CAPUT VI.

DE SUMMATIONE PROGRESSIONUM PER SERIES INFINITAS.

140.

Expressio generalis, quam in Capite praecedente pro termino summatorio cuiusque seriei, cuius terminus generalis seu indici x respondens est $= z$, invenimus:

$$Sz = \int z dx + \tfrac{1}{2} z + \frac{\mathfrak{A}\, dz}{1.2\, dx} - \frac{\mathfrak{B}\, d^3 z}{1.2.3.4\, dx^3} + \frac{\mathfrak{C}\, d^5 z}{1.2\ldots 6\, dx^5} - \&c.$$

proprie inservit seriebus summandis, quarum termini generales sunt functiones quaecunque rationales integrae indicis x, quoniam his casibus ad differentialia tandem evanescentia pervenitur. Sin autem z non fuerit eiusmodi functio ipsius x, tum eius differentialia in infinitum progrediuntur, sicque resultat series infinita summam seriei propositae exprimens, & quidem ad datum usque terminum, cuius index est $= x$. Quocirca progressionis propositae in infinitum continuatae summa prodibit, si ponatur $x = \infty$; hocque pacto alia invenitur series infinita priori aequalis.

141. Sin autem ponatur $x = 0$, tum expressio summam exhibens debet evanescere, uti iam annotavimus; quod nisi fiat, eiusmodi quantitas constans ad summam addi vel inde auferri debet, ut huic conditioni satisfiat. Quo facto si ponatur $x = 1$, summa inventa praebebit terminum primum seriei: si $x = 2$, aggregatum primi & secundi; si $x = 3$, orietur aggregatum trium terminorum initialium seriei, & ita porro. His igitur casibus, quia summa unius, vel duorum, vel trium, &c. terminorum est cognita, seriei infinitae, qua ista summa exprimitur; valor innotescet; ex hocque fonte innumerabiles series summari poterunt.

142. Quoniam, si eiusmodi constans summae fuerit adie-

cta, ut ea evanescat posito $x=0$, tum summa omnibus reliquis casibus, quicunque numeri pro x substituantur, satisfacit; manifestum est, dummodo summae inventae eiusmodi quantitas constans adiiciatur, ut uno quodam casu vera summa indicetur, tum omnibus reliquis casibus veram summam prodire debere. Quare si ponendo $x=0$, non pateat, cuiusmodi valorem expressio summae recipiat, neque igitur constans adiicienda hinc inveniri queat; tum alius quicunque numerus pro x statui poterit, adiiciendaque constante effici ut debita summa indicetur: quod quomodo fieri debeat, ex sequentibus magis fiet perspicuum.

Consideremus primum hanc progressionem harmonicam:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\quad\dots\quad+\frac{1}{x}=s,$$

cuius terminus generalis cum sit $=\frac{1}{x}$, fiet $z=\frac{1}{x}$, & terminus summatorius s ita invenietur. Primo erit $\int z dx=\int\frac{dx}{x}=lx$; deinde differentialia ita se habebunt:

$$\frac{dz}{dx}=-\frac{1}{x^2};\quad \frac{ddz}{2dx^2}=\frac{1}{x^3};\quad \frac{d^3z}{6dx^3}=-\frac{1}{x^4};$$

$$\frac{d^4z}{24dx^4}=\frac{1}{x^5};\quad \frac{d^5z}{120dx^5}=-\frac{1}{x^6}\ \&c.$$ Hinc itaque erit:

$$s=lx+\frac{1}{2x}-\frac{\mathfrak{A}}{2x^2}+\frac{\mathfrak{B}}{4x^4}-\frac{\mathfrak{C}}{6x^6}+\frac{\mathfrak{D}}{8x^8}-\&c.+\text{Const.}$$

Constans igitur hic addenda ex casu $x=0$ non potest definiri. Ponatur ergo $x=1$, quia tum fit $s=1$, erit

$$1=\frac{1}{2}-\frac{\mathfrak{A}}{2}+\frac{\mathfrak{B}}{4}-\frac{\mathfrak{C}}{6}+\frac{\mathfrak{D}}{8}+\text{Const.}$$ unde fit ista constans $=\frac{1}{2}+\frac{\mathfrak{A}}{2}-\frac{\mathfrak{B}}{4}+\frac{\mathfrak{C}}{6}-\frac{\mathfrak{D}}{8}+\&c.$ eritque ideo terminus summatorius quaesitus:

$s=$

$$s = lx + \frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \&c.$$
$$+ \frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \&c.$$

143. Quoniam numeri Bernoulliani $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. constituunt seriem divergentem, hic valor constantis cognosci nequit. Sin autem loco x substituatur numerus maior, atque summa totidem terminorum actu quaeratur, valor constantis commode investigabitur. Ponatur in hunc finem $x = 10$, decemque primis terminis colligendis reperietur eorum summa =

$$2,928968253968253968$$

cui aequalis esse debet expressio summae, si in ea ponatur $x = 10$, quae fit:

$$l10 + \frac{1}{20} - \frac{\mathfrak{A}}{200} + \frac{\mathfrak{B}}{40000} - \frac{\mathfrak{C}}{6000000} + \frac{\mathfrak{D}}{800000000} - \&c. + C.$$

sumto ergo pro $l10$ logarithmo hyperbolico denarii & loco $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. substitutis valoribus supra inventis, reperietur constans illa: $C = 0,5772156649015325$ qui numerus ergo exprimit summam seriei:

$$\frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \frac{\mathfrak{E}}{10} - \&c.$$

144. Si pro x numeri non nimis magni substituantur, quia summa seriei facile actu invenitur, obtinebitur summa seriei huius:

$$\frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \&c. = s - lx - C.$$

Sin autem x significet numerum valde magnum, quia tum valor huius expressionis in infinitum excurrentis facile in fractionibus decimalibus assignatur, vicissim summa seriei definietur. Ac primo quidem constat, si series in infinitum continuetur, eius summam futuram esse infinite magnam; facto enim $x = \infty$ fit lx quoque infinitus; etsi ∞ ad x rationem infinite parvam teneat. Quo autem commodius summa quot-

quotcunque terminorum seriei assignari queat, valores litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. in fractionibus decimalibus exprimamus:

$$\begin{aligned}
\mathfrak{A} &= 0,1666666666666 \\
\mathfrak{B} &= 0,0333333333333 \\
\mathfrak{C} &= 0,0238095238095 \\
\mathfrak{D} &= 0,0333333333333 \\
\mathfrak{E} &= 0,0757575757575 \\
\mathfrak{F} &= 0,2531135531135 \\
\mathfrak{G} &= 1,1666666666666 \\
\mathfrak{H} &= 7,0921568627451 \quad \&c.
\end{aligned}$$

unde ergo erit

$$\begin{aligned}
\frac{\mathfrak{A}}{2} &= 0,0833333333333 \\
\frac{\mathfrak{B}}{4} &= 0,0083333333333 \\
\frac{\mathfrak{C}}{6} &= 0,0039682539682 \\
\frac{\mathfrak{D}}{8} &= 0,0041666666666 \\
\frac{\mathfrak{E}}{10} &= 0,0075757575757 \\
\frac{\mathfrak{F}}{12} &= 0,0210927960928 \\
\frac{\mathfrak{G}}{14} &= 0,0833333333333 \\
\frac{\mathfrak{H}}{16} &= 0,4432598039216 \quad \&c.
\end{aligned}$$

EXEMPLUM I.

Invenire summam mille terminorum seriei

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c.$$

Po-

Ponatur ergo $x = 1000$, & cum ſit

$$\begin{array}{rcl} l10 & = & 2,302585092994045684 \quad \text{erit} \\ lx & = & 6,907755278982\text{1} \\ \text{Conſt.} & = & 0,577215664901\text{5} \\ \frac{1}{2x} & = & 0,0005000000000 \\ & & 7,4854709438836 \\ \text{ſubt.}\ \frac{\mathfrak{A}}{2xx} & = & 0,0000000833333 \\ & & 7,4854708605503 \\ \text{add.}\ \frac{\mathfrak{B}}{4x^4} & = & 0,0000000000000 \\ \text{Ergo} & & 7,4854708605503 \end{array}$$

eſt ſumma quaeſita mille terminorum, qui nequidem ſeptem unitates cum ſemiſſe conficiunt.

