DILUCIDATIONES

IN CAPITA POSTREMA CALCULI MEI DIFFERENTIALIS

DE FUNCTIONIBUS INEXPLICABILIBUS

1.

Cum hoc argumentum, utpote in Analysi prorsus novum neutiquam satis dilucide sit pertractatum, constitui hic, idem maiori studio retractare, atque omnia momenta, quibus innititur, ex primis principiis repetere; ubi plurimum iuvabit idonea signa in calculum introduxisse. Ita si proposita fuerit series quaecunque, eius terminos indicibus 1, 2, 3, 4, &c. respondentes his signis repraesentabo (1), (2), (3), (4), &c. hincque ergo terminus generalis huius seriei, indici indefinito x respondens, mihi erit (x), qui ergo pro quavis serie certa erit functio ipsius x, quam penitus cognitam assumo, ita scilicet comparatam, ut eius valores non solum pro numeris integris, loco x assumtis, sed etiam pro fractis, atque adeo surdis exhiberi queant.

2. Denotet porro $\Sigma : x$ terminum summatorium eiusdem seriei, qui exprimat summam terminorum a primo incipientium usque ad terminum (x), ita ut sit

$$\Sigma : x = (1) + (2) + (3) + (4) + \ldots\ldots + (x),$$

cuius ergo omnes valores, quoties x fuerit numerus integer positivus ex ipsa serie actu exhiberi poterunt; siquidem erit ut sequitur:

$$\Sigma : 1 = (1)$$
$$\Sigma : 2 = (1) + (2)$$
$$\Sigma : 3 = (1) + (2) + (3)$$
$$\Sigma : 4 = (1) + (2) + (3) + (4)$$
&c.

Cuiusmodi autem valores eadem formula $\Sigma : x$ sit acceptura quando loco x numeri fracti, vel adeo surdi, sive positivi, sive negativi tribuantur, hinc neutiquam apparet, unde istos valores ad peculiare functionum genus, quas *inexplicabiles* vocavi, refero. Quemadmodum igitur tales functiones per formulas analyticas determinatas exprimi queant, hic imprimis sum investigaturus

3. Totum autem hoc negotium commodissime expedietur per differentias continuas ex serie proposita derivatas, dum scilicet quilibet terminus a sequente subtrahitur, quo pacto oritur series primarum differentiarum, ex qua porro simili modo differentiae secundae, tertiae, quartae &c. formabuntur. Omnes autem has differentias sequentibus characteribus indicabo

DIFFERENTIAE I.	DIFFERENTIAE II.	DIFFERENTIAE III.
$(2)-(1)=\Delta 1$	$\Delta 2-\Delta 1=\Delta^2 1$	$\Delta^2 2-\Delta^2 1=\Delta^3 1$
$(3)-(2)=\Delta 2$	$\Delta 3-\Delta 2=\Delta^2 2$	$\Delta^2 3-\Delta^2 2=\Delta^3 2$
$(4)-(3)=\Delta 3$	$\Delta 4-\Delta 3=\Delta^2 3$	$\Delta^2 4-\Delta^2 3=\Delta^3 3$
$(5)-(4)=\Delta 4$	$\Delta 5-\Delta 4=\Delta^2 4$	$\Delta^2 5-\Delta^2 4=\Delta^3 4$
&c.	&c.	&c.

4. His characteribus constitutis singuli seriei termini ex primo (1), eiusque differentiis $\Delta 1$, $\Delta^2 1$, $\Delta^3 1$, $\Delta^4 1$, &c. exprimi poterunt. Cum enim sit $(2)=(1)+\Delta 1$, & $\Delta 2=\Delta 1+\Delta^2 1$, ob $(3)=(2)+\Delta 2$, erit $(3)=(1)+2\Delta 1+\Delta^2 1$: hinc iam fluit ista aequalitas $\Delta 3=\Delta 1+2\Delta^2 1+\Delta^3 1$. Quia nunc $(4)=(3)+\Delta 3$ habebimus $(4)=(1)+3\Delta 1+3\Delta^2 1+\Delta^3 1$. Inde porro sequitur: $\Delta 4=\Delta 1+3\Delta^2 1+3\Delta^3 1+\Delta^4 1$, ob $(5)=(4)+\Delta 4$, erit $(5)=(1)+4\Delta 1+6\Delta^2 1+4\Delta^3 1+\Delta^4 1$, & ita porro. Ex ipsa formatione harum formularum manifestum est hic eosdem coefficientes, qui in potestate binomiali habentur, cuius exponens est unitate minor quam index termini propositi, occurrere. Ita erit

$$(n)=(1)+\frac{n-1}{1}\Delta 1+\frac{n-1}{1}\cdot\frac{n-2}{2}\Delta^2 1+\frac{n-1}{1}\cdot\frac{n-2}{2}\cdot\frac{n-3}{3}\Delta^3 1+\&c.$$

5. Quod si hic numerum n unitate augeamus habebimus:

$$(n+1)=(1)+\frac{n}{1}\Delta 1+\frac{n}{1}.\frac{n-1}{2}\Delta^2 1+\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}\Delta^3 1+\&c.$$

Cum iam haec postrema expressio exprimat terminum qui a primo n gradibus est remotus, simili modo terminus, qui a secundo per totidem gradus est promotus $(n+2)$ ex secundo eiusque differentiis determinatur: Erit enim:

$$(n+2)=(2)+\frac{n}{1}\Delta 2+\frac{n}{1}.\frac{n-1}{2}\Delta^2 2+\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}\Delta^3 2+\&c.$$

Eodem modo evidens est fore protinus.

$$(n+3)=(3)+\frac{n}{1}\Delta 3+\frac{n}{1}.\frac{n-1}{2}\Delta^2 3+\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}\Delta^3 3+\&c.$$

$$(n+4)=(4)+\frac{n}{1}\Delta 4+\frac{n}{1}.\frac{n-1}{2}\Delta^2 4+\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}\Delta^3 4+\&c.$$

6. Hinc ergo patet, ipsum seriei nostrae terminum generalem (x) ex primo eiusque differentiis hoc modo definiri:

$$(x)=(1)+\frac{x-1}{1}\Delta 1+\frac{x-1}{1}.\frac{x-2}{2}\Delta^2 1+\frac{x-1}{1}.\frac{x-2}{2}.\frac{x-3}{3}\Delta^3 1+\&c.$$

Hinc terminus ultimum sequens $(x+1)$ erit:

$$(x+1)=(1)+\frac{x}{1}\Delta 1+\frac{x}{1}.\frac{x-1}{2}\Delta^2 1+\frac{x}{1}.\frac{x-1}{2}.\frac{x-2}{3}\Delta^3 1+\&c.$$

quae expressio, cum in sequentibus frequentissime occurrat, brevitatis gratia introducamus sequentes characteres

$$\frac{x}{1}=x$$

$$\frac{x}{1}.\frac{x-1}{2}=x'$$

$$\frac{x}{1}.\frac{x-1}{2}.\frac{x-2}{3}=x''$$

$$\frac{x}{1}.\frac{x-1}{2}.\frac{x-2}{3}.\frac{x-3}{4}=x'''$$

&c.

 qui-

quibus adhibitis habebimus sequentes aequationes:

$$(x+1)=(1)+x\Delta 1+x'\Delta^2 1+x''\Delta^3 1+\&c.$$
$$(x+2)=(2)+x\Delta 2+x'\Delta^2 2+x''\Delta^3 2+\&c.$$
$$(x+3)=(3)+x\Delta 3+x'\Delta^2 3+x''\Delta^3 3+\&c.$$
$$(x+4)=(4)+x\Delta 4+x'\Delta^2 4+x''\Delta^3 4+\&c.$$
$$\cdots\cdots\cdots$$
$$(x+n)=(n)+x\Delta n+x'\Delta^2 n+x''\Delta^3 n+\&c.$$

