

CAPUT VI.

DE SUMMATIONE PROGRESSIONUM PER SERIES INFINITAS.

140.

Expressio generalis, quam in Capite praecedente pro termino summatorio cuiusque seriei, cuius terminus generalis seu indici x respondens est $=z$, invenimus:

$$Sz = \int z dx + \frac{1}{2}z + \frac{A dz}{1.2 dx} - \frac{B d^3 z}{1.2.3.4 dx^3} + \frac{C d^5 z}{1.2 \dots 6 dx^5} - \&c.$$

proprie inservit seriebus summandis, quarum termini generales sunt functiones quaecunque rationales integrae indicis x , quoniam his casibus ad differentialia tandem evanescentia pervenitur. Sin autem z non fuerit eiusmodi functio ipsius x , tum eius differentialia in infinitum progrediuntur, sicque resultat series infinita summam seriei propositae exprimens, & quidem ad datum usque terminum, cuius index est $=x$. Quocirca progressionis propositae in infinitum continuatae summa prodibit, si ponatur $x = \infty$; hocque pacto alia invenitur series infinita priori aequalis.

141. Sin autem ponatur $x = 0$, tum expressio summam exhibens debet evanescere, uti iam annotavimus; quod nisi fiat, eiusmodi quantitas constans ad summam addi vel inferri debet, ut huic conditioni satisfiat. Quo facto si ponatur $x = 1$, summa inventa praebit terminum primum seriei: si $x = 2$, aggregatum primi & secundi; si $x = 3$, orietur aggregatum trium terminorum initialium seriei, & ita porro. His igitur casibus, quia summa unius, vel duorum, vel trium, &c. terminorum est cognita, seriei infinitae, qua ista summa exprimitur; valor innotescet; ex hocque fonte innumerabiles series summari poterunt.

142. Quoniam, si eiusmodi constans summae fuerit adiecta,

Zz

cta,

sta, ut ea evanescat posito $x=0$, tum summa omnibus reliquis casibus, quicunque numeri pro x substituuntur, satisfacit; manifestum est, dummodo summae inventae eiusmodi quantitas constans adiciatur, ut uno quodam casu vera summa indicetur, tum omnibus reliquis casibus veram summam prodire debere. Quare si ponendo $x=0$, non pateat, cuiusmodi valorem expressio summae recipiat, neque igitur constans adicienda hinc inveniri queat; tum alius quicunque numerus pro x statui poterit, adiciendaque constans effici ut debita summa indicetur: quod quomodo fieri debeat, ex sequentibus magis fiet perspicuum.

Consideremus primum hanc progressionem harmonicam:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{x} = s,$$

cuius terminus generalis cum sit $= \frac{1}{x}$, fiet $z = \frac{1}{x}$, & terminus summatorius s ita invenietur. Primo erit

$$\int z dx = \int \frac{dx}{x} = lx; \text{ deinde differentialia ita se habebunt:}$$

$$\frac{dz}{dx} = -\frac{1}{x^2}; \quad \frac{d^2z}{dx^2} = \frac{1}{x^3}; \quad \frac{d^3z}{dx^3} = -\frac{1}{x^4};$$

$$\frac{d^4z}{dx^4} = \frac{1}{x^5}; \quad \frac{d^5z}{dx^5} = -\frac{1}{x^6} \text{ \&c. Hinc itaque erit:}$$

$$s = lx + \frac{1}{2x} - \frac{1}{2x^2} + \frac{1}{4x^4} - \frac{1}{6x^6} + \frac{1}{8x^8} - \text{\&c.} + \text{Const.}$$

Constans igitur hic addenda ex casu $x=0$ non potest defini. Ponatur ergo $x=1$, quia tum fit $s=1$, erit

$$1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{8} + \text{Const. unde fit ista constans}$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \text{\&c. eritque ideo terminus summatorius quaesitus:}$$

$s =$

$$s = \ln x + \frac{1}{2x} - \frac{1}{2x^2} + \frac{1}{4x^4} - \frac{1}{6x^6} + \frac{1}{8x^8} - \dots$$

$$+ \frac{1}{2} - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \dots + \dots$$

143. Quoniam numeri Bernoulliani A, B, C, D, &c. constituunt seriem divergentem, hic valor constantis cognosci nequit. Sin autem loco x substituatur numerus maior, atque summa totidem terminorum actu quaeratur, valor constantis commode investigabitur. Ponatur in hunc finem x=10, decemque primis terminis colligendis reperietur eorum summa =

2,928968253968253968

cui aequalis esse debet expressio summae, si in ea ponatur x=10, quae fit:

$$1/10 + \frac{1}{20} - \frac{1}{200} + \frac{1}{4000} - \frac{1}{600000} + \frac{1}{80000000} - \dots + C.$$

sumto ergo pro 1/10 logarithmo hyperbolico denarii & loco A, B, C, &c. substitutis valoribus supra inventis, reperietur constans illa: C = 0,5772156649015325 qui numerus ergo exprimit summam seriei:

$$\frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \dots$$

144. Si pro x numeri non nimis magni substituuntur, quia summa seriei facile actu invenitur, obtinebitur, summa seriei huius:

$$\frac{1}{2x} - \frac{1}{2x^2} + \frac{1}{4x^4} - \frac{1}{6x^6} + \frac{1}{8x^8} - \dots = s - \ln x - C.$$

Sin autem x significet numerum valde magnum, quia tum valor huius expressionis in infinitum excurrentis facile in fractionibus decimalibus assignatur, vicissim summa seriei definitur. Ac primo quidem constat, si series in infinitum continetur, eius summam futuram esse infinite magnam; factum enim x = ∞ fit ln quoque infinitus; etsi ∞ ad x rationem infinite parvam teneat. Quo autem commodius summa quot-

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quotcunque terminorum seriei assignari queat, valores litterarum A, B, C, &c. in fractionibus decimalibus exprimamus :

$$\begin{aligned} A &= 0,166666666666 \\ B &= 0,033333333333 \\ C &= 0,0238095238095 \\ D &= 0,033333333333 \\ E &= 0,075757575757 \\ F &= 0,2531135531135 \\ G &= 1,166666666666 \\ H &= 7,0921568627451 \end{aligned} \quad \&c.$$

unde ergo erit

$$\begin{aligned} \frac{A}{2} &= 0,083333333333 \\ \frac{B}{4} &= 0,008333333333 \\ \frac{C}{6} &= 0,0039682539682 \\ \frac{D}{8} &= 0,004166666666 \\ \frac{E}{10} &= 0,0075757575757 \\ \frac{F}{12} &= 0,0210927960928 \\ \frac{G}{14} &= 0,083333333333 \\ \frac{H}{16} &= 0,4432598039216 \end{aligned} \quad \&c.$$

EXEMPLUM I.

Invenire summam mille terminorum seriei

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c.$$

Ponatur ergo $x = 1000$, & cum sit

$$l_{10} = 2,3025850929940456840 \quad \text{erit}$$

$$lx = 6,9077552789821$$

$$\text{Const.} = 0,5772156649015$$

$$\frac{1}{2x} = 0,0005000000000$$

$$7,4854709438836$$

subt. $\frac{2}{2xx} = 0,0000000833333$

$$7,4854708505503$$

add. $\frac{3}{4x^4} = 0,0000000000000$

$$\text{Ergo} \quad 7,4854708505503$$

est summa quaesita mille terminorum, qui nequidem septem unitates cum semisse conficiunt.

E X E M P L U M II.

Invenire summam millies mille terminorum seriei

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Quia est $x = 1000000$, erit $lx = 6.l 10$, ergo

$$lx = 13,8155105579642$$

$$\text{Const.} = 0,5772156649015$$

$$\frac{1}{2x} = 0,0000005000000$$

$$14,3927267228657 = \text{summae quaesitae.}$$

145. Si ergo pro x statuatur numerus vehementer magnus, summa satis exacte invenitur ex solo primo termino lx constante C aucto: unde egregia corollaria deduci possunt. Sic si x fuerit numerus vehementer magnus, ponaturque:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{x} = s$$

$$\& 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{x} + \dots + \frac{1}{x+y} = t$$

quia

quia est proxime $s = lx + C$, & $t = l(x+y) + C$;

erit $t - s = l(x+y) - lx = l \frac{x+y}{x}$, ideoque hic logarithmus proxime per seriem harmonicam finito terminorum numero constantem exprimetur hoc modo:

$$l \frac{x+y}{x} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \dots + \frac{1}{x+y}$$

Accuratius autem hic logarithmus exhibebitur, si superiores summae s & t exactius capiantur. Sic cum fit

$$s = lx + C + \frac{1}{2x} - \frac{1}{12xx}, \quad \&$$

$$t = l(x+y) + C + \frac{1}{2(x+y)} - \frac{1}{12(x+y)^2}; \quad \text{erit}$$

$$t - s = l \frac{x+y}{x} - \frac{1}{2x} + \frac{1}{2(x+y)} + \frac{1}{12xx} - \frac{1}{12(x+y)^2}, \quad \text{ideoque}$$

$$l \frac{x+y}{x} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \dots + \frac{1}{x+y} + \frac{1}{2x} - \frac{1}{2(x+y)} - \frac{1}{12xx} + \frac{1}{12(x+y)^2}$$

Sin autem fit numerus tam magnus, ut bini termini ultimi reijci queant, erit proxime:

$$l \frac{x+y}{x} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \dots + \frac{1}{x+y} + \frac{1}{2} \left(\frac{1}{x} - \frac{1}{x+y} \right)$$

Ex hac quoque serie harmonica derivare poterimus summam huius seriei, in qua tantum numeri impares occurrunt:

