CAPUT VIII.

DE USU CALCULI DIFFERENTIALIS IN FORMANDIS SERIEBUS.

198.

Unum adhuc calculi differentialis usum in doctrina serierum commemorabimus, qui in ipsa formatione serierum consistit, & ad quem iam supra provocavimus, cum quaestio esset de fractione, cuius denominator sit potestas quaecunque functionis cuiuspiam, in seriem evolvenda. Ista methodus autem similis est ei, qua iam aliquoties sumus usi, dum functio in seriem convertenda aequalis fingitur cuipiam seriei, in singulis terminis coefficientes indeterminatos habenti, qui deinceps aequalitate constituta determinentur. Haec autem determinatio saepenumero mirifice adiuvatur, si antequam ea suscipiatur ad differentialia cum prima, tum nonnunquam quoque ad secunda aequatio perducatur. Quae methodus cum in calculo integrali amplissimi sit usus, eam hic diligentius exponemus.

199. Primum igitur breviter repetamus, quae supra de evolutione fractionum in series sine calculi differentialis subsidio attulimus. Sit fractio quaecunque proposita:

$$\frac{A + Bx + Cx^2 + Dx^3 + \&c.}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.} = s$$

quam in seriem secundum potestates ipsius x procedentem converti oporteat. Fingatur pro s series indeterminata:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \mathfrak{G}x^6 + \&c.$$

Cum igitur fractione per multiplicationem sublata sit:

$$A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 \quad \&c.$$
$$= s(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \eta x^6 \quad \&c.)$$

si pro s series ficta substituatur prodibit sequens aequatio:

A

$$\begin{array}{lllllll} A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c. & = \\ \hline \mathfrak{A}\alpha & + \mathfrak{B}\alpha x & + \mathfrak{C}\alpha x^2 & + \mathfrak{D}\alpha x^3 & + \mathfrak{E}\alpha x^4 & + \mathfrak{F}\alpha x^5 & + \&c. \\ & + \mathfrak{A}\beta & + \mathfrak{B}\beta & + \mathfrak{C}\beta & + \mathfrak{D}\beta & + \mathfrak{E}\beta & + \&c. \\ & & + \mathfrak{A}\gamma & + \mathfrak{B}\gamma & + \mathfrak{C}\gamma & + \mathfrak{D}\gamma & + \&c. \\ & & & + \mathfrak{A}\delta & + \mathfrak{B}\delta & + \mathfrak{C}\delta & + \&c. \\ & & & & + \mathfrak{A}\varepsilon & + \mathfrak{B}\varepsilon & + \&c. \\ & & & & & + \mathfrak{A}\zeta & + \&c. \end{array}$$

Aequalitate ergo inter ſingulos terminos, qui easdem ipſius x poteſtates continent, conſtituta fiet:

$$\mathfrak{A}\alpha - A = 0$$
$$\mathfrak{B}\alpha + \mathfrak{A}\beta - B = 0$$
$$\mathfrak{C}\alpha + \mathfrak{B}\beta + \mathfrak{A}\gamma - C = 0$$
$$\mathfrak{D}\alpha + \mathfrak{C}\beta + \mathfrak{B}\gamma + \mathfrak{A}\delta - D = 0$$
$$\mathfrak{E}\alpha + \mathfrak{D}\beta + \mathfrak{C}\gamma + \mathfrak{B}\delta + \mathfrak{A}\varepsilon - E = 0 \quad \&c.$$

ex quibus aequationibus coefficientes ficti $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. determinantur, ſicque ſeries infinita invenitur:

$$\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \&c.$$

fractioni propoſitae s aequalis. Atque in hac forma ſi tam numerator quam denominator fractionis s finito terminorum numero conſtent, omnes ſeries recurrentes comprehenduntur, de quibus iam ſupra fuſius eſt tractatum.

200. Quodſi autem vel numerator vel denominator vel uterque ad dignitatem quamcunque fuerit elevatus, tum hoc modo ſeries difficulter obtinetur; propterea quod negotium, niſi functio elevata ſit binomium, perquam ſit laborioſum. Calculo autem differentiali iſte labor evitari poteſt. Adſit primum ſolus numerator, ſitque:

$$s = (A + Bx + Cxx)^n,$$

unde quaeratur ſeries ſecundum poteſtates ipſius x procedens huic trinomii dignitati aequalis; quam quidem finitam fore conſtat, ſi exponens n fuerit numerus integer affirmativus. Fingatur iterum pro s ſeries indefinita:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \mathfrak{G}x^6 + \&c.$$

cuius terminum primum $\mathfrak{A}$ conſtat eſſe $= A^n$: ſi enim ponatur

tur $x=0$, ex priori forma proposita fit $s = A^n$, ex serie autem ficta $s = \mathfrak{A}$. Haec autem primi termini determinatio ex ipsa rei natura est petenda, si ad differentialia descendere velimus, quia hinc primus terminus non determinatur, uti mox patebit.

201. Cum sit $s = (A + Bx + Cx^2)^n$, erit logarithmis sumendis $ls = nl(A + Bx + Cx^2)$, hincque sumtis differentialibus habebitur: $\frac{ds}{s} = \frac{nBdx + 2nCxdx}{A + Bx + Cx^2}$, seu

$$(A + Bx + Cx^2)\frac{ds}{dx} = ns(B + 2Cx).$$

Ex serie autem ficta est:

$$\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + 5\mathfrak{F}x^4 + \&c.$$

si igitur haec series loco $\frac{ds}{dx}$, & pro s ipsa series ficta substituatur, prodibit sequens aequatio:

$$\begin{array}{llllll} A\mathfrak{B} + & 2A\mathfrak{C}x + & 3A\mathfrak{D}x^2 + & 4A\mathfrak{E}x^3 + & 5A\mathfrak{F}x^4 + & \&c. \\ + & B\mathfrak{B} + & 2B\mathfrak{C} + & 3B\mathfrak{D} + & 4B\mathfrak{E} + & \&c. \\ & + & C\mathfrak{B} + & 2C\mathfrak{C} + & 3C\mathfrak{D} + & \&c. \end{array} =$$

$$\begin{array}{llllll} nB\mathfrak{A} + & nB\mathfrak{B} + & nB\mathfrak{C} + & nB\mathfrak{D} + & nB\mathfrak{E} + & \&c. \\ + & 2nC\mathfrak{A} + & 2nC\mathfrak{B} + & 2nC\mathfrak{C} + & 2nC\mathfrak{D} + & \&c. \end{array}$$

Aequalitate ergo hic inter terminos eiusdem ipsius x potestatis constituta erit:

$$\mathfrak{B} = \frac{nB\mathfrak{A}}{A}$$

$$\mathfrak{C} = \frac{(n-1)B\mathfrak{B} + 2nC\mathfrak{A}}{2A}$$

$$\mathfrak{D} = \frac{(n-2)B\mathfrak{C} + (2n-1)C\mathfrak{B}}{3A}$$

$$\mathfrak{E} = \frac{(n-3)B\mathfrak{D} + (2n-2)C\mathfrak{C}}{4A}$$

$$\mathfrak{F} = \frac{(n-4)B\mathfrak{E} + (2n-3)C\mathfrak{D}}{5A} \quad \&c.$$

Cum igitur ut ante vidimus sit $\mathfrak{A} = A^n$, erit $\mathfrak{B} = nA^{n-1}B$, hincque reliqui coefficientes omnes successive determinabuntur. Lex autem, quam ipsi sequuntur facillime ex his formulis patet, quae vehementer obscura mansisset, si trinomium actu elevare voluissemus.

