CAPUT VII.

METHODUS SUMMANDI SUPERIOR ULTERIUS PROMOTA.

167.

Ut defectum methodi summandi ante traditae suppleamus, in hoc Capite eiusmodi series considerabimus, quarum termini generales magis sint complexi. Cum igitur expressio ante inventa in progressionibus geometricis, etsi aliis methodis facillime summari possunt, veram summam finita formula contentam non praebeat, hic primum eiusmodi series contemplabimur, quarum termini sint producta ex terminis seriei geometricae & alius cuiuscunque. Sit igitur proposita haec series:

$$\overset{1}{ } \quad \overset{2}{ } \quad \overset{3}{ } \quad \overset{4}{ } \qquad\qquad\qquad \overset{x}{ }$$
$$s = ap + bp^2 + cp^3 + dp^4 + \ldots + yp^x$$

quae est composita ex geometrica p, p^2, p^3, &c. & alia quacunque serie $a + b + c + d +$ &c. cuius terminus generalis seu indici x respondens sit $= y$, atque expressionem generalem investigemus pro valore eius summae $s = S.yp^x$.

168. Instituamus ratiocinium eodem modo, quo supra usi sumus, sitque v terminus antecedens ipsi y in serie $a + b + c + d +$ &c. atque A praecedens ipsi a seu is qui indici o respondet, eritque vp^{x-1} terminus generalis huius seriei:

$$\overset{1}{ } \quad \overset{2}{ } \quad \overset{3}{ } \quad \overset{4}{ } \qquad\qquad\qquad \overset{x}{ }$$
$$A + ap + bp^2 + cp^3 + \ldots + vp^{x-1}$$

cuius summa, si indicetur per $S.vp^{x-1}$ erit:

$$S.vp^{x-1} = \frac{1}{p} S.vp^x = S.yp^x - yp^x + A.$$

Cum autem sit:

v =

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \frac{d^4y}{24dx^4} - \frac{d^5y}{120dx^5} + \&c.$$

erit:

$$S.yp^x - yp^x + A = \frac{1}{p} S.yp^x - \frac{1}{p} S \frac{dy}{dx} p^x + \frac{1}{2p} S \frac{ddy}{dx^2} p^x$$

$$- \frac{1}{6p} S \frac{d^3y}{dx^3} p^x + \frac{1}{24p} S \frac{d^4y}{dx^4} p^x - \&c.$$ Ex qua fit:

$$S.yp^x = \frac{1}{p-1}\left(yp^{x+1} - Ap - S\frac{dy}{dx}p^x + S\frac{ddy}{2dx^2}p^x - S\frac{d^3y}{6dx^3}p^x + \&c.\right)$$

Si ergo habeantur termini summatorii serierum, quarum termini generales sunt $\frac{dy}{dx}p^x$; $\frac{ddy}{dx^2}p^x$; $\frac{d^3y}{dx^3}p^x$; &c. ex iis definiri poterit terminus summatorius Syp^x.

169. Hinc iam summae inveniri poterunt serierum, quarum termini generales in hac forma $x^n p^x$ continentur. Sit enim $y = x^n$, erit $A = 0$, nisi sit $n = 0$, quo casu foret $A = 1$, & quia est:

$$\frac{dy}{dx} = nx^{n-1}; \frac{ddy}{2dx^2} = \frac{n(n-1)}{1 \cdot 2} x^{n-2}; \frac{d^3y}{dx^3} = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^{n-3};$$

erit:

$$S.x^n p^x = \frac{1}{p-1}\left(x^n p^{x+1} - Ap - nS.x^{n-1}p^x + \frac{n(n-1)}{1 \cdot 2} S.x^{n-2}p^x\right.$$

$$\left. - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} S.x^{n-3}p^x + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} S.x^{n-4}p^x - \&c.\right)$$

Ex hac forma nunc successive pro n substituendo numeros $0, 1, 2, 3$, &c. obtinebuntur sequentes summationes; ac primo quidem si $n = 0$, fit $A = 1$, in reliquis autem casibus erit $A = 0$:

$$S.x^0 p^x = S.p^x = \frac{1}{p-1}(p^{x+1} - p) = \frac{p^{x+1} - p}{p-1} = \frac{p(p^x - 1)}{p-1},$$

quae est summa progressionis geometricae cognita:

$$S.xp^x = \frac{1}{p-1}(xp^{x+1} - S.p^x) = \frac{xp^{x+1}}{p-1} - \frac{p^{x+1}-p}{(p-1)^2}$$

seu $$S.xp^x = \frac{pxp^x}{p-1} - \frac{p(p^x-1)}{(p-1)^2},$$

$$S.x^2p^x = \frac{1}{p-1}(x^2p^{x+1} - 2S.xp^x + S.p^x)$$ seu

$$S.x^2p^x = \frac{x^2p^{x+1}}{p-1} - \frac{2xp^{x+1}}{(p-1)^2} + \frac{p(p+1)(p^x-1)}{(p-1)^3}.$$

Porro est

$$S.x^3p^x = \frac{1}{p-1}(x^3p^{x+1} - S.x^2p^x + 3S.xp^x - S.p^x)$$ seu

$$S.x^3p^x = \frac{x^3p^{x+1}}{p-1} - \frac{3x^2p^{x+1}}{(p-1)^2} + \frac{3(p+1)xp^{x+1}}{(p-1)^3} - \frac{p(pp+4p+1)(p^x-1)}{(p-1)^4}$$

sicque ulterius progrediendo superiorum potestatum x^4p^x; x^5p^x; x^6p^x; &c. summae definiri poterunt, hoc vero commodius praestabitur ope expressionis generalis, quam nunc investigabimus.

170. Quoniam invenimus esse:

$$S.yp^x = \frac{1}{p-1}\left(yp^{x+1} - Ap - S.\frac{dy}{dx}p^x + S.\frac{ddy}{2dx^2}p^x - S.\frac{d^3y}{6dx^3}p^x + \&c.\right)$$

ubi A est eiusmodi constans, ut summa fiat $= 0$, si ponatur $x = 0$: namque hoc casu fit $y = A$, & $yp^{x+1} = Ap$; hanc constantem omittere poterimus, dummodo perpetuo meminerimus ad summam quamque semper eiusmodi constantem adiici oportere, ut facto $x = 0$, evanescat, seu ut alii cuipiam casui satisfiat. Statuamus ergo z loco y, eritque

$$S.p^xz = \frac{p^{x+1}z}{p-1} - \frac{1}{p-1}S.p^x\frac{dz}{dx} + \frac{1}{2(p-1)}S.p^x\frac{ddz}{dx^2} - \frac{1}{6(p-1)}S.p^x\frac{d^3z}{dx^3} + \frac{1}{24(p-1)}S.p^x\frac{d^4z}{dx^4} - \frac{1}{120(p-1)}S.p^x\frac{d^5z}{dx^5} + \&c.$$

Dein-

Deinde statuamus successive $\frac{dz}{dx}$; $\frac{ddz}{dx^2}$; $\frac{d^3z}{dx^3}$; &c. in locum y eritque:

$$S.\frac{p^x dz}{dx} = \frac{p^{x+1}}{p-1}.\frac{dz}{dx} - \frac{1}{p-1}S.\frac{p^x ddz}{dx^2} + \frac{1}{2(p-1)}S.\frac{p^x d^3z}{dx^3} - \&c.$$

$$S.\frac{p^x ddz}{dx^2} = \frac{p^{x+1}}{p-1}.\frac{ddz}{dx^2} - \frac{1}{p-1}S.\frac{p^x d^3z}{dx^3} + \frac{1}{2(p-1)}S.\frac{p^x d^4z}{dx^4} - \&c.$$

$$S.\frac{p^x d^3z}{dx^3} = \frac{p^{x+1}}{p-1}.\frac{d^3z}{dx^3} - \frac{1}{p-1}S.\frac{p^x d^4z}{dx^4} + \frac{1}{2(p-1)}S.\frac{p^x d^5z}{dx^5} - \&c.$$

&c.