7. Deinde etiam summas quotcunque terminorum nostrae seriei ex solo termino primo eiusque differentiis determinari poterit, quemadmodum sequens tabula declarat:

$$\begin{array}{ll}
 & \Sigma:1=(1) \\
\text{add.} & (2)=(1)+\Delta 1 \\
\hline
 & \Sigma:2=2(1)+\Delta 1 \\
 & (3)=(1)+2\Delta 1+\Delta^2 1 \\
\hline
 & \Sigma:3=3(1)+3\Delta 1+\Delta^2 1 \\
 & (4)=(1)+3\Delta 1+3\Delta^2 1+\Delta^3 1 \\
\hline
 & \Sigma:4=4(1)+6\Delta 1+4\Delta^2 1+\Delta^3 1 \\
 & (5)=(1)+4\Delta 1+6\Delta^2 1+4\Delta^3 1+\Delta^4 1 \\
\hline
 & \Sigma:5=5(1)+10\Delta 1+10\Delta^2 1+5\Delta^3 1+\Delta^4 1 \\
 & \&c.
\end{array}$$

Hic iterum evidens est coefficientes eosdem esse, qui in potestate binomiali eiusdem ordinis occurrunt

8. In usum igitur vocatis characteribus modo ante stabilitis ipsum terminum summatorium nostrae seriei $\Sigma:x$ exprimere valemus: Erit enim

$$\Sigma:x=x(1)+x'\Delta 1+x''\Delta^2 1+x'''\Delta^3 1+\&c.$$

quae forma iam ita est comparata, ut loco x non solum numeros integros, sed etiam fractos, imo surdos, quoscunque tam positivos quam negativos accipere liceat, quibus casibus utique ista expressio in infinitum progredietur, nisi forte series proposita deducat tandem ad differentias evanescentes, cuiusmodi series algebraicae vocari solent, quibus ergo casibus

non

non ad functiones inexplicabiles pervenitur. Interim tamen ista expressio pro termino summatorio inventa quando in infinitum porrigitur nihil adiumenti affert, quando differentiationes, vel etiam summationes sunt instituendae; quamobrem in id erit incumbendum, quemadmodum, saltem pro certis casibus terminus summatorius inventus in alias formas transfundi queat, quae neque differentiationi neque integrationi refragentur, atque huc pertinent omnia subsidia, quae in calculo differentiali fusius exposui, & quorum inventio non parum erat obstrusa: sequenti autem modo totum hoc negotium facile conficietur.

9. Ad expressionem pro termino summatorio $\Sigma : x$ modo ante inventam addantur plures formulae sub hac specie contentae.

$$(n) + x \Delta n + x' \Delta^2 n + x'' \Delta^3 n + \&c. \quad . \quad . \quad . \quad - (x+n)$$

quarum summae cum sint nihilo aequales, omnes, quotcunque fuerint, cum $\Sigma : x$ iunctim sumtae nihilominus terminum summatorium expriment. Sumantur ergo pro n successive omnes numeri 1, 2, 3, 4, &c., & tota expressio secundum columnas verticales singulis valoribus x, x', x'', &c. respondentes sequenti modo disponantur:

Expressio generalis pro termino summatorio

$$\begin{array}{l}
\qquad x(1) + x'\Delta 1 + x''\Delta^2 1 + x'''\Delta^3 1 + \&c. \\
(1) + x\Delta 1 + x'\Delta^2 1 + x''\Delta^3 1 + x'''\Delta^4 1 + \&c. \ldots - (x+1) \\
(2) + x\Delta 2 + x'\Delta^2 2 + x''\Delta^3 2 + x'''\Delta^4 2 + \&c. \ldots - (x+2) \\
(3) + x\Delta 3 + x'\Delta^2 3 + x''\Delta^3 3 + x'''\Delta^4 3 + \&c. \ldots - (x+3) \\
\quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
(n) + x\Delta n + x'\Delta^2 n + x''\Delta^3 n + x'''\Delta^4 n + \&c. \ldots - (x+n)
\end{array}$$

10. Etsi veritas huius expressionis nulli amplius dubio est obnoxia, tamen non parum iuvabit, eam ex ipsa forma con-

confirmaſſe. Colligantur nimirum in unam ſummam ſingulae columnae verticales, ac prima quidem ſumma erit

$$(1)+(2)+(3)+(4)+\;.\;.\;.\;.\;.\;.\;.\;.\;+(n)=\Sigma:n$$

ſecunda columna dat

$$x[(1)+\Delta 1+\Delta 2+\Delta 3+\;.\;.\;.\;.\;.\;+\Delta n]$$

Cum autem ſit

$$\Delta 1=(2)-(1);\quad \Delta 2=(3)-(2);\quad \Delta 3=(4)-(3);\ \&c.$$

tota haec ſumma contrahetur in $x(n+1)$.

Simili modo tertiae columnae ſumma erit

$$x'[\Delta 1+\Delta^2 1+\Delta^2 2+\Delta^2 3+\Delta^2 4+\;.\;.\;.\;.\;.\;+\Delta^2 n]$$

& quia

$$\Delta^2 1=\Delta 2-\Delta 1;\ \Delta^2 2=\Delta 3-\Delta 2\;.\;.\;.\;.\;\Delta^2 n=\Delta(n+1)-\Delta n$$

illa ſumma contrahitur in $x'\Delta(n+1)$. Eodem modo patet fore quartae columnae ſummam $x''\Delta^2(n+1)$, & quintae $= x'''\Delta^3(n+1)$, & ita porro. Ultimae vero columnae ſubtrahendae ſumma eſt

$$(x+1)+(x+2)+(x+3)+\;.\;.\;.\;.\;.\;+(x+n)=\Sigma:(x+n)-\Sigma:x.$$

11. Summa igitur omnium columnarum verticalium mediarum praeter primam & ultimam eſt ut vidimus

$$x(n+1)+x'\Delta(n+1)+x''\Delta^2(n+1)+x'''\Delta^3(n+1)+\&c.$$

Cum autem ſit $x(1)+x'\Delta 1+x''\Delta^2 1+x'''\Delta^3 1+\&c.=\Sigma:x$, ſingulis terminis numero n auctis erit ſumma noſtrae ſeriei

$$x(n+1)+x'\Delta(n+1)+x''\Delta^2(n+1)+\&c.=\Sigma:(x+n)-\Sigma:n$$

conſequenter omnium plane columnarum ſumma praeter ultimam eſt $=\Sigma:(x+n)$: unde ſi ſumma ultimae columnae, quae eſt $\Sigma:(x+n)-\Sigma:x$ ſubtrahatur, remanebit ſumma totius figurae $=\Sigma:x$, hoc eſt terminus ſummatorius quaeſitus.

12. Maxime hic mirum videbitur, quod valorem formulae $\Sigma:x$, quae ſerie ſatis ſimplici exprimitur, per congeriem innumerabilium ſerierum expreſſum & involutum dederimus; verum mox ſummus uſus huius formae complicatiſſimae patebit, quando numerum ſerierum horizontalium adeo in infinitum continuaverimus, quod fiet, ſi pro n numerum infinitum accipiamus, quemadmodum nunc clarius explicabimus.

13.