EXEMPLUM II.

Invenire ſummam millies mille terminorum ſeriei

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$$

Quia eſt $x = 1000000$, erit $lx = 6.l10$, ergo

$$\begin{array}{rcl} lx & = & 13,8155105579642 \\ \text{Conſt.} & = & 0,5772156649015 \\ \frac{1}{2x} & = & 0,0000005000000 \\ & & 14,3927267228657 = \text{ſummae quaeſitae.} \end{array}$$

145. Si ergo pro x ſtatuatur numerus vehementer magnus, ſumma ſatis exacte invenitur ex ſolo primo termino lx conſtante C aucto: unde egregia corollaria deduci poſſunt. Sic ſi x fuerit numerus vehementer magnus, ponaturque:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \frac{1}{x} = s$$

$$\& \ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots + \frac{1}{x} + \ldots \frac{1}{x+y} = t$$

quia

quia est proxime $s = lx + C$, & $t = l(x+y) + C$; erit $t - s = l(x+y) - lx = l\frac{x+y}{x}$, ideoque hic logarithmus proxime per seriem harmonicam finito terminorum numero constantem exprimetur hoc modo:

$$l\frac{x+y}{x} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \ldots\ldots + \frac{1}{x+y}.$$

Accuratius autem hic logarithmus exhibebitur, si superiores summae s & t exactius capiantur. Sic cum sit

$$s = lx + C + \frac{1}{2x} - \frac{1}{12xx}, \quad \&$$

$$t = l(x+y) + C + \frac{1}{2(x+y)} - \frac{1}{12(x+y)^2}; \text{ erit}$$

$$t - s = l\frac{x+y}{x} - \frac{1}{2x} + \frac{1}{2(x+y)} + \frac{1}{12xx} - \frac{1}{12(x+y)^2}, \text{ ideoque}$$

$$l\frac{x+y}{x} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \ldots + \frac{1}{x+y} + \frac{1}{2x} - \frac{1}{2(x+y)} - \frac{1}{12xx} + \frac{1}{12(x+y)^2}$$

Sin autem sit numerus tam magnus, ut bini termini ultimi reiici queant, erit proxime:

$$l\frac{x+y}{x} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \ldots + \frac{1}{x+y} + \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+y}\right)$$

Ex hac quoque serie harmonica derivare poterimus summam huius seriei, in qua tantum numeri impares occurrunt:

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} \ldots\ldots + \frac{1}{2x+1}.$$

Cum enim omnibus terminis capiendis sit:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots\ldots\ldots + \frac{1}{2x} + \frac{1}{2x+1} =$$

$$l(2x+1) + C + \frac{1}{2(2x+1)} - \frac{\mathfrak{A}}{2(2x+1)^2} + \frac{\mathfrak{B}}{4(2x+1)^4} - \frac{\mathfrak{C}}{6(2x+1)^6} + \&c.$$

termi-

terminorum vero ordine parium:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6} \dots \dots + \frac{1}{2x}$$

summa sit semissis superioris nempe:

$$\tfrac{1}{2}C+\tfrac{1}{2}lx+\frac{1}{4x}-\frac{\mathfrak{A}}{4x^2}+\frac{\mathfrak{B}}{8x^4}-\frac{\mathfrak{C}}{12x^6}+\frac{\mathfrak{D}}{16x^8}-\&c.$$

erit hac serie ab illa ablata:

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+ \dots \dots + \frac{1}{2x+1} =$$

$$\tfrac{1}{2}C+l\frac{2x+1}{\sqrt{x}}+\frac{1}{2(2x+1)}-\frac{\mathfrak{A}}{2(2x+1)^2}+\frac{\mathfrak{B}}{4(2x+1)^4}-\&c.$$

$$-\frac{1}{4x}+\frac{\mathfrak{A}}{4x^2}-\frac{\mathfrak{B}}{8x^4}+\&c.$$

146. Potest vero etiam per eandem expressionem generalem summa cuiusque seriei harmonicae inveniri; sit enim;

$$\frac{1}{m+n}+\frac{1}{2m+n}+\frac{1}{3m+n}+\frac{1}{4m+n}+ \dots + \frac{1}{mx+n}=s,$$

quia est terminus generalis $z=\frac{1}{mx+n}$, erit:

$$\int z dx=\frac{1}{m}l(mx+n); \quad \frac{dz}{dx}=-\frac{m}{(mx+n)^2}$$

$$\frac{ddz}{2dx^2}=\frac{mm}{(mx+n)^3}; \quad \frac{d^3z}{6dx^3}=-\frac{m^3}{(mx+n)^4}$$

$$\frac{d^4z}{24dx^4}=\frac{m^4}{(mx+n)^5}; \quad \frac{d^5z}{120dx^5}=-\frac{m^5}{(mx+n)^6}\ \&c.$$

Ex his ergo reperitur:

$$s=D+\frac{1}{m}l(mx+n)+\frac{1}{2(mx+n)}-\frac{\mathfrak{A}m}{2(mx+n)^2}+\frac{\mathfrak{B}m^3}{4(mx+n)^4}$$

$$-\frac{\mathfrak{C}m^5}{6(mx+n)^6}+\frac{\mathfrak{D}m^7}{8(mx+n)^8}-\&c.$$

Po-

Posito ergo $x = 0$, fiet constans illa addenda:

$$D = -\frac{1}{m}\, ln - \frac{1}{2n} + \frac{\mathfrak{A}m}{2n^2} - \frac{\mathfrak{B}m^3}{4n^4} + \frac{\mathfrak{C}n^5}{6n^6} - \&c.$$

147. Si vero sit $n = 0$, quoniam seriei:

$$\frac{1}{m} + \frac{1}{2m} + \frac{1}{3m} + \frac{1}{4m} + \ldots\ldots\ldots + \frac{1}{mx}$$

summa est $= \frac{1}{m}C + \frac{1}{m}lx + \frac{1}{2mx} - \frac{\mathfrak{A}}{2mx^2} + \frac{\mathfrak{B}}{4mx^4} - \&c.$

at vero huius seriei:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots\ldots + \frac{1}{mx}$$

summa est $= C + lmx + \frac{1}{2mx} - \frac{\mathfrak{A}}{2m^2x^2} + \frac{\mathfrak{B}}{4m^4x^4} - \&c.$

si ab hac serie illa m vicibus sumta subtrahatur ut prodeat haec series:

$$1 + \tfrac{1}{2} + \ldots + \frac{1}{m} + \ldots + \frac{1}{2m} + \ldots + \frac{1}{3m} + \ldots + \frac{1}{mx}$$
$$\quad - \frac{m}{m} \qquad - \frac{m}{2m} \qquad - \frac{m}{3m} \qquad - \frac{m}{mx}$$

eius summa erit $= lm + \frac{1}{2mx} - \frac{\mathfrak{A}}{2m^2x^2} + \frac{\mathfrak{B}}{4m^4x^4} - \&c.$

$$- \frac{1}{2x} + \frac{\mathfrak{A}}{2xx} - \frac{\mathfrak{B}}{4x^4} + \&c.$$

atque si statuatur $x = \infty$ summa erit $= lm$. Hinc pro m ponendo numeros 2, 3, 4, &c. erit:

$l2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \&c.$

$l3 = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9}\ \&c.$

$l4 = 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{3}{8} + \&c.$

$l5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{4}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{4}{10}\ \&c.$

&c.