Si igitur hi valores successive substituantur, $S.p^x z$ huiusmodi forma exprimetur:

$$S.p^x z = \frac{p^{x+1}z}{p-1} - \frac{\alpha p^{x+1}}{p-1}.\frac{dz}{dx} + \frac{\beta p^{x+1}}{p-1}.\frac{ddz}{dx^2} - \frac{\gamma p^{x+1}}{p-1}.\frac{d^3z}{dx^3} + \frac{\delta p^{x+1}}{p-1}.\frac{d^4z}{dx^4} - \frac{\varepsilon p^{x+1}}{p-1}.\frac{d^5z}{dx^5} + \&c.$$

171. Ad valores litterarum α, β, γ, δ, ε, &c. definiendos, substituantur pro quovis termino series ante inventae nempe:

$$\frac{p^{x+1}z}{p-1} = S.p^x z + \frac{1}{p-1}S.\frac{p^x dz}{dx} - \frac{1}{2(p-1)}S.\frac{p^x ddz}{dx^2} + \frac{1}{6(p-1)}S.\frac{p^x d^3z}{dx^3} - \&c.$$

$$\frac{p^{x+1}dz}{(p-1)dx} = S.\frac{p^x dz}{dx} + \frac{1}{p-1}S.\frac{p^x ddz}{dx^2} - \frac{1}{2(p-1)}S.\frac{p^x d^3z}{dx^3} + \&c.$$

$$\frac{p^{x+1}ddz}{(p-1)dx^2} = S.\frac{p^x ddz}{dx^2} + \frac{1}{p-1}S.\frac{p^x d^3z}{dx^3} - \&c.$$

$$\frac{p^{x+1}d^3z}{(p-1)dx^3} = S.\frac{p^x d^3z}{dx^3} + \&c.$$

Habebimus ergo:

$$
\begin{array}{llllll}
S.\,p^x z = & & & & & \\
S.\,p^x z + \frac{1}{p-1}S.\frac{p^x dz}{dx} & - \frac{1}{2(p-1)}S.\frac{p^x ddz}{dx^2} & + \frac{1}{6(p-1)}S.\frac{p^x d^3 z}{dx^3} & - \frac{1}{24(p-1)}S.\frac{p^x d^4 z}{dx^4} & + \&c. \\
\quad - \alpha & - \frac{\alpha}{p-1} & + \frac{\alpha}{2(p-1)} & - \frac{\alpha}{6(p-1)} & \\
 & + \beta & + \frac{\beta}{p-1} & - \frac{\beta}{2(p-1)} & \\
 & & - \gamma & - \frac{\gamma}{p-1} & \\
 & & & + \delta &
\end{array}
$$

unde coefficientium α, β, γ, δ, &c. valores sequentes obtinebuntur.

$$
\begin{aligned}
\alpha &= \frac{1}{p-1} \\
\beta &= \frac{1}{p-1}\left(\alpha + \frac{1}{2}\right) \\
\gamma &= \frac{1}{p-1}\left(\beta + \frac{\alpha}{2} + \frac{1}{6}\right) \\
\delta &= \frac{1}{p-1}\left(\gamma + \frac{\beta}{2} + \frac{\alpha}{6} + \frac{1}{24}\right) \\
\varepsilon &= \frac{1}{p-1}\left(\delta + \frac{\gamma}{2} + \frac{\beta}{6} + \frac{\alpha}{24} + \frac{1}{120}\right) \qquad \&c.
\end{aligned}
$$

172. Sit brevitatis gratia $\frac{1}{p-1} = q$, erit:

$$
\begin{aligned}
\alpha &= q \\
\beta &= \alpha q + \tfrac{1}{2} q = qq + \tfrac{1}{2} q \\
\gamma &= \beta q + \tfrac{1}{2}\alpha q + \tfrac{1}{6} q = q^3 + qq + \tfrac{1}{6} q \\
\delta &= \gamma q + \tfrac{1}{2}\beta q + \tfrac{1}{6}\alpha q + \tfrac{1}{24} q = q^4 + \tfrac{3}{2} q^3 + \tfrac{7}{12} q^2 + \tfrac{1}{24} q \\
\varepsilon &= \delta q + \tfrac{1}{2}\gamma q + \tfrac{1}{6}\beta q + \tfrac{1}{24}\alpha q + \tfrac{1}{120} q \qquad \&
\end{aligned}
$$

seu $\varepsilon = q^5 + 2q^4 + \tfrac{5}{4} q^3 + \tfrac{1}{4} q^2 + \tfrac{1}{120} q$ &c.

$\zeta = q^6 + \tfrac{5}{2} q^5 + \tfrac{13}{6} q^4 + \tfrac{3}{4} q^3 + \tfrac{31}{360} q^2 + \tfrac{1}{720} q$

seu

seu hoc modo exprimantur:

$$\alpha = \frac{q}{1}$$

$$\beta = \frac{2qq + q}{1.\,2}$$

$$\gamma = \frac{6q^3 + 6q^2 + q}{1.\,2.\,3}$$

$$\delta = \frac{24q^4 + 36q^3 + 14q^2 + q}{1.\,2.\,3.\,4}$$

$$\varepsilon = \frac{120q^5 + 240q^4 + 150q^3 + 30q^2 + q}{1.\,2.\,3.\,4.\,5}$$

$$\zeta = \frac{720q^6 + 1800q^5 + 1560q^4 + 540q^3 + 62q^2 + q}{1.\,2.\,3.\,4.\,5.\,6}$$

$$\eta = \frac{5040q^7 + 15120q^6 + 16800q^5 + 8400q^4 + 1806q^3 + 126q^2 + q}{1.\,2.\,3.\,4.\,5.\,6.\,7}$$

&c.

ubi quilibet coefficiens 16800 oritur, si summa binorum superiorum 1560 + 1800 per exponentem ipsius q, qui hic est 5, multiplicetur.

173. Restituamus autem loco q valorem $\frac{1}{p-1}$,

$$\alpha = \frac{1}{1(p-1)}$$

$$\beta = \frac{p+1}{1.2\,(p-1)^2}$$

$$\gamma = \frac{pp + 4p + 1}{1.2.3\,(p-1)^3}$$

$$\delta = \frac{p^3 + 11p^2 + 11p + 1}{1.\,2.\,3.\,4\,(p-1)^4}$$

$$\varepsilon = \frac{p^4 + 26p^3 + 66p^2 + 26p + 1}{1.\,2.\,3.\,4.\,5\,(p-1)^5}$$

$\zeta =$

$$\zeta = \frac{p^5 + 57p^4 + 302p^3 + 302p^2 + 57p + 1}{1.2.3.4.5.6(p-1)^6}$$

$$\eta = \frac{p^6 + 120p^5 + 1191p^4 + 2416p^3 + 1191p^2 + 120p + 1}{1.2.3.4.5.6.7(p-1)^7}$$

&c.

Lex harum quantitatum ita se habet, ut si ponatur terminus quicunque:

$$\frac{p^{n-2} + Ap^{n-3} + Bp^{n-4} + Cp^{n-5} + Dp^{n-6} + \&c.}{1.2.3 \ldots\ldots (n-1)(p-1)^{n-1}}$$

futurum sit:

$$A = 2^{n-1} - n$$

$$B = 3^{n-1} - n.2^{n-1} + \frac{n(n-1)}{1.2}$$

$$C = 4^{n-1} - n.3^{n-1} + \frac{n(n-1)}{1.2}2^{n-1} - \frac{n(n-1)(n-2)}{1.2.3}$$

$$D = 5^{n-1} - n.4^{n-1} + \frac{n(n-1)}{1.2}3^{n-1} - \frac{n(n-1)(n-2)}{1.2.3}2^{n-1} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$$

&c.

unde isti coefficientes α, β, γ, δ, &c. quousque libuerit, continuari possunt.

174. Quodsi vero legem, qua hi coefficientes inter se cohaerent, consideremus, facile patet, eos seriem recurrentem constituere, atque prodire si haec fractio evolvatur:

$$\frac{1}{1 - \frac{u}{p-1} - \frac{u^2}{2(p-1)} - \frac{u^3}{6(p-1)} - \frac{u^4}{24(p-1)} - \&c.}$$

prodibit enim haec series:

$$1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \&c.$$

Ponatur illa fractio $= V$, & cum sit:

$$V = \frac{p-1}{p - 1 - u - \frac{u^2}{2} - \frac{u^3}{6} - \frac{u^4}{24} - \&c.}$$

erit $V = \frac{p-1}{p-e^u}$; ubi e est numerus cuius logarithmus hyperbolicus est $= 1$.