202. Haec eadem methodus succedit, si polynomium quodcunque ad quampiam dignitatem elevari debeat. Sit $s = (A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.)^n$ fingaturque:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \&c.$$

erit $\mathfrak{A} = A^n$, qui valor colligitur, si ponatur $x = 0$. Sumtis iam ut ante logarithmis, eorumque differentialibus reperietur:

$$\frac{ds}{s} = \frac{nBdx + 2nCxdx + 3nDx^2dx + 4nEx^3dx + \&c.}{A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.}$$

seu $(A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.)\dfrac{ds}{dx} =$

$s(nB + 2nCx + 3nDx^2 + 4nEx^3 + \&c.)$. Cum igitur sit:

$$\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + 5\mathfrak{F}x^4 + \&c.$$

Erit his seriebus pro s & $\dfrac{ds}{dx}$ substitutis:

$$\begin{array}{llllll}
A\mathfrak{B} & + 2A\mathfrak{C}x & + 3A\mathfrak{D}x^2 & + 4A\mathfrak{E}x^3 & + 5A\mathfrak{F}x^4 & + \&c. \\
 & + B\mathfrak{B} & + 2B\mathfrak{C} & + 3B\mathfrak{D} & + 4B\mathfrak{E} & + \&c. \\
 & & + C\mathfrak{B} & + 2C\mathfrak{C} & + 3C\mathfrak{D} & + \&c. \\
 & & & + D\mathfrak{B} & + 2D\mathfrak{C} & + \&c. \\
 & & & & + E\mathfrak{B} & + \&c. \quad = \\
\hline
nB\mathfrak{A} & + nB\mathfrak{B} & + nB\mathfrak{C} & + nB\mathfrak{D} & + nB\mathfrak{E} & + \&c. \\
 & + 2nC\mathfrak{A} & + 2nC\mathfrak{B} & + 2nC\mathfrak{C} & + 2nC\mathfrak{D} & + \&c. \\
 & & + 3nD\mathfrak{A} & + 3nD\mathfrak{B} & + 3nD\mathfrak{C} & + \&c. \\
 & & & + 4nE\mathfrak{A} & + 4nE\mathfrak{B} & + \&c. \\
 & & & & + 5nF\mathfrak{A} & + \&c.
\end{array}$$

Unde derivantur sequentes determinationes:

$$\begin{aligned}
A\mathfrak{B} &= n\,B\mathfrak{A} \\
2A\mathfrak{C} &= (n-1)\,B\mathfrak{B} + 2nC\mathfrak{A} \\
3A\mathfrak{D} &= (n-2)\,B\mathfrak{C} + (2n-1)\,C\mathfrak{B} + 3nD\mathfrak{A} \\
4A\mathfrak{E} &= (n-3)\,B\mathfrak{D} + (2n-2)\,C\mathfrak{C} + (3n-1)\,D\mathfrak{B} + 4nE\mathfrak{A}
\end{aligned}$$

$5A\mathfrak{F} = (n-4)B\mathfrak{E} + (2n-3)C\mathfrak{D} + (3n-2)D\mathfrak{C} + (4n-1)E\mathfrak{B} + 5nF\mathfrak{A}$ &c.

unde quemadmodum coefficientes ficti $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. a se invicem pendeant, hincque determinentur, cum sit $\mathfrak{A} = A^n$, luculentissime apparet.

203. Quoniam, si quantitas $A + Bx + Cx^2 + Dx^3 +$ &c. ex finito terminorum numero constat, numerusque n fuerit integer affirmativus, quaecunque potestas finito etiam terminorum numero constare debet: manifestum est hoc casu, formulas modo inventas tandem evanescere debere, atque cum omnes termini adesse debeant, ut primum unus evanuerit, simul omnes sequentes evanescere debere. Ponamus formulam propositam $A + Bx + Cx^2$ esse trinomium, eiusque cubum quaeri, ut sit $n = 3$, erit

$$
\begin{array}{lll}
\mathfrak{A} = A^3 & \text{ideoque}; & \mathfrak{A} = A^3 \\
A\mathfrak{B} = 3B\mathfrak{A} & ; & \mathfrak{B} = 3A^2B \\
2A\mathfrak{C} = 2B\mathfrak{B} + 6C\mathfrak{A} & ; & \mathfrak{C} = 3AB^2 + 3A^2C \\
3A\mathfrak{D} = 1B\mathfrak{C} + 5C\mathfrak{B} & ; & \mathfrak{D} = B^3 + 6ABC \\
4A\mathfrak{E} = 0 + 4C\mathfrak{C} & ; & \mathfrak{E} = 3B^2C + 3AC^2 \\
5A\mathfrak{F} = -B\mathfrak{E} + 3C\mathfrak{D} & ; & \mathfrak{F} = 3BC^2 \\
6A\mathfrak{G} = -2B\mathfrak{F} + 2C\mathfrak{E} & ; & \mathfrak{G} = C^3 \\
7A\mathfrak{H} = -3B\mathfrak{G} + 1C\mathfrak{F} & ; & \mathfrak{H} = 0 \\
8A\mathfrak{I} = -4B\mathfrak{H} + 0 & ; & \mathfrak{I} = 0.
\end{array}
$$

Quoniam igitur iam bini sunt $= 0$, sequentiumque quilibet a duobus praecedentibus pendet, patet omnes sequentes pariter evanescere debere. Hancque ob causam lex, qua hi coefficientes a se invicem pendere sunt inventi, eo magis est notatu digna.

204. Si n fuerit numerus negativus, ita ut s aequale fiat fractioni, series in infinitum excurret. Sit igitur

$$s = \frac{1}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.)^n}$$

fingatur pro eius valore haec series:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \&c.$$

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Atque si in superioribus formulis pro litteris A, B, C, D, &c. ponantur α, β, γ, δ, &c. simulque fiat n negativum, sequentes determinationes coefficientium $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. prodibunt:

$$\mathfrak{A} = \alpha^{-n} = \frac{1}{\alpha^n}$$

$$\alpha\mathfrak{B} + n\beta\mathfrak{A} = 0$$
$$2\alpha\mathfrak{C} + (n+1)\beta\mathfrak{B} + 2n\gamma\mathfrak{A} = 0$$
$$3\alpha\mathfrak{D} + (n+2)\beta\mathfrak{C} + (2n+1)\gamma\mathfrak{B} + 3n\delta\mathfrak{A} = 0$$
$$4\alpha\mathfrak{E} + (n+3)\beta\mathfrak{D} + (2n+2)\gamma\mathfrak{C} + (3n+1)\delta\mathfrak{B} + 4n\varepsilon\mathfrak{A} = 0$$
$$5\alpha\mathfrak{F} + (n+4)\beta\mathfrak{E} + (2n+3)\gamma\mathfrak{D} + (3n+2)\delta\mathfrak{C} + (4n+1)\varepsilon\mathfrak{B} + 5n\zeta\mathfrak{A} = 0$$
&c.

Quae formulae eandem continent legem horum coefficientium numerorum, quam iam supra observavimus in Introductione; cuiusque adeo veritatem nunc demum rigide demonstrare licuit.

205. Haec ita se habent, si numerator fractionis fuerit unitas, vel etiam quaepiam ipsius x potestas, puta x^m; posteriori enim casu tantum oportebit seriem priori inventam $\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 +$ &c. multiplicare per x^m. At si numerator constet ex duobus pluribusve terminis, tum supra quidem legem progressionis non observavimus, quamobrem eam hic per differentiationem investigemus. Sit igitur:

$$s = \frac{A + Bx + Cx^2 + Dx^3 + \&c.}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.)^n}$$

fingaturque pro valore huius fractionis sequens series:

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \&c.$$

cuius primus terminus $\mathfrak{A}$ ut definiatur, ponatur $x = 0$, eritque ex priori expressione $s = \frac{A}{\alpha^n}$, ex ficta vero $s = \mathfrak{A}$, unde necesse est, ut sit $\mathfrak{A} = \frac{A}{\alpha^n}$. Quo termino determinato reliqui per differentiationem innotescent.