13. Denotante igitur n numerum infinite magnum, ſumma ſecundae columnae verticalis, quae eſt $x(n+1)$ continebit terminum ſeriei noſtrae infiniteſimum, qui ergo ſi evaneſcat, multo magis ſummae ſequentium columnarum verticalium evaneſcent; quamobrem hoc caſu ſufficiet ſolam primam columnam cum ultima in calculo retinuiſſe. Sin autem termini infiniteſimi non evaneſcant, ſed tamen inter ſe fuerint aequales, tum tertiam columnam cum ſequentibus abiicere licebit. Porro autem ſi demum differentiae ſecundae infiniteſimae evaneſcant, tres priores columnas verticales in calculo retineri debebunt, ſimilique modo quatuor ſi tertiae demum infiniteſimae evaneſcant. Secundum igitur hoc ſerierum diſcrimen, ipſas ſeries in ſequentes ſpecies diſtribuemus:

Species prima ſerierum,

quarum termini infiniteſimi evaneſcunt

14. Quoties igitur tales ſeries proponuntur, pro earum termino ſummatorio ſufficiet terminos primae & ultimae columnae verticalis in calculo retinuiſſe, ſicque nanciſcemur pro termino ſummatorio ſequentem expreſſionem

$$\begin{aligned} \Sigma : x = {} & (1) + (2) + (3) + (4) + \&c. \\ & -(x+1) - (x+2) - (x+3) - (x+4) - \&c. \end{aligned}$$

quae quidem in infinitum excurrit, atque eo magis converget quo minor fuerit index x, quandoquidem ſi evaneſcat, tota ſeries in nihilum abibit, ſive erit $\Sigma : 0 = 0$, id quod cum rei natura egregie congruit; quando enim numerus terminorum addendorum eſt nullus, etiam ſumma neceſſario debet eſſe nulla.

15. Quando autem index x numerus eſt praemagnus, haec ſeries utique parum converget, verum ſemper licebit huiuſmodi caſus ad indices minores reducere. Cum enim ſit $\Sigma : (x+1) = \Sigma : x + (x+1)$, ſimili erit modo

Σ :

$\Sigma : (x+2) = \Sigma : x + (x+1) + (x+2)$, atque adeo in genere, denotante i numerum integrum

$\Sigma : (x+i) = \Sigma : x + (x+1) + (x+2) + \ldots + (x+i)$

quamobrem si summa $x+i$ terminorum desideretur, sufficiet summam x terminorum, hoc est $\Sigma : x$ investigasse, hocque modo omnes huiusmodi quaestiones reduci poterunt ad casus, ubi index x est adeo unitate minor, quo casu series pro $\Sigma : x$ ante data vehementer converget.

16. Talis reductio imprimis est necessaria, quando index x est numerus negativus. Cum enim sit $\Sigma : x = \Sigma : (x-1) + (x)$, erit $\Sigma : (x-1) = \Sigma : x - (x)$, eodemque modo $\Sigma : (x-2) = \Sigma : x - (x) - (x-1)$, & $\Sigma : (x-3) = \Sigma : x - (x) - (x-1) - (x-2)$ & in genere $\Sigma : (x-i) = \Sigma : x - (x) - (x-1) \ldots - (x-i+1)$ hocque modo quantumvis numerus negativus $x-i$ fuerit magnus, resolutio semper ad $\Sigma : x$ reduci potest, ita ut sit $x < 1$.

EXEMPLUM.

17. Proposita sit haec series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots\ldots \frac{1}{x} = \Sigma : x$$

ita ut huius seriei harmonicae summa x terminorum desideretur, ubi pro x numeros quoscunque praeter integros positivos accipere licet, siquidem pro casibus, quibus x est numerus integer positivus, tota res nulla difficultate laborat. Hoc igitur casu ex forma ante data erit

$$\Sigma : x = \begin{cases} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{\&c.} \\ -\frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{x+4} - \text{\&c.} \end{cases}$$

quae duae series in hanc unicam contrahentur

$$\Sigma : x = \frac{x}{x+1} + \frac{x}{2(x+2)} + \frac{x}{3(x+3)} + \frac{x}{4(x+4)} + \text{\&c.}$$

cuius ergo seriei summa per se constat, quoties x fuerit numerus

merus integer positivus, ita erit:

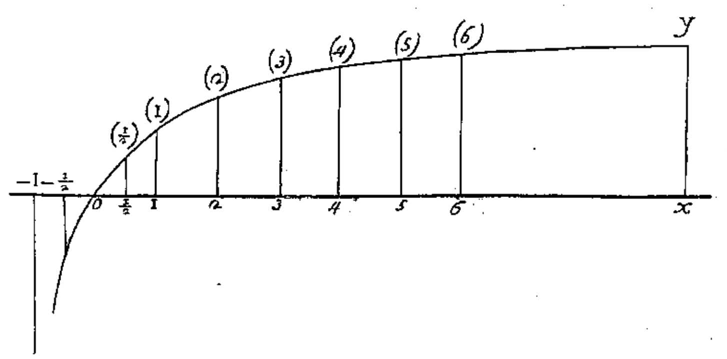
$$
\begin{array}{r|l}
\text{si } x = 1 & 1 = \frac{1}{2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \&c. \\
x = 2 & 1 + \frac{1}{2} = \frac{2}{1.3} + \frac{2}{2.4} + \frac{2}{3.5} + \frac{2}{4.6} + \&c. \\
x = 3 & 1 + \frac{1}{2} + \frac{1}{3} = \frac{3}{1.4} + \frac{3}{2.5} + \frac{3}{3.6} + \frac{3}{4.7} + \&c. \\
x = 4 & 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{4}{1.5} + \frac{4}{2.6} + \frac{4}{3.7} + \frac{4}{4.8} + \&c. \\
\&c. & \qquad \&c.
\end{array}
$$

quae quidem series sunt notissimae.

18. Quo haec clarius intelligantur construamus curvam, cuius abscissae $ox = x$ respondeat applicata $xy = y = \Sigma : x$; ita ut sumtis super axe ox intervallis unitate aequalibus $0, 1$; $1, 2$; $2, 3$; $3, 4$; &c. applicatae futurae sint

$1 \,.\,.\, (1) = 1$
$2 \,.\,.\, (2) = 1 + \frac{1}{2}$
$3 \,.\,.\, (3) = 1 + \frac{1}{2} + \frac{1}{3}$
$4 \,.\,.\, (4) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
&c. atque aequatio inter binas coordinatas erit

$$y = \frac{x}{x+1} + \frac{x}{2(x+2)} + \frac{x}{3(x+3)} + \frac{x}{4(x+4)} + \&c.$$



ex qua ergo aequatione omnes applicatae intermediae definiri poterunt, atque adeo sufficiet pro x valores unitate minores accepisse. Ita si applicata $\frac{1}{2} \ldots (\frac{1}{2})$ abscissae $0 \ldots \frac{1}{2} = \frac{1}{2}$ desideretur reperietur $\frac{1}{2} \ldots (\frac{1}{2}) = \frac{1}{3} + \frac{1}{2.5} + \frac{1}{3.7} + \frac{1}{4.9} + \frac{1}{5.11} +$ &c. cuius summa per logarithmos assignari poterit hoc modo: formetur haec series: $y = \frac{t^3}{1.3} + \frac{t^5}{2.5} + \frac{t^7}{3.7} + \frac{t^9}{4.9} +$ &c. quae ergo series sumto $t = 1$ dabit valorem quaesitum, at vero differentiando habebimus:

$\frac{dy}{dt} = \frac{tt}{1} + \frac{t^4}{2} + \frac{t^6}{3} + \frac{t^8}{4} +$ &c. & denuo differentiando:

$\frac{ddy}{2dt^2} = t + t^3 + t^5 + t^7 +$ &c. $= \frac{t}{1-tt}$. Hinc ergo vicissim erit $\frac{dy}{2dt} = \int \frac{t\,dt}{1-tt}$, & $y = 2\int dt \int \frac{t\,dt}{1-tt}$, quae duplex integratio reducitur more solito ad unicam, quo facto erit $y = 2t \int \frac{t\,dt}{1-tt} - 2\int \frac{tt\,dt}{1-tt}$. Quia autem post integrationem statui debet $t = 1$, erit $y = 2\int \frac{t\,dt}{1-tt} - 2\int \frac{tt\,dt}{1-tt} = 2\int \frac{t\,dt}{1+t}$; quamobrem integrando fiet $y = 2t - 2l(t+1)$, ideoque nostro casu $y = 2 - 2l2$, cuius valor proxime verus est $0,61370564$.