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} \dots + \frac{1}{2n+1}$$

Cum enim omnibus terminis capiendis fit:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots + \frac{1}{2n} + \frac{1}{2n+1} =$$

$$l(2n+1) + C + \frac{1}{2(2n+1)} - \frac{1}{2(2n+1)^2} + \frac{1}{4(2n+1)^4} - \frac{1}{6(2n+1)^6} + \&c.$$

termi-

terminorum vero ordine parium:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} \dots \dots \dots + \frac{1}{2n}$$

summa fit semiffis superioris nempe:

$$\frac{1}{2} C + \frac{1}{2} l n + \frac{1}{4n} - \frac{1}{4n^2} + \frac{1}{8n^4} - \frac{1}{12n^6} + \frac{1}{16n^8} - \&c.$$

erit hac serie ab illa ablata:

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots \dots \dots + \frac{1}{2n+1} =$$

$$\frac{1}{2} C + l \frac{2n+1}{\sqrt{n}} + \frac{1}{2(2n+1)} - \frac{1}{2(2n+1)^2} + \frac{1}{4(2n+1)^4} - \&c.$$

$$- \frac{1}{4n} + \frac{1}{4n^2} - \frac{1}{8n^4} + \&c.$$

146. Potest vero etiam per eandem expressionem generalem summa cuiusque seriei harmonicae inveniri; fit enim;

$$\frac{1}{m+n} + \frac{1}{2m+n} + \frac{1}{3m+n} + \frac{1}{4m+n} + \dots + \frac{1}{mx+n} = s,$$

quia est terminus generalis $z = \frac{1}{mx+n}$, erit:

$$fzdx = \frac{1}{m} l(mx+n); \quad \frac{dz}{dn} = - \frac{m}{(mx+n)^2}$$

$$\frac{ddz}{2dx^2} = \frac{mm}{(mx+n)^3}; \quad \frac{d^3z}{6dx^3} = - \frac{m^3}{(mx+n)^4}$$

$$\frac{d^4z}{24dx^4} = \frac{m^4}{(mx+n)^5}; \quad \frac{d^5z}{120dx^5} = - \frac{m^5}{(mx+n)^6} \&c.$$

Ex his ergo reperitur:

$$s = D + \frac{1}{m} l(mx+n) + \frac{1}{2(mx+n)} - \frac{1}{2(mx+n)^2} + \frac{1}{4(mx+n)^4}$$

$$- \frac{1}{6(mx+n)^6} + \frac{1}{8(mx+n)^8} - \&c.$$

Po-

Posito ergo $x=0$, fiet constans illa addenda:

$$D = -\frac{1}{m}ln - \frac{1}{2n} + \frac{1}{2n^2} - \frac{1}{4n^4} + \frac{1}{6n^6} - \&c.$$

147. Si vero fit $n=0$, quoniam seriei:

$$\frac{1}{m} + \frac{1}{2m} + \frac{1}{3m} + \frac{1}{4m} + \dots + \frac{1}{m^x} \text{ sum-}$$

$$\text{ma est} = \frac{1}{m}C + \frac{1}{m}lnx + \frac{1}{2mx} - \frac{1}{2mx^2} + \frac{1}{4mx^4} - \&c.$$

at vero huius seriei:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{m^x}$$

$$\text{summa est} = C + lmx + \frac{1}{2mx} - \frac{1}{2m^2x^2} + \frac{1}{4m^4x^4} - \&c.$$

si ab hac serie illa m vicibus sumpta subtrahatur ut prodeat haec series:

$$1 + \frac{1}{2} + \dots + \frac{1}{m} + \dots + \frac{1}{2m} + \dots + \frac{1}{3m} + \dots + \frac{1}{mx}$$

$$-\frac{1}{m} \qquad -\frac{1}{2m} \qquad -\frac{1}{3m} \qquad -\frac{1}{mx}$$

$$\text{eius summa erit} = lmx + \frac{1}{2mx} - \frac{1}{2m^2x^2} + \frac{1}{4m^4x^4} - \&c.$$

$$-\frac{1}{2x} + \frac{1}{2xx} - \frac{1}{4x^4} + \&c.$$

atque si statuatur $x=\infty$ summa erit $=lm$. Hinc pro m ponendo numeros 2, 3, 4, &c. erit:

$$l_2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \&c.$$

$$l_3 = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \&c.$$

$$l_4 = 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{3}{8} + \&c.$$

$$l_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{4}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{4}{10} + \&c.$$

148. Relicta autem serie harmonica progrediamur ad seriem quadratorum reciprocam, sitque:

s =

$$s = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots + \frac{1}{n^2},$$

in qua cum sit terminus generalis $z = \frac{1}{n^2}$, erit

$$\int z dx = -\frac{1}{x}, \text{ \& differentialia ipsius } z \text{ ita se habebunt}$$

$$\frac{dz}{dx} = -\frac{1}{x^3}; \frac{ddz}{2 \cdot 3 dx^2} = \frac{1}{x^4}; \frac{d^3z}{2 \cdot 3 \cdot 4 dx^3} = -\frac{1}{x^5}; \text{ \&c.}$$

unde erit summa

$$s = C - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{3x^3} + \frac{1}{4x^4} - \frac{1}{5x^5} + \frac{1}{6x^6} - \frac{1}{7x^7} + \dots + \text{\&c.}$$

in qua constans addenda C ex uno casu, quo summa constat, est definienda. Ponamus ergo $x = 1$, quia fit $s = 1$ debet esse: $C = 1 + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots + \text{\&c.}$ quae series autem cum sit maxime divergens, valorem constantis C non ostendit. Quia autem supra demonstravimus summam huius seriei in infinitum continuatae esse

$$= \frac{\pi\pi}{6}: \text{ facto } x = \infty, \text{ si ponatur } s = \frac{\pi\pi}{6}, \text{ fiet } C = \frac{\pi\pi}{6};$$

ob reliquos terminos omnes evanescentes. Erit ergo

$$1 + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots + \text{\&c.} = \frac{\pi\pi}{6}.$$

149. Sin autem summa huius seriei cognita non fuisset, valor constantis illius C ex alio quopiam casu, quo summa actu est inventa, determinari deberet. Hunc in finem ponamus $x = 10$, atque decem terminis actu addendis reperietur:

$$s = 1, 549767731166540690 \quad \text{tum est}$$

$$\text{add. } \frac{1}{x} = 0, 1$$

$$\text{subtr. } \frac{1}{2x^2} = 0, 005$$

$$1, 644767731166540690$$

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add.	$\frac{A}{x^3} =$	$1, 644767731166540690$ $0, 000166666666666666$
subtr.	$\frac{B}{x^5} =$	$1, 644934397833207356$ $0, 000000333333333333$
add.	$\frac{C}{x^7} =$	$1, 644934064499874023$ $0, 000000002380952381$
subtr.	$\frac{D}{x^9} =$	$1, 644934066880826404$ $0, 000000000033333333$
add.	$\frac{E}{x^{11}} =$	$1, 644934066847493071$ $0, 000000000000757575$
subtr.	$\frac{F}{x^{13}} =$	$1, 644934066848250646$ $0, 000000000000025311$
add.	$\frac{G}{x^{15}} =$	$1, 644934066848225335$ $0, 000000000000001166$
subtr.	$\frac{H}{x^{17}} =$	$\frac{71}{1, 644934066848226430} = C.$

Hicque numerus simul est valor expressionis $\frac{\pi\pi}{6}$, quemadmodum ex valore ipsius π cognito calculum instituenti patebit. Unde simul intelligitur, etiamsi series $A, B, C, \&c.$ divergat, tamen hoc modo veram prodire summam.

150. Sit nunc $z = \frac{1}{x^3}$; atque

$$s = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots + \frac{1}{x^3}, \quad \text{quia est}$$

$$sz dx = -\frac{1}{2x^4}; \quad \frac{dz}{1 \cdot 2 \cdot 3 dx} = -\frac{1}{2x^4}; \quad \frac{ddz}{1 \cdot 2 \cdot 3 \cdot 4 dx^2} = \frac{1}{2x^5}$$

$$\frac{d^3z}{1 \cdot 2 \dots 5 dx^3} = -\frac{1}{2x^6}; \quad \frac{d^5z}{1 \cdot 2 \dots 7 dx^5} = -\frac{1}{2x^8}; \quad \&c. \quad \text{erit}$$

$$s =$$

$$s = C - \frac{1}{2n} + \frac{1}{2n^3} - \frac{3A}{2n^4} + \frac{5B}{2n^6} - \frac{7C}{2n^8} + \&c.$$

hincque posito $n = 1$, ob $s = 1$, fiet:

$$C = 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{2}A - \frac{1}{2}B + \frac{1}{2}C - \frac{1}{2}D + \&c.$$

atque iste valor ipsius C simul ostendet summam seriei propositae in infinitum continuatae. Quoniam vero summae potestatum imparium non aequae ac parium constant, iste ipsius C valor ex cognita summa aliquot terminorum definiri debet. Sit ergo $n = 10$, erit:

$$C = s + \frac{1}{2n} - \frac{1}{2n^3} + \frac{3A}{2n^4} - \frac{5B}{2n^6} + \frac{7C}{2n^8} - \&c.$$

Est vero ad computum facilius instituendum:

$\frac{3A}{2}$	=	0,250000000000	
$\frac{5B}{2}$	=	0,083333333333	
$\frac{7C}{2}$	=	0,083333333333	
$\frac{9D}{2}$	=	0,150000000000	
$\frac{11E}{2}$	=	0,416666666666	
$\frac{13F}{2}$	=	1,645238095238	
$\frac{15G}{2}$	=	8,750000000000	
$\frac{17H}{2}$	=	60,283333333333	&c.

