CAPUT V.

INVESTIGATIO SUMMAE SERIERUM EX TERMINO GENERALI.

103.

Sit Seriei cuiusque terminus generalis $=y$, respondens indici x, ita ut y sit functio quaecunque ipsius x. Sit porro Sy summa seu terminus summatorius seriei, exprimens aggregatum omnium terminorum a primo seu alio termino fixo usque ad y inclusive. Computabimus autem summas serierum a termino primo, unde si sit $x=1$, dabit y terminum primum, atque Sy hunc y terminum primum exhibebit: sin autem ponatur $x=0$, terminus summatorius Sy in nihilum abire debet, propterea quod nulli termini summandi adsunt. Quocirca terminus summatorius Sy eiusmodi erit functio ipsius x, quae evanescat posito $x=0$.

104. Si terminus generalis y ex pluribus partibus constet, ut sit $y=p+q+r+$ &c. tum ipsa series considerari poterit tanquam conflata ex pluribus aliis seriebus, quarum termini generales sint p, q, r, &c. Hinc si singularum istarum serierum summae fuerint cognitae, simul seriei propositae summa poterit assignari; erit enim aggregatum ex summis singularum serierum. Hancobrem si sit $y=p+q+r+$ &c. erit $Sy=Sp+Sq+Sr+$ &c. Cum igitur supra exhibuerimus summas serierum, quarum termini generales sint quaecunque potestates ipsius x, habentes exponentes integros affirmativos, hinc cuiusque seriei, cuius terminus generalis est $ax^{\alpha}+bx^{\beta}+cx^{\gamma}+$ &c. denotantibus α, β, γ, &c. numeros integros affirmativos, seu cuius terminus generalis est functio rationalis integra ipsius x, terminus summatorius inveniri poterit.

105. Sit in serie, cuius terminus generalis seu exponenti x respondens est $=y$, terminus hunc praecedens seu

exponenti $x-1$ respondens $=v$, quoniam v oritur ex y, si loco x scribatur $x-1$; erit:

$$v=y-\frac{dy}{dx}+\frac{ddy}{2dx^2}-\frac{d^3y}{6dx^3}+\frac{d^4y}{24dx^4}-\frac{d^5y}{120dx^5}+\&c.$$

Si igitur y fuerit terminus generalis huius seriei

$$\overset{1}{a}+\overset{2}{b}+\overset{3}{c}+\overset{4}{d}+\ldots\ldots+\overset{x-1}{v}+\overset{x}{y}$$

huiusque seriei terminus indici o respondens fuerit $=A$, erit v, quatenus est functio ipsius x, terminus generalis huius seriei:

$$\overset{1}{A}+\overset{2}{a}+\overset{3}{b}+\overset{4}{c}+\overset{5}{d}+\ldots\ldots+\overset{x}{v}$$

unde si Sv denotet summam huius seriei, erit $Sv=Sy-y+A$. Sicque posito $x=0$, quia fit $Sy=0$ & $y=A$, quoque Sv evanescet.

106. Cum igitur sit $v=y-\frac{dy}{dx}+\frac{ddy}{2dx^2}-\frac{d^3y}{6dx^3}+\&c.$ erit per ante ostensa:

$$Sv=Sy-S\frac{dy}{dx}+S\frac{ddy}{2dx^2}-S\frac{d^3y}{6dx^3}+S\frac{d^4y}{24dx^4}-\&c.$$

atque ob $Sv=Sy-y+A$, erit:

$$y-A=S\frac{dy}{dx}-S\frac{ddy}{2dx^2}+S\frac{d^3y}{6dx^3}-S\frac{d^4y}{24dx^4}+\&c.$$

ideoque habebitur:

$$S\frac{dy}{dx}=y-A+S\frac{ddy}{2dx^2}-S\frac{d^3y}{6dx^3}+S\frac{d^4y}{24dx^4}-\&c.$$

Si ergo habeantur termini summatorii serierum, quarum termini generales sunt $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, &c. ex iis obtinebitur terminus summatorius seriei, cuius terminus generalis est $\frac{dy}{dx}$. Quantitas vero constans A ita debet esse comparata, ut fa-

facto $x = 0$ terminus summatorius $S\frac{dy}{dx}$ evanescat; hacque conditione facilius determinatur, quam si diceremus, eam esse terminum indici o respondentem in serie, cuius terminus generalis sit $= y$.