Atque si valor ipsius V per seriem exprimatur secundum potestates ipsius u, orietur:

$$V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \&c.$$

cuius coefficientes α, β, γ, δ, &c. erunt ii ipsi, quorum in praesenti negotio opus habemus. Iis igitur inventis erit:

$$S. p^x z = \frac{p^{x+1}}{p-1}\left(z - \frac{\alpha dz}{dx} + \frac{\beta ddz}{dx^2} - \frac{\gamma d^3 z}{dx^3} + \frac{\delta d^4 z}{dx^4} - \&c.\right)$$
$$\pm \text{Const.}$$

quae ergo expressio est terminus summatorius seriei huius:

$$ap + bp^2 + cp^3 + \ldots\ldots + p^x z$$

cuius terminus generalis est $= p^x z$.

175. Quoniam invenimus esse $V = \frac{p-1}{p-e^u}$, erit $e^u = \frac{pV - p + 1}{V}$, & logarithmis sumendis fiet $u = l(pV - p + 1) - lV$, hincque; differentiando $du = \frac{(p-1)dV}{pV^2 - (p-1)V}$ quocirca erit $pV^2 = (p-1)V + \frac{(p-1)dV}{du}$. Quoniam ergo est $V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \&c.$ erit:

$$\begin{aligned} pV^2 = p &+ 2\alpha pu + 2\beta pu^2 + 2\gamma pu^3 + 2\delta pu^4 + 2\varepsilon pu^5 + \&c. \\ &+ \alpha^2 pu^2 + 2\alpha\beta pu^3 + 2\alpha\gamma pu^4 + 2\alpha\delta pu^5 + \&c. \\ &+ \beta^2 pu^4 + 2\beta\gamma pu^5 + \&c. \end{aligned}$$

$$\begin{aligned}(p-1)V = p - 1 &+ \alpha(p-1)u + \beta(p-1)u^2 + \gamma(p-1)u^3 \\ &+ \delta(p-1)u^4 + \varepsilon(p-1)u^5 + \&c.\end{aligned}$$

$$\begin{aligned}\frac{(p-1)dV}{du} = (p-1)\alpha &+ 2(p-1)\beta u + 3(p-1)\gamma u^2 + 4(p-1)\delta u^3 \\ &+ 5(p-1)\varepsilon u^4 + 6(p-1)\zeta u^5 + \&c.\end{aligned}$$

quibus expressionibus inter se coaequatis reperietur:

(p

$$
\begin{aligned}
(p-1)\,\alpha &= 1\\
2\,(p-1)\,\beta &= \alpha\,(p+1)\\
3\,(p-1)\,\gamma &= \beta\,(p+1)+\alpha^2 p\\
4\,(p-1)\,\delta &= \gamma\,(p+1)+2\alpha\beta p\\
5\,(p-1)\,\varepsilon &= \delta\,(p+1)+2\alpha\gamma p+\beta^2 p\\
6\,(p-1)\,\zeta &= \varepsilon\,(p+1)+2\alpha\delta p+2\beta\gamma p\\
7\,(p-1)\,\eta &= \zeta\,(p+1)+2\alpha\varepsilon p+2\beta\delta p+\gamma^2 p\\
&\&c.
\end{aligned}
$$

ex quibus formulis, si pro p datus numerus assumatur, valores coefficientium α, β, γ, δ, &c. facilius determinari possunt, quam ex lege primum inventa.

176. Antequam ad casus speciales ratione valoris ipsius p descendamus, ponamus esse $z = x^n$, ita ut haec series summari debeat:

$$s = p + 2^n p^2 + 3^n p^3 + 4^n p^4 + \ldots + x^n p^x$$

eritque per expressionem ante inventam:

$$s = p^x\left(\frac{p}{p-1}\cdot x^n - \frac{p}{(p-1)^2}nx^{n-1} + \frac{pp+p}{(p-1)^3}\cdot\frac{n(n-1)}{1\cdot 2}x^{n-2} - \frac{(p^3+4p^2+p)}{(p-1)^4}\cdot\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3} + \&c.\right)$$

$\pm C$, quae reddat $s = 0$ si ponatur $x = 0$.

Hinc ponendo pro n successive numeros $0, 1, 2, 3, 4,$ &c. erit:

$$S.x^0p^x = p^x\cdot\frac{p}{p-1} - \frac{p}{p-1}$$

$$S.x^1p^x = p^x\left(\frac{px}{p-1} - \frac{p}{(p-1)^2}\right) + \frac{p}{(p-1)^2}$$

$$S.x^2p^x = p^x\left(\frac{px^2}{p-1} - \frac{2px}{(p-1)^2} + \frac{p(p+1)}{(p-1)^3}\right) - \frac{p(p+1)}{(p-1)^3}$$

$$S.x^3p^x = p^x\left(\frac{px^3}{p-1} - \frac{3px^2}{(p-1)^2} + \frac{3p(p+1)x}{(p-1)^3} - \frac{p(p^2+4p+1)}{(p-1)^4}\right) + \frac{p(p^2+4p+1)}{(p-1)^4}$$

S.x^4

$$S.x^4p^x = p^x\left(\frac{px^4}{p-1} - \frac{4px^3}{(p-1)^2} + \frac{6p(p+1)x^2}{(p-1)^3} - \frac{4p(p^2+4p+1)x}{(p-1)^4} + \frac{p(p^3+11p^2+11p+1)}{(p-1)^5}\right) - \frac{p(p^3+11p^2+11p+1)}{(p-1)^5}$$

$$S.x^5p^x = \frac{p^{x+1}x^5}{p-1} - \frac{5p^{x+1}x^4}{(p-1)^2} + \frac{10(p+1)p^{x+1}x^3}{(p-1)^3} - \frac{10(p^2+4p+1)p^{x+1}x^2}{(p-1)^4} + \frac{5(p^3+11p^2+11p+1)p^{x+1}x}{(p-1)^5} - \frac{(p^4+26p^3+66p^2+26p+1)(p^{x+1}-p)}{(p-1)^6}$$

$$S.x^6p^x = \frac{p^{x+1}x^6}{p-1} - \frac{6p^{x+1}x^5}{(p-1)^2} + \frac{15(p+1)p^{x+1}x^4}{(p-1)^3} - \frac{20(p^2+4p+1)p^{x+1}x^3}{(p-1)^4} + \frac{15(p^3+11p^2+11p+1)p^{x+1}x^2}{(p-1)^5} - \frac{6(p^4+26p^3+66p^2+26p+1)p^{x+1}x}{(p-1)^6} + \frac{(p^5+57p^4+302p^3+302p^2+57p+1)(p^{x+1}-p)}{(p-1)^7}$$
&c.

177. Hinc intelligitur, quoties z fuerit functio rationalis integra ipsius x, toties seriei, cuius terminus generalis est $p^x z$, summam exhiberi posse; propterea quod differentialia ipsius z sumendo, tandem ad evanescentia perveniatur. Ita si proponatur haec series:

$$p + 3p^2 + 6p^3 + 10p^4 + \ldots + \frac{(xx+x)}{2}p^x,$$

ob $z = \frac{xx+x}{2}$, & $\frac{dz}{dx} = x + \frac{1}{2}$; atque $\frac{ddz}{dx^2} = 1$;

erit terminus summatorius:

$$s = \frac{p^{x+1}}{p-1}\left(\tfrac{1}{2}xx + \tfrac{1}{2}x - \frac{2x+1}{2(p-1)} + \frac{p+1}{2(p-1)^2}\right) - \frac{p}{p-1}\left(\frac{p+1}{2(p-1)^2} - \frac{1}{2(p-1)}\right)$$

seu $s = p^{x+1}\left(\frac{xx}{2(p-1)} + \frac{(p-3)x}{2(p-1)^2} + \frac{1}{(p-1)^3}\right) - \frac{p}{(p-1)^3}$.
Sin autem z fuerit functio non rationalis integra, tum ista termini summatorii expressio in infinitum excurret. Ita si sit $z = \frac{1}{x}$, ut summanda sit haec series:

$$s = p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \frac{1}{4}p^4 + \dots + \frac{1}{x}p^x,$$ ob

$$\frac{dz}{dx} = -\frac{1}{xx};\ \frac{ddz}{dx^2} = \frac{2}{x^3};\ \frac{d^3z}{dx^3} = -\frac{2.3}{x^4};\ \frac{d^4z}{dx^4} = \frac{2.3.4}{x^5};\ \&c.$$

prodibit terminus summatorius:

$$s = \frac{p^{x+1}}{p-1}\left(\frac{1}{x} + \frac{1}{(p-1)x^2} + \frac{p+1}{(p-1)^2x^3} + \frac{pp+4p+1}{(p-1)^3x^4} + \frac{p^3+11p^2+11p+1}{(p-1)^4x^5} + \&c.\right) + C.$$