Hinc ergo fient termini ad s addendi:

$$\frac{1}{2N^2} = 0,005000000000000000$$

$$\frac{3N}{2N^4} = 0,000025000000000000$$

$$\frac{7C}{2N^8} = 0,000000000833333333$$

$$\frac{11E}{2N^{12}} = 0,000000000000416666$$

$$\frac{15G}{2N^{16}} = 0,000000000000000875$$

$0,005025000833750875$
termini autem subtrahendi sunt:

$$\frac{1}{2N^3} = 0,000500000000000000$$

$$\frac{5B}{2N^6} = 0,000000083333333333$$

$$\frac{9D}{2N^{10}} = 0,000000000150000000$$

$$\frac{13F}{2N^{14}} = 0,00000000000016452$$

$$\frac{17H}{2N^{18}} = 0,000000000000000060$$

$$\text{ab : } \begin{array}{r} 0,000500083348349845 \\ 0,005025000833750875 \\ \hline \end{array}$$

$$s = \begin{array}{r} 0,004524917485401030 \\ 1,197531985674193251 \\ \hline \end{array}$$

$$C = 1,202056903159594281.$$

151. Si hoc modo ulterius progrediamur, inveniemus summas omnium serierum potestatum reciprocarum in fractionibus decimalibus expressas:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c. = 1,6449340668482264 = \frac{2^2 \mathfrak{A}}{1.2} \pi^2$$

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \&c. = 1,2020569031595942$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \&c. = 1,0823232337111381 = \frac{2^3 \mathfrak{B}}{1.2.3.4} \pi^4$$

$$1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \&c. = 1,0369277551068632$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \&c. = 1,0173430619844491 = \frac{2^4 \mathfrak{C}}{1.2 \dots 6} \pi^6$$

$$1 + \frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{4^7} + \&c. = 1,0083492773866018$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \&c. = 1,0040773561979443 = \frac{2^5 \mathfrak{D}}{1.2 \dots 8} \pi^8$$

$$1 + \frac{1}{2^9} + \frac{1}{3^9} + \frac{1}{4^9} + \&c. = 1,0020083928260822$$

$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \&c. = 1,0009945751278180 = \frac{2^6 \mathfrak{E}}{1.2 \dots 10} \pi^{10}$$

$$1 + \frac{1}{2^{11}} + \frac{1}{3^{11}} + \frac{1}{4^{11}} + \&c. = 1,0004941886041094$$

$$1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{4^{12}} + \&c. = 1,0002460865533080 = \frac{2^{11} \mathfrak{F}}{1.2 \dots 12} \pi^{12}$$

$$1 + \frac{1}{2^{13}} + \frac{1}{3^{13}} + \frac{1}{4^{13}} + \&c. = 1,0001227233475857$$

$$1 + \frac{1}{2^{14}} + \frac{1}{3^{14}} + \frac{1}{4^{14}} + \&c. = 1,0000612481350587 = \frac{2^{13} \mathfrak{G}}{1.2 \dots 14} \pi^{14}$$

$$1 + \frac{1}{2^{15}} + \frac{1}{3^{15}} + \frac{1}{4^{15}} + \&c. = 1,0000305882363070$$

$$1 + \frac{1}{2^{16}} + \frac{1}{3^{16}} + \frac{1}{4^{16}} + \&c. = 1,0000152822594086 = \frac{2^{14} \mathfrak{H}}{1.2 \dots 16} \pi^{16}$$

&c. 152.

152. Ex his ergo vicissim summae illarum serierum infinitarum numeris Bernoullianis constantium exhiberi poterunt. Erit enim:

$$\begin{aligned}
 1 + 0 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{8} + \frac{1}{8} - \frac{1}{16} + \frac{1}{16} - \frac{1}{32} + \frac{1}{32} - \frac{1}{64} + \frac{1}{64} - \frac{1}{128} + \frac{1}{128} - \frac{1}{256} + \frac{1}{256} - \frac{1}{512} + \frac{1}{512} - \frac{1}{1024} + \frac{1}{1024} - \frac{1}{2048} + \frac{1}{2048} - \frac{1}{4096} + \frac{1}{4096} - \frac{1}{8192} + \frac{1}{8192} - \frac{1}{16384} + \frac{1}{16384} - \frac{1}{32768} + \frac{1}{32768} - \frac{1}{65536} + \frac{1}{65536} - \frac{1}{131072} + \frac{1}{131072} - \frac{1}{262144} + \frac{1}{262144} - \frac{1}{524288} + \frac{1}{524288} - \frac{1}{1048576} + \frac{1}{1048576} - \frac{1}{2097152} + \frac{1}{2097152} - \frac{1}{4194304} + \frac{1}{4194304} - \frac{1}{8388608} + \frac{1}{8388608} - \frac{1}{16777216} + \frac{1}{16777216} - \frac{1}{33554432} + \frac{1}{33554432} - \frac{1}{67108864} + \frac{1}{67108864} - \frac{1}{134217728} + \frac{1}{134217728} - \frac{1}{268435456} + \frac{1}{268435456} - \frac{1}{536870912} + \frac{1}{536870912} - \frac{1}{1073741824} + \frac{1}{1073741824} - \frac{1}{2147483648} + \frac{1}{2147483648} - \frac{1}{4294967296} + \frac{1}{4294967296} - \frac{1}{8589934592} + \frac{1}{8589934592} - \frac{1}{17179869184} + \frac{1}{17179869184} - \frac{1}{34359738368} + \frac{1}{34359738368} - \frac{1}{68719476736} + \frac{1}{68719476736} - \frac{1}{137438953472} + \frac{1}{137438953472} - \frac{1}{274877906944} + \frac{1}{274877906944} - \frac{1}{549755813888} + \frac{1}{549755813888} - \frac{1}{1099511627776} + \frac{1}{1099511627776} - \frac{1}{2199023255552} + \frac{1}{2199023255552} - \frac{1}{4398046511104} + \frac{1}{4398046511104} - \frac{1}{8796093022208} + \frac{1}{8796093022208} - \frac{1}{17592186044416} + \frac{1}{17592186044416} - \frac{1}{35184372088832} + \frac{1}{35184372088832} - \frac{1}{70368744177664} + \frac{1}{70368744177664} - \frac{1}{140737488355328} + \frac{1}{140737488355328} - \frac{1}{281474976710656} + \frac{1}{281474976710656} - \frac{1}{562949953421312} + \frac{1}{562949953421312} - \frac{1}{1125899906842624} + \frac{1}{1125899906842624} - \frac{1}{2251799813685248} + \frac{1}{2251799813685248} - \frac{1}{4503599627370496} + \frac{1}{4503599627370496} - \frac{1}{9007199254740992} + \frac{1}{9007199254740992} - \frac{1}{18014398509481984} + \frac{1}{18014398509481984} - \frac{1}{36028797018963968} + \frac{1}{36028797018963968} - \frac{1}{72057594037927936} + \frac{1}{72057594037927936} - \frac{1}{144115188075855872} + \frac{1}{144115188075855872} - \frac{1}{288230376151711744} + \frac{1}{288230376151711744} - \frac{1}{576460752303423488} + \frac{1}{576460752303423488} - \frac{1}{1152921504606846976} + \frac{1}{1152921504606846976} - \frac{1}{2305843009213693952} + \frac{1}{2305843009213693952} - \frac{1}{4611686018427387904} + \frac{1}{4611686018427387904} - \frac{1}{9223372036854775808} + \frac{1}{9223372036854775808} - \frac{1}{18446744073709551616} + \frac{1}{18446744073709551616} - \frac{1}{36893488147419103232} + \frac{1}{36893488147419103232} - \frac{1}{73786976294838206464} + \frac{1}{73786976294838206464} - \frac{1}{147573952589676412928} + \frac{1}{147573952589676412928} - \frac{1}{295147905179352825856} + \frac{1}{295147905179352825856} - \frac{1}{590295810358705651712} + \frac{1}{590295810358705651712} - \frac{1}{1180591620717411303424} + \frac{1}{1180591620717411303424} - \frac{1}{2361183241434822606848} + \frac{1}{2361183241434822606848} - \frac{1}{4722366482869645213696} + \frac{1}{4722366482869645213696} - \frac{1}{9444732965739290427392} + \frac{1}{9444732965739290427392} - \frac{1}{18889465931478580854784} + \frac{1}{18889465931478580854784} - \frac{1}{37778931862957161709568} + \frac{1}{37778931862957161709568} - \frac{1}{75557863725914323419136} + \frac{1}{75557863725914323419136} - \frac{1}{151115727451828646838272} + \frac{1}{151115727451828646838272} - \frac{1}{302231454903657293676544} + \frac{1}{302231454903657293676544} - \frac{1}{604462909807314587353088} + \frac{1}{604462909807314587353088} - \frac{1}{1208925819614629174706176} + \frac{1}{1208925819614629174706176} - \frac{1}{2417851639229258349412352} + \frac{1}{2417851639229258349412352} - \frac{1}{4835703278458516698824704} + \frac{1}{4835703278458516698824704} - \frac{1}{9671406556917033397649408} + \frac{1}{9671406556917033397649408} - \frac{1}{19342813113834066795298816} + \frac{1}{19342813113834066795298816} - \frac{1}{38685626227668133590597632} + \frac{1}{38685626227668133590597632} - \frac{1}{77371252455336267181195264} + \frac{1}{77371252455336267181195264} - \frac{1}{154742504910672534362390528} + \frac{1}{154742504910672534362390528} - \frac{1}{309485009821345068724781056} + \frac{1}{309485009821345068724781056} - \frac{1}{618970019642690137449562112} + \frac{1}{618970019642690137449562112} - \frac{1}{1237940039285380274899124224} + \frac{1}{1237940039285380274899124224} - \frac{1}{2475880078570760549798248448} + \frac{1}{2475880078570760549798248448} - \frac{1}{4951760157141521099596496896} + \frac{1}{4951760157141521099596496896} - \frac{1}{9903520314283042199192993792} + \frac{1}{9903520314283042199192993792} - \frac{1}{19807040628566084398385987584} + \frac{1}{19807040628566084398385987584} - \frac{1}{39614081257132168796771975168} + \frac{1}{39614081257132168796771975168} - \frac{1}{79228162514264337593543950336} + \frac{1}{79228162514264337593543950336} - \frac{1}{158456325028528675187087900672} + \frac{1}{158456325028528675187087900672} - \frac{1}{316912650057057350374175801344} + \frac{1}{316912650057057350374175801344} - \frac{1}{633825300114114700748351602688} + \frac{1}{633825300114114700748351602688} - \frac{1}{1267650600228229401496703205376} + \frac{1}{1267650600228229401496703205376} - \frac{1}{2535301200456458802993406410752} + \frac{1}{2535301200456458802993406410752} - \frac{1}{5070602400912917605986812821504} + \frac{1}{5070602400912917605986812821504} - \frac{1}{10141204801825835211973625643008} + \frac{1}{10141204801825835211973625643008} - \frac{1}{20282409603651670423947251286016} + \frac{1}{20282409603651670423947251286016} - \frac{1}{40564819207303340847894502572032} + \frac{1}{40564819207303340847894502572032} - \frac{1}{81129638414606681695789005144064} + \frac{1}{81129638414606681695789005144064} - \frac{1}{162259276829213363391578010288128} + \frac{1}{162259276829213363391578010288128} - \frac{1}{324518553658426726783156020576256} + \frac{1}{324518553658426726783156020576256} - \frac{1}{649037107316853453566312041152512} + \frac{1}{649037107316853453566312041152512} - \frac{1}{1298074214633706907132624082305024} + \frac{1}{1298074214633706907132624082305024} - \frac{1}{2596148429267413814265248164610048} + \frac{1}{2596148429267413814265248164610048} - \frac{1}{5192296858534827628530496329220096} + \frac{1}{5192296858534827628530496329220096} - \frac{1}{10384593717069655257060992658440192} + \frac{1}{10384593717069655257060992658440192} - \frac{1}{20769187434139310514121985316880384} + \frac{1}{20769187434139310514121985316880384} - \frac{1}{41538374868278621028243970633760768} + \frac{1}{41538374868278621028243970633760768} - \frac{1}{83076749736557242056487941267521536} + \frac{1}{83076749736557242056487941267521536} - \frac{1}{166153499473114484112975882535043072} + \frac{1}{166153499473114484112975882535043072} - \frac{1}{332306998946228968225951765070086144} + \frac{1}{332306998946228968225951765070086144} - \frac{1}{664613997892457936451903530140172288} + \frac{1}{664613997892457936451903530140172288} - \frac{1}{1329227995784915872903807060280344576} + \frac{1}{1329227995784915872903807060280344576} - \frac{1}{2658455991569831745807614120560689152} + \frac{1}{2658455991569831745807614120560689152} - \frac{1}{5316911983139663491615228241121378304} + \frac{1}{5316911983139663491615228241121378304} - \frac{1}{10633823966279326983230456482242756608} + \frac{1}{10633823966279326983230456482242756608} - \frac{1}{21267647932558653966460912964485513216} + \frac{1}{21267647932558653966460912964485513216} - \frac{1}{42535295865117307932921825928971026432} + \frac{1}{42535295865117307932921825928971026432} - \frac{1}{85070591730234615865843651857942052864} + \frac{1}{85070591730234615865843651857942052864} - \frac{1}{170141183460469231731687303715884105728} + \frac{1}{170141183460469231731687303715884105728} - \frac{1}{340282366920938463463374607431768211456} + \frac{1}{340282366920938463463374607431768211456} - \frac{1}{680564733841876926926749214863536422912} + \frac{1}{680564733841876926926749214863536422912} - \frac{1}{1361129467683753853853498429727072845824} + \frac{1}{1361129467683753853853498429727072845824} - \frac{1}{2722258935367507707706996859454145691648} + \frac{1}{2722258935367507707706996859454145691648} - \frac{1}{5444517870735015415413993718908291383296} + \frac{1}{5444517870735015415413993718908291383296} - \frac{1}{10889035741470030830827987437816582766592} + \frac{1}{10889035741470030830827987437816582766592} - \frac{1}{21778071482940061661655974875633165533184} + \frac{1}{21778071482940061661655974875633165533184} - \frac{1}{43556142965880123323311949751266331066368} + \frac{1}{43556142965880123323311949751266331066368} - \frac{1}{87112285931760246646623899502532662132736} + \frac{1}{87112285931760246646623899502532662132736} - \frac{1}{174224571863520493293247799005065324265472} + \frac{1}{174224571863520493293247799005065324265472} - \frac{1}{348449143727040986586495598010130648530944} + \frac{1}{348449143727040986586495598010130648530944} - \frac{1}{696898287454081973172991196020261297061888} + \frac{1}{696898287454081973172991196020261297061888} - \frac{1}{1393796574908163946345982392040522594123776} + \frac{1}{1393796574908163946345982392040522594123776} - \frac{1}{2787593149816327892691964784081045188247552} + \frac{1}{2787593149816327892691964784081045188247552} - \frac{1}{5575186299632655785383929568162090376495104} + \frac{1}{5575186299632655785383929568162090376495104} - \frac{1}{11150372599265311570767859136324180752990208} + \frac{1}{11150372599265311570767859136324180752990208} - \frac{1}{22300745198530623141535718272648361505980416} + \frac{1}{22300745198530623141535718272648361505980416} - \frac{1}{44601490397061246283071436545296723011960832} + \frac{1}{44601490397061246283071436545296723011960832} - \frac{1}{89202980794122492566142873090593446023921664} + \frac{1}{89202980794122492566142873090593446023921664} - \frac{1}{178405961588244985132285746181186892047843328} + \frac{1}{178405961588244985132285746181186892047843328} - \frac{1}{356811923176489970264571492362373784095686656} + \frac{1}{356811923176489970264571492362373784095686656} - \frac{1}{713623846352979940529142984724747568191373312} + \frac{1}{713623846352979940529142984724747568191373312} - \frac{1}{1427247692705959881058285969449495136382746624} + \frac{1}{1427247692705959881058285969449495136382746624} - \frac{1}{2854495385411919762116571938898990272765493248} + \frac{1}{2854495385411919762116571938898990272765493248} - \frac{1}{5708990770823839524233143877797980545530986496} + \frac{1}{5708990770823839524233143877797980545530986496} - \frac{1}{11417981541647679048466287755595961091061972992} + \frac{1}{11417981541647679048466287755595961091061972992} - \frac{1}{22835963083295358096932575511191922182123945984} + \frac{1}{22835963083295358096932575511191922182123945984} - \frac{1}{45671926166590716193865151022383844364247891968} + \frac{1}{45671926166590716193865151022383844364247891968} - \frac{1}{91343852333181432387730302044767688728495783936} + \frac{1}{91343852333181432387730302044767688728495783936} - \frac{1}{182687704666362864775460604089535377456991567872} + \frac{1}{182687704666362864775460604089535377456991567872} - \frac{1}{365375409332725729550921208179070754913983135744} + \frac{1}{365375409332725729550921208179070754913983135744} - \frac{1}{730750818665451459101842416358141509827966271488} + \frac{1}{730750818665451459101842416358141509827966271488} - \frac{1}{1461501637330902918203684832716283019655932542976} + \frac{1}{1461501637330902918203684832716283019655932542976} - \frac{1}{2923003274661805836407369665432566039311851085952} + \frac{1}{2923003274661805836407369665432566039311851085952} - \frac{1}{5846006549323611672814739330865132078623702171904} + \frac{1}{5846006549323611672814739330865132078623702171904} - \frac{1}{11692013098647223345629478661730264157247404343808} + \frac{1}{11692013098647223345629478661730264157247404343808} - \frac{1}{23384026197294446691258957323460528314494808687616} + \frac{1}{23384026197294446691258957323460528314494808687616} - \frac{1}{46768052394588893382517914646921056628989617375232} + \frac{1}{46768052394588893382517914646921056628989617375232} - \frac{1}{93536104789177786765035829293842113257979234750464} + \frac{1}{93536104789177786765035829293842113257979234750464} - \frac{1}{187072209578355573530071658587684226515958469500928} + \frac{1}{187072209578355573530071658587684226515958469500928} - \frac{1}{374144419156711147060143317175368453031916939001856} + \frac{1}{374144419156711147060143317175368453031916939001856} - \frac{1}{748288838313422294120286634350736906063833878003712} + \frac{1}{748288838313422294120286634350736906063833878003712} - \frac{1}{1496577676626844588240573268701473812127667756007424} + \frac{1}{1496577676626844588240573268701473812127667756007424} - \frac{1}{2993155353253689176481146537402947624255335512014848} + \frac{1}{2993155353253689176481146537402947624255335512014848} - \frac{1}{5986310706507378352962293074805895248510671024029696} + \frac{1}{5986310706507378352962293074805895248510671024029696} - \frac{1}{11972621413014756705924586149611790497021342048059392} + \frac{1}{11972621413014756705924586149611790497021342048059392} - \frac{1}{23945242826029513411849172299223580994042684096118784} + \frac{1}{23945242826029513411849172299223580994042684096118784} - \frac{1}{47890485652059026823698344598447161988085368192237568} + \frac{1}{47890485652059026823698344598447161988085368192237568} - \frac{1}{95780971304118053647396689196894323976170736384475136} + \frac{1}{95780971304118053647396689196894323976170736384475136} - \frac{1}{191561942608236107294793378393788647952341472768950272} + \frac{1}{191561942608236107294793378393788647952341472768950272} - \frac{1}{3831238852164722145895867$$

$$\begin{aligned}
 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \&c. & . . & = \frac{\pi^5}{295,1215} & \text{prox.} \\
 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \&c. & . . & = \frac{\pi^6}{945,000} & \text{vere} \\
 1 + \frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{4^7} + \&c. & . . & = \frac{\pi^7}{2995,286} & \text{prox.} \\
 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \&c. & . . & = \frac{\pi^8}{9450,000} & \text{vere} \\
 1 + \frac{1}{2^9} + \frac{1}{3^9} + \frac{1}{4^9} + \&c. & . . & = \frac{\pi^9}{29749,35} & \text{prox. } \&c.
 \end{aligned}$$