19. Inventa iam applicata abscissae $\frac{1}{2}$ respondente, scilicet $\Sigma : \frac{1}{2} = 2 - 2l2$, ex ea sequentes per formulas supra datas facile derivantur; scilicet

$$\Sigma : (1+\tfrac{1}{2}) = \frac{2}{3} + \Sigma : \tfrac{1}{2}$$

$$\Sigma : (2+\tfrac{1}{2}) = \frac{2}{3} + \frac{2}{5} + \Sigma : \tfrac{1}{2}$$

$$\Sigma : (3+\tfrac{1}{2}) = \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \Sigma : \tfrac{1}{2} \quad \text{\&c.}$$

Quin etiam praecedentes applicatae in figura non expressae ex formula $\Sigma:(x-i)$ determinari poterunt, quam invenimus $\Sigma:(x-i)=\Sigma:x-(x)-(x-1)-(x-2)\ldots-(x-i+1)$ quia nostro casu $x=\frac{1}{2}$, erit applicata

$\Sigma:\left(-\frac{1}{2}\right)=\Sigma:\frac{1}{2}-2=-2l2$, erit scilicet negativa, quae adeo sumto $x=-1$ fit infinita. Infinita vero etiam evadet casibus $x=-2$, $x=-3$, $x=-4$ &c.: intra autem haec intervalla erit

$$\Sigma:-\left(1+\frac{1}{2}\right)=\Sigma:\frac{1}{2}-2+2$$

$$\Sigma:-\left(2+\frac{1}{2}\right)=\Sigma:\frac{1}{2}-2+2-\frac{2}{3}$$

$$\Sigma:-\left(3+\frac{1}{2}\right)=\Sigma:\frac{1}{2}-2+2-\frac{2}{3}+\frac{2}{5}$$

&c.

20. Differentiemus nunc seriem pro applicata y inventam fietque

$$\frac{dy}{dx}=\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\&c.$$

quae ergo series exprimit tangentem anguli, sub quo elementum curvae in y ad axem inclinatur; unde patet pro abscissa infinita hanc inclinationem fore nullam, sive tractum curvae in infinito axi esse parallelum. Hinc ergo sumto $x=0$ innotescet inclinatio curvae ad ipsum initium =

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\&c.=\frac{\pi\pi}{6}=1,644,$$

ideoque angulus $=58^\circ, 42'$. Tum vero sumto $x=1$ erit

$$\frac{dy}{dx}=\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\&c.=\frac{\pi\pi}{6}-1=0,644,$$

ubi ergo inclinatio erit $=32^\circ, 48'$, hincque ulterius continuando inclinatio continuo decrescet.

 21.

21. Retrogrediendo vero ad abſciſſas negativas ſupra vidimus, caſibus, quibus $x = -1$; vel $x = -2$; vel $x = -3$ applicatas fieri infinite magnas, & totidem curvae aſſymptotas conſtituere. Nunc vero videbimus, iiſdem locis fieri $\frac{dy}{dx} = \infty$, ibique inclinationem curvae eſſe $90°$, ſive tangentes ad axem fore perpendiculares. Praeterea quoniam ſeries pro $\frac{dy}{dx}$ inventa ſemper habet ſummam poſitivam, ſequitur omnes partes curvae dextrorſum ſemper aſcendere, contra vero ſiniſtrorſum deſcendere.

22. Quin etiam poterimus integrationem adhibere, atque aream curvae ab initio uſque ad applicatam xy aſſignare. Ex prima enim forma, ad quam ſumus perducti immediate manifeſto fiet

$$\int y dx = \begin{matrix} x & + & \frac{1}{2}x & + & \frac{1}{3}x & + \&c. \\ -l(1+x) & - & l(2+x) & - & l(3+x) & - \&c. \end{matrix} + \text{Conſt.}$$

quae conſtans ita debet determinari, ut caſu $x = 0$ tota area evaneſcat, unde illa rite ita exprimetur

$$\int y dx = \begin{matrix} x & + & \frac{1}{2}x & + & \frac{1}{3}x & + \&c. \\ -l(1+x) & - & l(1+\frac{1}{2}x) & - & l(1+\frac{1}{3}x) & - \&c. \end{matrix}$$

Cum igitur ſit

$$l\left(1+\frac{x}{n}\right) = \frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4} + \&c. - \&c.$$

ſuperior expreſſio per ſeries ſequentes exprimi poterit:

$$\begin{aligned}\int y dx = & \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \frac{x^6}{6} - \&c. \\ & + \frac{x^2}{2.4} - \frac{x^3}{3.8} + \frac{x^4}{4.16} - \frac{x^5}{5.32} + \frac{x^6}{6.64} - \&c. \\ & + \frac{x^2}{2.9} - \frac{x^3}{3.27} + \frac{x^4}{4.81} - \frac{x^5}{5.243} + \frac{x^6}{6.729} - \&c. \\ & + \frac{x^2}{2.16} - \frac{x^3}{3.64} + \frac{x^4}{4.256} - \frac{x^5}{5.1224} + \frac{x^6}{6.4896} - \&c. \\ & \&c.\end{aligned}$$

23.

23. Quod si iam has series verticaliter colligamus habebimus

$$\int y\,dx = \frac{1}{2}x^2\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\&c.\right) = +0,822467\cdot x^2$$

$$-\frac{1}{3}x^3\left(1+\frac{1}{8}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\&c.\right) = -0,400685\cdot x^3$$

$$+\frac{1}{4}x^4\left(1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\frac{1}{625}+\&c.\right) = +0,270581\cdot x^4$$

$$-\frac{1}{5}x^5\left(1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1224}+\quad+\&c.\right) = -0,207385\cdot x^5$$

Ponamus nunc $x = 1$, ut prodeat area $o1(1)$, & quia fractiones decimales hic datae parum convergunt, notetur seriei cuiuscunque, ubi signa alternantur, scilicet:

$$s = a - b + c - d + e - \&c.$$

summam per differentias continuas ita exprimi, ut sit

$$s = \tfrac{1}{2}a - \tfrac{1}{4}\Delta a + \tfrac{1}{8}\Delta^2 a - \tfrac{1}{16}\Delta^3 a + \&c.$$

cuius ergo regulae ope calculus sequenti modo institui poterit:

		$-\Delta$	$+\Delta^2$	$-\Delta^3$
$a)$	o, 822467	o, 421782	o, 291678	o, 224770
$b)$	o, 400685	o, 130104	o, 066908	o, 041540
$c)$	o, 270581	o, 063196	o, 025368	o, 013047
$d)$	o, 207385	o, 037828	o, 012321	o, 005355
$e)$	o, 169557	o, 025507	o, 006966	o, 002600
$f)$	o, 144050	o, 018541	o, 004366	o, 001426
$g)$	o, 125509	o, 014175	o, 002940	...
$h)$	o, 111334	o, 011235		
$i)$	o, 100099			

$+\Delta^4$	$-\Delta^5$	$+\Delta^6$	$-\Delta^7$	$+\Delta^8$	
o, 183230	o, 154737	o, 133936	o, 118072	o, 105564	&c.
o, 028493	o, 020801	o, 015864	o, 012508		
o, 007692	o, 004937	o, 003356			
o, 002755	o, 001581				
o, 001174					