148. Relicta autem serie harmonica progrediamur ad seriem quadratorum reciprocam, sitque:

$s =$

$$s = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots\ldots + \frac{1}{xx},$$

in qua cum sit terminus generalis $z = \frac{1}{xx}$, erit

$\int z dx = -\frac{1}{x}$, & differentialia ipsius z ita se habebunt

$$\frac{dz}{2dx} = -\frac{1}{x^3};\ \frac{ddz}{2.3dx^2} = \frac{1}{x^4};\ \frac{d^3z}{2.3.4dx^3} = -\frac{1}{x^5};\ \&c.$$

unde erit summa

$$s = C - \frac{1}{x} + \frac{1}{2xx} - \frac{\mathfrak{A}}{x^3} + \frac{\mathfrak{B}}{x^5} - \frac{\mathfrak{C}}{x^7} + \frac{\mathfrak{D}}{x^9} - \frac{\mathfrak{E}}{x^{11}} + \&c.$$

in qua constans addenda C ex uno casu, quo summa constat, est definienda. Ponamus ergo $x = 1$, quia fit $s = 1$ debet esse: $C = 1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \&c.$ quae series autem cum sit maxime divergens, valorem constantis C non ostendit. Quia autem supra demonstravimus summam huius seriei in infinitum continuatae esse $= \frac{\pi\pi}{6}$: facto $x = \infty$, si ponatur $s = \frac{\pi\pi}{6}$, fiet $C = \frac{\pi\pi}{6}$; ob reliquos terminos omnes evanescentes. Erit ergo

$$1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \&c. = \frac{\pi\pi}{6}.$$

149. Sin autem summa huius seriei cognita non fuisset, valor constantis illius C ex alio quopiam casu, quo summa actu est inventa, determinari deberet. Hunc in finem ponamus $x = 10$, atque decem terminis actu addendis reperietur:

$s = 1,5497677311665406\,90$ tum est

add. $\frac{1}{x} = 0,1$

subtr. $\frac{1}{2xx} = 0,005$

$1,6447677311665406\,90$

		1,644767731166540690
add.	$\frac{\mathfrak{A}}{x^3} =$	0,000166666666666666
		1,644934397833207356
subtr.	$\frac{\mathfrak{B}}{x^5} =$	0,000000333333333333
		1,644934064499874023
add.	$\frac{\mathfrak{C}}{x^7} =$	0,000000002380952381
		1,644934066880826404
subtr.	$\frac{\mathfrak{D}}{x^9} =$	0,000000000033333333
		1,644934066847493071
add.	$\frac{\mathfrak{E}}{x^{11}} =$	0,000000000000757575
		1,644934066848250646
subtr.	$\frac{\mathfrak{F}}{x^{13}} =$	0,000000000000025311
		1,644934066848225335
add.	$\frac{\mathfrak{G}}{x^{15}} =$	0,000000000000001166
subtr.	$\frac{\mathfrak{H}}{x^{17}} =$	71
		1,644934066848226430 = C.

Hicque numerus simul est valor expressionis $\frac{\pi\pi}{6}$, quemadmodum ex valore ipsius π cognito calculum instituenti patebit. Unde simul intelligitur, etiamsi series $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. divergat, tamen hoc modo veram prodire summam.

150. Sit nunc $z = \frac{1}{x^3}$; atque

$s = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots\ldots + \frac{1}{x^3}$, quia est

$\int z dx = -\frac{1}{2xx}$; $\frac{dz}{1.2.3\,dx} = -\frac{1}{2x^4}$; $\frac{ddz}{1.2.3.4\,dx^2} = \frac{1}{2x^5}$

$\frac{d^3z}{1.2\ldots5dx^3} = -\frac{1}{2x^6}$; $\frac{d^5z}{1.2\ldots7dx^5} = -\frac{1}{2x^8}$; &c. erit

$s =$

$$s = C - \frac{1}{2xx} + \frac{1}{2x^3} - \frac{3\mathfrak{A}}{2x^4} + \frac{5\mathfrak{B}}{2x^6} - \frac{7\mathfrak{C}}{2x^8} + \&c.$$

hincque posito $x = 1$, ob $s = 1$, fiet:

$$C = 1 + \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{3}{2}\mathfrak{A} - \tfrac{5}{2}\mathfrak{B} + \tfrac{7}{2}\mathfrak{C} - \tfrac{9}{2}\mathfrak{D} + \&c.$$

atque iste valor ipsius C simul ostendet summam seriei propositae in infinitum continuatae. Quoniam vero summae potestatum imparium non aeque ac parium constant, iste ipsius C valor ex cognita summa aliquot terminorum definiri debet. Sit ergo $x = 10$, erit:

$$C = s + \frac{1}{2xx} - \frac{1}{2x^3} + \frac{3\mathfrak{A}}{2x^4} - \frac{5\mathfrak{B}}{2x^6} + \frac{7\mathfrak{C}}{2x^8} - \&c.$$

Est vero ad computum facilius instituendum:

$$\frac{3\mathfrak{A}}{2} = 0,250000000000$$
$$\frac{5\mathfrak{B}}{2} = 0,083333333333$$
$$\frac{7\mathfrak{C}}{2} = 0,083333333333$$
$$\frac{9\mathfrak{D}}{2} = 0,150000000000$$
$$\frac{11\mathfrak{E}}{2} = 0,416666666666$$
$$\frac{13\mathfrak{F}}{2} = 1,645238095238$$
$$\frac{15\mathfrak{G}}{2} = 8,750000000000$$
$$\frac{17\mathfrak{H}}{2} = 60,283333333333 \quad \&c.$$

 Hinc

Hinc ergo fient termini ad s addendi:

$$\frac{1}{2xx} = 0,005000000000000000$$
$$\frac{3\mathfrak{A}}{2x^4} = 0,000025000000000000$$
$$\frac{7\mathfrak{C}}{2x^8} = 0,000000000833333333$$
$$\frac{11\mathfrak{E}}{2x^{12}} = 0,000000000000416666$$
$$\frac{15\mathfrak{G}}{2x^{16}} = \underline{0,000000000000000875}$$
$$0,005025000833750875$$

termini autem ſubtrahendi ſunt:

$$\frac{1}{2x^3} = 0,000500000000000000$$
$$\frac{5\mathfrak{B}}{2x^6} = 0,000000083333333333$$
$$\frac{9\mathfrak{D}}{2x^{10}} = 0,000000000015000000$$
$$\frac{13\mathfrak{F}}{2x^{14}} = 0,000000000000016452$$
$$\frac{17\mathfrak{H}}{2x^{18}} = \underline{0,000000000000000060}$$
$$0,000500083348349845$$
$$\text{ab}: \underline{0,005025000833750875}$$
$$0,004524917485401030$$
$$s = \underline{1,197531985674193251}$$
$$C = 1,202056903159594281.$$

151. Si hoc modo ulterius progrediamur, inveniemus ſummas omnium ſerierum poteſtatum reciprocarum in fractionibus decimalibus expreſſas:

1 +

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c.=1,6449340668482264=\frac{2\mathfrak{A}}{1.2}\pi^2,$$

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\&c.=1,2020569031595942$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\&c.=1,0823232337111381=\frac{2^3\mathfrak{B}}{1.2.3.4}\pi^4$$

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\&c.=1,0369277551068632$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\&c.=1,0173430619844491=\frac{2^5\mathfrak{C}}{1.2\ldots 6}\pi^6$$

$$1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\&c.=1,0083492773866018$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\&c.=1,0040773561979443=\frac{2^7\mathfrak{D}}{1.2\ldots 8}\pi^8$$

$$1+\frac{1}{2^9}+\frac{1}{3^9}+\frac{1}{4^9}+\&c.=1,0020083928260822$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\&c.=1,0009945751278180=\frac{2^9\mathfrak{E}}{1.2\ldots 10}\pi^{10}$$

$$1+\frac{1}{2^{11}}+\frac{1}{3^{11}}+\frac{1}{4^{11}}+\&c.=1,0004941886041094$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\&c.=1,0002460865533080=\frac{2^{11}\mathfrak{F}}{1.2\ldots 12}\pi^{12}$$

$$1+\frac{1}{2^{13}}+\frac{1}{3^{13}}+\frac{1}{4^{13}}+\&c.=1,0001227233475857$$

$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\&c.=1,0000612481350587=\frac{2^{13}\mathfrak{G}}{1.2\ldots 14}\pi^{14}$$

$$1+\frac{1}{2^{15}}+\frac{1}{3^{15}}+\frac{1}{4^{15}}+\&c.=1,0000305882363070$$

$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\&c.=1,0000152822594086=\frac{2^{15}\mathfrak{H}}{1.2\ldots 16}\pi^{16}$$

&c.