153. Ex hoc fonte series numerorum Bernoullianorum

$1, 2, 3, 4, 5, 6, 7, 8, 9$
 $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}, \&c.$
 quantumvis irregularis videatur interpolari, seu termini in medio binorum quorumcunque constituti assignari poterunt: si enim terminus medium interiacens inter primum \mathcal{A} & secundum \mathcal{B} , seu indici $1\frac{1}{2}$ respondens fuerit $= p$; erit utique:

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \&c. = \frac{2^2 p}{1.2.3} \pi^3$$

ideoque $p = \frac{3}{2\pi^3} \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \&c. \right) = 0,05815227$

Simili modo si terminus inter \mathcal{B} & \mathcal{C} medium interiacens seu indicem habens $2\frac{1}{2}$ ponatur $= q$, quia erit:

$$1 + \frac{1}{2^5} + \frac{1}{3^5} + \&c. = \frac{2^4 q}{1.2.3.4.5} \pi^5$$

fiet $q = \frac{15}{2\pi^5} \left(1 + \frac{1}{2^5} + \frac{1}{3^5} + \&c. \right) = 0,02541327$

Si ergo istarum serierum, in quibus exponentes potestatum sunt numeri impares, summae exhiberi possent, tum quoque series numerorum Bernoullianorum interpolari posset.

154. Ponamus nunc $x = \frac{1}{m + nn}$, & quaeratur

summa huius seriei:

$$s = \frac{1}{m+1} + \frac{1}{m+4} + \frac{1}{m+9} + \dots + \frac{1}{m+nx}$$

Quia est $\int z dx = \int \frac{dx}{m+nx}$; erit $\int z dx = \frac{1}{n} A \operatorname{tang} \frac{x}{n}$.

Ponatur $A \cot \frac{x}{n} = u$, erit $\int z dx = \frac{1}{n} \left(\frac{\pi}{2} - u \right)$, &

$$\frac{x}{n} = \cot u = \frac{\operatorname{cof} u}{\sin u}, \quad \& \quad \frac{m+nx}{n} = \frac{1}{\sin u^2}, \quad \& \quad z = \frac{\sin u^2}{m},$$

$$\& \quad \frac{dx}{n} = -\frac{du}{\sin u^2}, \quad \text{unde fit} \quad du = -\frac{dx \sin u^2}{n}$$

Hinc differentialia ipsius z invenientur hoc modo:

$$\frac{dz}{dx} = \frac{2 du \sin u \operatorname{cof} u}{m} = -\frac{dx \sin u^2 \cdot \sin 2u}{n^3} \quad \& \quad \frac{dz}{dx} = -\frac{\sin u^2 \cdot \sin 2u}{n^3}$$

$$\frac{d^2 z}{dx^2} = \frac{du (\sin u \operatorname{cof} u \sin 2u + \sin u^2 \cdot \operatorname{cof} 2u)}{n^3} = \frac{dx \sin u^3 \cdot \sin 3u}{n^4}$$

$$\& \quad \frac{d^2 z}{dx^2} = \frac{\sin u^3 \cdot \sin 3u}{n^4}$$

Simili modo erit, uti iam supra pro eodem casu invenimus:

$$\frac{d^3 z}{dx^3} = \frac{\sin u^4 \cdot \sin 4u}{n^5}; \quad \frac{d^4 z}{dx^4} = \frac{\sin u^5 \cdot \sin 5u}{n^6} \quad \&c.$$

ex quibus formabitur summa quaesita:

$$s = \frac{\pi}{2n} - \frac{u}{n} + \frac{\sin u \cdot \sin u}{2nn} - \frac{1}{2} \cdot \frac{\sin u^2 \cdot \sin 2u}{n^3} + \frac{3}{4} \cdot \frac{\sin u^4 \cdot \sin 4u}{n^5} \\ - \frac{5}{6} \cdot \frac{\sin u^6 \cdot \sin 6u}{n^7} + \frac{7}{8} \cdot \frac{\sin u^8 \cdot \sin 8u}{n^9} - \&c. \quad + \operatorname{Const.}$$

Si hic ad constantem determinandam ponatur $x=0$, quo fiat $s=0$, erit $\cot u=0$, ideoque u angulus 90° , ac propterea $\sin u=1$, $\sin 2u=0$, $\sin 4u=0$, $\sin 6u=0$, &c.

$$\text{videtur ergo fore } 0 = \frac{\pi}{2n} - \frac{\pi}{2n} + \frac{1}{2nn} + C, \quad \text{hinc } C = -\frac{1}{2nn}$$

at vero notandum est, etiam si reliqui termini evanescent, tamen quia coefficientes \mathcal{A} , \mathcal{B} , \mathcal{C} , &c. tandem in infinitum excrefcunt, eorum summam posse esse finitam.

155. Ad hanc ergo constantem rite determinandam ponamus esse $x = \infty$, summam enim huius seriei in infinitum excurrentis supra iam in introductione definivimus, ostendimusque esse eam

$$= -\frac{1}{2nn} + \frac{\pi}{2n} + \frac{\pi}{n(e^{2n\pi} - 1)}.$$

Posito autem $x = \infty$, fiet $u = 0$, ideoque $\sin u = 0$, simulque sinus omnium arcuum multiplo- rum evanescent. Cum autem in hac serie potestates ipsius $\sin u$ crescant, divergentia seriei impedire nequit, quominus valor seriei hoc casu evanescat. Fiet ergo $s = \frac{\pi}{2n} + C$; unde erit

$$\frac{\pi}{2n} + C = -\frac{1}{2nn} + \frac{\pi}{2n} + \frac{\pi}{n(e^{2n\pi} - 1)}, \quad \&$$

$$C = -\frac{1}{2nn} + \frac{\pi}{n(e^{2n\pi} - 1)}. \quad \text{Quare summa seriei quaesita}$$

$$\text{erit } s = \frac{\pi}{2n} - \frac{u}{n} + \frac{\cos u^2}{2nn} - \frac{\mathcal{A}}{2} \cdot \frac{\sin u^2 \cdot \sin 2u}{n^5} +$$

$$\frac{\mathcal{B}}{4} \cdot \frac{\sin u^4 \cdot \sin 4u}{n^5} - \frac{\mathcal{C}}{6} \cdot \frac{\sin u^6 \cdot \sin 6u}{n^7} + \&c. + \frac{\pi}{n(e^{2n\pi} - 1)}.$$

Ubi notandum est, si n fuerit numerus mediocriter magnus, ultimum terminum $\frac{\pi}{n(e^{2n\pi} - 1)}$ tantopere fieri exiguum, ut negligi queat.

156. Ponamus esse $x = n$, ita ut denotet:

$$s = \frac{1}{nn+1} + \frac{1}{nn+4} + \frac{1}{nn+9} + \dots + \frac{1}{nn+nn}.$$

Tum vero erit $\cot u = 1$, & $u = 45^\circ = \frac{\pi}{4}$. Quamobrem

Bbb

habe-

habebitur $\sin u = \frac{1}{\sqrt{2}}$; $\sin 2u = 1$; $\sin 4u = 0$; $\sin 6u = -1$;
 $\sin 8u = 0$; $\sin 10u = 1$; &c. Hancobrem erit:

$$s = \frac{\pi}{1n} - \frac{1}{4nn} - \frac{1}{2.2n^3} + \frac{1}{6.8n^7} - \frac{1}{10.2^5 n^{11}} \\ + \frac{1}{14.2^7 n^{15}} - \&c. + \frac{\pi}{n(e^{2n\pi} - 1)},$$

in qua expressione tantum numeri alterni ex Bernoullianis occurrunt. Si igitur valor ipsius s per computum actu institutum iam fuerit inventus, hinc quantitas π definiri poterit, erit enim:

$$\pi = 4ns + \frac{1}{n} + \frac{1}{1.n^2} - \frac{1}{3.2^2 n^6} + \frac{1}{5.2^4 n^{10}} \\ - \frac{1}{7.2^6 . n^{14}} + \&c. - \frac{\pi}{e^{2n\pi} - 1}.$$

Etsi enim in termino ultimo inest π , tamen quia is tantopere est parvus, sufficit valorem ipsius π proxime nosse.