107. Ex hoc fonte summae potestatum numerorum naturalium investigari solent. Sit enim $y = x^{n+1}$; quoniam fit $\frac{dy}{dx} = (n+1)x^n$; $\frac{ddy}{2dx^2} = \frac{(n+1)n}{1.2}x^{n-1}$; $\frac{d^3y}{6dx^3} = \frac{(n+1)n(n-1)}{1.2.3}x^{n-2}$; $\frac{d^4y}{24dx^4} = \frac{(n+1)n(n-1)(n-2)}{1.2.3.4}x^{n-3}$; &c. erit his valoribus substitutis:

$$(n+1)Sx^n = x^{n+1} - A + \frac{(n+1)n}{1.2}Sx^{n-1} - \frac{(n+1)n(n-1)}{1.2.3}Sx^{n-2} + \&c.$$

atque si utrinque per $n+1$ dividatur; erit:

$$Sx^n = \frac{1}{n+1}x^{n+1} + \frac{n}{2}Sx^{n-1} - \frac{n(n-1)}{2.3}Sx^{n-2} + \frac{n(n-1)(n-2)}{2.3.4}Sx^{n-3} + \&c. - \text{Const.}$$

quae constans ita accipi debet, ut posito $x = 0$, totus terminus summatorius evanescat. Ope huius ergo formulae ex iam cognitis summis potestatum inferiorum, quarum termini generales sunt x^{n-1}, x^{n-2}, &c. inveniri poterit summa potestatum superiorum termino generali x^n expressarum.

108. Si in hac expressione n denotet numerum integrum affirmativum, numerus terminorum erit finitus. Atque adeo hinc summa infinitarum potestatum si $n = 0$, absolute cognoscetur; erit enim: $S.x^0 = x$. Hacque cognita ad superiores progredi licebit, posito enim $n = 1$; fiet:

$S.x^1 = \frac{1}{2}x^2 + \frac{1}{2}Sx^0 = \frac{1}{2}x^2 + \frac{1}{2}x$

si porro ponatur $n = 2$ prodibit:

$S.x^2 = \frac{1}{3}x^3 + Sx - \frac{1}{3}Sx^0 = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x$; deinde

$S.x^3 = \frac{1}{4}x^4 + \frac{3}{2}Sx^2 - Sx + \frac{1}{4}Sx^0 = \frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2$

Tt 2 S.

$S.x^4 = \frac{1}{5}x^5 + \frac{4}{2}Sx^3 - \frac{4}{2}Sx^2 + Sx - \frac{1}{5}Sx^0$ ſive
$S.x^4 = \frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{1}{30}x.$
Sicque porro quarumvis poteſtatum ſuperiorum ſummae ſucceſſivae ex inferioribus colligentur ; hoc autem facilius per ſequentes modos praeſtabitur.

109. Quoniam ſupra invenimus eſſe:

$$S\frac{dy}{dx} = y + \frac{1}{2}S\frac{ddy}{dx^2} - \frac{1}{6}S\frac{d^3y}{dx^3} + \frac{1}{24}S\frac{d^4y}{dx^4} - \frac{1}{120}S\frac{d^5y}{dx^5} + \&c.$$

Si ponamus $\frac{dy}{dx} = z$; fiet $\frac{ddy}{dx^2} = \frac{dz}{dx}$; $\frac{d^3y}{dx^3} = \frac{ddz}{dx^2}$; &c. tum vero ob $dy = zdx$, erit y quantitas, cuius differentiale eſt $= zdx$, quod hoc modo indicamus, ut ſit $y = \int zdx$. Quanquam autem haec inventio ipſius y ex dato z a calculo integrali pendet, tamen hic iam iſta forma $\int zdx$ uti poterimus, ſi quidem pro z alias ipſius x functiones non ſubſtituamus, niſi eiuſmodi, ut functio illa, cuius differentiale eſt $= zdx$, ex praecedentibus exhiberi queat. His igitur valoribus ſubſtitutis erit:

$$Sz = \int zdx + \frac{1}{2}S\frac{dz}{dx} - \frac{1}{6}S\frac{ddz}{dx^2} + \frac{1}{24}S\frac{d^3z}{dx^3} - \&c.$$

adiiciendo eiuſmodi conſtantem, ut poſito $x = 0$ ipſa ſumma Sz evaneſcat.