Hoc ergo casu constans C non ex casu $x = 0$ definiri potest: ad eam igitur definiendam ponatur $x = 1$, & quia fit $s = p$, erit:

$$C = p - \frac{pp}{p-1}\left(1 + \frac{1}{p-1} + \frac{p+1}{(p-1)^2} + \frac{pp+4p+1}{(p-1)^3} + \&c.\right)$$

178. Ex his perspicuum est, nisi p determinatum numerum significet, parum utilitatis hinc ad summas serierum proxime exhibendas redundare. Primum autem patet pro p non posse scribi 1, propterea quod omnes coefficientes α, β, γ, δ, &c. fierent infinite magni. Quare cum series, quam nunc tractamus, abeat in eam quam ante iam sumus contemplati si ponatur $p = 1$, mirum est, quod ille casus tamquam facillimus ex hoc erui nequeat. Tum vero quoque notabile est, quod casu $p = 1$ summatio requirat integrale $\int z dx$, cum tamen generaliter summa sine ullo integrali exhiberi queat. Sic igitur fit, ut dum omnes coefficientes α, β, γ, δ, &c. in infinitum excrescunt, simul formula illa integralis invehatur. Hicque adeo casus, quo $p = 1$, est solus, ad quem

quem generalis expressio hic inventa applicari nequeat. Neque vero hoc casu generalis forma a vero recedere censenda est; nam etsi singuli termini fiunt infiniti, tamen revera omnia infinita se destruunt, restatque quantitas finita summae aequalis, & congruens cum ea, quae per priorem methodum invenitur, quod infra fusius sumus declaraturi.

179. Sit igitur $p = -1$, atque signa in serie summanda alternatim se excipient:

$$\overset{1}{-a} + \overset{2}{b} - \overset{3}{c} + \overset{4}{d} \ldots \ldots \ldots \pm \overset{x}{z}$$

ubi z erit affirmativum si x fuerit numerus par, negativum autem, si x sit numerus impar. Posito ergo

$$-a + b - c + d \ldots \ldots \pm z = s, \text{ erit}$$

$$s = \frac{\pm 1}{2}\left(z - \frac{\alpha dz}{dx} + \frac{\beta ddz}{dx^2} - \frac{\gamma d^3 z}{dx^3} + \frac{\delta d^4 z}{dx^4} - \&c.\right) + C.$$

ubi signorum ambiguorum superius valet, si x sit numerus par, contra vero si x sit numerus impar. Mutandis ergo signis erit:

$$a - b + c - d + e - f + \ldots \mp z =$$

$$\mp \frac{1}{2}\left(z - \frac{\alpha dz}{dx} + \frac{\beta ddz}{dx^2} - \frac{\gamma d^3 z}{dx^3} + \frac{\delta d^4 z}{dx^4} - \&c.\right) + C.$$

ubi signorum ambiguitas eandem sequitur legem.

180. Hoc casu coefficientes α, β, γ, δ, ε, ζ, &c. inveniri possunt ex valoribus ante traditis ponendo ubique $p = -1$. Facilius autem eruentur ex formulis generalibus §. 175. datis, ex quibus simul perspicietur alternos istos coefficientes evanescere. Facto enim $p = -1$ istae formulae abibunt in

$$\begin{aligned}
-2\alpha &= 1 \\
-4\beta &= 0 \\
-6\gamma &= 0 - \alpha^2 \\
-8\delta &= 0 - 2\alpha\beta \\
-10\varepsilon &= 0 - 2\alpha\gamma - \beta\beta \\
-12\zeta &= 0 - 2\alpha\delta - 2\beta\gamma \qquad \&c.
\end{aligned}$$

un-

unde cum sit $\beta = 0$, erit quoque $\delta = 0$, porroque $\zeta = 0$, $\theta = 0$, &c. & reliquae litterae ita determinabuntur, ut sit:

$$\alpha = -\frac{1}{2}; \quad \gamma = \frac{\alpha^2}{6}; \quad \varepsilon = \frac{2\alpha\gamma}{10};$$

$$\eta = \frac{2\alpha\varepsilon + \gamma\gamma}{14}; \quad \iota = \frac{2\alpha\eta + 2\gamma\varepsilon}{18}; \quad \&c.$$

181. Quo iste calculus commodius absolvi possit introducamus novas litteras sitque:

$$\alpha = -\frac{A}{1.2}; \quad \gamma = \frac{B}{1.2.3.4}; \quad \varepsilon = -\frac{C}{1.2.3.4.5.6};$$

$$\eta = \frac{D}{1.2.3\ldots 8}; \quad \iota = -\frac{E}{1.2.3\ldots 10}; \quad \&c.$$

Eritque summa ante exhibita:

$$\mp \frac{1}{2}\left(z + \frac{A\,dz}{1.2\,dx} - \frac{B\,d^3z}{1.2.3.4\,dx^3} + \frac{C\,d^5z}{1.2\ldots 6\,dx^5} - \frac{D\,d^7z}{1.2\ldots 8\,dx^7} + \&c.\right)$$
$$+ \text{Const.}$$

Coefficientes vero ex sequentibus formulis definientur:

$$A = 1$$

$$3B = \frac{4.3}{1.2}\cdot\frac{AA}{2}$$

$$5C = \frac{6.5}{1.2}\,AB$$

$$7D = \frac{8.7}{1.2}\,AC + \frac{8.7.6.5}{1.2.3.4}\cdot\frac{BB}{2}$$

$$9E = \frac{10.9}{1.2}\,AD + \frac{10.9.8.7}{1.2.3.4}\cdot BC$$

$$11F = \frac{12.11}{1.2}\,AE + \frac{12.11.10.9}{1.2.3.4}\cdot BD + \frac{12.11.10.9.8.7}{1.2.3.4.5.6}\cdot\frac{CC}{2}$$

&c.

quae hoc modo facilius atque ad calculum accommodatius repraesentari possunt:

A

$$A = 1$$
$$B = 2.\frac{AA}{2}$$
$$C = 3.AB$$
$$D = 4.AC + 4.\frac{6.5}{3.4}.\frac{BB}{2}$$
$$E = 5.AD + 5.\frac{8.7}{3.4}.BC$$
$$F = 6.AE + 6.\frac{10.9}{3.4}.BD + 6.\frac{10.9.8.7}{3.4.5.6}.\frac{CC}{2}$$
$$G = 7.AF + 7.\frac{12.11}{3.4}.BE + 7.\frac{12.11.10.9}{3.4.5.6}.CD$$

&c.

Hinc igitur calculo inſtituto reperietur:

$$A = 1$$
$$B = 1$$
$$C = 3$$
$$D = 17$$
$$E = 155 = 5.31$$
$$F = 2073 = 691.3$$
$$G = 38227 = 7.5461 = 7.\frac{127.129}{3}$$
$$H = 929569 = 3617.257$$
$$I = 28820619 = 43867.9.73 \quad \&c.$$

182. Si hos numeros attentius perpendamus, ex factoribus 691, 3617, 43867, facile concludere licet, hos numeros cum ſupra exhibitis Bernoullianis nexum habere, indeque determinari poſſe. Hanc igitur relationem inveſtiganti mox patebit hos numeros ex Bernoullianis 𝔄, 𝔅, ℭ, 𝔇, 𝔈, &c. ſequenti modo formari poſſe:

A

$$A = 2.\ 1.\ 3\ \mathfrak{A} = 2(2^2-1)\mathfrak{A}$$
$$B = 2.\ 3.\ 5\ \mathfrak{B} = 2(2^4-1)\mathfrak{B}$$
$$C = 2.\ 7.\ 9\ \mathfrak{C} = 2(2^6-1)\mathfrak{C}$$
$$D = 2.\ 15.\ 17\ \mathfrak{D} = 2(2^8-1)\mathfrak{D}$$
$$E = 2.\ 31.\ 33\ \mathfrak{E} = 2(2^{10}-1)\mathfrak{E}$$
$$F = 2.\ 63.\ 65\ \mathfrak{F} = 2(2^{12}-1)\mathfrak{F}$$
$$G = 2.127.129\ \mathfrak{G} = 2(2^{14}-1)\mathfrak{G}$$
$$H = 2.255.257\ \mathfrak{H} = 2(2^{16}-1)\mathfrak{H}$$

&c.