152. Ex his ergo viciſſim ſummae illarum ſerierum infinitarum numeris Bernoullianis conſtantium exhiberi poterunt. Erit enim:

$$1 + 0 - \tfrac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \&c. = 0,57721\ \&c.$$

$$1 + 1 - \tfrac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \&c. = \frac{2\,\mathfrak{A}}{1.2}\pi^2$$

$$1 + \tfrac{1}{2} - \tfrac{1}{2} + \frac{3\mathfrak{A}}{2} - \frac{5\mathfrak{B}}{2} + \frac{7\mathfrak{C}}{2} - \frac{9\mathfrak{D}}{2} + \&c. = 1,2020\ \&c.$$

$$1 + \tfrac{1}{3} - \tfrac{1}{2} + \frac{3.4\mathfrak{A}}{2.\,3} - \frac{5.6\mathfrak{B}}{2.\,3} + \frac{7.8\mathfrak{C}}{2.\,3} - \frac{9.10\mathfrak{D}}{2.\,3} + \&c. = \frac{2^3\mathfrak{B}}{1.2.3.4}\pi^4$$

$$1 + \tfrac{1}{4} - \tfrac{1}{2} + \frac{3.4.5\mathfrak{A}}{2.3.4} - \frac{5.6.7.\mathfrak{B}}{2.3.4} + \frac{7.8.9\mathfrak{C}}{2.3.4} - \frac{9.10.11\mathfrak{D}}{2.3.4.} + \&c. = 1,0369\ \&c.$$

$$1 + \tfrac{1}{5} - \tfrac{1}{2} + \frac{3.4.5.6\mathfrak{A}}{2.3.4.5} - \frac{5.6.7.8\mathfrak{B}}{2.3.4.5} + \frac{7.8.9.10\,\mathfrak{C}}{2.3.4.5.} - \&c. = \frac{2^5\mathfrak{C}}{1.2\ldots 6}\pi^6$$

Harum ergo ſerierum alternae ope quadraturae circuli ſummari poſſunt; a quanam vero quantitate tranſcendente reliquae pendeant, adhuc non conſtat: neque enim ad poteſtates ipſius π exponentes impares habentes revocari poſſunt, ita ut coefficientes eſſent numeri rationales. Quo autem ſaltem proxime appareat, quales futuri ſint coefficientes poteſtatum ipſius π pro exponentibus imparibus, tabellam ſequentem adiunximus:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.\ \text{in infin.} = \frac{\pi}{0,0000} = \infty$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\ \ldots = \frac{\pi^2}{6,0000}\ \text{vere}$$

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \&c.\ \ldots = \frac{\pi^3}{25,79436}\ \text{prox.}$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \&c.\ \ldots = \frac{\pi^4}{90,00000}\ \text{vere}$$

1 +

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\&c. \quad . \quad . = \frac{\pi^5}{295,1215} \quad \text{prox.}$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\&c. \quad . \quad . = \frac{\pi^6}{945,000} \quad \text{vere}$$

$$1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\&c. \quad . \quad . = \frac{\pi^7}{2995,286} \quad \text{prox.}$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\&c. \quad . \quad . = \frac{\pi^8}{9450,000} \quad \text{vere}$$

$$1+\frac{1}{2^9}+\frac{1}{3^9}+\frac{1}{4^9}+\&c. \quad . \quad . = \frac{\pi^9}{29749,35} \quad \text{prox.} \quad \&c.$$

153. Ex hoc fonte series numerorum Bernoullianorum

1	2	3	4	5	6	7	8	9	
𝔄,	𝔅,	ℭ,	𝔇;	𝔈	𝔉,	𝔊,	ℌ,	ℑ,	&c.

quantumvis irregularis videatur interpolari, seu termini in medio binorum quorumcunque constituti assignari poterunt: si enim terminus medium interiacens inter primum 𝔄 & secundum 𝔅, seu indici $1\frac{1}{2}$ respondens fuerit $= p$; erit utique:

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\&c. = \frac{2^2 p}{1.2.3}\pi^3$$

ideoque $p = \frac{3}{2\pi^3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\&c.\right) = 0,05815227$

Simili modo si terminus inter 𝔅 & ℭ medium interiacens seu indicem habens $2\frac{1}{2}$ ponatur $= q$, quia erit:

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\&c. = \frac{2^4 q}{1.2.3.4.5}\pi^5$$

fiet $q = \frac{15}{2\pi^5}\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\&c.\right) = 0,02541327$

Si ergo istarum serierum, in quibus exponentes potestatum sunt numeri impares, summae exhiberi possent, tum quoque series numerorum Bernoullianorum interpolari posset.

154. Ponamus nunc $z = \frac{1}{nn+xx}$, & quaeratur

summa huius seriei:

$$s = \frac{1}{nn+1} + \frac{1}{nn+4} + \frac{1}{nn+9} + \ldots + \frac{1}{nn+xx}.$$

Quia est $\int z dx = \int \frac{dx}{nn+xx}$; erit $\int z dx = \frac{1}{n} \text{A tang} \frac{x}{n}$.

Ponatur $\text{A cot} \frac{x}{n} = u$, erit $\int z dx = \frac{1}{n}\left(\frac{\pi}{2} - u\right)$, &

$\frac{x}{n} = \cot u = \frac{\cos u}{\sin u}$, & $\frac{nn+xx}{nn} = \frac{1}{\sin u^2}$, & $z = \frac{\sin u^2}{nn}$,

& $\frac{dx}{n} = -\frac{du}{\sin u^2}$, unde fit $du = -\frac{dx \sin u^2}{n}$.

Hinc differentialia ipsius z invenientur hoc modo:

$$dz = \frac{2 du \sin u . \cos u}{nn} = -\frac{dx \sin u^2 . \sin 2u}{n^3} \ \& \ \frac{dz}{dx} = -\frac{\sin u^2 . \sin 2u}{n^3}$$

$$\frac{ddz}{2dx} = -\frac{du(\sin u \cos u \sin 2u + \sin u^2 . \cos 2u)}{n^3} = \frac{dx \sin u^3 . \sin 3u}{n^4}$$

$$\& \quad \frac{ddz}{2dx^2} = \frac{\sin u^3 . \sin 3u}{n^4}.$$

Simili modo erit, uti iam supra pro eodem casu invenimus:

$$\frac{d^3 z}{2.3 dx^3} = -\frac{\sin u^4 . \sin 4u}{n^5}; \quad \frac{d^4 z}{2.3.4 dx^4} = \frac{\sin u^5 . \sin 5u}{n^6} \ \&c.$$

ex quibus formabitur summa quaesita:

$$s = \frac{\pi}{2n} - \frac{u}{n} + \frac{\sin u . \sin u}{2nn} - \frac{\mathfrak{A}}{2} . \frac{\sin u^2 . \sin 2u}{n^3} + \frac{\mathfrak{B}}{4} . \frac{\sin u^4 . \sin 4u}{n^5}$$

$$- \frac{\mathfrak{C}}{6} . \frac{\sin u^6 . \sin 6u}{n^7} + \frac{\mathfrak{D}}{8} . \frac{\sin u^8 . \sin 8u}{n^9} - \&c. \quad + \text{Const.}$$

Si hic ad constantem determinandam ponatur $x = 0$, quo fiat $s = 0$, erit $\cot u = 0$, ideoque u angulus 90°, ac praeterea $\sin u = 1$, $\sin 2u = 0$, $\sin 4u = 0$, $\sin 6u = 0$, &c. videtur ergo fore $0 = \frac{\pi}{2n} - \frac{\pi}{2n} + \frac{1}{2nn} + C$, hinc $C = -\frac{1}{2nn}$: at at vero notandum eſt, etiamſi reliqui termini evaneſcant, tamen quia coefficientes 𝔄, 𝔅, ℭ, &c. tandem in infinitum excreſcunt, eorum ſummam poſſe eſſe finitam.