24. Harum columnarum, quarum prima ex Calculi Differentialis Cap. VI. Part. II. pag. 365 est definita, numeri supremi referunt terminum primum a cum suis differentiis continuis, secundi vero descendendo referunt terminum b cum suis differentiis, tertii terminum c cum suis differentiis. Quia nunc supremi termini parum convergunt, duos primos $a - b$ actu colligamus, eritque $a - b = 0,421782$: sequentium vero $c - d + e - f +$ &c. summam

$$= \frac{1}{2}c - \frac{1}{4}\Delta c + \frac{1}{8}\Delta^2 c - \frac{1}{16}\Delta^3 c + \&c.$$

secundum datam legem computemus eritque

$$\begin{array}{rl}
\frac{1}{2}c & = 0,135290 \\
-\frac{1}{4}\Delta\, c & = 0,015799 \\
+\frac{1}{8}\Delta^2 c & = 0,003171 \\
-\frac{1}{16}\Delta^3 c & = 0,000815 \\
+\frac{1}{32}\Delta^4 c & = 0,000220 \\
-\frac{1}{64}\Delta^5 c & = 0,000077 \\
+\frac{1}{128}\Delta^6 c & = 0,000026 \\
\text{seqq.} & = 0,000010 \\
\hline
\text{summa} & = 0,155408 \\
a - b & = 0,421782 \\
\hline
\text{Area} & = 0,577190
\end{array}$$

Spero autem, fusiorem evolutionem huius Lineae Curvae satis memorabilis nemini fore ingratam praecipue cum aequatio pro hac curva pertineat ad functiones inexplicabiles, atque idcirco ista ad casum specialiorem digressio a nostro scopo haud aliena sit existimanda.

Species secunda serierum

quarum differentiae infinitesimae primae evanescunt.

25. Ad hanc ergo speciem pertinent omnes series, quarum termini infinitesimi inter se sunt aequales. Ut ergo termi-

minum summatorium harum serierum $\Sigma : x$ exprimamus, nil aliud opus est, nisi ut ad expressionem praecedentis speciei insuper termini secundae columnae verticalis formae generalis §. 9. exhibitae adiungantur, cuius quidem terminus supremus seorsim erit exhibendus, & quia columnae singulae horizontales iam tribus terminis constant, terminus summatorius quaesitus $\Sigma : x$ sequenti serie triplicata definietur.

$$\Sigma : x = x(1) + \left.\begin{array}{l} +(1) \quad -(2) \quad +(3) \quad -(4) \\ x\Delta 1 + x\Delta 2 + x\Delta 3 + x\Delta 4 \\ -(x+1) - (x+2) - (x+3) - (x+4) \end{array}\right\} \&c.$$

quae forma ob $\Delta 1 = (2) - (1)$; $\Delta 2 = (3) - (2)$; $\Delta 3 = (4) - (3)$; &c. transfundetur in hanc:

$$\Sigma : x = x(1) + \left.\begin{array}{l} +\overline{1-x}(1) + \overline{1-x}(2) + \overline{1-x}(3) \\ \quad x(2) + x(3) + x(4) \\ -(x+1) - (x+2) - (x+3) \end{array}\right\} \&c.$$

quae series eo magis convergit, quo minor x accipiatur. Supra autem docuimus omnes casus semper eo reduci posse ubi x sit fractio unitate minor.

26. Consideremus primo casum simplicissimum, quo omnes seriei termini sunt inter se aequales, scilicet $(x) = a$; per se enim patet eius terminum summatorium esse $a x$, quem eundem valorem nostra expressio statim declarabit.

Erit enim $\Sigma x = x a$.

27. Nunc consideretur casus, quo $(x) = \frac{x+1}{x}$, ita ut nostra series sit $\Sigma : x = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \ldots + \frac{x+1}{x}$. cuius termini infinitesimi omnes unitate aequantur. Nostra igitur formula nobis dabit

$$\Sigma : x = 2x + \left.\begin{array}{l} +\overline{1-x}\,\frac{2}{1} + \overline{1-x}\,\frac{3}{2} + \overline{1-x}\,\frac{4}{3} \\ \quad x\,\frac{3}{2} + x\,\frac{4}{3} + x\,\frac{5}{4} \\ -\frac{(x+2)}{x+1} - \frac{(x+3)}{x+2} - \frac{(x+4)}{x+3} \end{array}\right\} \&c.$$

unde patet ſumto $x = 1$ fore $\Sigma : x = \frac{2}{1}$; at ſumto $x = 2$ fiet :

$$\Sigma : x = 4 \left.\begin{array}{l} -\ 1 . \frac{2}{1} - 1 . \frac{3}{2} - 1 . \frac{4}{3} \\ +\ 2 . \frac{3}{2} + 2 . \frac{4}{3} + 2 . \frac{5}{4} \\ -\ \quad \frac{4}{3} - \quad \frac{5}{4} - \quad \frac{6}{5} \end{array}\right\} \&c. = 4 - \frac{2}{1} + \frac{3}{2}$$

28. At vero iſte caſus facile reduci poteſt ad ſpeciem praecedentem. Cum enim terminus generalis ſit $(x) = \frac{x+1}{x}$, is in partes reſolutus dabit $(x) = 1 + \frac{1}{x}$; quamobrem duae formentur ſeries, prior ſcilicet ex termino generali 1, altera vero ex termino generali $\frac{1}{x}$, haeque duae ſeries iunctim ſumtae dabunt ſummam quaeſitam $\Sigma : x$, erit ſcilicet

$$\Sigma : x = \begin{array}{l} 1 + 1 + 1 + 1 + \ldots\ldots + x \\ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{x} \end{array}$$

Iam ſuperioris feriei ſumma eſt x, inferior vero per ſpeciem primam evolvi poteſt, indeque habebitur :

$$\Sigma : x = \begin{array}{l} x + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c. \\ - \frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{x+4} - \&c. \end{array}$$

quae expreſſio multo eſt ſimplicior praecedente, nihilo vero minus eundem valorem exhibet. Ita ſi ſumatur $x = \frac{1}{2}$ prior expreſſio nobis dabit $\Sigma : x = 1 \left.\begin{array}{l} + \frac{1}{2} . \frac{2}{1} + \frac{1}{2} . \frac{3}{2} + \frac{1}{2} . \frac{4}{3} + \frac{1}{2} . \frac{5}{4} \\ + \frac{1}{2} . \frac{3}{2} + \frac{1}{2} . \frac{4}{3} + \frac{1}{2} . \frac{5}{4} + \frac{1}{2} . \frac{6}{5} \\ - \frac{5}{3} - \frac{7}{5} - \frac{9}{7} - \frac{11}{9} \end{array}\right\} \&c.$

ter-

terminisque secundum ordinem collectis fiet

$$\Sigma : \tfrac{1}{2} = 1 + \frac{1}{3.4} + \frac{1}{5.12} + \frac{1}{7.24} + \frac{1}{9.40} + \frac{1}{11.60} + \&c.$$

cuius ordo clarius patescet ex sequenti forma

$$\Sigma : \tfrac{1}{2} = 1 + \frac{1}{1.3.4} + \frac{1}{2.5.6} + \frac{1}{3.7.8} + \frac{1}{4.9.10} + \frac{1}{5.11.12} + \&c.$$

altera vero expressio dat has series

$$\Sigma : \tfrac{1}{2} = \tfrac{1}{2} \begin{array}{l} + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \&c. \\ - \tfrac{2}{3} - \tfrac{2}{5} - \tfrac{2}{7} - \tfrac{2}{9} - \&c. \end{array}$$

quae collectae dabunt

$$\Sigma : \frac{1}{2} = \frac{1}{2} + \frac{1}{3} + \frac{1}{2.5} + \frac{1}{3.7} + \frac{1}{4.9} + \&c.$$