EXEMPLUM. Sit $n = 5$; erit:

$$s = \frac{1}{26} + \frac{1}{29} + \frac{1}{34} + \frac{1}{41} + \frac{1}{50}$$

qui termini actu additi dabunt:

$$s = 0,146746305690549494$$

unde erunt termini illi;

$$\begin{array}{r} 4ns = 2,93492611381098988 \\ \frac{1}{n} = 0,2 \\ \frac{1}{nn} = 0,006666666666666666 \\ \frac{1}{3.2^2 . n^6} = 3,14159278047765654 \\ \frac{1}{5.2^4 . n^{10}} = 0,00000012698412698 \\ \hline 3,14159265349352956 \end{array}$$

⊗

$$\begin{array}{r}
 \text{E} \\
 \hline
 5 \cdot 2^4 \cdot n^{10} \\
 \hline
 \text{G} \\
 \hline
 7 \cdot 2^6 \cdot n^{14} \\
 \hline
 \text{J} \\
 \hline
 9 \cdot 2^8 \cdot n^{18}
 \end{array}
 =
 \begin{array}{r}
 3, 14159265349352956 \\
 \hline
 0, 00000000009696969 \\
 \hline
 3, 14159265359049925 \\
 \hline
 0, 00000000000042666 \\
 \hline
 3, 14159265359007259 \\
 \hline
 625 \\
 \hline
 3, 14159265359007884
 \end{array}$$

Hic valor iam tam prope ad veritatem accedit, ut mirandum fit tam levi calculo eousque perveniri posse. Est vero haec expressio aliquantillum iusto maior, subtrahi enim adhuc inde debet $\frac{4\pi}{e^{2n\pi} - 1}$, cuius valor, dummodo π prope fit cognitum, exhiberi potest; quod per logarithmos ita expeditur.

Quia est $\pi l e = 1,3643763538$
 erit $l e^{2n\pi} = 10 \pi l e = 13,6437635$.

Cam iam fit $\frac{4\pi}{e^{2n\pi} - 1} = \frac{4\pi}{e^{2n\pi}} + \frac{4\pi}{e^{4n\pi}} + \&c.$

facile intelligitur ad nostrum calculum sufficere primum terminorum sumisse. Augeamus ergo characteristicam numero 17, quia habemus totidem figuras decimales, erit:

$$l\pi = 17,4971498$$

$$l4 = 0,6020600$$

$$18,0992098$$

Subtr. $l e^{2n\pi} = 13,6437635$
 $4,4554463$

Ergo $\frac{4\pi}{e^{2n\pi}} = 28539$ subtrahatur

ab $3,14159265359007884$ erit

$$\pi = 3,14159265358979345$$

quae expressio in figura demum penultima a veritate recedit;
 Bbb 2 quod

quod mirandum non est, cum adhuc terminum $\frac{1}{1 \cdot 1 \cdot 2^{10} \cdot n^{22}}$, qui dat 22, subtrahere debuiffemus, ficque ne ultima quidem figura aberrasset. Ceterum intelligitur, si pro n maiorem numerum uti 10 assumiffemus, tum facili negotio peripheriam π ad 25 pluresque figuras inveniri potuiffe.

157. Ponamus nunc quoque pro z functiones transcendentis ipsius x , fitque $z = lx$, fumendo logarithmos hyperbolicos, quoniam vulgares facile eo revocantur; fitque:

$$s = l_1 + l_2 + l_3 + l_4 + \dots + lx.$$

Quia igitur est $z = lx$, erit $\int z dx = xlx - x$, huius enim differentiale dat $dxlx$. Deinde est

$$\frac{dz}{dx} = \frac{1}{x}; \quad \frac{d^2z}{dx^2} = -\frac{1}{x^2}; \quad \frac{d^3z}{1 \cdot 2 dx^3} = \frac{1}{x^3}; \quad \frac{d^4z}{1 \cdot 2 \cdot 3 \cdot 4 dx^4} = \frac{1}{x^4}; \quad \&c.$$

Hinc itaque concluditur fore:

$$s = xlx - x + \frac{1}{2}lx + \frac{A}{1 \cdot 2 x} - \frac{B}{3 \cdot 4 x^3} + \frac{C}{5 \cdot 6 x^5} - \frac{D}{7 \cdot 8 x^7} + \&c. \\ + \text{Const.}$$

Haec autem constans ponendo $x = 1$, quia fit $ls = l1 = 0$ ita definitur, ut fit: $C = 1 - \frac{A}{1 \cdot 2} + \frac{B}{3 \cdot 4} - \frac{C}{5 \cdot 6} + \frac{D}{7 \cdot 8} - \&c.$ quae series ob nimiam divergentiam est inepta ad valorem ipsius C faltem proxime eruendum.

158. Non solum autem proximum sed etiam ipsum verum valorem ipsius C inveniemus, si consideremus expressio- nem Wallisianam pro valore ipsius π inventam, atque in introductione demonstratam: quae erat:

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot \&c.}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot \&c.}$$

hinc enim logarithmis fumendis erit:

$$l\pi - l2 = 2l2 + 2l4 + 2l6 + 2l8 + 2l10 + \&c. \\ - l1 - 2l3 - 2l5 - 2l7 - 2l9 - 2l11 - \&c.$$

Ponamus ergo in serie assumta $x = \infty$ & cum fit:

l1

$$\begin{aligned} & l_1 + l_2 + l_3 + l_4 + \dots + l_x = C + (x + \frac{1}{2})l_x - x \\ \text{erit } & l_1 + l_2 + l_3 + l_4 + \dots + l_{2x} = C + (2x + \frac{1}{2})l_{2x} - 2x \\ \& \quad l_2 + l_4 + l_6 + l_8 + \dots + l_{2x} = C + (x + \frac{1}{2})l_x + xl_{2-x} \\ \text{hinc } & l_1 + l_3 + l_5 + l_7 + \dots + l_{(2x-1)} = xl_x + (x + \frac{1}{2})l_{2-x} \end{aligned}$$

Cum igitur sit:

$$\begin{aligned} l \frac{\pi}{2} &= 2l_2 + 2l_4 + 2l_6 + \dots + 2l_{2x} - l_{2x} \\ &\quad - 2l_1 - 2l_3 - 2l_5 - \dots - 2l_{(2x-1)} \end{aligned}$$

posito $x = \infty$, erit:

$$\begin{aligned} l \frac{\pi}{2} &= 2C + (2x + 1)l_x + 2xl_2 - 2x - l_2 - l_x \\ &\quad - 2xl_x - (2x + 1)l_2 + 2x, \end{aligned}$$

ideoque $l \frac{\pi}{2} = 2C - 2l_2$, ergo $2C = l_{2\pi}$, & $C = \frac{1}{2}l_{2\pi}$,

unde in fractionibus decimalibus reperitur:

$$C = 0,9189385332046727417803297$$

atque simul sequens series summatur:

$$1 - \frac{A}{1.2} + \frac{B}{3.4} - \frac{C}{5.6} + \frac{D}{7.8} - \frac{E}{9.10} + \&c. = \frac{1}{2}l_{2\pi}.$$

159. Cognita nunc ista constante $C = \frac{1}{2}l_{2\pi}$, summa quotcunque logarithmorum ex hac serie $l_1 + l_2 + l_3 + \&c.$ exhiberi potest. Si enim ponatur:

$$s = l_1 + l_2 + l_3 + l_4 + \dots + l_x, \text{ erit}$$

$$s = \frac{1}{2}l_{2\pi} + (x + \frac{1}{2})l_x - x + \frac{A}{1.2x} - \frac{B}{3.4x^3} + \frac{C}{5.6x^5} - \frac{D}{7.8x^7} + \&c.$$

si quidem logarithmi propositi fuerint hyperbolici; sin autem proponantur logarithmi vulgares, tum in terminis $\frac{1}{2}l_{2\pi} + (x + \frac{1}{2})l_x$ pro $l_{2\pi}$ & l_x summi debebunt logarithmi vulgares, reliqui

$$\text{autem seriei termini } -x + \frac{A}{1.2x} - \frac{B}{3.4x^3} + \&c.$$

multiplicari debent per $0,434294481903251827 = \pi$.

Erit igitur hoc casu pro logarithmis vulgaribus:

$l\pi$

$$\begin{aligned}
 l\pi &= 0,497149872694133854351268 \\
 l2 &= 0,301029995663981195213738 \\
 l2\pi &= 0,798179868358115049565006 \\
 \frac{1}{2}l2\pi &= 0,399089934179057524782503
 \end{aligned}$$

E X E M P L U M

Quaeratur aggregatum mille logarithmorum tabularium

$$s = l1 + l2 + l3 + \dots + l1000.$$

Erit ergo $n = 1000$, & $ln = 3,00000000000000$
unde fit $nlx = 3000,00000000000000$
 $\frac{1}{2}ln = 1,50000000000000$
 $\frac{1}{2}l2\pi = 0,3990899341790$
subtr. $nx = 3001,8990899341790$
 $434,2944319032518$
 $2567,6046080309272$

Deinde est $\frac{n^2x}{1.2n} = 0,0000361912068$

subtr. $\frac{n^3}{3.4n^3} = 0,0000000000012$
 $0,0000361912056$
addatur $2567,6046080309272$

summa quaesita $s = 2567,6046442221328.$

Cum igitur s sit logarithmus producti numerorum:

1. 2. 3. 4. 5. 6. . . . 1000

patet hoc productum, si actu multiplicetur, constare ex 2568 figuris, atque notas a leva initiales fore 4023872 quas insuper 2561 figurae sequentur.