Cum igitur numeri Bernoulliani sint fracti, coefficientes vero nostri integri, patet hos factores semper tollere fractiones; eruntque ergo:

$$A = 1$$
$$B = 1$$
$$C = 3$$
$$D = 17$$
$$E = 5.31 = 155$$
$$F = 3.691 = 2073$$
$$G = 7.43.127 = 38227$$
$$H = 257.3617 = 929569$$
$$I = 9.73.43867 = 28820619$$
$$K = 5.31.41.174611 = 1109652905$$
$$L = 89.683.854513 = 51943281731$$
$$M = 3.4097.236364091 = 2905151042481$$
$$N = 2731.8191.8553103 = 191329672483963$$

&c.

Ex his ergo numeris integris vicissim numeri Bernoulliani inveniri poterunt.

183. Adhibendo igitur numeros Bernoullianos seriei propositae:

$$\overset{1}{a} - \overset{2}{b} + \overset{3}{c} - \overset{4}{d} + \overset{5}{e} - \ldots \mp \overset{x}{z},$$

summa erit:

$$\mp\left(\frac{1}{2}z + \frac{(2^2-1)\mathfrak{A}dz}{1.2dx} - \frac{(2^4-1)\mathfrak{B}d^3z}{1.2.3.4dx^3} + \frac{(2^6-1)\mathfrak{C}d^5z}{1.2\ldots6dx^5} - \frac{(2^8-1)\mathfrak{D}d^7z}{1.2\ldots8dx^7} + \&c.\right)$$
$$+ \text{Const.}$$

Hinc

Hinc autem perspicitur istos numeros non casu in hanc expressionem ingredi; quemadmodum enim series proposita oritur, si ab ista: $a + b + c + d + \ldots\ldots + z$, ubi omnes termini signum habent $+$ subtrahatur summa alternorum $b + d + f +$ &c. bis sumta; ita quoque expressio inventa in duas resolvi potest partes, quarum altera est summa omnium terminorum signo $+$ affectorum, quae erit:

$$\int z dx + \frac{1}{2} z + \frac{\mathfrak{A} dz}{1.2 dx} - \frac{\mathfrak{B} d^3 z}{1.2.3.4 dx^3} + \frac{\mathfrak{C} d^5 z}{1.2 \ldots 6 dx^5} - \&c.$$

Summa vero alternorum pari modo invenietur, quo supra usi sumus. Cum enim ultimus terminus sit z indici x respondens, antecedens indici $x - 2$ respondens erit;

$$z - \frac{2dz}{dx} + \frac{2^2 ddz}{1.2 dx^2} - \frac{2^3 d^3 z}{1.2.3 dx^3} + \frac{2^4 d^4 z}{1.2.3.4 dx^4} - \&c.$$

quae forma ex illa, qua ante terminus antecedens exprimebatur, oritur, si loco x scribatur $\frac{x}{2}$. Habebitur ergo summa alternorum, si in summa omnium ubique loco x scribatur $\frac{x}{2}$, quae propterea erit:

$$\frac{1}{2} \int z dx + \frac{1}{2} z + \frac{2 \mathfrak{A} dz}{1.2 dx} - \frac{2^3 \mathfrak{B} d^3 z}{1.2.3.4 dx^3} + \frac{2^5 \mathfrak{C} d^5 z}{1.2 \ldots 6 dx^5} - \&c.$$

cuius duplum si a summa praecedente subtrahatur, existente x numero pari, vel si praecedens summa a duplo huius si x est numerus impar subtrahatur, residuum ostendet summam seriei:

$$\overset{1}{a} - \overset{2}{b} + \overset{3}{c} - \overset{4}{d} + \overset{5}{e} \;\ldots\ldots\; \mp \overset{x}{z}$$

quae ergo erit:

$$\mp \left(\frac{1}{2} z + \frac{(2^2 - 1) \mathfrak{A} dz}{1.2 dx} - \frac{(2^4 - 1) \mathfrak{B} d^3 z}{1.2.3.4 dx^3} + \&c. \right) + C.$$

quae est eadem expressio, quam modo inveneramus.

184.

184. Sumatur pro z poteſtas ipſius x, nempe x^n, ut reperiatur ſumma feriei:

$$1 - 2^n + 3^n - 4^n + \ldots \mp x^n.$$

Ob $\frac{dz}{1\,dx} = \frac{n}{1} x^{n-1}$; $\frac{d^3 z}{1.2.3\,dx^3} = \frac{n(n-1)(n-2)}{1.\ 2.\ 3} x^{n-3}$; &c. erit adhibendis coefficientibus A, B, C, &c. ſumma quaeſita:

$$\mp \frac{1}{2}\Big(x^n + \frac{A}{2} n x^{n-1} - \frac{B}{4}\cdot\frac{n(n-1)(n-2)}{1.\ 2.\ 3} x^{n-3} + \frac{C}{6}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)}{1.\ 2.\ 3.\ 4.\ 5} x^{n-5}$$
$$- \frac{D}{8}\cdot\frac{n(n-1)\ldots\ldots(n-6)}{1.\ 2\ldots\ldots 7} x^{n-7} + \&c.\Big) + \text{Conſt.}$$

ubi ſignum ſuperius valet ſi ſit x numerus par, inferius vero ſi impar. Conſtans autem ita definiri debet, ut ſumma evaneſcat, ſi $x = 0$, quo caſu ſignum ſuperius valet. Pro n ergo ſucceſſive numeros 0, 1, 2, 3, &c. ſubſtituendo ſequentes prodibunt ſummationes:

I. $1 - 1 + 1 - 1 + \ldots \mp 1 = \mp \frac{1}{2}(1) + \frac{1}{2}$
ſcilicet ſi numerus terminorum fuerit par, ſumma erit $= 0$, ſin impar erit $= +1$.

II. $1 - 2 + 3 - 4 + \ldots \mp x = \mp \frac{1}{2}(x + \frac{1}{2}) + \frac{1}{4}$
ſcilicet ſi numerus terminorum ſit par, ſumma erit $= -\frac{1}{2}x$
& pro numero terminorum impari $= +\frac{1}{2}x + \frac{1}{2}$.

III. $1 - 2^2 + 3^2 - 4^2 + \ldots \mp x^2 = \mp \frac{1}{2}(x^2 + x)$
ſcilicet pro pari numero $= -\frac{1}{2}xx - \frac{1}{2}x$
& pro impari numero $= +\frac{1}{2}xx + \frac{1}{2}x$

IV. $1 - 2^3 + 3^3 - 4^3 + \ldots \mp x^3 = \mp \frac{1}{2}(x^3 + \frac{3}{2}xx - \frac{1}{4}) - \frac{1}{8}$
ſcilicet pro pari $= -\frac{1}{2}x^3 - \frac{3}{4}x^2$
& pro impari $= \frac{1}{2}x^3 + \frac{3}{4}x^2 - \frac{1}{4}$.

V. $1 - 2^4 + 3^4 - 4^4 + \ldots \mp x^4 = \mp \frac{1}{2}(x^4 + 2x^3 - x)$
ſcilicet pro numero pari $= -\frac{1}{2}x^4 - x^3 + \frac{1}{2}x$
& pro numero impari $= \frac{1}{2}x^4 + x^3 - \frac{1}{2}x$

&c.