155. Ad hanc ergo conſtantem rite determinandam ponamus eſſe $x = \infty$, ſummam enim huius ſeriei in infinitum excurrentis ſupra iam in introductione definivimus, oſtendimuſque eſſe eam $= -\frac{1}{2nn} + \frac{\pi}{2n} + \frac{\pi}{n(e^{2n\pi} - 1)}$. Poſito autem $x = \infty$, fiet $u = 0$, ideoque ſin $u = 0$, ſimulque ſinus omnium arcuum multiplorum evaneſcent. Cum autem in hac ſerie poteſtates ipſius ſin u creſcant, divergentia ſeriei impedire nequit, quominus valor ſeriei hoc caſu evaneſcat. Fiet ergo $s = \frac{\pi}{2n} + C$; unde erit

$$\frac{\pi}{2n} + C = -\frac{1}{2nn} + \frac{\pi}{2n} + \frac{\pi}{n(e^{2n\pi} - 1)}, \quad \&$$

$C = -\frac{1}{2nn} + \frac{\pi}{n(e^{2n\pi} - 1)}$. Quare ſumma ſeriei quaeſita erit

$$s = \frac{\pi}{2n} - \frac{u}{n} - \frac{\cos u^2}{2nn} - \frac{𝔄}{2} \cdot \frac{\sin u^2 \,.\, \sin 2u}{n^3} + \frac{𝔅}{4} \cdot \frac{\sin u^4 \,.\, \sin 4u}{n^5} - \frac{ℭ}{6} \cdot \frac{\sin u^6 \,.\, \sin 6u}{n^7} + \&c. + \frac{\pi}{n(e^{2n\pi} - 1)}.$$

Ubi notandum eſt, ſi n fuerit numerus mediocriter magnus, ultimum terminum $\frac{\pi}{n(e^{2n\pi} - 1)}$ tantopere fieri exiguum, ut negligi queat.

156. Ponamus eſſe $x = n$, ita ut denotet:

$$s = \frac{1}{nn+1} + \frac{1}{nn+4} + \frac{1}{nn+9} + \ldots\ldots + \frac{1}{nn+nn}.$$

Tum vero erit cot $u = 1$, & $u = 45° = \frac{\pi}{4}$. Quamobrem habebitur $\sin u = \frac{1}{\sqrt{2}}$; $\sin 2u = 1$; $\sin 4u = 0$; $\sin 6u = -1$; $\sin 8u = 0$; $\sin 10u = 1$; &c. Hancobrem erit:

$$s = \frac{\pi}{4n} - \frac{1}{4nn} - \frac{\mathfrak{A}}{2.2n^3} + \frac{\mathfrak{C}}{6.8n^7} - \frac{\mathfrak{E}}{10.2^5 n^{11}}$$
$$+ \frac{\mathfrak{G}}{14.2^7 n^{15}} - \&c. + \frac{\pi}{n(e^{2n\pi}-1)},$$

in qua expressione tantum numeri alterni ex Bernoullianis occurrunt. Si igitur valor ipsius s per computum actu institutum iam fuerit inventus, hinc quantitas π definiri poterit, erit enim:

$$\pi = 4ns + \frac{1}{n} + \frac{\mathfrak{A}}{1.n^2} - \frac{\mathfrak{C}}{3.2^2 n^6} + \frac{\mathfrak{E}}{5.2^4 n^{10}}$$
$$- \frac{\mathfrak{G}}{7.2^6.n^{14}} + \&c. - \frac{\pi}{e^{2n\pi}-1}.$$

Etsi enim in termino ultimo inest π, tamen quia is tantopere est parvus, sufficit valorem ipsius π proxime nosse.

EXEMPLUM. Sit $n = 5$; erit:

$$s = \frac{1}{26} + \frac{1}{29} + \frac{1}{34} + \frac{1}{41} + \frac{1}{50}$$

qui termini actu additi dabunt:

$$s = 0,146746305690549494$$

unde erunt termini illi;

$$4ns = 2,93492611381098988$$
$$\frac{1}{n} = 0,2$$
$$\frac{\mathfrak{A}}{nn} = 0,00666666666666666$$
$$3,14159278047765654$$
$$\frac{\mathfrak{C}}{3.2^2.n^6} = 0,00000012698412698$$
$$3,14159265349352956$$

ℭ

$$3,14159265349352956$$

$$\frac{\mathfrak{E}}{5\cdot 2^4\cdot n^{10}} = 0,0000000009696969$$

$$3,1415926535904992 5$$

$$\frac{\mathfrak{G}}{7\cdot 2^6\cdot n^{14}} = 0,000000000000042666$$

$$3,14159265359007259$$

$$\frac{\mathfrak{I}}{9\cdot 2^8\cdot n^{18}} = \quad 625$$

$$3,14159265359007884$$

Hic valor iam tam prope ad veritatem accedit, ut mirandum ſit tam levi calculo eousque perveniri poſſe. Eſt vero adhaec expreſſio aliquantillum iuſto maior, ſubtrahi enim adhuc inde debet $\frac{4\pi}{e^{2n\pi}-1}$, cuius valor, dummodo π prope ſit cognitum, exhiberi poteſt; quod per logarithmos ita expedietur.

Quia eſt $\pi\, l e = 1,3643763538$

erit $l e^{2n\pi} = 10\pi\, l e = 13,64437635.$

Cum iam ſit $\frac{4\pi}{e^{2n\pi}-1} = \frac{4\pi}{e^{2n\pi}} + \frac{4\pi}{e^{4n\pi}} + \&c.$

facile intelligitur ad noſtrum calculum ſufficere primum terminorum ſumſiſſe. Augeamus ergo characteriſticam numero 17, quia habemus totidem figuras decimales, erit:

$$l\pi = 17,4971498$$
$$l4 = 0,6020600$$
$$18,0992098$$

Subtr. $l e^{2n\pi} = 13,6437635$

$$4,4554463$$

Ergo $\frac{4\pi}{e^{2n\pi}} = 28539$ ſubtrahatur

ab $3,14159265359007884$ erit

$$\pi = 3,14159265358979345$$

quae expreſſio in figura demum penultima a veritate recedit;

 quod

quod mirandum non est, cum adhuc terminum $\frac{\mathfrak{K}}{11.2^{10}.n^{22}}$, qui dat 22, subtrahere debuissemus, sicque ne ultima quidem figura aberrasset. Ceterum intelligitur, si pro n maiorem numerum uti 10 assumsissemus, tum facili negotio peripheriam π ad 25 pluresque figuras inveniri potuisse.

157. Ponamus nunc quoque pro z functiones transcendentes ipsius x, sitque $z = lx$, sumendo logarithmos hyperbolicos, quoniam vulgares facile eo revocantur; sitque:

$$s = l1 + l2 + l3 + l4 + \ldots + lx.$$

Quia igitur est $z = lx$, erit $\int z dx = xlx - x$, huius enim differentiale dat $dxlx$. Deinde est

$$\frac{dz}{dx} = \frac{1}{x};\ \frac{ddz}{dx^2} = -\frac{1}{x^2};\ \frac{d^3 z}{1.2 dx^3} = \frac{1}{x^3};\ \frac{d^5 z}{1.2.3.4 dx^5} = \frac{1}{x^5};\ \&c.$$

Hinc itaque concluditur fore:

$$s = xlx - x + \tfrac{1}{2} lx + \frac{\mathfrak{A}}{1.2x} - \frac{\mathfrak{B}}{3.4x^3} + \frac{\mathfrak{C}}{5.6x^5} - \frac{\mathfrak{D}}{7.8x^7} + \&c. + \text{Const.}$$

Haec autem constans ponendo $x = 1$, quia fit $s = l1 = 0$ ita definietur, ut sit: $C = 1 - \frac{\mathfrak{A}}{1.2} + \frac{\mathfrak{B}}{3.4} - \frac{\mathfrak{C}}{5.6} + \frac{\mathfrak{D}}{7.8} - \&c.$ quae series ob nimiam divergentiam est inepta ad valorem ipsius C saltem proxime eruendum.

