayy + b}$$

Iam ex aequatione assumta primo determinetur  $xy$  per  $x$   
 sicque fiet  $xy = \sqrt{\frac{c - bxx - x^4}{2a}}$ ; tum vero  $xx + ayy + b$   
 per  $y$ , at ob  $(xx + ayy + b)^2 = c + (ayy + b)^2$ , erit

$$xx + ayy + b = \sqrt{c + (ayy + b)^2}$$

Quocirca habebitur aequatio differentialis ista

$$\frac{dx\sqrt{2a}}{\sqrt{(c - 2bxx - x^4)}} = \frac{a dy}{\sqrt{(c + bb + ayy + ayy^2)}} \quad \text{cuius propterea integralis est assumta seu } y = \frac{\sqrt{(c - 2bxx - x^4)}}{x\sqrt{2a}}$$

§. 24. Et si hoc integrale non est completum,  
 tamen ex superioribus facile completum reddetur. Po-  
 natur enim:

$$\frac{ady}{\sqrt{(c + b + 2aby + ayy^2)}} = \frac{adz}{\sqrt{(c + bb + 2abzz + azz^2)}} \quad \text{ob } f = c + bb; g = 2ab; h = aa, \text{ erit}$$

$$y = \frac{z\sqrt{(c + bb)(c + bb + 2abze + aze^2)} + \sqrt{(c + bb)(c + bb + 2abzz + azz^2)}}{c + bb - azezz}$$

hic ergo valor aequalis statuatur ipsi  $\frac{\sqrt{(c - 2bxx - x^4)}}{x\sqrt{2a}}$ , et  
 aequatio hinc inter  $x$  et  $z$  resultans integralis erit com-  
 plera huius aequationi differentialis

$$\frac{dx\sqrt{2a}}{\sqrt{(c - 2bxx - x^4)}} = \frac{adz}{\sqrt{(c + b + 2abzz + azz^2)}}$$

Quin etiam ex allatis patet, si haec bina membra in-  
 super per numeros rationales quoscumque multiplicentur,  
 quemadmodum integrale completum inueniri oporteat.

§. 25. Verum missa membrorum disparitate formationem parium membrorum generalius concipimus, ponatur ergo:

$$(1) 0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy$$

vnde differentiando obtinetur:

$$dx(\beta + \gamma x + \delta y + 2\varepsilon xy + \zeta xxy) + dy(\gamma + \gamma y + \delta x + 2\varepsilon xy + \zeta xxy) = 0$$

ideoque.

$$(2) \frac{\beta + \gamma x + \delta y + 2\varepsilon xy + \zeta xxy}{\beta + \gamma y + \delta x + \varepsilon xy + \zeta xxy} = \frac{dy}{dx}$$

Ex resolutione autem aequationis assumtae elicitur.

$$\frac{-\beta - \delta x - \varepsilon xy + \sqrt{(\beta\delta - \alpha\gamma + 2(\beta\delta - \alpha\varepsilon - \beta\gamma)x + (\delta\delta - \gamma\gamma - \alpha\delta - 2\beta\varepsilon)xy + 2(\delta\varepsilon - \beta\zeta - \gamma\delta)x^2 + (\varepsilon\varepsilon - \gamma\zeta)x^4)}}{\gamma + 2\varepsilon x + \zeta xx}$$

Ponatur breuitatis gratia

$$\beta\beta - \alpha\gamma = A; \quad \beta\delta - \alpha\varepsilon - \beta\gamma = B; \quad \delta\delta - \gamma\gamma - \alpha\zeta - 2\beta\varepsilon = C$$

$$\varepsilon\varepsilon - \gamma\zeta = E; \quad \delta\varepsilon - \beta\zeta - \gamma\varepsilon = D;$$

eritque

$$\beta + \delta x + \varepsilon xy + \gamma y + 2\varepsilon xy + \zeta xxy = \pm \sqrt{(A + 2Bx + Cxx + 2Dx^2 + Ex^4)}$$

$$\beta + \delta y + \varepsilon y + \gamma x + 2\varepsilon xy + \zeta xxy = \mp \sqrt{(A + 2By + Cyy + 2Dy^2 + Ey^4)}$$

§. 26. Hinc itaque concludimus huius aequationis differentialis:

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx + 2Dx^2 + Ex^4)}} = \frac{dy}{\sqrt{(A + 2By + Cyy + 2Dy^2 + Ey^4)}}$$

aequationem integram eamque completam esse

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy$$

adhi-

adhibita scilicet superiori horum coefficientium determinatione. Primum autem definiatur  $\beta$  vel  $\varepsilon$  ex hac aequatione

$$\frac{B\varepsilon(\varepsilon-E)-D\varepsilon(\beta\beta-A)}{A\varepsilon\varepsilon-E\beta\beta} + \frac{2ADE-2BE\beta}{B\varepsilon-D\beta} = C$$

tum vero erit:

$$\gamma = \frac{A\varepsilon-E\beta\beta}{B\varepsilon-D\beta}; \alpha = \frac{\beta\beta-A}{\gamma}; \zeta = \frac{\varepsilon E-E}{\gamma} \text{ et}$$

$$\delta = \frac{B\beta(\varepsilon-E)-D\varepsilon(\beta\beta-A)}{A\varepsilon\varepsilon-E\beta\beta} + \gamma \text{ seu } \delta = \gamma + \frac{B+\alpha\varepsilon}{\beta}$$

§. 27. Hinc ergo perspicuum est etiam hanc aequationem differentialem:

$$\frac{dx}{\sqrt{(A+2Dx^2)}} = \frac{dy}{\sqrt{(A+2Dy^2)}}$$

integrari posse: nam ob  $B=0$ ,  $C=0$  et  $E=0$  erit

$$-\frac{D\varepsilon(\beta\beta-A)}{A\varepsilon\varepsilon} - \frac{2A\varepsilon}{\beta} = 0 \text{ seu } \varepsilon = \sqrt[3]{\frac{DD}{2AA}}\beta(A-\beta\beta)$$

at hinc valores nimis prodeunt complicati. Facilius negotium absoluetur, resolvendo valores litterarum evanescentium B, C et E; nam  $E=0$  dat:  $\zeta = \frac{\varepsilon E}{\gamma}$ ; tum  $B=0$  dat:  $\delta = \gamma + \frac{\alpha\varepsilon}{\beta}$ ; atque  $C=0$  dat  $\delta\delta - \gamma\gamma = \alpha\zeta + 2\beta\varepsilon = \frac{\alpha\varepsilon\varepsilon}{\gamma} + 2\beta\varepsilon = \frac{\alpha^2\varepsilon\varepsilon}{\beta\beta} + \frac{2\alpha\gamma\varepsilon}{\beta}$  cuius factores sunt  $\beta\beta = \alpha\gamma$  et  $\alpha\varepsilon + 2\beta\gamma\varepsilon = 0$ . At si esset  $\beta\beta = \alpha\gamma$  foret  $A=0$ , sin autem esset  $\varepsilon=0$  foret et  $\zeta=0$  et  $D=0$ , contra scopum. Fieri ergo oportet  $\alpha\varepsilon = -2\beta\gamma$ ; vnde fiet  $\alpha = -\frac{2\beta\gamma}{\varepsilon}$ ;  $\delta = -\gamma$ ; et  $\zeta = \frac{\varepsilon E}{\gamma}$ . Denique fieri debet  $\beta\beta + \frac{2\beta\gamma\gamma}{\varepsilon} = A$  et  $-2\gamma\varepsilon = \frac{\beta\varepsilon\varepsilon}{\gamma} = D$ . Inde fit  $\varepsilon = \frac{2\beta\gamma\gamma}{A-\beta\beta}$ ; et ob  $\frac{\gamma D}{\varepsilon} = - (2\gamma\gamma + \beta\varepsilon)$  et  $2\gamma\gamma + \beta\varepsilon = \frac{A\varepsilon}{\beta}$ , erit  $\frac{\gamma D}{\varepsilon} = -\frac{A\varepsilon}{\beta}$ ; ideoque  $\varepsilon\varepsilon = -\frac{\beta\gamma D}{A}$ . Ergo  $\frac{+\beta\gamma^2}{(A-\beta\beta)^2} + \frac{1}{A} = 0$ .