160. Ope ergo huius logarithmorum summationis producta ex quocumque factoribus, qui secundum numeros naturales procedunt, proxime assignari poterunt. Huc potissimum referri potest problema, quo quaeritur uncia media seu maxima in potestate binomii quacumque $(a + b)^m$; ubi quidem notandum est, si m sit numerus impar, binas dari medias inter se aequales, quae iunctim sumtae praebeant unciam mediam

diam in potestate sequente pari. Quare cum uncia maxima in quaque potestate pari sit duplo maior quam uncia media in potestate praecedente impari, sufficiet pro potestatibus paribus unciam mediam maximam determinasse. Sit igitur $m = 2n$, & uncia media ita exprimetur ut fit:

$$\frac{2n(2n-1)(2n-2)(2n-3) \dots (n+1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n}$$

Vocetur ista uncia media quae quaeritur $= u$, atque eo hoc modo repraesentari poterit, ut fit:

$$u = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \dots \cdot 2n}{(1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n)^2}$$

sumtisque logarithmis erit:

$$lu = l1 + l2 + l3 + l4 + l5 + \dots + l2n - 2l1 - 2l2 - 2l3 - 2l4 - 2l5 \dots - 2ln$$

161. Iam vero sumendis his logarithmis hyperbolicis erit:

$$l1 + l2 + l3 + l4 + \dots + l2n = \frac{1}{2} l2\pi + (2n + \frac{1}{2}) ln + (2n + \frac{1}{2}) l2 - 2n + \frac{A}{1 \cdot 2 \cdot 2n} - \frac{B}{3 \cdot 4 \cdot 2^3 n^3} + \frac{C}{5 \cdot 6 \cdot 2^5 n^5} - \&c.$$

$$\& 2l1 + 2l2 + 2l3 + 2l4 + \dots + 2ln = l2\pi + (2n + 1) ln - 2n + \frac{2A}{1 \cdot 2n} - \frac{2B}{3 \cdot 4n^3} + \frac{2C}{5 \cdot 6n^5} - \&c.$$

qua expressione ab illa sublata relinquetur:

$$lu = -\frac{1}{2} l\pi - \frac{1}{2} ln + 2nl2 + \frac{A}{1 \cdot 2 \cdot 2n} - \frac{B}{3 \cdot 4 \cdot 2^3 n^3} + \frac{C}{5 \cdot 6 \cdot 2^5 n^5} - \&c. - \frac{2A}{1 \cdot 2n} + \frac{2B}{3 \cdot 4n^3} - \frac{2C}{5 \cdot 6n^5} + \&c.$$

his vero binis terminis colligendis, erit:

$$lu = l \frac{2^{2n}}{\sqrt{2n}} - \frac{3A}{1 \cdot 2 \cdot 2n} + \frac{15B}{3 \cdot 4 \cdot 2^3 n^3} - \frac{63C}{5 \cdot 6 \cdot 2^5 n^5} + \frac{255D}{7 \cdot 8 \cdot 2^7 n^7} - \&c.$$

Sit

$$\text{Sit } \frac{3\mathcal{A}}{1.2.2^2 n^2} - \frac{15\mathcal{B}}{3.4.2^4 n^4} + \frac{63\mathcal{C}}{5.6.2^6 n^6} - \frac{255\mathcal{D}}{7.8.2^8 n^8} + \&c.$$

$$= l \left(1 + \frac{A}{2^2 n^2} + \frac{B}{2^4 n^4} + \frac{C}{2^6 n^6} + \frac{D}{2^8 n^8} + \&c. \right); \text{ erit}$$

$$lu = l \frac{2^{2n}}{\sqrt{n\pi}} - 2nl \left(1 + \frac{A}{2^2 n^2} + \frac{B}{2^4 n^4} + \frac{C}{2^6 n^6} + \&c. \right);$$

$$\text{ideoque } u = \frac{2^{2n}}{\left(1 + \frac{A}{2^2 n^2} + \frac{B}{2^4 n^4} + \frac{C}{2^6 n^6} + \&c. \right)^{2n} \sqrt{n\pi}}$$

Erit vero posito $2n = m$

$$l \left(1 + \frac{A}{2^2 n^2} + \frac{B}{2^4 n^4} + \frac{C}{2^6 n^6} + \frac{D}{2^8 n^8} + \&c. \right) =$$

$$\frac{A}{m^2} + \frac{B}{m^4} + \frac{C}{m^6} + \frac{D}{m^8} + \frac{E}{m^{10}} + \&c.$$

$$- \frac{A^2}{2m^4} - \frac{AB}{m^6} - \frac{AC}{m^8} - \frac{AD}{m^{10}} - \&c.$$

$$- \frac{BB}{2m^8} - \frac{BC}{m^{10}} - \&c.$$

$$+ \frac{A^3}{3m^6} + \frac{A^2 B}{m^8} + \frac{A^2 C}{m^{10}} + \&c.$$

$$+ \frac{AB^2}{m^{10}} + \&c.$$

$$- \frac{A^4}{4m^8} - \frac{A^3 B}{m^{10}} - \&c.$$

$$+ \frac{A^5}{5m^{10}} + \&c.$$

quae expressio cum aequalis esse debeat huic:

$$\frac{3\mathcal{A}}{1.2m^2} - \frac{15\mathcal{B}}{3.4m^4} + \frac{63\mathcal{C}}{5.6m^6} - \frac{255\mathcal{D}}{7.8m^8} + \&c. \text{ fiet:}$$

A

$$A = \frac{3\mathfrak{A}}{1.2}$$

$$B = \frac{A^2}{2} - \frac{15\mathfrak{B}}{3.4}$$

$$C = AB - \frac{1}{3}A^3 + \frac{63\mathfrak{C}}{5.6}$$

$$D = AC + \frac{1}{2}B^2 - A^2B + \frac{1}{4}A^4 - \frac{255\mathfrak{D}}{7.8}$$

$$E = AD + BC - A^2C - AB^2 + A^3B - \frac{1}{5}A^5 + \frac{1023\mathfrak{E}}{9.10}$$

&c.

162. Cum iam sit $\mathfrak{A} = \frac{1}{6}$; $\mathfrak{B} = \frac{1}{30}$; $\mathfrak{C} = \frac{1}{42}$;

$\mathfrak{D} = \frac{1}{30}$; $\mathfrak{E} = \frac{5}{66}$; erit :

$A = \frac{1}{4}$; $B = -\frac{1}{96}$; $C = \frac{27}{640}$; $D = -\frac{90031}{2^{11} \cdot 3^2 \cdot 5 \cdot 7}$ &c.

Hinc efficitur :

$$u = \frac{\left(1 + \frac{1}{2^4 n^2} - \frac{1}{2^9 \cdot 3 n^4} + \frac{27}{2^{13} \cdot 5 n^6} - \frac{90031}{2^{19} \cdot 3^2 \cdot 5 \cdot 7 n^8} + \&c. \right)^{2n} \sqrt{n\pi}}{2^{2n}}$$

vel si ista seriei elevatio actu instituat, erit proxime :

$$u = \frac{\sqrt{n\pi} \left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{5}{16 \cdot 128n^4} + \&c. \right)}{2^{2n}}$$

hinc terminus medius in $(1 + \frac{1}{4n})^{2n}$, erit ad summam omnium terminorum 2^{2n}

uti 1 ad $\sqrt{n\pi} \left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{5}{16 \cdot 128n^4} + \&c. \right)$

vel posito brevitatis gratia $4n = v$, erit ista ratio :

Ccc

ut

ut 1 ad $\sqrt{n\pi} \left(1 + \frac{1}{y} + \frac{1}{2y^2} - \frac{1}{2y^3} - \frac{5}{8y^4} + \frac{23}{8y^5} + \frac{53}{16y^6} - \&c. \right)$

E X E M P L U M I.

Quaeratur uncia media in binomio $(a+b)^{20}$ evoluto,

$$\text{quam constat esse} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 252.$$

Adhibendo ultimam formulam pro n inventam erit $n = 5$

$$\frac{1}{4n} = 0,0500000$$

$$\frac{1}{32n^2} = 0,0012500$$

$$\text{Subtr. } \frac{1}{128n^3} = \underline{\quad 625}$$

$$\text{Subtr. } \frac{5}{16 \cdot 128n^4} = \underline{\quad 39}$$

$$\text{Ergo } 1 + \frac{1}{4n} + \&c. = \underline{1,0511836}$$

$$\text{Huius log.} = 0,0216784$$

$$\ln = 0,6989700$$

$$\ln\pi = 0,4971498$$

$$1,2177982$$

$$\sqrt{n\pi}(1 + \&c.) = 0,6088991$$

$$a \cdot l2^{2n} = \underline{3,0102999}$$

$$lu = 2,4014008$$

$$\text{unde fit } u = 252.$$

E X E M P L U M II.

Investigetur ratio, quam in potestate centesima binomi
 $1 + 1$ terminus medius ad summam omnium 2^{100} tenet.

Utamur ad hoc formula primum inventa:

$$lu = l \frac{2^{2n}}{\sqrt{n\pi}} - \frac{3u}{1 \cdot 2 \cdot 2n} + \frac{15B}{3 \cdot 4 \cdot 2^3 n^3} - \frac{63C}{5 \cdot 6 \cdot 2^5 n^5} + \&c.$$

in qua posito $2n = m$ ut habeatur ista potestas $(1 + 1)^m$.
 & loco A, B, C, D , substitutis valoribus, fiet:

$$lu = l \frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{1}{4m} + \frac{1}{24m^3} - \frac{1}{20m^5} + \frac{17}{112m^7} - \frac{31}{36m^9} + \frac{691}{88m^{11}} - \&c.$$

qui logarithmi cum sint hyperbolici multiplicentur ii per
 $k = 0,434294481903251$,
 ut transmutentur in tabulares, eritque:

$$lu = l \frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{k}{4m} + \frac{k}{24m^3} - \frac{k}{20m^5} + \frac{17k}{112m^7} - \frac{31k}{36m^9} + \&c.$$

unde cum uncia media sit u , erit $2^m : u$ ratio quaesita, ideoque

$$\frac{2^m}{u} = l \sqrt{\frac{1}{2}m\pi} + \frac{k}{4m} - \frac{k}{24m^3} + \frac{k}{20m^5} - \frac{17k}{112m^7} + \frac{31k}{36m^9} - \frac{691k}{88m^{11}} + \&c.$$

Quare cum sit ob exponentem $m = 100$

$$\frac{k}{m} = 0,0043429448; \quad \frac{k}{m^3} = 0,0000004343;$$

$$\frac{k}{m^5} = 0,0000000000; \quad \text{erit:}$$

$$\frac{k}{4m} = 0,0010857362$$

$$\frac{k}{24m^3} = 0,0000000181$$

$$0,0010857181$$

$$\text{Tum est } l\pi = 0,4971498726$$

$$l\frac{1}{2}m = 1,6989700043$$

$$l\frac{1}{2}m\pi = 2,1961198769$$

$$l\sqrt{\frac{1}{2}m\pi} = 1,0980599384$$

$$\frac{k}{4m} - \frac{k}{24m^3} + \&c. = 0,0010857181$$

$$1,0991456565 = l \frac{2^{100}}{u}$$

Erit ergo $\frac{2^{100}}{u} = 12,56451$, atque adeo in potestate

(1 + 1)¹⁰⁰ evoluta terminus medius se habebit ad summam omnium 2¹⁰⁰ uti 1 ad 12, 56451.