206. Sumtis logarithmis erit;

$$\begin{array}{rl} ls = & l(A + Bx + Cx^2 + Dx^3 + \&c.) \\ - nl & (\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.) \end{array}$$

hincque differentiando orietur:

$$\frac{ds}{s} = \frac{Bdx + 2Cxdx + 3Dx^2dx + \&c.}{A + Bx + Cx^2 + Dx^3 + \&c.} - \frac{n\beta dx - 2n\gamma xdx - 3n\delta x^2 dx - \&c.}{\alpha + \beta x + \gamma x^2 + \delta x^3 + \&c.}$$

sublatisque per multiplicationem fractionibus erit:

$$\left(\begin{array}{lllll} A\alpha & + A\beta x & + A\gamma x^2 & + A\delta x^3 & + \&c. \\ & + B\alpha & + B\beta & + B\gamma & + \&c. \\ & & + C\alpha & + C\beta & + \&c. \\ & & & + D\alpha & + \&c. \end{array}\right\} \frac{ds}{dx} =$$

$$\left(\begin{array}{llllll} B\alpha & + B\beta x & + B\gamma x^2 & + B\delta x^3 & + B\varepsilon x^4 & + \&c. \\ & + 2C\alpha & + 2C\beta & + 2C\gamma & + 2C\delta & + \&c. \\ & & + 3D\alpha & + 3D\beta & + 3D\gamma & + \&c. \\ & & & + 4E\alpha & + 4E\beta & + \&c. \\ & & & & + 5F\alpha & + \&c. \end{array}\right\} s$$

$$-\left(\begin{array}{llllll} A\beta & + 2A\gamma x & + 3A\delta x^2 & + 4A\varepsilon x^3 & + 5A\zeta x^4 & + \&c. \\ & + B\beta & + 2B\gamma & + 3B\delta & + 4B\varepsilon & + \&c. \\ & & + C\beta & + 2C\gamma & + 3C\delta & + \&c. \\ & & & + D\beta & + 2D\gamma & + \&c. \\ & & & & + E\beta & + \&c. \end{array}\right\} ns.$$

Cum nunc sit $\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + \&c.$

erit factis substitutionibus:

$$A\alpha\mathfrak{B} + nA\beta\mathfrak{A} - B\alpha\mathfrak{A} = 0$$

$$2A\alpha\mathfrak{C} + (n+1)A\beta\mathfrak{B} + 2nA\gamma\mathfrak{A} + (n-1)B\beta\mathfrak{A} - 2C\alpha\mathfrak{A} = 0$$

$$\left.\begin{array}{l} 3A\alpha\mathfrak{D} + (n+2)A\beta\mathfrak{C} + (2n+1)A\gamma\mathfrak{B} + 3nA\delta\mathfrak{A} \\ \qquad B\alpha\mathfrak{C} + nB\beta\mathfrak{B} + (2n-1)B\gamma\mathfrak{A} \\ \qquad - C\alpha\mathfrak{B} + (n-2)C\beta\mathfrak{A} \\ \qquad - 3D\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{l} 4A\alpha\mathfrak{E} + (n+3)A\beta\mathfrak{D} + (2n+2)A\gamma\mathfrak{C} + (3n+1)A\delta\mathfrak{B} + 4nA\varepsilon\mathfrak{A} \\ \qquad 2B\alpha\mathfrak{D} + (n+1)B\beta\mathfrak{C} + 2nB\gamma\mathfrak{B} + (3n-1)B\delta\mathfrak{A} \\ \qquad 0\,C\alpha\mathfrak{C} + (n-1)C\beta\mathfrak{B} + (2n-2)C\gamma\mathfrak{A} \\ \qquad - 2D\alpha\mathfrak{B} + (n-3)D\beta\mathfrak{A} \\ \qquad - 4E\alpha\mathfrak{A} \end{array}\right\} = 0$$

Hinc lex; qua iſtae formulae progrediuntur, facile perſpicitur: prima enim cuiuſque aequationis linea eandem ſequitur legem, quam §. 204. habuimus. Tum vero coefficientes ſecundarum linearum oriuntur, ſi a coefficientibus ſuperioribus ſubtrahatur $n+1$, ſimilique modo ex linea ſecunda formatur linea tertia & ſequentes, a coefficientibus ſuperioribus continuo ſubtrahendo $n+1$; ipſae autem litterae quemvis terminum componentes per ſolam inſpectionem facillime formantur.

207. Sin autem quoque numerator fractionis fuerit quaepiam poteſtas: ſcilicet

$$s = \frac{(A + Bx + Cx^2 + Dx^3 + \&c.)^m}{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \&c.)^n}$$

fingaturque $s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \&c.$

erit $\mathfrak{A} = \frac{A^m}{\alpha^n}$; reliqui vero coefficientes ex ſequentibus formulis determinabuntur:

$$\left.\begin{array}{l} A\alpha\mathfrak{B} + nA\beta\mathfrak{A} \\ \quad - mB\alpha\mathfrak{A} \end{array}\right\} = 0.$$

$$\left.\begin{array}{l} 2A\alpha\mathfrak{C} + (n+1)A\beta\mathfrak{B} + 2n\,A\gamma\mathfrak{A} \\ \quad -(m-1)B\alpha\mathfrak{B} + (n-m)B\beta\mathfrak{A} \\ \qquad - 2m\,C\alpha\mathfrak{A} \end{array}\right\} = 0.$$

$$\left.\begin{array}{l} 3A\alpha\mathfrak{D} + (n+2)A\beta\mathfrak{C} + (2n+1)A\gamma\mathfrak{B} + 3n\,A\delta\mathfrak{A} \\ \quad -(m-2)B\alpha\mathfrak{C} + (n-m+1)B\beta\mathfrak{B} + (2n-m)B\gamma\mathfrak{A} \\ \qquad -(2m-1)C\alpha\mathfrak{B} + (n-2m)C\beta\mathfrak{A} \\ \qquad\quad - 3m\,D\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{l} 4A\alpha\mathfrak{E} + (n+3)A\beta\mathfrak{D} + (2n+2)A\gamma\mathfrak{C} + (3n+1)A\delta\mathfrak{B} + 4n\,A\varepsilon\mathfrak{A} \\ \quad -(m-3)B\alpha\mathfrak{D} + (n-m+2)B\beta\mathfrak{C} + (2n-m+1)B\gamma\mathfrak{B} + (3n-m)B\delta\mathfrak{A} \\ \qquad -(2m-2)C\alpha\mathfrak{C} + (n-2m+1)C\beta\mathfrak{B} + (2n-2m)C\gamma\mathfrak{A} \\ \qquad\quad -(3m-1)D\alpha\mathfrak{B} + (n-3m)D\beta\mathfrak{A} \\ \qquad\qquad - 4m\,E\alpha\mathfrak{A} \end{array}\right\} = 0$$

Lex, qua iſtae formulae ulterius continuantur, ex inſpectione facilius apparet, quam verbis deſcribi queat. Deſcendendo autem coefficientes diminuuntur differentia $n+m$; horizontaliter autem progrediendo augentur continuo differentia $n-1$,

208. Hoc igitur modo doctrina de ſeriebus recurrentibus amplificatur, dum iſtum defectum ſupplevimus, atque legem coefficientium definivimus, ſi non ſolum denominator fractionis fuerit poteſtas quaecunque, ſed etiam numerator ex quotlibet terminis conſtet, ad quam legem detegendam ſola inductio non ſufficiebat. Praeter plurimos autem uſus ſerierum recurrentium, quos iam expoſuimus, maximam quoque afferunt utilitatem ad ſummas quarumvis ſerierum proxime inveniendas: cuius ſpecimen iam in Capite primo huius ſectionis exhibuimus, dum ſeriem ſubſtitutione $x = \frac{y}{1 + ny}$ in aliam tranſmutavimus, quae ſaepenumero terminorum numero finito conſtet. Eaque methodus ulterius extendi potuiſſet, ſi pro x aliae functiones ſubſtitutae fuiſſent. Quoniam vero tum lex progreſſionis ſerierum, quae loco poteſtatum ipſius x poni deberent, non ſatis luculenter conſtabat, in hunc locum iſtam amplificationem reſervare viſum eſt; cum memorata lex iam penitus eſſet detecta. Interim tamen re diligentius perpenſa comperimus idem negotium ſine hac progreſſionis lege expediri poſſe, in ſubſidium tantum vocando methodum, qua hic ad hanc ipſam legem inveſtigandam ſumus uſi.