§. 28. Cum autem tantum ratio litterarum  $A$  et  $D$  in censum veniat, aequatio vltima valori ab-  
soluto ipsius  $A$  inueniendo inseruit, quem autem nosse  
non est opus. Manebunt ergo litterae  $\gamma$  et  $\beta$  inde-  
terminatae. Ponatur ergo  $\gamma = -Ac$  et  $\beta = Dc$ , erit  
 $\varepsilon\varepsilon = DDcc$ , seu  $\varepsilon = Dc$ , hincque  $\delta = Ac$ ;  $\zeta = -\frac{DDc}{A}$ ;  
et  $\alpha = 2Ac$ . Quare huius aequationis differentialis:

$$\frac{dx}{\sqrt{(A+2Dx^2)}} = \frac{dy}{\sqrt{(A+2Dy^2)}}$$

integrale est.

$$\alpha = 2A + 2D(x+y) - A(xx+yy) + 2Axy + 2Dxy(x+y) - \frac{DD}{A}xyy'$$

Hoc autem integrale non est completum, tale autem  
reddetur ponendo  $\gamma = -A$  et  $\beta = Dcc$ , vnde fit  
 $\varepsilon\varepsilon = DDcc$  et  $\varepsilon = Dc$ ; porro erit  $\delta = A$ ;  $\zeta = -\frac{DDcc}{A}$ ;  
 $\alpha = 2Ac$ ; ita vt integrale completum fit:

$$\alpha = 2Ac + 2Dcc(x+y) - A(xx+yy) + 2Axy + 2Dcxy(x+y) - \frac{DDcc}{A}xyy'$$

vbi  $c$  est constans ab arbitrio pendens, vnde fit:

$$y = \frac{Dcc + Ax + Dcxy + \sqrt{c(2A + \frac{D}{A}c^2)}(A + 2Dx)}{A - 2Dcx + \frac{DDcc}{A}xx}$$

§. 29. Hic casus notari meretur, quo  $A = x$   
et  $D = \frac{1}{2}$ , vt habeatur haec aequatio differentialis.

$$\frac{dx}{\sqrt{(1+x^2)}} = \frac{dy}{\sqrt{(1+y^2)}}$$

vbi ad fractionem tollendas loco  $c$  scribatur  $2c$  eritque  
integrale completum:

$$\alpha = 4c + 4cc(x+y) - xx - yy + 2xy + 2cxy(x+y) - ccxyy'$$

$$\text{seu } y = \frac{2cc + x + cxy + \sqrt{c(1+c^2)}\sqrt{(1+x^2)}}{1 - 2cx + ccxx}$$

integra-



Integralia ergo particularia erunt

I. si  $c = 0$ ;  $y = x$ ;

II. si  $c = \infty$ ;  $y = \frac{x \pm \sqrt{1+x^2}}{x}$ ;

III. si  $c = -1$ ;  $y = \frac{2+x}{1+x^2}$ ;

§. 30. Ex eodem principio si in §. 29. loco litterarum A, B, C, D, E, eadem per quantitatem quam  $p$  multiplicentur, nihilo minus aequatio differentialis erit

$$\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^2+Ex^4)}} = \frac{dy}{\sqrt{(A+2By+Cy^2+2Dy^2+Ey^4)}}$$

inveniturque

$$p = \frac{BBEE-DD\beta\beta}{BBE-ADD} + 2 \frac{(ADE-BE\beta)(AEE-E\beta\beta)}{(BE-D\beta)(BBE-ADD)} - \frac{C(AEE-E\beta\beta)}{BBE-ADD}$$

sum erit  $\gamma = \frac{AEE-E\beta\beta}{BE-D\beta}$ ;  $\alpha = \frac{\beta\beta - Ap}{\gamma}$ ;  $\zeta = \frac{EE-E\beta}{\gamma}$  atque

$\delta = \gamma + \frac{AE+BP}{\beta}$ ; ita ut litterae  $\beta$  et  $\epsilon$  maneant indeter-

minatae, setque propterea aequatio integralis completa:

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\epsilon xy(x+y) + \zeta xyy$$

unde fit:

$$y = \frac{-\beta - \delta x - \epsilon xx + \sqrt{\frac{1}{2}(A+2Bx+Cxx+2Dx^2+Ex^4)}}{\gamma + 2\epsilon x + \zeta xx}$$