29. Ex hoc exemplo apparet, seriem ex specie secunda deductam magis convergere quam posteriorem ex specie prima derivatam; quare operae pretium erit prioris seriei convergentiam attentius considerare. Quilibet scilicet huius seriei terminus oritur ex his tribus partibus

$$\frac{1}{2} \cdot \frac{n+1}{n} + \frac{1}{2} \cdot \frac{n+2}{n+1} - \frac{2n+3}{2n+1},$$

quae cum se mutuo proxime destruant, summa duorum priorum proxime aequalis erit tertiae, unde sequitur haec formula satis memorabilis $\frac{n+1}{n} + \frac{n+2}{n+1} = \frac{2(n+3)}{2n+1}$, quod eo propius ad veritatem accedit, quo maior fuerit numerus n. Hinc utrinque subtrahendo 2 erit proxime

$$\frac{1}{n} + \frac{1}{n+1} = \frac{4}{2n+1}.$$

30. Talis autem reductio ad speciem primam semper locum habere potest, quando series proposita tandem ad valorem finitum convergit; verum si seriei termini tandem in infinitum crescant, haec reductio non amplius locum habere potest, ideoque necessario ad speciem secundam erit recurrendum.

dum. Talis est casus, quo $(x) = \sqrt{x}$, denotante enim n numerum infinitum bini termini infinitesimi contigui erunt $\sqrt{n}$ & $\sqrt{n+1}$, quorum differentia est $\frac{1}{2\sqrt{n}}$, ideoque evanescens. Hoc ergo casu series nostra erit

$$\Sigma : x = \sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \ldots\ldots + \sqrt{x}.$$

Hinc ergo per praecepta data habebimus hanc expressionem

$$\Sigma : x = x + \left.\begin{array}{l} +(1-x)\sqrt{1} + (1-x)\sqrt{2} + (1-x)\sqrt{3} \\ \quad\;\; + x\sqrt{2} \quad\;\; + x\sqrt{3} \quad\;\; + x\sqrt{4} \\ - \sqrt{x+1} \quad - \sqrt{x+2} \quad - \sqrt{x+3} \end{array}\right\} \&c.$$

quae series quantopere convergat videamus casu $x = \frac{1}{2}$, eritque

$$\Sigma : \tfrac{1}{2} = \tfrac{1}{2} + \left.\begin{array}{l} +\frac{1}{2}\sqrt{1} + \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{4} \\ +\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{4} + \frac{1}{2}\sqrt{5} \\ -\sqrt{\frac{3}{2}} - \sqrt{\frac{5}{2}} - \sqrt{\frac{7}{2}} - \sqrt{\frac{9}{2}} \end{array}\right\} \&c.$$

cuius terminus quicunque erit $\frac{1}{2}\sqrt{n} + \frac{1}{2}\sqrt{n+1} - \sqrt{\frac{2n+1}{2}}$ qui eo propius ad nihilum accedere debet, quo maior fuerit numerus n, quocirca proxime erit $\sqrt{n} + \sqrt{n+1} = \sqrt{2(2n+1)}$. Sumtis enim quadratis, habebimus $2n+1+2\sqrt{n(n+1)} = 2(2n+1)$, ideoque $2\sqrt{n(n+1)} = 2n+1$. Sumtis denuo quadratis fiet $4nn+4n = 4nn+4n+1$, quae ratio utique proxime ad aequalitatem accedit. Ceterum hic notari meretur, veros valores pro fractionibus loco x assumtis tantopere esse transcendentes, ut nullis plane formulis analyticis exprimi queant. Quin etiam quilibet valor pro x assumtus ad peculiare transcendentium genus pertinebit.

31. Antequam hanc speciem deseramus, adiungamus adhuc insigne Theorema circa convergentiam formularum multo generalius eo, quod modo ante attulimus.

Theo-

Theorema

Sequens aequalitas

$$(\beta-\alpha)\sqrt[\mu]{n^{\nu}}+\alpha\sqrt[\mu]{(n+1)^{\nu}}=\beta\sqrt[\mu]{\left(n+\frac{\alpha}{\beta}\right)^{\nu}}$$

eo propius ad veritatem accedet, quo maior ſumatur numerus n, *ſimulque quo minor fuerit fractio* $\frac{\alpha}{\beta}$, *ſi modo exponens* $\frac{\nu}{\mu}$ *unitate fuerit minor. At vero ſumto* ν *negativo iſta aequalitas*

$$\frac{\beta-\alpha}{\sqrt[\mu]{n^{\nu}}}+\frac{\alpha}{\sqrt[\mu]{(n+1)^{\nu}}}=\frac{\beta}{\sqrt[\mu]{\left(n+\frac{\alpha}{\beta}\right)^{\nu}}}$$

ſine poſteriore conditione ad veritatem eo propius accedet, quo maior fuerit numerus n *&* *quo minor fuerit fractio* $\frac{\alpha}{\beta}$. *Quin etiam ſub iiſdem conditionibus ad logarithmos transferri poteſt, ita ut ſit tam*

$$(\beta-\alpha)\,ln+\alpha\,l(n+1)=\beta\,l\left(n+\frac{\alpha}{\beta}\right)$$

quam $$\frac{\beta-\alpha}{ln}+\frac{\alpha}{l(n+1)}=\frac{\beta}{l\left(n+\frac{\alpha}{\beta}\right)}$$

Demonſtratio

32. Sequitur hoc Theorema ex ſolutione generali pro hac ſpecie data, cuius terminus quicunque conſiſtit his partibus $\overline{1-x}\,(n)+x(n+1)-(n+x)$, atque eo minor evadit, quo maior ſumatur numerus n exiſtente x fractione unitate minore. Quod ſi iam ponamus $x=\frac{\alpha}{\beta}$ & $(x)=\sqrt[\mu]{x^{\nu}}$, ideoque

etiam

etiam $(n) = \sqrt[\mu]{n^{\nu}}$, necesse est ut sit $\frac{\mu}{\nu} < 1$, quia alioquin termini infinitesimi non haberent differentias evanescentes. Hae autem substitutiones praebent formulas priores in theoremate datas. Quando vero fractio $\frac{\mu}{\nu}$ negativa accipiatur, tum series proposita adeo in specie prima continebitur, siquidem ipsi termini infinitesimi in nihilum abeunt.

33. Quo vis huius Theorematis clarius intelligatur, notasse iuvabit, has formulas quatuor casibus exacte cum veritate convenire, quorum primus est si $\alpha = 0$; secundus quo $\alpha = \beta$; tertius est quo $\nu = 0$; quartus denique locum habet si pro n accipiatur numerus infinitus; praeterea vero datur casus quintus, quo in forma priore est $\mu = \nu$ sive $\sqrt[\mu]{n^{\nu}} = n$.

Species tertia serierum

quarum differentiae demum infinitesimae secundae evanescunt.

34. Hoc igitur eveniet, quoties ipsi termini infinitesimi progressionem arithmeticam constituunt; formula igitur ante pro $\Sigma : x$ in superiore specie inventa ad hunc casum accommodabitur, si insuper singuli termini columnae tertiae verticalis adiungantur; hoc modo terminus summatorius sequenti modo exprimetur

$$\Sigma : x = \begin{array}{l} x(1) + (1) + (2) + (3) \ldots\ldots + (n) \\ \quad + x\Delta 1 + x\Delta 2 + x\Delta 3 \ldots\ldots + x\Delta n \\ x'\Delta 1 + x'\Delta^2 1 + x'\Delta^2 2 + x'\Delta^2 3 \ldots\ldots + x'\Delta^2 n \\ \quad - (x+1) - (x+2) - (x+3) \ldots\ldots - (x+n) \end{array}$$

35. Transmutemus nunc hanc expressionem in formam ad usum magis accommodatam, ac primo quidem loco x' scribamus eius valorem $\frac{xx - x}{2}$; tum vero ob $\Delta n = (n+1) - (n)$

& $\Delta^2 n = (n+2) - 2(n+1) + (n)$, his valoribus ſubſtitutis poſtrema columna praecedentis formulae abibit in hanc formam:

$$(n) + x(n+1) + \frac{xx-x}{2}(n+2)$$

$$-x(n) - \overline{xx-x}(n+1)$$

$$+\frac{xx-x}{2}(n)$$

qui termini collecti praebent

$$\frac{xx-3x+2}{2}(n) - \overline{xx-2x}(n+1) + \frac{xx-x}{2}(n+2).$$

Ponamus igitur brev. gr. $\frac{xx-3x+2}{2} = p$; $xx - 2x = q$ & $\frac{xx-x}{2} = r$, ſicque terminus ſummatorius quaeſitus ſequenti forma exprimetur:

$$\begin{aligned} \Sigma : x = \frac{3x - xx}{2}(1) &+ \frac{xx-x}{2}(2) \\ &+ p(1) - q(2) + r(3) - (x+1) \\ &+ p(2) - q(3) + r(4) - (x+2) \\ &+ p(3) - q(4) + r(5) - (x+3) \end{aligned}$$

quae ſeries iam vehementer converget.