209. Sit igitur propoſita ſeries quaecunque

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c.$$

quam in aliam transformari oporteat, cuius termini ſinguli ſint fractiones, quarum denominatores ſecundum poteſtates formulae huiuſmodi $\alpha + \beta x + \gamma x^2 + \delta x^3 + \&c.$ procedant. Quo igitur a ſimplicioribus incipiamus, ponamus eſſe:

$$s = \frac{\mathfrak{A}}{\alpha + \beta x} + \frac{\mathfrak{B}x}{(\alpha + \beta x)^2} + \frac{\mathfrak{C}x^2}{(\alpha + \beta x)^3} + \frac{\mathfrak{D}x^3}{(\alpha + \beta x)^4} + \&c.$$

aequalitate illius ſeriei cum hac expreſſione conſtituta, multiplicetur ubique per $\alpha + \beta x$, fietque:

$$\begin{matrix} A\alpha + B\alpha x + C\alpha x^2 + D\alpha x^3 + \&c. \\ + A\beta + B\beta \quad + C\beta \quad + \&c. \end{matrix} = \mathfrak{A} + \frac{\mathfrak{B}x}{\alpha + \beta x} + \frac{\mathfrak{C}x^2}{(\alpha + \beta x)^2} + \&c.$$

ſtatua-

statuatur $\mathfrak{A} = A\alpha$; fiatque:

$$A\beta + B\alpha = A^{I}$$
$$B\beta + C\alpha = B^{I}$$
$$C\beta + D\alpha = C^{I}$$
$$D\beta + E\alpha = D^{I} \quad \&c.$$

erit divisione per x instituta:

$$A^{I} + B^{I}x + C^{I}x^2 + D^{I}x^3 + \&c. = \frac{\mathfrak{B}}{\alpha + \beta x} + \frac{\mathfrak{C}x}{(\alpha + \beta x)^2} + \frac{\mathfrak{D}x^2}{(\alpha + \beta x)^3} + \&c.$$

Multiplicetur denuo per $\alpha + \beta x$, positoque

$$A^{I}\beta + B^{I}\alpha = A^{II}$$
$$B^{I}\beta + C^{I}\alpha = B^{II}$$
$$C^{I}\beta + D^{I}\alpha = C^{II} \quad \&c. \quad \text{fiet}$$

$$A^{I}\alpha + A^{II}x + B^{II}x^2 + C^{II}x^3 + \&c. = \mathfrak{B} + \frac{\mathfrak{C}x}{\alpha + \beta x} + \frac{\mathfrak{D}x^2}{(\alpha + \beta x)^2} + \&c.$$

Sit igitur $\mathfrak{B} = A^{I}\alpha$; atque operationem ut ante instituendo, si fiat:

$A^{II}\beta + B^{II}\alpha = A^{III}$	$A^{III}\beta + B^{III}\alpha = A^{IV}$
$B^{II}\beta + C^{II}\alpha = B^{III}$	$B^{III}\beta + C^{III}\alpha = B^{IV}$
$C^{II}\beta + D^{II}\alpha = C^{III}$	$C^{III}\beta + D^{III}\alpha = C^{IV}$
&c.	&c.

erit $\mathfrak{C} = A^{II}\alpha$; $\mathfrak{D} = A^{III}\alpha$; $\mathfrak{E} = A^{IV}\alpha$; &c. unde summa seriei propositae hoc modo exprimetur, ut sit:

$$s = \frac{A\alpha}{\alpha + \beta x} + \frac{A^{I}\alpha x}{(\alpha + \beta x)^2} + \frac{A^{II}\alpha x^2}{(\alpha + \beta x)^3} + \frac{A^{III}\alpha x^3}{(\alpha + \beta x)^4} + \&c.$$

Quae eadem series orta fuisset ex substitutione $\frac{x}{\alpha + \beta x} = y$ seu $x = \frac{\alpha y}{1 - \beta y}$.

210. Haec transformatio optimo cum successu adhibetur, si series proposita $A + Bx + Cx^2 + \&c.$ ita fuerit comparata, ut tandem confundatur cum serie recurrente seu potius geometrica ex fractione $\frac{P}{\alpha + \beta x}$ orta. Tum enim valores A^{I},

A^{I}, B^{I}, C^{I}, D^{I}, &c. tandem evanescent; hincque multo magis litterae A^{II}, A^{III}, A^{IV}, &c. constituent seriem maxime convergentem. Poterimus autem simili modo denominatores trinomiales & polynomiales quoscunque adhibere, qui usum habebunt eximium, si series proposita tandem cum recurrente confundatur. Proposita ergo serie:

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c.$$

statuatur $s = \frac{\mathfrak{A} + \mathfrak{B}x}{\alpha + \beta x + \gamma x^2} + \frac{\mathfrak{A}^{I}x^2 + \mathfrak{B}^{I}x^3}{(\alpha + \beta x + \gamma x^2)^2}$

$$+ \frac{\mathfrak{A}^{II}x^4 + \mathfrak{B}^{II}x^5}{(\alpha + \beta x + \gamma x^2)^3} + \frac{\mathfrak{A}^{III}x^6 + \mathfrak{B}^{III}x^7}{(\alpha + \beta x + \gamma x^2)^4} + \&c.$$

Multiplicetur ubique per $\alpha + \beta x + \gamma x^2$, positoque

$$\begin{array}{lcl} A\gamma + B\beta + C\alpha = A^{I} & & \mathfrak{A} = A\alpha \\ B\gamma + C\beta + D\alpha = B^{I} & \& & \mathfrak{B} = A\beta + B\alpha \\ C\gamma + D\beta + E\alpha = C^{I} & & \\ \&c. & & \end{array}$$

orietur aequatio priori similis, divisione per xx instituta.

$$A^{I} + B^{I}x + C^{I}x^2 + D^{I}x^3 + E^{I}x^4 + \&c. =$$

$$\frac{\mathfrak{A}^{I} + \mathfrak{B}^{I}x}{\alpha + \beta x + \gamma xx} + \frac{\mathfrak{A}^{II}x^2 + \mathfrak{B}^{II}x^3}{(\alpha + \beta x + \gamma xx)^2} + \frac{\mathfrak{A}^{III}x^4 + \mathfrak{B}^{III}x^5}{(\alpha + \beta x + \gamma xx)^3} + \&c.$$

Si igitur ut ante operatio instituatur faciendo

$$\begin{array}{ll} A^{I}\gamma + B^{I}\beta + C^{I}\alpha = A^{II} & \mathfrak{A}^{I} = A^{I}\alpha \\ B^{I}\gamma + C^{I}\beta + D^{I}\alpha = B^{II} & \mathfrak{B}^{I} = A^{I}\beta + B^{I}\alpha \\ C^{I}\gamma + D^{I}\beta + E^{I}\alpha = C^{II} & \\ \&c. & \text{porroque:} \\ A^{II}\gamma + B^{II}\beta + C^{II}\alpha = A^{III} & \mathfrak{A}^{II} = A^{II}\alpha \\ B^{II}\gamma + C^{II}\beta + D^{II}\alpha = B^{III} & \mathfrak{B}^{II} = A^{II}\beta + B^{II}\alpha \\ C^{II}\gamma + D^{II}\beta + E^{II}\alpha = C^{III} & \\ \&c. & \end{array}$$

sique ulterius valores similes investigando erit: $s =$

$$\frac{A\alpha + (A\beta + B\alpha)x}{\alpha + \beta x + \gamma xx} + \frac{[A^{I}\alpha + (A^{I}\beta + B^{I}\alpha)x]x^2}{(\alpha + \beta x + \gamma xx)^2} + \frac{[A^{II} + (A^{II}\beta + B^{II}\alpha)x]x^4}{(\alpha + \beta x + \gamma xx)^3} + \&c.$$