§. 31. Notandum denique est, non solum hanc aequationem differentialem, cuius integrale completum modo exhibui, sed etiam hanc multo latius patentem

$$\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^2+Ex^4)}} = \frac{m \, dy}{\sqrt{(A+2By+Cy^2+2Dy^2+Ey^4)}}$$

semper algebraice et quidem complete integrari posse, dummodo coefficientium  $m$  et  $n$  ratio fuerit rationalis; haec enim integratio simili modo instituitur, quo supra vius sum ad aequationem, quae mihi hic praecipue erat proposita, integrandam. Methodus autem, cuius

hic specimina attuli, ita mihi videtur comparata, ut inaequalem eius diligentius excolendo, ad insignes usus apta reddi queat, vnde haud contemnenda commoda in Analyſi ſunt redundatura.

§ 32. Hic autem obſervo, formulam §. 28 aſſertam latius extendendo, cuiusmodi differentialia inter ſe comparari poſſe, quae ſunt diſparia, atque adeo exemplum diſparitatis §. 26. allatum hoc modo obſiderari poſſe; ita ut omnia, quae haecenus ſunt tradita, in hac generali inueſtigatione contineantur. Pingatur ſcilicet haec aequatio integralis:

$$(1) \dots axxyy + 2\beta xxy + \delta xy + \epsilon y + 2\zeta xy + 2\eta x + 2\theta y + x = 0$$

ex qua fit.

$$(2) \dots y = \frac{-\beta xy - \zeta x - \theta + \sqrt{(\beta xy + \zeta x - \theta)^2 - (axx + 2\eta x + \delta xy + \epsilon y + 2\theta y + x)}}{axx + 2\eta x + \delta xy + \epsilon y}$$

$$(3) \dots x = \frac{-yy - \zeta y - \eta - \sqrt{(yy + \zeta y + \eta)^2 - (axy + \beta y + \delta)(eyy + 2\theta y + x)}}{axy + 2\beta y + \delta}$$

Ponatur iam breuitatis gratia:

$$App = \beta\beta - a\delta$$

$$2Bpp = 2\beta\zeta - 2a\eta - 2\gamma\delta$$

$$Cpp = \zeta\zeta + 2\beta\theta - ax - \delta\epsilon - 4\gamma\eta$$

$$2Dpp = 2\zeta\theta - 2\gamma x - 2\epsilon\eta$$

$$Epp = \theta\theta - \epsilon x$$

$$2Aqq = \gamma\gamma - a\epsilon$$

$$2Bqq = 2\gamma\zeta - 2a\theta - 2\beta\epsilon$$

$$Cqq = \zeta\zeta + 2\gamma\eta - ax - \delta\epsilon - 4\beta\theta$$

$$2Dqq = 2\zeta\eta - 2\beta x - 2\delta\theta$$

$$Eqq = \eta\eta - \delta x$$

eritque:

$$(4) \dots pV(Ax^4 + 2Bx^3 + Cx^2 + 2Dx + E) = axxy + 2\gamma xy + \epsilon y + \beta xx + \zeta x + \theta$$

$$(5) \dots -qV(21y^4 + 22y^3 + 6yy^2 + 2Dy + E) = axyy + 2\beta xy + \delta x + \gamma y + \zeta y + \eta$$

§. 33.

§. 33. At si aequatio integralis assumta differentietur, fiet

$$(6) \dots dx (\alpha xy + 2\beta yy + \gamma yy + \delta x + \zeta y + \eta) + dy (\alpha xy + \beta xx + 2\gamma xy + \epsilon y + \zeta x + \theta) = 0$$

vade si istorum factorum valores (4) et (5) reperti substituantur, orietur ista aequatio differentialis:

$$(7) \dots \frac{pdy}{\sqrt{(Ax^2 + 2Bxy + Cyy^2 + 2Dx + E)}} = \frac{pdy}{\sqrt{(2y^2 + 2By^2 + Cy + 2Dy + E)}}$$

cuius propterea integralis est aequatio assumta (1). Cum autem supra habeantur 10 aequationes, coefficientium autem  $\alpha, \beta, \gamma, \delta$ , etc. numerus sit 9, quorum vnus pro lubitu assumi potest, octo remanebunt litterae determinandae. Porro autem insuper definiendae accedunt binae litterae  $p$  et  $q$ , ita vt nunc decem quantitates adsint incognitae; ex quo coefficients vtriusque formulae A, B, C, D, E, et  $\mathcal{M}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$  videntur pro lubitu assumi posse. Verum perspicuum est, cum alteri iam fuerint ad libitum assumti, alteros non omnino ab arbitrio nostro pendere, alias enim quaeuis formula ad algebraicam reduci posset.