110. Subſtituendo autem loco y in ſuperiori expreſſione litteram z, vel quod eodem redit differentiando iſtam aequationem erit:

$$S\frac{dz}{dx} = z + \frac{1}{2}S\frac{ddz}{dx^2} - \frac{1}{6}S\frac{d^3z}{dx^3} + \frac{1}{24}S\frac{d^4z}{dx^4} - \&c.$$

ſin autem loco y ponatur $\frac{dz}{dx}$; erit:

$$S\frac{ddz}{dx^2} = \frac{dz}{dx} + \frac{1}{2}S\frac{d^3z}{dx^3} - \frac{1}{6}S\frac{d^4z}{dx^4} + \frac{1}{24}S\frac{d^5z}{dx^5} - \&c.$$

Similique modo ſi pro y ſucceſſive ponantur valores

ddz

$\frac{ddz}{dx^2}$; $\frac{d^3z}{dx^3}$; &c. reperietur:

$$S\frac{d^3z}{dx^3} = \frac{ddz}{dx^2} + \tfrac{1}{2}S\frac{d^4z}{dx^4} - \tfrac{1}{6}S\frac{d^5z}{dx^5} + \tfrac{1}{24}S\frac{d^6z}{dx^6} - \&c.$$

$$S\frac{d^4z}{dx^4} = \frac{d^3z}{dx^3} + \tfrac{1}{2}S\frac{d^5z}{dx^5} - \tfrac{1}{6}S\frac{d^6z}{dx^6} + \tfrac{1}{24}S\frac{d^7z}{dx^7} - \&c.$$

ſicque porro in infinitum.

111. Si nunc iſti valores pro $S\frac{dz}{dx}$; $S\frac{ddz}{dx^2}$; $S\frac{d^3z}{dx^3}$; &c. ſucceſſive ſubſtituantur in expreſſione:

$$Sz = \int z dx + \tfrac{1}{2}S\frac{dz}{dx} - \tfrac{1}{6}S\frac{ddz}{dx^2} + \tfrac{1}{24}S\frac{d^3z}{dx^3} - \&c.$$

invenietur expreſſio pro Sz, quae conſtabit ex his terminis $\int z dx$; z; $\frac{dz}{dx}$; $\frac{ddz}{dx^2}$; $\frac{d^3z}{dx^3}$; &c. quorum coefficientes facilius ſequenti modo inveſtigabuntur. Ponatur

$$Sz = \int z dz + \alpha z + \frac{\beta dz}{dx} + \frac{\gamma ddz}{dx^2} + \frac{\delta d^3z}{dx^3} + \frac{\varepsilon d^4z}{dx^4} + \&c.$$

atque pro his terminis ſui valores ſubſtituantur, quos obtinent ex praecedentibus ſeriebus, ex quibus eſt:

$$\int z dx = Sz - \frac{1}{2}S\frac{dz}{dx} + \frac{1}{6}S\frac{ddz}{dx^2} - \frac{1}{24}S\frac{d^3z}{dx^3} + \frac{1}{120}S\frac{d^4z}{dx^4} - \&c.$$

$$\alpha z = +\alpha S\frac{dz}{dx} - \frac{\alpha}{2}S\frac{ddz}{dx^2} + \frac{\alpha}{6}S\frac{d^3z}{dx^3} - \frac{\alpha}{24}S\frac{d^4z}{dx^4} + \&c.$$

$$\frac{\beta dz}{dx} = \beta S\frac{ddz}{dx^2} - \frac{\beta}{2}S\frac{d^3z}{dx^3} + \frac{\beta}{6}S\frac{d^4z}{dx^4} - \&c.$$

$$\frac{\gamma ddz}{dx^2} = \gamma S\frac{d^3z}{dx^3} - \frac{\gamma}{2}S\frac{d^4z}{dx^4} + \&c.$$