158. Non solum autem proximum sed etiam ipsum verum valorem ipsius C inveniemus, si consideremus expressionem Wallisianam pro valore ipsius π inventam, atque in introductione demonstratam: quae erat:

$$\frac{\pi}{2} = \frac{2.2.4.4.6.6.8.8.10.10.12.\&c.}{1.3.3.5.5.7.7.9.9.11.11.\&c.}$$

hinc enim logarithmis sumendis erit:

$$\begin{aligned} l\pi - l2 = {} & 2l2 + 2l4 + 2l6 + 2l8 + 2l10 + \&c. \\ & - l1 - 2l3 - 2l5 - 2l7 - 2l9 - 2l11 - \&c. \end{aligned}$$

Ponamus ergo in serie assumta $x = \infty$ & cum sit:

$$l1+l2+l3+l4+\ldots+lx = C+(x+\tfrac{1}{2})lx-x$$
erit $l1+l2+l3+l4+\ldots+l2x = C+(2x+\tfrac{1}{2})l2x-2x$
& $l2+l4+l6+l8+\ldots+l2x = C+(x+\tfrac{1}{2})lx+xl2-x$
hinc $l1+l3+l5+l7+\ldots+l(2x-1) = xlx+(x+\tfrac{1}{2})l2-x$

Cum igitur sit:

$$l\frac{\pi}{2} = 2l2+2l4+2l6+\ldots+2l2x-l2x$$
$$-2l1-2l3-2l5-\ldots-2l(2x-1)$$

posito $x=\infty$, erit:

$$l\frac{\pi}{2} = 2C+(2x+1)lx+2xl2-2x-l2-lx$$
$$-2xlx-(2x+1)l2+2x,$$

ideoque $l\frac{\pi}{2} = 2C-2l2$, ergo $2C = l2\pi$, & $C = \frac{1}{2}l2\pi$, unde in fractionibus decimalibus reperitur:

$$C = 0,91893853320467274178032\,97$$

atque simul sequens series summatur:

$$1-\frac{\mathfrak{A}}{1.2}+\frac{\mathfrak{B}}{3.4}-\frac{\mathfrak{C}}{5.6}+\frac{\mathfrak{D}}{7.8}-\frac{\mathfrak{E}}{9.10}+\&c. = \tfrac{1}{2}l2\pi.$$

159. Cognita nunc ista constante $C=\frac{1}{2}l2\pi$, summa quotcunque logarithmorum ex hac serie $l1+l2+l3+$ &c. exhiberi potest. Si enim ponatur:

$$s = l1+l2+l3+l4+\ldots+lx, \text{ erit}$$

$$s = \tfrac{1}{2}l2\pi+(x+\tfrac{1}{2})lx-x+\frac{\mathfrak{A}}{1.2x}-\frac{\mathfrak{B}}{3.4x^3}+\frac{\mathfrak{C}}{5.6x^5}-\frac{\mathfrak{D}}{7.8x^7}+\&c.$$

si quidem logarithmi propositi fuerint hyperbolici; sin autem proponantur logarithmi vulgares, tum in terminis $\frac{1}{2}l2\pi+(x+\frac{1}{2})lx$ pro $l2\pi$ & lx summi debebunt logarithmi vulgares, reliqui autem seriei termini $-x+\frac{\mathfrak{A}}{1.2x}-\frac{\mathfrak{B}}{3.4x^3}+$ &c. multiplicari debent per $0,4342944819032518\,27 = n$.

Erit igitur hoc casu pro logarithmis vulgaribus:

$l\pi$

$$l\pi = 0,497149872694133854351268$$
$$l2 = 0,301029995663981195213738$$
$$l2\pi = 0,798179868358115049565006$$
$$\tfrac{1}{2}l2\pi = 0,399089934179057524782503$$

EXEMPLUM

Quaeratur aggregatum mille logarithmorum tabularium

$$s = l1 + l2 + l3 + \quad . \quad . \quad . \quad . \quad + l1000.$$

Erit ergo $x = 1000$, & $lx = 3,000000000000$

unde fit $xlx = 3000,000000000000$

$\tfrac{1}{2}lx = 1,500000000000$

$\tfrac{1}{2}l2\pi = 0,3990899341790$

$3001,8990899341790$

subtr. $nx = 434,2944819032518$

$2567,6046080309272$

Deinde est $\dfrac{n\mathfrak{A}}{1.2x} = 0,0000361912068$

subtr. $\dfrac{n\mathfrak{B}}{3.4x^3} = 0,0000000000012$

$0,0000361912056$

addatur $2567,6046080309272$

summa quaesita $s = 2567,6046442221328$.

Cum igitur s sit logarithmus producti numerorum:

1. 2. 3. 4. 5. 6. 1000

patet hoc productum, si actu multiplicetur, constare ex 2568 figuris, atque notas a leva initiales fore 4023872 quas insuper 2561 figurae sequentur.

160. Ope ergo huius logarithmorum summationis producta ex quotcumque factoribus, qui secundum numeros naturales procedunt, proxime assignari poterunt. Huc potissimum referri potest problema, quo quaeritur uncia media seu maxima in potestate binomii quacumque $(a+b)^m$; ubi quidem notandum est, si m sit numerus impar, binas dari medias inter se aequales, quae iunctim sumtae praebeant unciam mediam

diam in potestate sequente pari. Quare cum uncia maxima in quaque potestate pari sit duplo maior quam uncia media in potestate praecedente impari, sufficiet pro potestatibus paribus unciam mediam maximam determinasse. Sit igitur $m = 2n$, & uncia media ita exprimetur ut sit:

$$\frac{2n(2n-1)(2n-2)(2n-3) \;.\;.\;.\;.\; (n+1)}{1.\;\;2.\;\;3.\;\;4.\;.\;.\;.\;.\;.\;.\;.\;.\; n}$$

Vocetur ista uncia media quae quaeritur $= u$, atque eo hoc modo repraesentari poterit, ut sit:

$$u = \frac{1.\;2.\;3.\;4.\;5.\;.\;.\;.\;.\;.\;.\;2n}{(1.\;2.\;3.\;4.\;.\;.\;.\;n)^2}$$

sumtisque logarithmis erit:

$$lu = l1 + l2 + l3 + l4 + l5 + \;.\;.\;.\; + l2n$$
$$-2l1 - 2l2 - 2l3 - 2l4 - 2l5 \;.\;.\;.\; - 2ln$$

161. Iam vero sumendis his logarithmis hyperbolicis erit:

$$l1 + l2 + l3 + l4 + \;.\;.\;.\;.\;.\;. + l2n =$$
$$= \tfrac{1}{2} l 2\pi + (2n + \tfrac{1}{2}) ln + (2n + \tfrac{1}{2}) l2 - 2n$$
$$+ \frac{\mathfrak{A}}{1.2.2n} - \frac{\mathfrak{B}}{3.4.2^3 n^3} + \frac{\mathfrak{C}}{5.6.2^5 n^5} - \&c.$$

$$\& \quad 2l1 + 2l2 + 2l3 + 2l4 + \;.\;.\;.\; + 2ln =$$

$$l2\pi + (2n+1) ln - 2n + \frac{2\mathfrak{A}}{1.2n} - \frac{2\mathfrak{B}}{3.4n^3} + \frac{2\mathfrak{C}}{5.6n^5} - \&c.$$

qua expressione ab illa sublata relinquetur:

$$lu = -\tfrac{1}{2} l\pi - \tfrac{1}{2} ln + 2nl2 + \frac{\mathfrak{A}}{1.2.2n} - \frac{\mathfrak{B}}{3.4.2^3 n^3} + \frac{\mathfrak{C}}{5.6.2^5 n^5} - \&c.$$
$$- \frac{2\mathfrak{A}}{1.2n} + \frac{2\mathfrak{B}}{3.4n^3} - \frac{2\mathfrak{C}}{5.6n^5} + \&c.$$

his vero binis terminis colligendis, erit:

$$lu = l\frac{2^{2n}}{\sqrt{n\pi}} - \frac{3\mathfrak{A}}{1.2.2n} + \frac{15\mathfrak{B}}{3.4.2^3 n^3} - \frac{63\mathfrak{C}}{5.6.2^5 n^5} + \frac{255\mathfrak{D}}{7.8.2^7 n^7} - \&c.$$

Sit

Sit $\frac{3\mathfrak{A}}{1.2.2^2n^2} - \frac{15\mathfrak{B}}{3.4.2^4n^4} + \frac{63\mathfrak{C}}{5.6.2^6n^6} - \frac{255\mathfrak{D}}{7.8.2^8n^8} + \&c.$
$= l\left(1 + \frac{A}{2^2n^2} + \frac{B}{2^4n^4} + \frac{C}{2^6n^6} + \frac{D}{2^8n^8} + \&c.\right)$; erit

$$lu = l\,\frac{2^{2n}}{\sqrt{n\pi}} - 2nl\left(1 + \frac{A}{2^2n^2} + \frac{B}{2^4n^4} + \frac{C}{2^6n^6} + \&c.\right);$$

ideoque $u = \dfrac{2^{2n}}{\left(1 + \frac{A}{2^2n^2} + \frac{B}{2^4n^4} + \frac{C}{2^6n^6} + \&c.\right)^{2n}\sqrt{n\pi}}$