211. Si ponatur $x = 1$, qua positione amplitudini nihil decedit, cum α, β, γ pro lubitu accipi possint, fueritque

$$s = A + B + C + D + E + F + G + \&c.$$

Computentur successive sequentes valores:

$$\begin{array}{l|l} A\gamma + B\beta + C\alpha = A' & A'\gamma + B'\beta + C'\alpha = A'' \\ B\gamma + C\beta + D\alpha = B' & B'\gamma + C'\beta + D'\alpha = B'' \\ C\gamma + D\beta + E\alpha = C' & C'\gamma + D'\beta + E'\alpha = C'' \\ \&c. & \&c. \end{array}$$

sicque porro

insuper vero brevitatis ergo ponatur: $\alpha + \beta + \gamma = m$ obtinebitur summa seriei propositae hoc modo expressa

$$s = \begin{array}{l} (\alpha + \beta)\left(\frac{A}{m} + \frac{A'}{m^2} + \frac{A''}{m^3} + \frac{A'''}{m^4} + \&c.\right) \\ + \alpha\left(\frac{B}{m} + \frac{B'}{m^2} + \frac{B''}{m^3} + \frac{B'''}{m^4} + \&c.\right) \end{array}$$

212. Eodem modo denominatores ex pluribus terminis constantes accipi possunt; & quoniam operatio ex praecedentibus facile perspicitur, hic tantum casum pro quadrinomio evolvamus: Sit ergo

$$s = A + B + C + D + E + F + G + \&c.$$

Quaerantur valores sequentes:

$$\begin{array}{l} A\delta + B\gamma + C\beta + D\alpha = A' \\ B\delta + C\gamma + D\beta + E\alpha = B' \\ C\delta + D\gamma + E\beta + F\alpha = C' \\ \&c. \end{array}$$

$$\begin{array}{l} A'\delta + B'\gamma + C'\beta + D'\alpha = A'' \\ B'\delta + C'\gamma + D'\beta + E'\alpha = B'' \\ C'\delta + D'\gamma + E'\beta + F'\alpha = C'' \\ \&c. \end{array}$$

$$\begin{array}{l} A''\delta + B''\gamma + C''\beta + D''\alpha = A''' \\ B''\delta + C''\gamma + D''\beta + E''\alpha = B''' \\ C''\delta + D''\gamma + E''\beta + F''\alpha = C''' \\ \&c. \end{array}$$

Tum vero sit $\alpha + \beta + \gamma + \delta = m$; eritque

$$s = \begin{array}{l} (\alpha + \beta + \gamma)\left(\frac{A}{m} + \frac{A'}{m^2} + \frac{A''}{m^3} + \frac{A'''}{m^4} + \&c.\right) \\ (\alpha + \beta)\left(\frac{B}{m} + \frac{B'}{m^2} + \frac{B''}{m^3} + \frac{B'''}{m^4} + \&c.\right) + \alpha\left(\frac{C}{m} + \frac{C'}{m^2} + \frac{C''}{m^3} + \frac{C'''}{m^4} + \&c.\right) \end{array}$$

unde simul progressio, si adhuc plures partes denominatori m tribuantur, clarissime perspicitur.

213. Neque vero absolute opus est, ut denominatores fractionum, ad quas summam seriei reducimus, sint potestates eiusdem formulae $\alpha + \beta x + \gamma x^2 +$ &c. sed haec ipsa in singulis terminis variari potest. Quo hoc clarius pateat, sumamus primo tantum duos terminos, fingaturque series

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c.$$

in hanc seriem fractionum converti:

$$s = \frac{\mathfrak{A}}{\alpha + \beta x} + \frac{\mathfrak{A}'x}{(\alpha + \beta x)(\alpha' + \beta' x)} + \frac{\mathfrak{A}''x^2}{(\alpha + \beta x)(\alpha' + \beta' x)(\alpha'' + \beta'' x)} + \&c.$$

Multiplicetur primum utrinque per $\alpha + \beta x$, ponaturque

$$\begin{aligned} A\beta + B\alpha &= A' \\ B\beta + C\alpha &= B' \quad \& \quad \mathfrak{A} = A\alpha \\ C\beta + D\alpha &= C' \\ \&c. \end{aligned}$$

fietque per x diviso

$$A' + B'x + C'x^2 + D'x^3 + \&c. = \frac{\mathfrak{A}'}{\alpha' + \beta' x} + \frac{\mathfrak{A}''x}{(\alpha' + \beta' x)(\alpha'' + \beta'' x)} + \&c.$$

Deinde simili modo multiplicando per $\alpha' + \beta' x$; tumque per $\alpha'' + \beta'' x$, & ita porro, si statuatur:

$A'\beta' + B'\alpha' = A''$	$A''\beta'' + B''\alpha'' = A'''$	$A'''\beta''' + B'''\alpha''' = A''''$
$B'\beta' + C'\alpha' = B''$	$B''\beta'' + C''\alpha'' = B'''$	$B'''\beta''' + C'''\alpha''' = B''''$
$C'\beta' + D'\alpha' = C''$	$C''\beta'' + D''\alpha'' = C'''$	$C'''\beta''' + D'''\alpha''' = C''''$
&c.	&c.	&c.

fiet $\mathfrak{A}' = A'\alpha'$; $\mathfrak{A}'' = A''\alpha''$; $\mathfrak{A}''' = A'''\alpha'''$; &c.

atque hinc series proposita convertetur in hanc:

$$s = \frac{A\alpha}{\alpha + \beta x} + \frac{A'\alpha' x}{(\alpha + \beta x)(\alpha' + \beta' x)} + \frac{A''\alpha'' xx}{(\alpha + \beta x)(\alpha' + \beta' x)(\alpha'' + \beta'' x)} \&c.$$

ubi valores α, β, α', β', α'', β'', α''', β''' &c. sunt arbitrarii, quovis autem casu ita accipi possunt, ut ista nova series maxime convergat.

214. Applicemus hoc quoque ad factores trinomiales, sitque proposita serie quacunque:

$s =$

$$s = A + B + C + D + E + F + G + \&c.$$

$$\begin{array}{l|l}
A\gamma + B\beta + C\alpha = A' & A'\gamma' + B'\beta' + C'\alpha' = A'' \\
B\gamma + C\beta + D\alpha = B' & B'\gamma' + C'\beta' + D'\alpha' = B'' \\
C\gamma + D\beta + E\alpha = C' & C'\gamma' + D'\beta' + E'\alpha' = C'' \\
\&c. & \&c.
\end{array}$$

$$\begin{array}{l|l}
A''\gamma'' + B''\beta'' + C''\alpha'' = A''' & A'''\gamma''' + B'''\beta''' + C'''\alpha''' = A'''' \\
B''\gamma'' + C''\beta'' + D''\alpha'' = B''' & B'''\gamma''' + C'''\beta''' + D'''\alpha''' = B'''' \\
C''\gamma'' + D''\beta'' + E''\alpha'' = C''' & C'''\gamma''' + D'''\beta''' + E'''\alpha''' = C'''' \\
\&c. & \&c.
\end{array}$$

Deinde statuatur brevitatis gratia:

$$\begin{aligned}
\alpha + \beta + \gamma &= m \\
\alpha' + \beta' + \gamma' &= m' \\
\alpha'' + \beta'' + \gamma'' &= m'' \\
\alpha''' + \beta''' + \gamma''' &= m''' \qquad \&c.
\end{aligned}$$

eritque seriei propositae summa:

$$s = \frac{\alpha(A+B)}{m} + \frac{\alpha'(A'+B')}{mm'} + \frac{\alpha''(A''+B'')}{mm'm''} + \frac{\alpha'''(A'''+B''')}{mm'm''m'''} + \&c.$$

$$+ \frac{\beta A}{m} + \frac{\beta' A'}{mm'} + \frac{\beta'' A''}{mm'm''} + \frac{\beta''' A'''}{mm'm''m'''} + \&c.$$

215. Quoniam haec tam late patent, ut usus minus clare percipi possit, restringamus transformationem §. 213. traditam, sitque $x = -1$, ut habeatur haec series:

$s = A - B + C - D + E - F + G - \&c.$ statuaturque:

$$\begin{array}{l|l|l|l}
B - A = A' & B' - 2A' = A'' & B'' - 3A'' = A''' & B''' - 4A''' = A'''' \\
C - B = B' & C' - 2B' = B'' & C'' - 3B'' = B''' & C''' - 4B''' = B'''' \\
D - C = C' & D' - 2C' = C'' & D'' - 3C'' = C''' & D''' - 4C''' = C'''' \\
E - D = D' & E' - 2D' = D'' & E'' - 3D'' = D''' & E''' - 4D''' = D'''' \\
\&c. & \&c. & \&c. & \&c.
\end{array}$$

Quibus valoribus inventis, erit summa seriei propositae aequalis sequenti seriei:

$$s = \frac{A}{2} - \frac{A'}{2.3} + \frac{A''}{2.3.4} - \frac{A'''}{2.3.4.5} + \frac{A''''}{2.3.4.5.6} - \&c.$$

Simili igitur modo series quaecunque proposita in innumerabiles

biles alias sibi aequales transmutari potest, inter quas sine dubio series maxime convergentes reperientur, quarum ope summa proposita vero proxime indagari queat.