36. Hinc igitur novum Theorema ſimile praecedenti, ſed multo latius patens poſſumus derivare, ponendo ut ante $x = \frac{\alpha}{\beta}$, $(n) = \sqrt[\mu]{n^\nu}$, ubi iam ſufficit ut exponens $\frac{\nu}{\mu}$ binario ſit minor, multo magis autem hunc exponentem negativum ſtatuere licebit. Theorema hoc eſt

Iſta aequalitas

$$(\alpha\alpha - 3\alpha\beta + 2\beta\beta)\sqrt[\mu]{n^\nu} - (2\alpha\alpha - 4\alpha\beta)\sqrt[\mu]{(n+1)^\nu} + (\alpha\alpha - \alpha\beta)\sqrt[\mu]{(n+2)^\nu} =$$

$$2\beta\beta\sqrt[\mu]{\left(n + \frac{\alpha}{\beta}\right)^\nu}$$

eo

eo propius ad veritatem accedet quo maior capiatur numerus n *& fractio* $\frac{\alpha}{\beta}$ *parum ab unitate discrepet, dummodo* $\frac{\nu}{\mu}$ *binario sit minus. Tum vero sumto* μ *negativo erit plerumque multo accuratius:*

$$\frac{\alpha\alpha-3\alpha\beta+2\beta\beta}{\sqrt[\mu]{n^\nu}}-\frac{2\alpha\alpha-4\alpha\beta}{\sqrt[\mu]{(n+1)^\nu}}+\frac{\alpha\alpha-\alpha\beta}{\sqrt[\mu]{(n+2)^\nu}}=\frac{2\beta\beta}{\sqrt[\mu]{(n+\frac{\alpha}{\beta})^\nu}}$$

Quin etiam pro formulis radicalibus logarithmi accipi poterunt.

37. Veritas huius Theorematis etiam exacte subsistit his quatuor casibus I. $\alpha=0$; II. $\alpha=\beta$; III. $\nu=0$ & IV. $n=\infty$. Praeterea vero idem evenit, quando in forma priori est vel $\nu=\mu$, vel $\nu=2\mu$, ita ut sit $\sqrt[\mu]{n^\nu}$ vel n vel nn. Habemus igitur sex casus, quibus hoc Theorema nihil plane a veritate aberrat, unde facile intelligitur etiam reliquis casibus omnibus errorem non esse posse notabilem.

38. Possumus etiam hoc theorema adhuc generalius reddere, loco n scribendo $\frac{n}{c}$ & ubique per debitam potestatem ipsius c multiplicando, quo fractiones tollantur. Sicque prior forma fiet $(\alpha\alpha-3\alpha\beta+2\beta\beta)\sqrt[\mu]{n^\nu}-(2\alpha\alpha-4\alpha\beta)\sqrt[\mu]{(n+c)^\nu}$ $+(\alpha\alpha-\alpha\beta)\sqrt[\mu]{(n+2c)^\nu}=2\beta\beta\sqrt[\mu]{\left(n+\frac{\alpha c}{\beta}\right)^\nu}$

altera autem forma ab hac non discrepat, nisi quod radicalia in denominatorem ingrediuntur, id quod etiam de logarithmis est intelligendum.

39. Operae pretium erit hoc theorema aliquo exemplo illustrare. Sumatur igitur $\alpha=1$ & $\beta=2$, fientque aequalitates in theoremate exhibitae

$3\sqrt{n}$

$$3\sqrt[\mu]{n^\nu} + 6\sqrt[\mu]{(n+c)^\nu} - \sqrt[\mu]{(n+2c)^\nu} = 8\sqrt[\mu]{\left(n+\frac{1}{2}c\right)^\nu}$$

$$\frac{3}{\sqrt[\mu]{n^\nu}} + \frac{6}{\sqrt[\mu]{(n+1)^\nu}} - \frac{1}{\sqrt[\mu]{(n+2)^\nu}} = \frac{8}{\sqrt[\mu]{(n+\frac{1}{2})^\nu}}$$

Applicemus formam priorem ad logarithmos, fietque $3\,l n + 6\,l\,(n+c) - l(n+2c) = 8\,l\,(n+\frac{1}{2}c)$ ſit nunc $n = 10$ & $c = 2$, ut prodeat $3\,l\,10 + 6\,l\,12 - l\,14 = 8\,l\,11$. Facta igitur evolutione

$$\begin{array}{lcl} 3\,l\,10 = 3,\ 0000000 & & l\,14 = 1,\ 1461280 \\ 6\,l\,12 = \underline{6,\ 4750872} & & 8\,l\,11 = \underline{8,\ 3311416} \\ \quad 9,\ 4750872 & = & \quad 9,\ 4772696 \end{array}$$

quarum differentia eſt 0, 0021824, quae multo minor prodiret, ſi numero n maiorem valorem tribuerimus

40. Circa ipſum autem terminum ſummatorium ſeriei propoſitae imprimis notari convenit, tam differentiationem quam integrationem facile inſtitui poſſe, ſumto ſcilicet indice x variabili, quemadmodum hoc iam in ſpecie prima fuſius eſt oſtenſum, ubi ipſe terminus ſummatorius $\Sigma : x$ tanquam applicata cuiuſdam curvae eſt conſideratus, dum index x referebat abſciſſam, hocque reſpectu in calculo differentiali potiſſimum functiones inexplicabiles ſum perſcrutatus.

41. Ex formula autem generali pro termino ſummatorio $\Sigma : x$ ſupra data evolvamus hic quoque caſum ſeriei harmonicae, quo eſt

$$\Sigma : x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{x}$$

& quaeramus eius valorem pro indice $x = \frac{1}{2}$, atque ob

$$(x) = \frac{1}{x}\ ;\ p = \frac{3}{8}\ ;\ q = -\frac{3}{4}\ ;\ r = -\frac{1}{8}$$

habebimus

$$\Sigma:\tfrac{1}{2}=\frac{5}{8}-\frac{1}{16}\left\{\begin{array}{l} +\frac{3}{8}+\frac{3}{16}+\frac{1}{8}+\frac{3}{32} \\ +\frac{3}{8}+\frac{1}{4}+\frac{3}{16}+\frac{3}{20} \\ -\frac{1}{24}-\frac{1}{32}-\frac{1}{40}-\frac{1}{48} \\ -\frac{2}{3}-\frac{2}{5}-\frac{2}{7}-\frac{2}{9} \end{array}\right\}\ \&c.$$

Sive erit $8\,\Sigma:\frac{1}{2}=\frac{9}{2}\left\{\begin{array}{l} +\frac{3}{1}+\frac{3}{2}+\frac{3}{3}+\frac{3}{4} \\ +\frac{6}{2}+\frac{6}{3}+\frac{6}{4}+\frac{6}{5} \\ -\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6} \\ -\frac{16}{3}-\frac{16}{5}-\frac{16}{7}-\frac{16}{9} \end{array}\right\}\ \&c.$

Contrahamus singulas columnas in unam summam eritque

$$8\,\Sigma:\tfrac{1}{2}=\frac{6}{1.2.3.5}+\frac{6}{2.3.4.5}+\frac{6}{3.4.5.7}+\frac{6}{4.5.6.9}+\&c.$$

quae series utique magis convergit ea, quam specie secunda invenimus.