Erit vero posito $2n = m$

$$l\left(1 + \frac{A}{2^2n^2} + \frac{B}{2^4n^4} + \frac{C}{2^6n^6} + \frac{D}{2^8n^8} + \&c.\right) =$$
$$\frac{A}{m^2} + \frac{B}{m^4} + \frac{C}{m^6} + \frac{D}{m^8} + \frac{E}{m^{10}} + \&c.$$
$$- \frac{A^2}{2m^4} - \frac{AB}{m^6} - \frac{AC}{m^8} - \frac{AD}{m^{10}} - \&c.$$
$$- \frac{BB}{2m^8} - \frac{BC}{m^{10}} - \&c.$$
$$+ \frac{A^3}{3m^6} + \frac{A^2B}{m^8} + \frac{A^2C}{m^{10}} + \&c.$$
$$+ \frac{AB^2}{m^{10}} + \&c.$$
$$- \frac{A^4}{4m^8} - \frac{A^3B}{m^{10}} - \&c.$$
$$+ \frac{A^5}{5m^{10}} + \&c.$$

quae expressio cum aequalis esse debeat huic:

$$\frac{3\mathfrak{A}}{1.2m^2} - \frac{15\mathfrak{B}}{3.4m^4} + \frac{63\mathfrak{C}}{5.6m^6} - \frac{255\mathfrak{D}}{7.8m^8} + \&c. \quad \text{fiet:}$$

A

$$A = \frac{3\mathfrak{A}}{1.2}$$

$$B = \frac{A^2}{2} - \frac{15\mathfrak{B}}{3.4}$$

$$C = AB - \tfrac{1}{3}A^3 + \frac{63\mathfrak{C}}{5.6}$$

$$D = AC + \tfrac{1}{2}B^2 - A^2B + \tfrac{1}{4}A^4 - \frac{255\mathfrak{D}}{7.8}$$

$$E = AD + BC - A^2C - AB^2 + A^3B - \tfrac{1}{5}A^5 + \frac{1023\mathfrak{E}}{9.10}$$

&c.

162. Cum iam fit $\mathfrak{A} = \frac{1}{6}$; $\mathfrak{B} = \frac{1}{30}$; $\mathfrak{C} = \frac{1}{42}$; $\mathfrak{D} = \frac{1}{30}$; $\mathfrak{E} = \frac{5}{66}$; erit:

$$A = \frac{1}{4};\; B = -\frac{1}{96};\; C = \frac{27}{640};\; D = -\frac{90031}{2^{11}.3^2.5.7} \;\&c.$$

Hinc efficitur:

$$u = \frac{2^{2n}}{\left(1 + \frac{1}{2^4 n^2} - \frac{1}{2^9.3n^4} + \frac{27}{2^{13}.5n^6} - \frac{90031}{2^{19}.3^2.5.7n^8} + \&c.\right)^{2n}\sqrt{n\pi}}$$

vel fi ifta feriei elevatio actu inftituatur, erit proxime:

$$u = \frac{2^{2n}}{\sqrt{n\pi}\left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{5}{16.128n^4} \&c.\right)}$$

hinc terminus medius in $(1+1)^{2n}$, erit ad fummam omnium terminorum 2^{2n}

uti 1 ad $\sqrt{n\pi}\left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{5}{16.128n^4} - \&c.\right)$

vel pofito brevitatis gratia $4n = \nu$, erit ifta ratio:

ut 1 ad $\sqrt{n\pi}\left(1+\frac{1}{\nu}+\frac{1}{2\nu^2}-\frac{1}{2\nu^3}-\frac{5}{8\nu^4}+\frac{23}{8\nu^5}+\frac{53}{16\nu^6}-\&c.\right)$

EXEMPLUM I.

Quaeratur uncia media in binomio $(a+b)^{10}$ *evoluto, quam constat esse* $=\frac{10.9.8.7.6}{1.2.3.4.5}=252.$

Adhibendo ultimam formulam pro u inventam erit $n=5$

$$\frac{1}{4n}=0,0500000$$

$$\frac{1}{32n^2}=0,0012500$$

$$0,0512500$$

$$\text{subtr.}\ \frac{1}{128n^3}=\ \ 625$$

$$0,0511875$$

$$\text{subtr.}\ \frac{5}{16.128n^4}=\ \ 39$$

$$\text{Ergo } 1+\frac{1}{4n}+\&c.=1,0511836$$

$$\text{Huius log.}=0,0216784$$

$$ln=0,6989700$$

$$l\pi=0,4971498$$

$$1,2177982$$

$$l\sqrt{n\pi}(1+\&c.)=0,6088991$$

$$\text{a}\quad l2^{21}=3,0102999$$

$$lu=2,4014008$$

unde fit $u=252.$

EXEMPLUM II.

Investigetur ratio, quam in potestate centesima binomii $1+1$ *terminus medius ad summam omnium* 2^{100} *tenet.*

Utamur ad hoc formula primum inventa:

$$lu=l\frac{2^{2n}}{\sqrt{n\pi}}-\frac{3\mathfrak{A}}{1.2.2n}+\frac{15\mathfrak{B}}{3.4.2^3n^3}-\frac{63\mathfrak{C}}{5.6.2^5n^5}+\&c.$$

in qua posito $2n = m$ ut habeatur ista potestas $(1+1)^m$. & loco 𝔄, 𝔅, ℭ, 𝔇, substitutis valoribus, fiet:

$$lu = l\frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{1}{4m} + \frac{1}{24m^3} - \frac{1}{20m^5} + \frac{17}{112m^7} - \frac{31}{36m^9} + \frac{691}{88m^{11}} - \&c.$$

qui logarithmi cum sint hyperbolici multiplicentur ii per

$$k = 0,434294481903251,$$

ut transmutentur in tabulares, eritque

$$lu = l\frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{k}{4m} + \frac{k}{24m^3} - \frac{k}{20m^5} + \frac{17k}{112m^7} - \frac{31k}{36m^9} + \&c.$$

unde cum uncia media sit u, erit $2^m : u$ ratio quaesita, ideoque

$$l\frac{2^m}{u} = l\sqrt{\tfrac{1}{2}m\pi} + \frac{k}{4m} - \frac{k}{24m^3} + \frac{k}{20m^5} - \frac{17k}{112m^7} + \frac{31k}{36m^9} - \frac{691k}{88m^{11}} + \&c.$$

Quare cum sit ob exponentem $m = 100$

$$\frac{k}{m} = 0,0043429448;\quad \frac{k}{m^3} = 0,0000004343;$$

$$\frac{k}{m^5} = 0,0000000000;$$ erit:

$$\frac{k}{4m} = 0,0010857362$$

$$\frac{k}{24m^3} = 0,0000000181$$

$$0,0010857181$$

Tum est $l\pi = 0,4971498726$

$$l\tfrac{1}{2}m = 1,6989700043$$

$$l\tfrac{1}{2}m\pi = 2,1961198769$$

$$l\sqrt{\tfrac{1}{2}m\pi} = 1,0980599384$$

$$\frac{k}{4m} - \frac{k}{24m^3} + \&c. = 0,0010857181$$

$$1,0991456565 = l\frac{2^{100}}{u}.$$

Erit ergo $\frac{2^{100}}{u} = 12,56451$, atque adeo in potestate

 (1+

$(1+1)^{100}$ evoluta terminus medius ſe habebit ad ſummam omnium 2^{100} uti 1 ad 12, 56451.