$$\frac{\delta d^3z}{dx^3} = \delta S\frac{d^4z}{dx^4} - \&c.$$

qui

qui valores additi, cum producere debeant Sz, coefficientes $\alpha, \beta, \gamma, \delta$, &c. ex sequentibus aequationibus definientur:

$$\alpha - \frac{1}{2} = 0$$

$$\beta - \frac{\alpha}{2} + \frac{1}{6} = 0$$

$$\gamma - \frac{\beta}{2} + \frac{\alpha}{6} - \frac{1}{24} = 0$$

$$\delta - \frac{\gamma}{2} + \frac{\beta}{6} - \frac{\alpha}{24} + \frac{1}{120} = 0$$

$$\varepsilon - \frac{\delta}{2} + \frac{\gamma}{6} - \frac{\beta}{24} + \frac{\alpha}{120} - \frac{1}{720} = 0$$

$$\zeta - \frac{\varepsilon}{2} + \frac{\delta}{6} - \frac{\gamma}{24} + \frac{\beta}{120} - \frac{\alpha}{720} + \frac{1}{5040} = 0$$

112. Ex his ergo aequationibus successive valores omnium litterarum $\alpha, \beta, \gamma, \delta$, &c. definiri poterunt, reperietur autem:

$$\alpha = \frac{1}{2}$$

$$\beta = \frac{\alpha}{2} - \frac{1}{6} = \frac{1}{12}$$

$$\gamma = \frac{\beta}{2} - \frac{\alpha}{6} + \frac{1}{24} = 0$$

$$\delta = \frac{\gamma}{2} - \frac{\beta}{6} + \frac{\alpha}{24} - \frac{1}{120} = -\frac{1}{720}$$

$$\varepsilon = \frac{\delta}{2} - \frac{\gamma}{6} + \frac{\beta}{24} - \frac{\alpha}{120} + \frac{1}{720} = 0$$

&c.

sicque ulterius progrediendo reperientur continuo termini alterni evanescentes. Litterae ergo ordine tertia, quinta, septima, ma, &c. omnesque impares erunt $=0$, excepta prima, quo ipso haec valorum series contra legem continuitatis impingere videtur. Quamobrem eo magis necesse est, ut rigide demonstretur, omnes terminos impares praeter primum necessario evanescere.

113. Quoniam singulae litterae secundum legem constantem ex praecedentibus determinantur, eae seriem recurrentem inter se constituent. Ad quam explicandam concipiatur ista series: $1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+$ &c. cuius valor sit $=V$; atque manifestum est hanc seriem recurrentem oriri ex evolutione huius fractionis:

$$V=\frac{1}{1-\frac{1}{2}u+\frac{1}{6}u^2-\frac{1}{24}u^3+\frac{1}{120}u^4-\text{\&c.}}$$

atque si ista fractio alio modo in seriem infinitam secundum potestates ipsius u progredientem resolvi queat, necesse est, ut semper eadem series:

$$V=1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\text{\&c.}$$

resultet: hocque modo alia lex, qua isti iidem valores α, β, γ, δ, &c. determinantur, eruetur.

114. Quoniam, si e denotet numerum, cuius logarithmus hyperbolicus unitati aequatur, erit:

$$e^{-u}=1-u+\frac{1}{2}u^2-\frac{1}{6}u^3+\frac{1}{24}u^4-\frac{1}{120}u^5+\text{\&c.}$$

erit: $$\frac{1-e^{-u}}{u}=1-\frac{1}{2}u+\frac{1}{6}u^2-\frac{1}{24}u^3+\frac{1}{120}u^4-\text{\&c.}$$

ideoque $V=\frac{u}{1-e^{-u}}$. Nunc extinguatur ex serie secundus terminus $\alpha u=\frac{1}{2}u$, ut fit:

$V-\frac{1}{2}u=1+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+$ &c. erit:

$V-\frac{1}{2}u=\frac{\frac{1}{2}u(1+e^{-u})}{1-e^{-u}}$. Multiplicentur numerator ac denominator per $e^{\frac{1}{2}u}$, eritque $V-\frac{1}{2}u=\frac{u(e^{\frac{1}{2}u}+e^{-\frac{1}{2}u})}{2(e^{\frac{1}{2}u}-e^{-\frac{1}{2}u})}$,