42. Quodsi autem terminos non contrahamus, sed eos qui eundem habent denominatorem colligamus, omissa serie infima habebimus

$$8\,\Sigma:\tfrac{1}{2}=\frac{9}{2}+\frac{3}{1}+\frac{9}{2}\begin{array}{l} +8\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\&c.\right) \\ -16\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\&c.\right) \end{array}$$

sive loco superioris seriei scribendo $16\left(\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}+\&c.\right)$
habebimus $\frac{1}{2}\,\Sigma:\frac{1}{2}-\frac{3}{4}=-\frac{1}{3}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{8}-\frac{1}{9}+\frac{1}{10}-\frac{1}{11}+$ &c. ad-
damus utrinque $l\,2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+$ &c. fiet
$\frac{1}{2}\,\Sigma:\frac{1}{2}-\frac{3}{4}+l\,2=1-\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$; consequenter
$\Sigma:\frac{1}{2}=2-2\,l\,2$, qui valor egregie convenit cum eo, qui
in specie prima est datus.

SUP-

SUPPLEMENTUM
DE FUNCTIONIBUS INEXPLICABILIBUS FORMAE

$$\pi : x = A.\ B.\ C.\ D.\ E.\ \ldots\ldots X.$$

1. Hic factores A, B, C, D, &c. sunt termini cuiusdam seriei indicibus 1, 2, 3, 4, &c. respondentes, & X terminus indici x respondens; factores autem, qui indicibus sequentibus $x+1$; $x+2$; $x+3$; &c. respondent per X', X'', X''' &c. designabo. Hinc iam statim patet fore $\pi:(x+1) = X'.\pi:x$, & $\pi:(x+2) = X'.X''.\pi:x$, & ita porro. Praecedentes vero erunt $\pi:(x-1) = \frac{\pi:x}{X'}$, &c. Unde intelligitur sufficere, dummodo hae formulae pro valoribus ipsius x unitate minoribus assignentur.

2. Quoties fuerit x numerus integer positivus, valores ipsius $\pi:x$ sponte se produnt. Erit nempe

$$\pi:1 = A\ ;\quad \pi:2 = AB\ ;\quad \pi:3 = ABC\ ;\quad \&c.$$

Quando autem x non est numerus integer positivus, productum, quod charactere $\pi:x$ designamus erit functio inexplicabilis ipsius x, nisi forte factores A, B, C, D, &c. ita fuerint comparati, ut praecedentes a sequentibus destruantur, veluti evenit in hac forma: $\pi:x = \frac{1}{2}.\frac{2}{3}.\frac{3}{4}.\frac{4}{5}\ldots\ldots\frac{x}{x+1}$: quandoquidem hic manifesto est $\pi:x = \frac{1}{x+1}$, vel etiam in hoc exemplo: $\pi:x = \frac{3}{4}.\frac{8}{9}.\frac{15}{16}.\frac{24}{25}\ldots\ldots\frac{xx+2x}{(x+1)^2}$; hinc enim erit

$$\pi:1 = \frac{3}{2.2};\ \pi:2 = \frac{2}{3} = \frac{4}{2.3};\ \pi:3 = \frac{5}{8} = \frac{5}{2.4};\ \pi:4 = \frac{3}{5} = \frac{6}{2.5};$$

$\pi : 5 = \frac{4}{2.6}$; &c. unde patet in genere fore $\pi : x = \frac{x+2}{2(x+1)}$.

3. Casus autem inexplicabiles sumendis logarithmis ad praecedentem dissertationem revocabuntur. Erit enim

$$l\pi : x = lA + lB + lC \quad . \quad . \quad . \quad . \quad + lX,$$

quae forma cum supra tractata comparata nobis dabit sequentes valores

$\Sigma : x = l\pi : x$; $(1) = lA$; $(2) = lB$; $(3) = lC$; &c. & $(x) = lX$ tum vero erit $(x+1) = lX'$; $(x+2) = lX''$; &c. hocque consensu observato species supra tractatas ad praesentem casum accommodemus.

Species prima

ubi logarithmi factorum infinitesimorum evanescunt, sive ubi factores infinitesimi unitati aequantur.

4. Cum igitur pro hac specie introductis valoribus modo datis habeamus:

$$l\pi : x = \begin{matrix} lA & + lB & + lC & + lD & + \&c. \\ -lX' & - lX'' & - lX''' & - lX^{IV} & - \&c. \end{matrix}$$

ad numeros ascendendo erit

$$\pi : x = \frac{A}{X'} \cdot \frac{B}{X''} \cdot \frac{C}{X'''} \cdot \frac{D}{X^{IV}} \cdot \&c.$$

Hic nulla exempla subiungo, quia iam plura in Calculo Differentiali sunt evoluta.

Species secunda

ubi factores infinitesimi inter se sunt aequales.

5. Tum enim eorum logarithmi etiam inter se erunt aequales, ideoque differentiae primae omnes evanescent. Huc igitur

igitur accommodemus formam supra §. 25. inventam eritque

$$l\pi : x = x\,lA \begin{matrix} +\overline{1-x}\,lA + \overline{1-x}\,lB + \overline{1-x}\,lC \\ +\quad x\,lB + \quad x\,lC + \quad x\,lD \\ -\quad lX' \;-\quad lX'' \;-\quad lX''' \end{matrix} \Bigg\} \text{ \&c.}$$

unde ad numeros ascendendo habebimus

$$\pi : x = A^x . \frac{A^{1-x} B^x}{X'} . \frac{B^{1-x} C^x}{X''} . \frac{C^{1-x} D^x}{X'''} . \text{ \&c.}$$

Species tertia

ubi termini infinitesimi constituunt progressionem geometricam.

6. Tum enim logarithmi horum terminorum progressionem arithmeticam constituent, cuius ergo differentiae secundae evanescent. Ut iam expressionem supra §. 35. inventam ad hunc casum accommodemus, notandum est br. gr. positum fuisse $p = \frac{xx - 3x + 2}{2}$; $q = xx - 2x$; & $r = \frac{xx - x}{2}$, unde habebimus

$$l\pi : x = \left. \begin{matrix} + p\,lA + p\,lB + p\,lC \\ \frac{3x - xx}{2}\, lA - q\,lB - q\,lC - q\,lD \\ \frac{xx - x}{2}\, lB + r\,lC + r\,lD + r\,lE \\ - lX' - lX'' - lX''' \end{matrix} \right\} \text{ \&c.}$$

Ponamus autem hic porro compendii caussa $\frac{xx - 3x}{2} = m$ & $\frac{xx - x}{2} = n$, atque ad numeros ascendendo habebimus hanc expressionem.

$$\pi : x = \frac{B^n}{A^m} \cdot \frac{A^p C^r}{B^q X'} \cdot \frac{B^p D^r}{C^q X''} \cdot \frac{C^p E^r}{D^q X'''} \cdot \text{\&c.}$$

7. Hoc modo confido, doctrinam de Functionibus Inexplicabilibus, quae in Calculo Differentiali non satis accurate & luculenter est exposita, fere penitus exhausisse, ita ut nihil amplius desiderari possit, id quod eo magis necessarium videbatur, cum hoc argumentum plane sit novum & a nemine adhuc tractatum, praecipue autem eius summus usus in interpolatione serierum, atque hinc adeo symptomata linearum curvarum, quarum applicatae per functiones inexplicabiles exprimuntur investiganda erant.