§. 34. Hinc autem aliae datae formulae transformationes non inelegantes obtineri possunt, si loco  $y$  alii valores substituantur. Veluti si ponatur  $\mathcal{C} = 0$ , seu  $\eta\eta = \delta x$ , statuatursque  $y = z$  sequens prodibit aequatio differentialis.

$$(8) \dots \frac{qdx}{\sqrt{(Ax^2 + 2Bxz + Cxz^2 + 2Dx + E)}} = \frac{zpdz}{\sqrt{(2z^2 + 2Bz^2 + Cz^2 + 2E)}}$$

cuius propterea integralis est aequatio assumta, si ponatur  $y = z$ , statuatursque  $\eta\eta = \delta x$ , ac reliquae litterae rite deter-

determinentur. Integrale etiam completum nulla difficultate reperietur, nam etiam si fortasse integrale inueniatur non iam non inuoluat constantem, ponatur

$$\sqrt{(Ax^4 + 2Bx^3 + Cx^2 + Dx + E)} = \sqrt{(Au^4 + 2Bu^3 + Cu^2 + Du + E)}$$

et huius aequationis integrale completum ex antecedentibus assignare licebit; atque hinc integrale quoque completum aequationis ex formulis disparibus constantis colligetur.

§. 35. Quemadmodum huius aequationis differentialis, ut a simplicissimis incipiam:

$$\frac{dx}{\sqrt{(f + gx)}} = \frac{dy}{\sqrt{(f + gy)}}$$

integrale completum est:

$$gg(xx + yy) - 2ggxy - 2ccg(x + y) + c^2 - 4ccf = 0$$

Deinde vero huius aequationis differentialis

$$\frac{dx}{\sqrt{(f + gx)}} = \frac{dy}{\sqrt{(f + gy)}}$$

integrale completum est:

$$xx + yy - 2xy\sqrt{(f + fgc)} - ccff = 0$$

Tertio vero huius aequationis differentialis

$$\frac{dx}{\sqrt{(f + gx^2)}} = \frac{dy}{\sqrt{(f + gy^2)}}$$

integrale completum est

$$f(xx + yy) + \frac{g^2 cc}{f} xxyy - gxy(x + y) - 2fxy - gcc(x + y) - 2fc = 0$$

Quarto porro huius aequationis differentialis

$$\frac{dx}{\sqrt{(f + gx^4)}} = \frac{dy}{\sqrt{(f + gy^4)}}$$

integrale completum reperitur est

$$f(xx + yy) - fcc - gccxxyy - 2xy\sqrt{(f + gc^4)} = 0$$

Ita

Ita etiam integrale completum huius aequationis

$$\frac{dx}{\sqrt{(f+g-x^2)}} = \frac{dy}{\sqrt{(f+g-y^2)}}$$

reperiri poterit :

§. 36. Determinentur primo in §. 33. valores, ita ut prodeat haec aequatio

$$\frac{dx}{\sqrt{(f+g-x^2)}} = \frac{dy}{\sqrt{(f+g-y^2)}}$$

cuius integralis completa reperitur:

$$gg(xx+yy) - Aggexxy - Afgccxy(x+y) - 2fgc(x+y) + fcc = 0$$

Ponatur nunc  $x=tt$  et  $y=uu$ , ut prodeat haec aequatio differentialis

$$\frac{dt}{\sqrt{(f+g-t^2)}} = \frac{du}{\sqrt{(f+g-u^2)}}$$

cuius propterea integralis completa erit

$$gg(t^2+u^2) - Aggct^2t - Afgccuu(tt+uu) - 2ggttuu - 2fgc(tt+uu) + fcc = 0$$

vnde notari meretur casus ex hypothesi  $c = \infty$  resultans, qui dat  $Agttuu(tt+uu) = f$ .