& quantitatibus $e^{\frac{1}{2}u}$ & $e^{-\frac{1}{2}u}$ in series conversis fiet

$$V - \tfrac{1}{2}u = \frac{1 + \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} + \frac{u^6}{2.4.6.8.10.12} + \&c.}{2(\frac{1}{2} + \frac{u^2}{2.4.6} + \frac{u^4}{2.4.6.8.10} + \&c.)}$$

five

$$V - \tfrac{1}{2}u = \frac{1 + \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} + \frac{u^6}{2.4\ldots 12} + \frac{u^8}{2.4\ldots 16} + \&c.}{1 + \frac{u^2}{4.6} + \frac{u^4}{4.6.8.10} + \frac{u^6}{4.6\ldots 14} + \frac{u^8}{4.6\ldots 18} + \&c.}$$

115. Cum igitur in hac fractione poteſtates impares prorſus deſint, in eius quoque evolutione poteſtates impares omnino nullae ingredientur; quare cum $V - \frac{1}{2}u$ aequetur iſti feriei:

$$1 + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \&c.$$

coefficientes imparium poteſtatum γ, ε, η, ι, &c. omnes evaneſcent. Sicque ratio manifeſta eſt, cur in ſerie:

$$1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \&c.$$

termini ordine pares omnes praeter ſecundum ſint $= 0$, neque tamen lex continuitatis vim patiatur. Erit ergo

$$V = 1 + \tfrac{1}{2}u + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \varkappa u^{10} + \&c.$$

litteriſque β, δ, ζ, θ, $\varkappa$, &c. per evolutionem ſuperioris fractionis determinatis, obtinebimus terminum ſummatorium Sz feriei, cuius terminus generalis eſt $= z$, indici x reſpondens, hoc modo expreſſum:

$$Sz = \int z dx + \tfrac{1}{2}z + \frac{\beta dz}{dx} + \frac{\delta d^3 z}{dx^3} + \frac{\zeta d^5 z}{dx^5} + \frac{\theta d^7 z}{dx^7} + \&c.$$

116. Quia ſeries $1 + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \&c.$ oritur ex evolutione huius fractionis:

$$\frac{1 + \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} + \frac{u^6}{2.4.6.8.10.12} + \&c.}{1 + \frac{u^2}{4.6} + \frac{u^4}{4.6.8.10} + \frac{u^6}{4.6.8.10.12.14} + \&c.}$$

lit-

litterae β, δ, ζ, θ, &c. hanc legem tenebunt, ut sit:

$$\beta = \frac{1}{2.4} - \frac{1}{4.6}$$

$$\delta = \frac{1}{2.4.6.8} - \frac{\beta}{4.6} - \frac{1}{4.6.8.10}$$

$$\zeta = \frac{1}{2.4.6\ldots12} - \frac{\delta}{4.6} - \frac{\beta}{4.6.8.10} - \frac{1}{4.6\ldots14}$$

$$\theta = \frac{1}{2.4\ldots16} - \frac{\zeta}{4.6} - \frac{\delta}{4.6.8.10} - \frac{\beta}{4.6\ldots14} - \frac{1}{4.6\ldots18}.$$

Hi autem valores alternative fiunt affirmativi & negativi.

117. Si igitur harum litterarum alternae capiantur negative, ita ut sit:

$$Sz = \int z dx + \tfrac{1}{2} z - \frac{\beta dz}{dx} + \frac{\delta d^3 z}{dx^3} - \frac{\zeta d^5 z}{dx^5} + \frac{\theta d^7 z}{dx^7} - \&c.$$

litterae β, δ, ζ, θ, &c. definientur ex hac fractione:

$$\frac{1 - \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} - \frac{u^6}{2.4\ldots12} + \frac{u^8}{2.4\ldots16} - \&c.}{1 - \frac{u^2}{4.6} + \frac{u^4}{4.6.8.10} - \frac{u^6}{4.6\ldots14} + \frac{u^8}{4.6\ldots18} - \&c.}$$

eam evolvendo in seriem

$1 + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \&c.$ quocirca erit;

$$\beta = \frac{1}{4.6} - \frac{1}{2.4}$$

$$\delta = \frac{\beta}{4.6} - \frac{1}{4.6.8.10} + \frac{1}{2.4.6.8}$$

$$\zeta = \frac{\delta}{4.6} - \frac{\beta}{4.6.8.10} + \frac{1}{4.6\ldots14} - \frac{1}{2.4\ldots12}$$

&c.

nunc autem omnes termini fient negativi.