185. Apparet ergo in poteſtatibus paribus praeter $n = 0$, conſtantem adiiciendam evaneſcere, hiſque caſibus ſum-

summam terminorum numero sive parium sive imparium tantum ratione signi discrepare. Quodsi ergo n fuerit numerus infinitus, quoniam is est neque par neque impar, haec consideratio cessare debet, ac propterea in summa termini ambigui sunt reiiciendi: unde sequitur huiusmodi serierum in infinitum continuatarum summam exprimi per solam quantitatem constantem adiiciendam. Hancobrem erit:

$$1 - 1 + 1 - 1 + \&c. \text{ in infinitum} = \frac{1}{2}$$

$$1 - 2 + 3 - 4 + \&c. \ldots = \frac{A}{4} = + \frac{(2^2-1)\mathfrak{A}}{2}$$

$$1 - 2^2 + 3^2 - 4^2 + \&c. \ldots = 0$$

$$1 - 2^3 + 3^3 - 4^3 + \&c. \ldots = - \frac{B}{8} = - \frac{(2^4-1)\mathfrak{B}}{4}$$

$$1 - 2^4 + 3^4 - 4^4 + \&c. \ldots = 0$$

$$1 - 2^5 + 3^5 - 4^5 + \&c. \ldots = \frac{C}{12} = + \frac{(2^6-1)\mathfrak{C}}{6}$$

$$1 - 2^6 + 3^6 - 4^6 + \&c. \ldots = 0$$

$$1 - 2^7 + 3^7 - 4^7 + \&c. \ldots = - \frac{D}{16} = - \frac{(2^8-1)\mathfrak{D}}{8}$$

&c.

Quae eaedem summae per methodum supra traditam series, in quibus signa + & − alternantur, summandi inveniuntur.

186. Si pro n statuantur numeri negativi, expressio summae in infinitum excurret. Sit $n = -1$, erit summa seriei:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \mp \frac{1}{x} =$$

$$\mp \frac{1}{2}\left(\frac{1}{x} - \frac{A}{2x^2} + \frac{B}{4x^4} - \frac{C}{6x^6} + \frac{D}{8x^8} - \&c.\right) + \text{Const.}$$

Hic autem quia constans non ex casu $x = 0$ definiri potest, ex ex alio casu erit definienda. Ponatur $x = 1$, atque ob summam $= 1$ & signum inferius erit:

$$\text{Const.} = 1 - \frac{1}{2}\left(\frac{1}{1} - \frac{A}{2} + \frac{B}{4} - \frac{C}{6} + \&c.\right) \quad \text{seu}$$

$$\text{Const.} = \frac{1}{2} + \frac{A}{4} - \frac{B}{8} + \frac{C}{12} - \frac{D}{16} + \&c.$$

Vel ponatur $x = 2$, ob summam $= \frac{1}{2}$, & signum superius reperietur:

$$\text{Const.} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{A}{2 \cdot 2^2} + \frac{B}{4 \cdot 2^4} - \frac{C}{6 \cdot 2^6} + \&c.\right)$$

$$\text{seu} \quad \text{Const.} = \frac{3}{4} - \frac{A}{4 \cdot 2^2} + \frac{B}{8 \cdot 2^4} - \frac{C}{12 \cdot 2^6} + \frac{D}{16 \cdot 2^8} - \&c.$$

sin autem ponatur $x = 4$, erit:

$$\text{Const.} = \frac{17}{24} - \frac{A}{4 \cdot 4^2} + \frac{B}{8 \cdot 4^4} - \frac{C}{12 \cdot 4^6} + \frac{D}{16 \cdot 4^8} - \&c.$$

Utcumque autem constans definiatur, idem prodibit valor, qui simul summam seriei in infinitum continuatae, quae est $= l\,2$, indicabit.

187. Ceterum ex his novis numeris A, B, C, D, E, &c. summae serierum potestatum reciprocarum parium, in quibus tantum numeri impares occurrunt, commode summari poterunt. Si enim ponatur:

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \&c. = s \quad \text{erit}$$

$$\frac{1}{2^{2n}} + \frac{1}{4^{2n}} + \frac{1}{6^{2n}} + \&c. = \frac{s}{2^{2n}},$$

quae ab illa subtracta relinquet:

$$1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \&c. = \frac{(2^{2n} - 1)s}{2^{2n}}.$$

Cum

Cum igitur valores ipsius s pro singulis numeris n iam supra exhibuerimus: (125), erit:

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\&c.=\frac{A}{1.2}\cdot\frac{\pi^2}{4}$$

$$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\&c.=\frac{B}{1.2.3.4}\cdot\frac{\pi^4}{4}$$

$$1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\&c.=\frac{C}{1.2.3\ldots6}\cdot\frac{\pi^6}{4}$$

$$1+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\&c.=\frac{D}{1.2.3\ldots8}\cdot\frac{\pi^8}{4}$$

$$1+\frac{1}{3^{10}}+\frac{1}{5^{10}}+\frac{1}{7^{10}}+\&c.=\frac{E}{1.2.3\ldots10}\cdot\frac{\pi^{10}}{4}$$

&c.

Sin autem omnes numeri ingrediantur, signaque alternentur quia erit:

$$1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\&c.=\frac{(2^{2n}-1)s-s}{2^{2n}}$$

habebitur:

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\&c.=\frac{(A-2\mathfrak{A})}{1.2}\cdot\frac{\pi^2}{4}=\frac{(2-1)\mathfrak{A}}{1.2}\cdot\pi^2$$

$$1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\&c.=\frac{(B-2\mathfrak{B})}{1.2.3.4}\cdot\frac{\pi^4}{4}=\frac{(2^3-1)\mathfrak{B}}{1.2.3.4}\cdot\pi^4$$

$$1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\&c.=\frac{(C-2\mathfrak{C})}{1.2\ldots6}\cdot\frac{\pi^6}{4}=\frac{(2^5-1)\mathfrak{C}}{1.2\ldots6}\cdot\pi^6$$

$$1-\frac{1}{2^8}+\frac{1}{3^8}-\frac{1}{4^8}+\frac{1}{5^8}-\&c.=\frac{(D-2\mathfrak{D})}{1.2\ldots8}\cdot\frac{\pi^8}{4}=\frac{(2^7-1)\mathfrak{D}}{1.2\ldots8}\cdot\pi^8$$

$$1-\frac{1}{2^{10}}+\frac{1}{3^{10}}-\frac{1}{4^{10}}+\frac{1}{5^{10}}-\&c.=\frac{(E-2\mathfrak{E})}{1.2\ldots10}\cdot\frac{\pi^{10}}{4}=\frac{(2^9-1)\mathfrak{E}}{1.2\ldots10}\cdot\pi^{10}$$

&c.

188. Quemadmodum hactenus seriem sumus contemplati, cuius termini erant producta ex terminis progressionis

geometricae p, p^2, p^3, &c. & ex terminis seriei cuiuscunque a, b, c, &c. ita poterimus simili ratione prosequi seriem, cuius termini sint producta ex terminis duarum quarumcunque serierum, quarum altera tanquam cognita assumatur.

Sit series cognita: $\overset{1}{A}+\overset{2}{B}+\overset{3}{C}+ \ldots + \overset{x}{Z}$

altera vero incognita $a+b+c+ \ldots +z$

atque quaeratur summa huius seriei:

$Aa+Bb+Cc+ \ldots + Zz$

quae ponatur $= Zs$. Sit in serie cognita terminus penultimus $= Y$, atque posito $x-1$ loco x expressio summae $S.Zs$ abibit in

$$Y\left(s-\frac{ds}{dx}+\frac{dds}{2dx^2}-\frac{d^3s}{6dx^3}+\frac{d^4s}{24dx^4}-\&c.\right)$$

Quae cum exprimat summam seriei Zs termino ultimo Zz multatae erit:

$$Zs-Zz=Ys-\frac{Yds}{dx}+\frac{Ydds}{2dx^2}-\frac{Yd^3s}{6dx^3}+\&c.$$

quae aequatio continet relationem, qua summa Zs pendet ab Y, Z, & z.