216. Revertamur autem ad inventionem serierum, quarum progressionis legem calculus differentialis declarat. Cum igitur hoc in quantitatibus algebraicis iam sit praestitum, progrediamur ad transcendentes, quaeraturque series huic logarithmo aequalis:

$$s = l\,(1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \&c.)$$

fingatur quaesito satisfacere haec series:

$$s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \mathfrak{F}x^6 + \&c.$$

Cum igitur ex illius aequationis differentiatione sequatur

$$\frac{ds}{dx} = \frac{\alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + 5\varepsilon x^4 + \&c.}{1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \&c.} \quad \text{erit:}$$

$$(1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \&c.)\frac{ds}{dx} = \alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + \&c.$$

Quia vero ex ficta aequatione est:

$$\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + 5\mathfrak{E}x^4 + \&c.$$

facta hac substitutione oritur haec aequatio:

$$\begin{array}{llllll}
\mathfrak{A} & + 2\mathfrak{B}x & + 3\mathfrak{C}x^2 & + 4\mathfrak{D}x^3 & + 5\mathfrak{E}x^4 & + \&c. \\
 & + \mathfrak{A}\alpha & + 2\mathfrak{B}\alpha & + 3\mathfrak{C}\alpha & + 4\mathfrak{D}\alpha & + \&c. \\
 & & + \mathfrak{A}\beta & + 2\mathfrak{B}\beta & + 3\mathfrak{C}\beta & + \&c. \\
 & & & + \mathfrak{A}\gamma & + 2\mathfrak{B}\gamma & + \&c. \\
 & & & & + \mathfrak{A}\delta & + \&c. = \\
\hline
\alpha & + 2\beta x & + 3\gamma x^2 & + 4\delta x^3 & + 5\varepsilon x^4 & + \&c.
\end{array}$$

Ex qua sequentes determinationes obtinentur:

$$\begin{aligned}
\mathfrak{A} &= \alpha \\
\mathfrak{B} &= -\tfrac{1}{2}\mathfrak{A}\alpha + \beta \\
\mathfrak{C} &= -\tfrac{2}{3}\mathfrak{B}\alpha - \tfrac{1}{3}\mathfrak{A}\beta + \gamma \\
\mathfrak{D} &= -\tfrac{3}{4}\mathfrak{C}\alpha - \tfrac{2}{4}\mathfrak{B}\beta - \tfrac{1}{4}\mathfrak{A}\gamma + \delta \\
\mathfrak{E} &= -\tfrac{4}{5}\mathfrak{D}\alpha - \tfrac{3}{5}\mathfrak{C}\beta - \tfrac{2}{5}\mathfrak{B}\gamma - \tfrac{1}{5}\mathfrak{A}\delta + \varepsilon \qquad \&c.
\end{aligned}$$

217. Proposita nunc sit quantitas exponentialis:

$s =$

$$s = e^{\alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \&c.}$$

in qua e denotet numerum, cuius logarithmus hyperbolicus eſt $= 1$, atque fingatur ſeries quaeſita:

$$s = 1 + \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \&c.$$

iam enim ex caſu $x = 0$ patet, primum terminum eſſe debere unitatem. Cum igitur ſumendis logarithmis ſit

$$ls = \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \zeta x^6 + \&c.$$

erit differentialibus ſumtis:

$$\frac{ds}{dx} = s(\alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + 5\varepsilon x^4 + \&c.)$$

At vero ex aequatione ficta erit:

$$\begin{aligned} \frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 &+ 4\mathfrak{D}x^3 + 5\mathfrak{E}x^4 + \&c. = \\ \alpha + \mathfrak{A}\alpha x + \mathfrak{B}\alpha x^2 &+ \mathfrak{C}\alpha x^3 + \mathfrak{D}\alpha x^4 + \&c. \\ + 2\beta + 2\mathfrak{A}\beta &+ 2\mathfrak{B}\beta + 2\mathfrak{C}\beta + \&c. \\ + 3\gamma &+ 3\mathfrak{A}\gamma + 3\mathfrak{B}\gamma + \&c. \\ &+ 4\delta + 4\mathfrak{A}\delta + \&c. \\ &+ 5\varepsilon + \&c. \end{aligned}$$

ex quibus ſequentes prodeunt litterarum $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, &c. determinationes

$$\begin{aligned} \mathfrak{A} &= \alpha \\ \mathfrak{B} &= \beta + \tfrac{1}{2}\mathfrak{A}\alpha \\ \mathfrak{C} &= \gamma + \tfrac{2}{3}\mathfrak{A}\beta + \tfrac{1}{3}\mathfrak{B}\alpha \\ \mathfrak{D} &= \delta + \tfrac{3}{4}\mathfrak{A}\gamma + \tfrac{2}{4}\mathfrak{B}\beta + \tfrac{1}{4}\mathfrak{C}\alpha \\ \mathfrak{E} &= \varepsilon + \tfrac{4}{5}\mathfrak{A}\delta + \tfrac{3}{5}\mathfrak{B}\gamma + \tfrac{2}{5}\mathfrak{C}\beta + \tfrac{1}{5}\mathfrak{D}\alpha \qquad \&c. \end{aligned}$$

218. Si quoque arcus, cuius ſinus vel coſinus quaeritur, exprimatur binomio vel polynomio, vel etiam ſerie infinita, hoc modo quoque eius ſinus & coſinus per ſeriem infinitam exprimi poſſunt. At vero quo hoc commodiſſime fiat, non ſufficit ad differentialia prima proceſſiſſe, ſed opus eſt ut differentialia ſecundi gradus in ſubſidium vocemus. Sit igitur

$$s = \sin(\alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \&c.)$$

fingaturque ſeries quae quaeritur:

$$s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \&c.$$

primum enim terminum conſtat evaneſcere: quia vero ad differentialia ſecunda deſcendendum eſt, coefficientem $\mathfrak{A}$ quoque aliunde definiri oportet, quod fiet si x ponamus infinite parvum. Tum enim ob arcum $= \alpha x$ sinus ipsi fiet aequalis, eritque ergo $\mathfrak{A} = \alpha$. Ponamus nunc brevitatis gratia $z = \alpha x + \beta x^2 + \gamma x^3 + \&c.$ ut sit $s = \sin z$, erit differentiando $ds = dz \cos z$ denuoque differentiando $dds = ddz \cos z - dz^2 \sin z$.