118. Ponamus ergo $\beta = -A$; $\delta = -B$; $\zeta = -C$; &c. ut sit:

$$Sz = \int z dx + \tfrac{1}{2} z + \frac{A dz}{dx} - \frac{B d^3 z}{dx^3} + \frac{C d^5 z}{dx^5} - \frac{D d^7 z}{dx^7} + \&c.$$

atque, ad litteras A, B, C, D, &c., definiendas consideretur haec series:

$$1 - Au^2 - Bu^4 - Cu^6 - Du^8 - Eu^{10} - \&c.$$

quae oritur ex evolutione huius fractionis:

$$\frac{1 - \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} - \frac{u^6}{2.4 \ldots 12} + \frac{u^8}{2.4 \ldots 16} - \&c.}{1 - \frac{u^2}{4.6} + \frac{u^4}{4.6.8.10} - \frac{u^6}{4.6 \ldots 14} + \frac{u^8}{4.6 \ldots 18} - \&c.}$$

vel consideretur ista series:

$$\frac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \&c. = s$$

quae oritur ex evolutione huius fractionis:

$$s = \frac{1 - \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} - \frac{u^6}{2.4 \ldots 12} + \&c.}{u - \frac{u^3}{4.6} + \frac{u^5}{4.6.8.10} - \frac{u^7}{4.6 \ldots 14} + \&c.}$$

Cum autem sit:

$$\cos \tfrac{1}{2} u = 1 - \frac{u^2}{2.4} + \frac{u^4}{2.4.6.8} - \frac{u^6}{2.4 \ldots 12} + \&c.$$

$$\sin \tfrac{1}{2} u = \frac{u}{2} - \frac{u^3}{2.4.6} + \frac{u^5}{2.4.6.8.10} - \frac{u^7}{2.4 \ldots 14} + \&c.$$

sequitur fore: $s = \dfrac{\cos \frac{1}{2} u}{2 \sin \frac{1}{2} u} = \tfrac{1}{2} \cot. \tfrac{1}{2} u.$

Quare si cotangens arcus $\frac{1}{2} u$ in seriem convertatur, cuius termini secundum potestates ipsius u procedant, ex ea cognoscentur valores litterarum A, B, C, D, E, &c.

119. Cum igitur sit $s = \frac{1}{2} \cot \frac{1}{2} u$; erit $\frac{1}{2} u =$ A cot $2s$, & differentiando erit $\frac{1}{2} du = \dfrac{-2ds}{1 + 4ss}$ seu

$$4ds + du + 4ssdu = 0, \quad \text{sive} \quad \frac{4ds}{du} + 1 + 4ss = 0.$$

Quia autem est: $s = \frac{1}{u} - Au - Bu^3 - Cu^5 - \&c.$ erit

$$\frac{4ds}{du} = -\frac{4}{uu} - 4A - 3.4Bu^2 - 5.4Cu^4 - 7.4Du^6 - \&c.$$

$$1 = \qquad 1$$

$$4ss = \frac{4}{uu} - 8A - 8Bu^2 - 8Cu^4 - 8Du^6 - \&c.$$
$$+ 4A^2u^2 + 8ABu^4 + 8ACu^6 + \&c.$$
$$+ 4BBu^6 + \&c.$$

perductis his terminis homogeneis ad cyphram fiet:

$$A = \frac{1}{12}$$

$$B = \frac{A^2}{5}$$

$$C = \frac{2AB}{7}$$

$$D = \frac{2AC + BB}{9}$$

$$E = \frac{2AD + 2BC}{11}$$

$$F = \frac{2AE + 2BD + CC}{13}$$

$$G = \frac{2AF + 2BE + 2CD}{15}$$

$$H = \frac{2AG + 2BF + 2CE + DD}{17}$$

&c.