163. Denotet nunc terminus generalis z functionem exponentialem a^x , ita ut summari debeat haec series geometrica:

$$s = a + a^2 + a^3 + a^4 + \dots + a^x$$

quae cum sit geometrica, eius summa iam constat, erit enim

$$s = \frac{(a^x - 1)a}{a - 1}. \text{ Modo autem hic expōsito hanc summam in-}$$

vestigemus. Quia est $z = a^x$, erit $\int z dx = \frac{a^x}{\ln a}$, huius enim differentiale est $a^x dx$, tum vero erit:

$$\frac{dz}{dx} = a^x \ln a; \quad \frac{d^2 z}{dx^2} = a^x (\ln a)^2; \quad \frac{d^3 z}{dx^3} = a^x (\ln a)^3; \quad \&c.$$

unde sequitur fore:

$$s = a^x \left(\frac{1}{\ln a} + \frac{1}{2} + \frac{1}{1.2} \ln a - \frac{1}{1.2.3.4} (\ln a)^2 + \frac{1}{1.2.3\dots 6} (\ln a)^3 - \&c. \right) + C.$$

Ad constantem C definiendam ponatur $x = 0$, & ob $s = 0$,

$$\text{erit } C = -\frac{1}{\ln a} - \frac{1}{2} - \frac{1}{1.2} \ln a + \frac{1}{1.2.3.4} (\ln a)^2 - \&c.$$

ideoque fiet:

$$s = (a^x - 1) \left(\frac{1}{\ln a} + \frac{1}{2} + \frac{1}{1.2} \ln a - \frac{1}{1.2.3.4} (\ln a)^2 + \frac{1}{1.2.3\dots 6} (\ln a)^3 - \&c. \right)$$

Cum igitur summa sit $\frac{(a^x - 1)a}{a - 1}$, erit:

$$\frac{a}{a-1} = \frac{1}{\ln a} + \frac{1}{2} + \frac{1}{1.2} \ln a - \frac{1}{1.2.3.4} (\ln a)^2 + \frac{1}{1.2\dots 6} (\ln a)^3 - \&c.$$

ubi $\ln a$ denotat logarithmum hyperbolicum ipsius a : hinc

$$\text{fit } \frac{(a+1)\ln a}{2(a-1)} = 1 + \frac{1}{1.2} (\ln a)^2 - \frac{1}{1.2.3.4} (\ln a)^4 + \frac{1}{1.2\dots 6} (\ln a)^6 - \&c.$$

sicque istius seriei summa exhiberi poterit.

164. Sit terminus generalis $z = \sin ax$, &
 $s = \sin a + \sin 2a + \sin 3a + \dots + \sin ax$
 quae series cum fit recurrens quoque summari potest;
 erit enim

$$s = \frac{\sin a + \sin ax - \sin(ax+a)}{1 - 2 \cos a + 1} = \frac{\sin a + (1 - \cos a)\sin ax - \sin a \cos ax}{2(1 - \cos a)}$$

Erit vero $\int z dx = \int dx \sin ax = -\frac{1}{a} \cos ax$, & $\frac{dz}{dx} = a \cos ax$;

$$\frac{ddz}{dx^2} = -aa \sin ax; \quad \frac{d^3z}{dx^3} = -a^3 \cos ax; \quad \frac{d^5z}{dx^5} = a^5 \cos ax \text{ \&c.}$$

$$s = C - \frac{1}{a} \cos ax + \frac{1}{2} \sin ax + \frac{A a \cos ax}{1 \cdot 2} + \frac{B a^3 \cos ax}{1 \cdot 2 \cdot 3 \cdot 4} \\ + \frac{C a^5 \cos ax}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{D a^7 \cos ax}{1 \cdot 2 \dots 8} + \text{\&c.}$$

Ponatur $x = 0$ ut fiat $s = 0$; eritque:

$$C = \frac{1}{a} - \frac{A a}{1 \cdot 2} - \frac{B a^3}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{C a^5}{1 \cdot 2 \dots 6} - \text{\&c.} \quad \text{ergo}$$

$$s = \frac{1}{2} \sin ax + (1 - \cos ax) \left(\frac{1}{a} - \frac{A a}{1 \cdot 2} - \frac{B a^3}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{C a^5}{1 \cdot 2 \dots 6} - \text{\&c.} \right)$$

At cum fit $s = \frac{1}{2} \sin ax + \frac{(1 - \cos ax) \sin a}{2(1 - \cos a)}$, fiet:

$$\frac{\sin a}{2(1 - \cos a)} = \frac{1}{2} \cot \frac{1}{2} a = \frac{1}{a} - \frac{A a}{1 \cdot 2} - \frac{B a^3}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{C a^5}{1 \cdot 2 \dots 6} - \text{\&c.}$$

quam eandem seriem iam supra §. 127. habuimus.

165. Sit nunc $z = \cos ax$, ac series summanda:
 $s = \cos a + \cos 2a + \cos 3a + \dots + \cos ax$
 cuius seriei, quia est recurrens, erit summa:

$$s = \frac{\cos a - 1 + \cos ax - \cos(ax+a)}{1 - 2 \cos a + 1} = -\frac{1}{2} + \frac{1}{2} \cos ax + \frac{1}{2} \cot \frac{1}{2} a \cdot \sin ax.$$

At vero ad summam nostra methodo exprimendam, erit:
 $\int z dx$

$$\int x dx = \int dx \cos ax = \frac{1}{a} \sin ax, \quad \& \quad \frac{dz}{dx} = -a \sin ax;$$

$$\frac{d^3 z}{dx^3} = a^3 \sin ax; \quad \frac{d^5 z}{dx^5} = -a^5 \sin ax; \quad \&c. \quad \text{Ergo}$$

$$s = C + \frac{1}{a} \sin ax + \frac{1}{2} \cos ax - \frac{1}{1 \cdot 2} a \sin ax - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} a^3 \sin ax - \&c.$$

Sit $x = 0$, erit $s = 0$, & $C = -\frac{1}{2}$, hincque erit:

$$s = -\frac{1}{2} + \frac{1}{2} \cos ax + \frac{1}{a} \sin ax - \frac{1}{1 \cdot 2} a \sin ax - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} a^3 \sin ax - \&c.$$

Quare cum fit $s = -\frac{1}{2} + \frac{1}{2} \cos ax + \frac{1}{2} \cot \frac{1}{2} a \cdot \sin ax$,
erit uti iam modo invenimus:

$$\frac{1}{2} \cot \frac{1}{2} a = \frac{1}{a} - \frac{1}{1 \cdot 2} a - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} a^3 - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} a^5 - \&c.$$

166. Quoniam supra invenimus, si a denotet arcum quemcunque, esse

$$\frac{\pi}{2} = \frac{a}{2} + \sin a + \frac{1}{2} \sin 2a + \frac{1}{3} \sin 3a + \frac{1}{4} \sin 4a + \&c.$$

consideremus hanc seriem, sitque $z = \frac{1}{x} \sin ax$, ut fit

$$s = \sin a + \frac{1}{2} \sin 2a + \frac{1}{3} \sin 3a + \dots + \frac{1}{x} \sin ax.$$

Hoc autem casu fit $\int z dx = \int \frac{dx}{x} \sin ax$, quod integrale exhiberi nequit.

$$\text{Erit vero} \quad \frac{dz}{dx} = \frac{a}{x} \cos ax - \frac{1}{x^2} \sin ax;$$

$$\frac{ddz}{dx^2} = -\frac{a^2}{x} \sin ax - \frac{2a}{x^2} \cos ax + \frac{2}{x^3} \sin ax;$$

$$\frac{d^3 x}{dx^3} = -\frac{a^3}{x} \cos ax + \frac{3a^2}{x^2} \sin ax + \frac{6a}{x^3} \cos ax - \frac{6}{x^4} \sin ax;$$

$$\frac{d^4 x}{dx^4} = \frac{a^4}{x} \sin ax + \frac{4a^3}{xx} \cos ax - \frac{12a^2}{x^3} \sin ax - \frac{24a}{x^4} \cos ax + \frac{24}{x^5} \sin ax.$$

Quia igitur neque formulam integram $\int x dx$ exhibere, neque haec differentialia satis commode exprimere licet, summam huius seriei per hanc methodum definire non possumus, ita ut quicquam inde concludi posset. Idem incommodum in multis aliis seriebus occurrit, quoties terminus generalis non satis est simplex, ut eius differentialia ad commodam legem exprimi queant. Quamobrem in sequenti Capite alias expressiones generales pro summis serierum, quarum termini generales vel nimis sunt compositi vel prorsus dari nequeunt, eliciemus; quae feliciori successu in usum vocari poterunt. Imprimis autem insufficientia methodi hic traditae elucet, si signa terminorum seriei propositae alternentur, tum enim quantumvis termini generales sint simplices, tamen termini summatorii hac methodo exhiberi commode nequeunt.