163. Denotet nunc terminus generalis z functionem exponentialem a^x, ita ut ſummari debeat haec ſeries geometrica:

$$s = a + a^2 + a^3 + a^4 + \dots + a^x$$

quae cum ſit geometrica, eius ſumma iam conſtat, erit enim $s = \frac{(a^x - 1)a}{a-1}$. Modo autem hic expoſito hanc ſummam inveſtigemus. Quia eſt $z = a^x$, erit $\int z dx = \frac{a^x}{la}$, huius enim differentiale eſt $a^x dx$, tum vero erit:

$$\frac{dz}{dx} = a^x la; \quad \frac{ddz}{dx^2} = a^x (la)^2; \quad \frac{d^3 z}{dx^3} = a^x (la)^3; \quad \&c.$$

unde ſequitur fore:

$$s = a^x\left(\frac{1}{la} + \frac{1}{2} + \frac{\mathfrak{A}}{1.2} la - \frac{\mathfrak{B}}{1.2.3.4}(la)^3 + \frac{\mathfrak{C}}{1.2.3\dots6}(la)^5 - \&c.\right) + C.$$

Ad conſtantem C definiendam ponatur $x = 0$, & ob $s = 0$, erit $C = -\frac{1}{la} - \frac{1}{2} - \frac{\mathfrak{A}}{1.2} la + \frac{\mathfrak{B}}{1.2.3.4}(la)^3 - \&c.$

ideoque fiet:

$$s = (a^x - 1)\left(\frac{1}{la} + \frac{1}{2} + \frac{\mathfrak{A}}{1.2} la - \frac{\mathfrak{B}}{1.2.3.4}(la)^3 + \frac{\mathfrak{C}}{1.2.3\dots6}(la)^5 - \&c.\right)$$

Cum igitur ſumma ſit $\frac{(a^x - 1)a}{a-1}$, erit:

$$\frac{a}{a-1} = \frac{1}{la} + \frac{1}{2} + \frac{\mathfrak{A}}{1.2} la - \frac{\mathfrak{B}}{1.2.3.4}(la)^3 + \frac{\mathfrak{C}(la)^5}{1.2\dots6} - \&c.$$

ubi la denotat logarithmum hyperbolicum ipſius a: hinc fit $\frac{(a+1)la}{2(a-1)} = 1 + \frac{\mathfrak{A}(la)^2}{1.2} - \frac{\mathfrak{B}(la)^4}{1.2.3.4} + \frac{\mathfrak{C}(la)^6}{1.2\dots6} - \&c.$ ſicque iſtius ſeriei ſumma exhiberi poterit.

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164. Sit terminus generalis $z = \sin ax$, & $s = \sin a + \sin 2a + \sin 3a + \ldots + \sin ax$ quae series cum sit recurrens quoque summari potest; erit enim

$$s = \frac{\sin a + \sin ax - \sin(ax+a)}{1 - 2\cos a + 1} = \frac{\sin a + (1-\cos a)\sin ax - \sin a \cos ax}{2(1-\cos a)}$$

Erit vero $\int z dx = \int dx \sin ax = -\frac{1}{a}\cos ax$, & $\frac{dz}{dx} = a\cos ax$;

$\frac{ddz}{dx^2} = -aa\sin ax$; $\frac{d^3z}{dx^3} = -a^3\cos ax$; $\frac{d^5z}{dx^5} = a^5 \cos ax$ &c.

$$s = C - \frac{1}{a}\cos ax + \tfrac{1}{2}\sin ax + \frac{\mathfrak{A}\,a\cos ax}{1.2} + \frac{\mathfrak{B}\,a^3\cos ax}{1.2.3.4} + \frac{\mathfrak{C}\,a^5\cos ax}{1.2.3.4.5.6} + \frac{\mathfrak{D}\,a^7\cos ax}{1.2\ldots 8} + \&c.$$

Ponatur $x = 0$ ut fiat $s = 0$; eritque:

$$C = \frac{1}{a} - \frac{\mathfrak{A}a}{1.2} - \frac{\mathfrak{B}a^3}{1.2.3.4} - \frac{\mathfrak{C}a^5}{1.2\ldots 6} - \&c. \quad \text{ergo}$$

$$s = \tfrac{1}{2}\sin ax + (1-\cos ax)\left(\frac{1}{a} - \frac{\mathfrak{A}a}{1.2} - \frac{\mathfrak{B}a^3}{1.2.3.4} - \frac{\mathfrak{C}a^5}{1.2\ldots 6} - \&c.\right)$$

At cum sit $s = \tfrac{1}{2}\sin ax + \frac{(1-\cos ax)\sin a}{2(1-\cos a)}$, fiet:

$$\frac{\sin a}{2(1-\cos a)} = \tfrac{1}{2}\cot\tfrac{1}{2}a = \frac{1}{a} - \frac{\mathfrak{A}a}{1.2} - \frac{\mathfrak{B}a^3}{1.2.3.4} - \frac{\mathfrak{C}a^5}{1.2\ldots 6} - \&c.$$

quam eandem seriem iam supra §. 127. habuimus.

165. Sit nunc $z = \cos ax$, ac series summanda: $s = \cos a + \cos 2a + \cos 3a + \ldots + \cos ax$ cuius seriei, quia est recurrens, erit summa:

$$s = \frac{\cos a - 1 + \cos ax - \cos(ax+a)}{1 - 2\cos a + 1} = -\tfrac{1}{2} + \tfrac{1}{2}\cos ax + \tfrac{1}{2}\cot\tfrac{1}{2}a.\sin ax.$$

At vero ad summam nostra methodo exprimendam, erit:

$\int z dx$

$\int z dx = \int dx \cos ax = \frac{1}{a} \sin ax$, & $\frac{dz}{dx} = -a \sin ax$;

$\frac{d^3 z}{dx^3} = a^3 \sin ax$; $\frac{d^5 z}{dx^5} = -a^5 \sin ax$; &c. Ergo

$$s = C + \frac{1}{a} \sin ax + \tfrac{1}{2} \cos ax - \frac{\mathfrak{A} a \sin ax}{1.2} - \frac{\mathfrak{B} a^3 \sin ax}{1.2.3.4} - \&c.$$

Sit $x = 0$, erit $s = 0$, & $C = -\frac{1}{2}$, hincque erit:

$$s = -\tfrac{1}{2} + \tfrac{1}{2} \cos ax + \frac{1}{a} \sin ax - \frac{\mathfrak{A} a \sin ax}{1.2} - \frac{\mathfrak{B} a^3 \sin ax}{1.2.3.4} - \&c.$$

Quare cum sit $s = -\frac{1}{2} + \frac{1}{2} \cos ax + \frac{1}{2} \cot \frac{1}{2} a . \sin ax$, erit uti iam modo invenimus:

$$\tfrac{1}{2} \cot \tfrac{1}{2} a = \frac{1}{a} - \frac{\mathfrak{A} a}{1.2} - \frac{\mathfrak{B} a^3}{1.2.3.4} - \frac{\mathfrak{C} a^5}{1.2.3.4.5.6} - \&c.$$

166. Quoniam supra invenimus, si a denotet arcum quemcunque, esse

$$\frac{\pi}{2} = \frac{a}{2} + \sin a + \tfrac{1}{2} \sin 2a + \tfrac{1}{3} \sin 3a + \tfrac{1}{4} \sin 4a + \&c.$$

consideremus hanc seriem, sitque $z = \frac{1}{x} \sin ax$, ut sit

$$s = \sin a + \tfrac{1}{2} \sin 2a + \tfrac{1}{3} \sin 3a + \dots + \frac{1}{x} \sin ax.$$

Hoc autem casu sit $\int z dx = \int \frac{dx}{x} \sin ax$, quod integrale exhiberi nequit.

Erit vero $\frac{dz}{dx} = \frac{a}{x} \cos ax - \frac{1}{xx} \sin ax$;

$$\frac{ddz}{dx^2} = -\frac{a^2}{x} \sin ax - \frac{2a}{xx} \cos ax + \frac{2}{x^3} \sin ax;$$

$$\frac{d^3z}{dx^3} = -\frac{a^3}{x}\cos ax + \frac{3a^2}{x^2}\sin ax + \frac{6a}{x^3}\cos ax - \frac{6}{x^4}\sin ax;$$

$$\frac{d^4z}{dx^4} = \frac{a^4}{x}\sin ax + \frac{4a^3}{xx}\cos ax - \frac{12a^2}{x^3}\sin ax - \frac{24a}{x^4}\cos ax + \frac{24}{x^5}\sin ax.$$

Quia igitur neque formulam integralem $\int z dx$ exhibere, neque haec differentialia satis commode exprimere licet, summam huius seriei per hanc methodum definire non possumus, ita ut quicquam inde concludi posset. Idem incommodum in multis aliis seriebus occurrit, quoties terminus generalis non satis est simplex, ut eius differentialia ad commodam legem exprimi queant. Quamobrem in sequenti Capite alias expressiones generales pro summis serierum, quarum termini generales vel nimis sunt compositi vel prorsus dari nequeunt, eliciemus; quae feliciori successu in usum vocari poterunt. Imprimis autem insufficientia methodi hic traditae elucet, si signa terminorum seriei propositae alternentur, tum enim quantumvis termini generales sint simplices, tamen termini summatorii hac methodo exhiberi commode nequeunt.