Quia igitur est $\sin z = s$ & $\cos z = \frac{ds}{dz}$; erit

$$dds = \frac{dsddz}{dz} - sdz^2, \text{ seu } dzdds + sdz^3 = dsddz.$$

219. Ponamus arcum z tantum binomio exprimi esseque $z = \alpha x + \beta x^2$; erit $dz = (\alpha + 2\beta x)dx$, & posito dx constante, $ddz = 2\beta dx^2$; atque $dz^3 = (\alpha^3 + 6\alpha^2\beta x + 12\alpha\beta^2 x^2 + 8\beta^3 x^3)dx^3$. Deinde ob $s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \&c.$

erit $\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + \&c.$

& $\frac{dds}{dx^2} = 2\mathfrak{B} + 6\mathfrak{C}x + 12\mathfrak{D}x^2 + \&c.$

Quibus valoribus in aequatione differentio-differentiali substitutis fiet:

$$\begin{array}{lllllll}
\frac{dzdds}{dx^3} = & 1.2\mathfrak{B}\alpha + & 2.3\mathfrak{C}\alpha x + & 3.4\mathfrak{D}\alpha x^2 + & 4.5\mathfrak{E}\alpha x^3 + \&c. \\
 & & + 2.1.2\mathfrak{B}\beta & + 2.2.3\mathfrak{C}\beta & + 2.3.4\mathfrak{D}\beta & + \&c. \\
+\frac{sdz^3}{dx^3} = & & + & \mathfrak{A}\alpha^3 + & \mathfrak{B}\alpha^3 + & \mathfrak{C}\alpha^3 + \&c. \\
 & & & + & 6\mathfrak{A}\alpha^2\beta + & 6\mathfrak{B}\alpha^2\beta + \&c. \\
 & & & & + & 12\mathfrak{A}\alpha\beta^2 + \&c. \\
\hline
\frac{dsddz}{dx^3} = & 2\mathfrak{A}\beta + & 4\mathfrak{B}\beta + & 6\mathfrak{C}\beta & + 8\mathfrak{D}\beta & + \&c.
\end{array}$$

Unde coefficientes sequenti modo definientur:

$\mathfrak{B} =$

$$\mathfrak{B} = \frac{2\mathfrak{A}\beta}{2\alpha}$$

$$\mathfrak{C} = \quad 0 \quad - \frac{\mathfrak{A}\alpha^2}{2.3}$$

$$\mathfrak{D} = -\frac{2\mathfrak{C}\beta}{4\alpha} - \frac{6\mathfrak{A}\alpha\beta}{3.4} - \frac{\mathfrak{B}\alpha^2}{3.4}$$

$$\mathfrak{E} = -\frac{4\mathfrak{D}\beta}{5\alpha} - \frac{12\mathfrak{A}\beta^2}{4.5} - \frac{6\mathfrak{B}\alpha\beta}{4.5} - \frac{\mathfrak{C}\alpha^2}{4.5}$$

$$\mathfrak{F} = -\frac{6\mathfrak{E}\beta}{6\alpha} - \frac{8\mathfrak{A}\beta^3}{5.6\alpha} - \frac{12\mathfrak{B}\beta\beta}{5.6} - \frac{6\mathfrak{C}\alpha\beta}{5.6} - \frac{\mathfrak{D}\alpha^2}{5.6}$$

$$\mathfrak{G} = -\frac{8\mathfrak{F}\beta}{7\alpha} - \frac{8\mathfrak{B}\beta^3}{6.7\alpha} - \frac{12\mathfrak{C}\beta\beta}{6.7} - \frac{6\mathfrak{D}\alpha\beta}{6.7} - \frac{\mathfrak{E}\alpha^2}{6.7}$$

&c. Quibus valoribus inventis erit:
$\sin(\alpha x + \beta x^2) = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \&c.$ existente $\mathfrak{A} = \alpha$

220. Pari modo cosinus cuiusque anguli in seriem convertitur, quia autem arcus rarissime per polynomium exprimitur, ostendamus usum differentio-differentialium in invenienda serie pro cosinu arcus x. Sit ergo $s = \cos x$, & fingatur:

$$s = 1 - \mathfrak{A}x^2 + \mathfrak{B}x^4 - \mathfrak{C}x^6 + \mathfrak{D}x^8 - \&c.$$

Quia est $ds = -dx \sin x$ & $dds = -dx^2 \cos x = -sdx^2$. erit $dds + sdx^2 = 0$; substitutione ergo facta fiet:

$$\frac{dds}{dx^2} = -1.2\,\mathfrak{A} + 3.4\,\mathfrak{B}x^2 - 5.6\,\mathfrak{C}x^4 + 7.8\,\mathfrak{D}x^6 - \&c.$$

$$s = \quad 1 - \mathfrak{A}x^2 + \mathfrak{B}x^4 - \mathfrak{C}x^6 + \&c.$$

& ex coaequatione terminorum sequitur:

$$\mathfrak{A} = \frac{1}{1.2} \quad ; \quad \mathfrak{B} = \frac{\mathfrak{A}}{3.4} = \frac{1}{1.2.3.4}$$

$$\mathfrak{C} = \frac{\mathfrak{B}}{5.6} = \frac{1}{1.2.3\ldots6} \quad ; \quad \mathfrak{D} = \frac{\mathfrak{C}}{7.8} = \frac{1}{1.2.3\ldots8}; \&c.$$

Patet ergo quod iam supra fusius demonstravimus esse:

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3\ldots6} + \frac{x^8}{1.2.3\ldots8} - \&c.$$

prior vero series pro sinu posito $\beta = 0$ & $\alpha = 1$ dabit:

$$\sin x = \frac{x}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3\ldots7} + \frac{x^9}{1.2.3\ldots9} - \&c.$$

221. Ex his seriebus pro sinu & cosinu notissimis deducuntur series pro tangente, cotangente, secante & cosecante cuiusvis anguli. Tangens enim prodit si sinus per cosinum, cotangens si cosinus per sinum, secans si radius 1 per cosinum, & cosecans si radius per sinum dividatur. Series autem ex his divisionibus ortae maxime videntur irregulares; verum excepta serie secantem exhibente reliquae per numeros Bernoullianos supra definitos $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, &c. ad facilem progressionis legem reduci possunt. Quoniam enim supra §. 127. invenimus esse:

$$\frac{\mathfrak{A}u^2}{1.2} + \frac{\mathfrak{B}u^4}{1.2.3.4} + \frac{\mathfrak{C}u^6}{1.2.3\ldots6} + \frac{\mathfrak{D}u^8}{1.2.3\ldots8} + \&c. = 1 - \frac{u}{2}\cot\tfrac{1}{2}u$$

erit posito $\frac{1}{2}u = x$;

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1.2} - \frac{2^4\mathfrak{B}x^3}{1.2.3.4} - \frac{2^6\mathfrak{C}x^5}{1.2.3\ldots6} - \frac{2^8\mathfrak{D}x^7}{1.2\ldots8} - \&c.$$

atque si ponatur $\frac{1}{2}x$ pro x, erit:

$$\cot\tfrac{1}{2}x = \frac{2}{x} - \frac{2\mathfrak{A}x}{1.2} - \frac{2\mathfrak{B}x^3}{1.2.3.4} - \frac{2\mathfrak{C}x^5}{1.2.3\ldots6} - \frac{2\mathfrak{D}x^7}{1.2.3\ldots8} - \&c.$$

222. Hinc autem tangens cuiusvis arcus sequenti modo per seriem exprimetur. Cum sit $\operatorname{tang} 2x = \frac{2\operatorname{tang} x}{1 - \operatorname{tang} x^2}$, erit:

$$\operatorname{cotang} 2x = \frac{1}{2\operatorname{tang} x} - \frac{\operatorname{tang} x}{2} = \tfrac{1}{2}\cot x - \tfrac{1}{2}\operatorname{tang} x:$$

ideoque $\operatorname{tang} x = \cot x - 2\cot 2x$. Cum igitur sit

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1.2} - \frac{2^4\mathfrak{B}x^3}{1.2.3.4} - \frac{2^6\mathfrak{C}x^5}{1.2\ldots6} - \frac{2^8\mathfrak{D}x^7}{1.2\ldots8} - \&c.$$

$$2\cot 2x = \frac{1}{x} - \frac{2^4\mathfrak{A}x}{1.2} - \frac{2^8\mathfrak{B}x^3}{1.2.3.4} - \frac{2^{12}\mathfrak{C}x^5}{1.2\ldots6} - \frac{2^{16}\mathfrak{D}x^7}{1.2\ldots8} - \&c.$$

erit hanc seriem ab illa subtrahendo:

$$\operatorname{tg} x = \frac{2^2(2^2-1)\mathfrak{A}x}{1.2} + \frac{2^4(2^4-1)\mathfrak{B}x^3}{1.2.3.4} + \frac{2^6(2^6-1)\mathfrak{C}x^5}{1.2\ldots6} + \frac{2^8(2^8-1)\mathfrak{D}x^7}{1.2\ldots8} + \&c.$$