189. Ad hanc aequationem resolvendam negligantur primum termini differentiales, eritque $s=\frac{Zz}{Z-Y}$, ponatur iste valor $\frac{Zz}{Z-Y}=P^1$, sitque revera $s=P^1+p$, quo valore in aequatione substituto fiet:

$$(Z-Y)p=-\frac{YdP^1}{dx}+\frac{YddP^1}{2dx^2}-\&c.$$
$$-\frac{Ydp}{dx}+\frac{Yddp}{2dx^2}-\&c.$$

addatur utrinque YP^1, & cum $P^1-\frac{dP^1}{dx}+\frac{ddP^1}{2dx^2}-\&c.$

sit valor ipsius P^1, qui prodit si loco x ponatur $x-1$, sit iste valor $= P$, eritque

$$(Z-Y)p + YP^1 = YP - \frac{Ydp}{dx} + \frac{Yddp}{2dx^2} - \&c.$$

unde neglectis differentialibus erit: $p = \frac{Y(P-P^1)}{Z-Y}$.

Ponatur $\frac{Y(P-P^1)}{Z-Y} = Q^1$, sitque $p = Q^1 + q$; fiet

$$(Z-Y)q = -\frac{Y(dQ^1+dq)}{dx} + \frac{Y(ddQ^1+ddq)}{2dx^2} - \&c.$$

positoque Q pro valore ipsius Q^1, quem induit si loco x scribatur $x-1$, erit:

$$(Z-Y)q + YQ^1 = YQ - \frac{Ydq}{dx} + \frac{Yddq}{2dx^2} - \&c.$$

unde neglectis differentialibus fit $q = \frac{Y(Q-Q^1)}{Z-Y}$.

Ponatur $\frac{Y(Q-Q^1)}{Z-Y} = R^1$, sitque revera $q = R^1 + r$;

ac simili modo reperitur $r = \frac{Y(R-R^1)}{Z-Y}$, sicque procedendo erit summa quaesita: $Zs = Z(P^1 + Q^1 + R^1 + \&c.)$.

190. Proposita ergo serie quacunque:

$$Aa + Bb + Cc + \ldots\ldots + Yy + Zz$$

eius summa sequenti modo definietur:

Ponatur   posito $x-1$ loco x

$\frac{Zz}{Z-Y} = P^1$; abeatque P^1 in P

$\frac{Y(P-P^1)}{Z-Y} = Q^1$; abeatque Q^1 in Q.

$\frac{Y(Q-Q^1)}{Z-Y} = R^1$; abeatque R^1 in R

Y

$$\frac{Y(R - R^1)}{Z - Y} = S^1; \quad \text{abeatque} \quad S^1 \quad \text{in} \quad S$$

&c.

His valoribus inventis erit summa seriei =

$$ZP^1 + ZQ^1 + ZR^1 + ZS^1 + \&c.$$

+ Constante, quae reddat summam $= 0$, si ponatur $x = 0$, seu quod eodem redit, quae efficiat, ut cuipiam casui satisfiat.

191. Formula haec, quia nullis differentialibus est implicata, in plurimis casibus facillime adhibetur, atque etiam veram summam saepenumero exhibet. Sic si proponatur haec series:

$$p + 4p^2 + 9p^3 + 16p^4 + \dots + x^2 p^x$$

fiat $Z = p^x$ & $z = x^2$, erit $Y = p^{x-1}$, atque

$$\frac{Z}{Z - Y} = \frac{p}{p - 1}, \quad \& \quad \frac{Y}{Z - Y} = \frac{1}{p - 1}, \quad \text{Hinc fiet}$$

$$P^1 = \frac{px^2}{p - 1}; \qquad P = \frac{pxx - 2px + p}{p - 1}$$

$$Q^1 = -\frac{2px + p}{(p - 1)^2}; \qquad Q = -\frac{2px + 3p}{(p - 1)^2}$$

$$R^1 = \frac{2p}{(p - 1)^3}; \qquad R = \frac{2p}{(p - 1)^3}$$

$$S^1 = 0; \qquad \text{\& reliqui evanescunt omnes:}$$

unde erit summa =

$$p^x \left(\frac{px^2}{p - 1} - \frac{2px + p}{(p - 1)^2} + \frac{2p}{(p - 1)^3}\right) - \frac{p}{(p - 1)^2} - \frac{2p}{(p - 1)^3}$$

$$= p^{x+1}\left(\frac{x^2}{p - 1} - \frac{2x}{(p - 1)^2} + \frac{p + 1}{(p - 1)^3}\right) - \frac{p + 1}{(p - 1)^3},$$

quemadmodum iam supra invenimus.

192. Simili modo, quo ad hanc summae expressionem pervenimus, aliam invenire poterimus expressionem, si series proposita non ex duabus aliis sit composita: quae illis potissimum

simum casibus in usum vocari poterit, cum in praecedente expressione ad denominatores evanescentes pervenitur. Sit igitur proposita haec series:

$$s = \overset{1}{a} + \overset{2}{b} + \overset{3}{c} + \overset{4}{d} + \ldots\ldots + \overset{x}{z}$$

quoniam posito $x-1$ loco x, summa ultimo termino truncatur, erit:

$$s - z = s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} - \&c.$$

$$\text{seu} \quad z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \&c.$$

Quia hic ipsa summa s non occurrit, negligantur differentialia altiora, fietque $s = \int z dx$, ponatur $\int z dx = P^1$, cuius valor abeat in P si pro x scribatur $x-1$: sitque revera $s = P^1 + p$, erit:

$$z = \frac{dP^1}{dx} - \frac{ddP^1}{2dx^2} + \&c. + \frac{dp}{dx} - \frac{ddp}{2dx^2} + \&c.$$

$$\text{quia est} \quad P = P^1 - \frac{dP^1}{dx} + \frac{ddP^1}{2dx^2} - \&c.$$

$$\text{erit} \quad z - P^1 + P = \frac{dp}{dx} - \frac{ddp}{2dx^2} + \&c. \quad \text{unde fit}$$

$p = \int (z - P^1 + P) dx$. Si porro ponatur $\int (z - P^1 + P) dx = Q^1$, hicque valor abeat in Q posito $x-1$ loco x, fit $\int (z - P^1 + P - Q^1 + Q) dx = R^1 = Q^1 - \int (Q^1 - Q) dx$ porro $R^1 - \int (R^1 - R) dx = S^1$; &c. erit summa quaesita: $s = P^1 + Q^1 + R^1 + S^1 + \&c. + \text{Con.}$ qua uni casui satisfiat.

193. Mutatis aliquantum litteris ista summatio huc redit. Proposita serie summanda:

$$s = \overset{1}{a} + \overset{2}{b} + \overset{3}{c} + \overset{4}{d} + \ldots\ldots + \overset{x}{z}$$

ponatur	*posito* x − 1 , *loco* x		
$\int z dx = P$	abeatque	P	in p
$P - \int (P - p) dx = Q$	abeatque	Q	in q
$Q - \int (Q - q) dx = R$	abeatque	R	in r

quibus valoribus inventis erit summa quaesita:

$$s = P + Q + R + S + \&c.$$

haecque expressio expedite ostendit summam, si formulae istae integrales exhiberi queant. Sit, ut usum eius exemplo illustremus, $z = xx + x$, eritque

$$P = \tfrac{1}{3}x^3 + \tfrac{1}{2}xx; \quad p = \tfrac{1}{3}x^3 - \tfrac{1}{2}xx + \tfrac{1}{6}$$

$$P - p = xx - \tfrac{1}{6} \;\&\; \int(P - p)\,dx = \tfrac{1}{3}x^3 - \tfrac{1}{6}x$$

$$Q = \tfrac{1}{2}xx + \tfrac{1}{6}x; \quad q = \tfrac{1}{2}xx - \tfrac{5}{6}x + \tfrac{1}{3}$$

$$Q - q = x - \tfrac{1}{3} \;\&\; \int(Q - q)\,dx = \tfrac{1}{2}xx - \tfrac{1}{3}x$$

$$R = \tfrac{1}{2}x \quad ; \quad r = \tfrac{1}{2}x - \tfrac{1}{2}$$

$$R - r = \tfrac{1}{2} \quad ; \quad \int(R - r)\,dx = \tfrac{1}{2}x$$

$S = 0$, reliquique valores evanescunt. Quare summa quaesita erit:

$$\begin{aligned}&\tfrac{1}{3}x^3 + \tfrac{1}{2}xx \\ &\quad + \tfrac{1}{2}xx + \tfrac{1}{6}x = \tfrac{1}{3}x^3 + xx + \tfrac{2}{3}x = \tfrac{1}{3}x(x+1)(x+2) \\ &\qquad + \tfrac{1}{2}x.\end{aligned}$$

Hocque ergo modo omnium serierum, quarum termini generales sunt functiones rationales integrae ipsius x, summae ope integrationum continuarum inveniri possunt. Ex quibus facile perspicitur, quam amplum occupet campum doctrina de summatione serierum, neque omnibus methodis, quae tum habentur tum adhuc excogitari possunt, capiendis plura volumina sufficere.

194. Hactenus summas serierum investigavimus a termino primo usque ad eum cuius index est x, quibus cognitis ponendo $x = \infty$ ipsius seriei in infinitum continuatae summa innotescet. Saepenumero autem hoc expeditius praestatur, si non summa terminorum a primo usque ad eum cuius index est x, sed summa omnium terminorum ab isto, cuius index est x, in infinitum usque quaeratur, hocque casu imprimis expressiones ultimae fiunt tractabiliores. Sit igitur proposita series cuius terminus generalis seu indici x respondens sit $= z$, sequens indici $x + 1$ respondens sit $= z^{I}$, huncque ultra sequentes sint z^{II}, z^{III}, &c. quaeraturque summa huius seriei infinitae:

x,

$$x,\ x+1,\ x+2,\ x+3,\ \&c.$$

$$s = z + z^{I} + z^{II} + z^{III} + \&c. \text{ in infinitum.}$$

Haec igitur summa s erit functio ipsius x, in qua si ponatur $x+1$ loco x, orietur summa prior termino z truncata. Cum ergo hac mutatione s abeat in

$s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \&c.$ erit:

$$s - z = s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} + \&c.$$

$$\text{seu } 0 = z + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} + \&c.$$

195. Si nunc ut ante ratiocinium instituamus, fiet neglectis differentialibus superioribus, $s = C - \int z dx$. Ponatur ergo $\int z dx = P$, sitque revera $s = C - P + p$, erit

$$0 = z - \frac{dP}{dx} - \frac{ddP}{2dx^2} - \frac{d^3P}{6dx^3} - \&c.$$
$$+ \frac{dp}{dx} + \frac{ddp}{2dx^2} + \frac{d^3p}{6dx^3} + \&c.$$

Abeat P in P^{I}, si loco x ponatur $x+1$, eritque

$$0 = z + P - P^{I} + \frac{dp}{dx} + \frac{ddp}{2dx^2} + \frac{d^3p}{6dx^3} + \&c.$$

Hinc neglectis differentialibus altioribus fiet: $p = \int(P^{I} - P)dx - P$. Statuatur $\int(P^{I} - P)dx - P = -Q$, sitque $p = -Q + q$, erit:

$$0 = z + P - P^{I} - \frac{dQ}{dx} - \frac{ddQ}{2dx^2} - \&c. + \frac{dq}{dx} + \frac{ddq}{2dx^2} + \&c.$$

Abeat Q in Q^{I}, si loco x ponatur $x+1$, eritque:

$$0 = z + P - P^{I} + Q - Q^{I} + \frac{dq}{dx} + \frac{ddq}{2dx^2} + \&c.$$

unde sequitur $q = \int(Q^{I} - Q)\,dx - Q$. Quamobrem si comma cuique quantitati infixum denotet eius valorem, quem induit posito $x+1$ loco x, ponaturque

$$\int z\,dx = P$$
$$P - \int(P^{I} - P)\,dx = Q$$
$$Q - \int(Q^{I} - Q)\,dx = R$$
$$R - \int(R^{I} - R)\,dx = S \qquad \&c.$$

erit seriei propositae $z + z^{I} + z^{II} + z^{III} + z^{IV} + \&c.$ summa $= C - P - Q - R - S - \&c.$ ubi constans C ita debet definiri, ut posito $x = \infty$ tota summa evanescat. Quia autem applicatio huius expressionis integrationes requirit, hoc loco eius usum declarare non licet.

196. Ut autem formulas integrales evitemus, statuamus summam seriei $= y\,s$, existente y functione ipsius x quacunque cognita, cuius valores y^{I}, y^{II}, &c. qui prodeunt ponendo $x + 1$, $x + 2$, &c. loco x, erunt noti. Si iam ponatur $x + 1$ loco x prodibit superior series termino primo mulctata, cuius summa propterea erit

$$y'\left(s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \&c.\right) = ys - z$$

seu $$z + \frac{y'ds}{dx} + \frac{y'dds}{2dx^2} + \frac{y'd^3s}{6dx^3} + \&c. = (y - y')\,s$$

unde neglectis differentialibus oritur $s = \frac{z}{y - y'}$. Statuatur $\frac{z}{y' - y} = P$, sitque revera $s = -P + p$, erit

$$\left.\begin{aligned} &-\frac{y'dP}{dx} - \frac{y'ddP}{2dx^2} - \frac{y'd^3P}{6dx^3} - \&c. \\ &+\frac{y'dp}{dx} + \frac{y'ddp}{2dx^2} + \frac{y'd^3p}{6dx^3} + \&c. \end{aligned}\right\} = (y - y')\,p$$

ideoque $$\frac{y'dp}{dx} + \frac{y'ddp}{2dx^2} + \frac{y'd^3p}{6dx^3} + \&c. = y'(P' - P) - (y' - y)\,p.$$

Statuatur $Q = \frac{y'(P' - P)}{y' - y}$, sitque $p = Q + q$; erit:

$$y'(Q' - Q) + y'\left(\frac{dq}{dx} + \frac{ddq}{2dx^2} + \&c.\right) = -(y' - y)\,q.$$

Statuatur $R = \frac{y'(Q' - Q)}{y' - y}$, sitque $q = -R + r$.

Hocque modo si ulterius progrediamur. Seriei propositae:

$$z + z^{\mathrm{I}} + z^{\mathrm{II}} + z^{\mathrm{III}} + z^{\mathrm{IV}} + \&c.$$

summa sequenti modo invenietur.

Sumta pro lubitu functione ipsius x, quae sit $= y$, statuatur:

$$P = \frac{z}{y' - y} = \frac{z}{\Delta y}$$

$$Q = \frac{y'(P' - P)}{y' - y} = \frac{y\Delta P}{\Delta y} + \Delta P$$

$$R = \frac{y'(Q' - Q)}{y' - y} = \frac{y\Delta Q}{\Delta y} + \Delta Q$$

$$S = \frac{y'(R' - R)}{y' - y} = \frac{y\Delta R}{\Delta y} + \Delta R \qquad \&c.$$

Hincque erit summa quaesita:

$$= C - Py + Qy - Ry + Sy - \&c.$$

Sumta pro C eiusmodi constante, ut posito $x = \infty$ summa evanescat.

197. Sumatur $y = a^x$, ob $y' = a^{x+1}$, erit: $y' - y = a^x(a - 1)$, unde fiet:

$$P = \frac{z}{a^x(n - 1)} \;;\quad P' = \frac{z'}{a^{x+1}(a - 1)}$$

$$Q = \frac{a(P' - P)}{a - 1} = \frac{z' - az}{a^x(a-1)^2} \;;\quad Q' = \frac{z'' - az'}{a^{x+1}(a - 1)^2}$$

$$R = \frac{a(Q' - Q)}{a - 1} = \frac{z'' - 2az' + aaz}{a^x(a - 1)^3}$$

$$S = \frac{a(R' - R)}{a - 1} = \frac{z''' - 3az'' + 3a^2z' - a^3z}{a^x(a - 1)^4} \qquad \&c.$$

Quocirca summa seriei propositae erit:

$$C - \frac{z}{a-1} + \frac{z' - az}{(a-1)^2} - \frac{z'' + 2az' - a^2z}{(a-1)^3} + \frac{z''' - 3az'' + 3a^2z' - a^3z}{(a-1)^4} \&c.$$

Haec vero eadem summae expressio iam supra est inventa Capite primo. Hinc autem aliis pro y valoribus accipiendis infinitae aliae expressiones erui poterunt; unde ea, quae cuique casui maxime sit accommodata, eligi potest.