Si ergo hic introducantur numeri A, B, C, &c. §. 182. inventi, erit: $\tang x = \frac{2Ax}{1.2} + \frac{2^3 Bx^3}{1.2.3.4} + \frac{2^5 Cx^5}{1.2\ldots6} + \frac{2^7 Dx^7}{1.2\ldots8} + \&c.$

223. Cosecans autem sequenti modo invenietur. Quia est $\cot x = \tang x + 2\cot 2x = \frac{1}{\cot x} + 2\cot 2x$; erit $\cot x^2 = 2\cot x.\cot 2x + 1$, & radice extracta: $\cot x = \cot 2x + \operatorname{cosec} 2x$, unde fit $\operatorname{cosec} 2x = \cot x - \cot 2x$, & x pro $2x$, posito $\operatorname{cosec} x = \cot\frac{1}{2}x - \cot x$. Quare cum cotangentes habeamus scilicet:

$$\cot\tfrac{1}{2}x = \frac{2}{x} - \frac{2\mathfrak{A}x}{1.2} - \frac{2\mathfrak{B}x^3}{1.2.3.4} - \frac{2\mathfrak{C}x^5}{1.2\ldots6} - \&c.$$

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1.2} - \frac{2^4\mathfrak{B}x^3}{1.2.3.4} - \frac{2^6\mathfrak{C}x^5}{1.2\ldots6} - \&c.$$

erit hac serie ab illa subtracta:

$$\operatorname{cosec} x = \frac{1}{x} + \frac{2(2-1)\mathfrak{A}x}{1.2} + \frac{2(2^3-1)\mathfrak{B}x^3}{1.2.3.4} + \frac{2(2^5-1)\mathfrak{C}x^5}{1.2.\ldots.6} + \&c.$$

224. Per hos autem numeros Bernoullianos secans exprimi non potest, sed requirit alios numeros, qui in summas potestatum reciprocarum imparium ingrediuntur. Si enim ponatur:

$$1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \&c. = \alpha.\ \frac{\pi}{2^2}$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{9^3} - \&c. = \frac{\beta}{1.2}.\frac{\pi^3}{2^4}$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{9^5} - \&c. = \frac{\gamma}{1.2.3.4}.\frac{\pi^5}{2^6}.$$

$$1 - \frac{1}{3^7} + \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{9^7} - \&c. = \frac{\delta}{1.2\ldots6}.\frac{\pi^7}{2^8}$$

$$1 - \frac{1}{3^9} + \frac{1}{5^9} + \frac{1}{7^9} + \frac{1}{9^9} - \&c. = \frac{\varepsilon}{1.2\ldots8}.\frac{\pi^9}{2^{10}}$$

$$1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} + \frac{1}{7^{11}} + \frac{1}{9^{11}} - \&c. = \frac{\zeta}{1.2\ldots10}.\frac{\pi^{11}}{2^{12}}$$

&c.

erit:

erit:

$\alpha = 1$	$\eta = 2702765$
$\beta = 1$	$\theta = 199360981$
$\gamma = 5$	$\iota = 19391512145$
$\delta = 61$	$\varkappa = 2404879661671$
$\varepsilon = 1385$	&c.
$\zeta = 50521$	

ex hisque valoribus obtinebitur:

$$\sec x = \alpha + \frac{\beta}{1.2}xx + \frac{\gamma}{1.2.3.4}x^4 + \frac{\delta}{1.2\ldots6}x^6 + \frac{\varepsilon}{1.2\ldots8}x^8 + \&c.$$

225. Ut autem nexum huius seriei cum numeris α, β, γ, δ, &c. ostendamus, consideremus seriem supra tractatam:

$$\frac{\pi}{n \sin\frac{m}{n}\pi} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \&c.$$

Ponatur $m = \frac{1}{2}n - k$, eritque

$$\frac{\pi}{2n\cos\frac{k}{n}\pi} = \frac{1}{n-2k} + \frac{1}{n+2k} - \frac{1}{3n-2k} - \frac{1}{3n+2k} + \frac{1}{5n-2k} + \&c.$$

Sit $\frac{k\pi}{n} = x$, seu $k\pi = nx$, erit

$$\frac{\pi}{2n}\sec x = \frac{\pi}{n\pi - 2nx} + \frac{\pi}{n\pi + 2nx} - \frac{\pi}{3n\pi - 2nx} - \&c. \text{ seu}$$

$$\sec x = \frac{2}{\pi - 2x} + \frac{2}{\pi + 2x} - \frac{2}{3\pi - 2x} - \frac{2}{3\pi + 2x} + \frac{2}{5\pi - 2x} + \&c.$$

$$\sec x = \frac{4\pi}{\pi^2 - 4x^2} - \frac{4.3\pi}{9\pi^2 - 4xx} + \frac{4.5\pi}{25\pi^2 - 4xx} - \frac{4.7\pi}{49\pi^2 - 4xx} + \&c.$$

si nunc singuli termini in series convertantur, fiet:

$$\sec x = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.\right)$$
$$+ \frac{2^4 x^2}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \&c.\right)$$
$$+ \frac{2^6 x^4}{\pi^5}\left(1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \&c.\right)$$
&c.

qua-

quarum serierum loco si valores supra assignati substituantur, prodibit eadem series pro secante, quam dedimus.

226. Hinc simul patet lex, qua numeri α, β, γ, &c. quibus summae potestatum imparium constituuntur, procedunt. Cum enim sit

$$\sec. x = \frac{1}{\cos x} = \alpha + \frac{\beta}{1.2}x^2 + \frac{\gamma}{1.2.3.4}x^4 + \frac{\delta}{1.2\ldots6}x^6 + \&c.$$

necesse est ut haec series aequalis sit fractioni

$$\frac{1}{1 - \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2\ldots6} + \frac{x^8}{1.2\ldots8} - \&c.}$$

aequalitate ergo constituta fiet $1 =$

$$\begin{array}{l}
\alpha + \frac{\beta}{1.2}x^2 + \frac{\gamma}{1.2.3.4}x^4 + \frac{\delta}{1.2\ldots6}x^6 + \frac{\varepsilon}{1.2\ldots8}x^8 + \&c. \\
\quad - \frac{\alpha}{1.2} - \frac{\beta}{1.2.1.2} - \frac{\gamma}{1.2.1\ldots4} - \frac{\delta}{1.2.1\ldots6}\ \&c. \\
\qquad + \frac{\alpha}{1.2.3.4} + \frac{\beta}{1\ldots4.1.2} + \frac{\gamma}{1\ldots4.1\ldots4}\ \&c. \\
\qquad\qquad - \frac{\alpha}{1\ldots6} - \frac{\beta}{1\ldots6.1.2}\ \&c. \\
\qquad\qquad\qquad + \frac{\alpha}{1\ldots8}\ \&c.
\end{array}$$

unde sequuntur hae aequationes:

$$\alpha = 1\ ; \qquad \beta = \frac{2.1}{1.2}\alpha\ ;$$

$$\gamma = \frac{4.3}{1.2}\beta - \frac{4.3.2.1}{1.2.3.4}\alpha\ ; \quad \delta = \frac{6.5}{1.2}\gamma - \frac{6.5.4.3}{1.2.3.4}\beta + \frac{6\ldots1}{1\ldots6}\alpha\ ;$$

$$\varepsilon = \frac{8.7}{1.2}\delta - \frac{8.7.6.5}{1.2.3.4}\gamma + \frac{8\ldots3}{1\ldots6}\beta - \frac{8\ldots1}{1\ldots8}\alpha\ ; \quad \&c.$$

Ex hisque formulis inventi sunt istarum litterarum valores, quos in §. 224. exhibuimus; & quorum ope summae serierum in hac forma contentarum,

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \&c.$$

si n fuerit numerus impar, exprimi possunt.