Ex quibus formulis iam manifesto liquet, singulos hos valores esse affirmativos.

120. Quoniam vero denominatores horum valorum fiunt vehementer magni, calculumque non mediocriter impediunt; loco litterarum A, B, C, D, &c. has novas introducamus:

$$A = \frac{\alpha}{1.2.3}$$

$$B = \frac{\beta}{1.2.3.4.5}$$

$$C = \frac{\gamma}{1.2.3\ldots\ldots 7}$$

$$D = \frac{\delta}{1.2.3\ldots\ldots 9}$$

$$E = \frac{\varepsilon}{1.2.3\ldots\ldots\ldots 11} \qquad \&c.$$

Atque reperietur fore:

$$\alpha = \frac{1}{2}$$

$$\beta = \frac{2}{3}\alpha^2$$

$$\gamma = 2.\frac{3}{3}\alpha\beta$$

$$\delta = 2.\frac{4}{3}\alpha\gamma + \frac{8.7}{4.5}\beta^2$$

$$\varepsilon = 2.\frac{5}{3}\alpha\delta + 2.\frac{10.9.8}{1\ldots 5}\beta\gamma$$

$$\zeta = 2.\frac{12}{1.2.3.}\alpha\varepsilon + 2.\frac{12.11.10}{1\ldots\ldots 5}\beta\delta + \frac{12.11.10.9.8}{1\ldots\ldots\ldots 7}\gamma\gamma$$

$$\eta = 2.\frac{14}{1.2.3}\alpha\zeta + 2.\frac{14.13.12}{1\ldots\ldots 5}\beta\varepsilon + 2.\frac{14.13.12.11.10}{1\ldots\ldots\ldots\ldots 7}\gamma\delta$$

&c.

121. Commodius autem utemur his formulis:

$$\alpha = \frac{1}{2}$$

$$\beta = \frac{4}{3} \cdot \frac{\alpha\alpha}{2}$$

$$\gamma = \frac{6}{3} \cdot \alpha\beta$$

$$\delta = \frac{8}{3} \cdot \alpha\gamma + \frac{8.\ 7.\ 6}{3.\ 4.\ 5} \cdot \frac{\beta\beta}{2}$$

$$\varepsilon = \frac{10}{3} \cdot \alpha\delta + \frac{10.\ 9.\ 8}{3.\ 4.\ 5} \cdot \beta\gamma$$

$$\zeta = \frac{12}{3} \cdot \alpha\varepsilon + \frac{12.11.10}{3.\ 4.\ 5} \cdot \beta\delta + \frac{12.11.10.9.8}{3.\ 4.\ 5.\ 6.\ 7} \cdot \frac{\gamma\gamma}{2}$$

$$\eta = \frac{14}{3} \cdot \alpha\zeta + \frac{14.13.12}{3.\ 4.\ 5} \cdot \beta\varepsilon + \frac{14.13 \ldots 10}{3.\ 4 \ldots 7} \cdot \gamma\delta$$

$$\theta = \frac{16}{3} \cdot \alpha\eta + \frac{16.15.14}{3.\ 4.\ 5} \cdot \beta\zeta + \frac{16.15 \ldots 12}{3.\ 4 \ldots 7} \cdot \gamma\varepsilon + \frac{16.15 \ldots 10}{3.\ 4 \ldots 9} \cdot \frac{\delta\delta}{2}$$

&c.

Ex hac igitur lege, ſecundum quam calculus non difficulter inſtituitur, ſi inventi fuerint valores litterarum α, β, γ, δ, &c. tum ſeriei cuiuſcunque; cuius terminus generalis ſeu indici x conveniens fuerit $= z$, terminus ſummatorius ita exprimetur, ut ſit:

$$Sz = \int z dx + \tfrac{1}{2} z + \frac{\alpha dz}{1.2.3\, dx} - \frac{\beta d^3 z}{1.2.3.4.5\, dx^3} + \frac{\gamma d^5 z}{1.2 \ldots 7\, dx^5}$$

$$- \frac{\delta d^7 z}{1.2 \ldots\ldots 9 dx^7} + \frac{\varepsilon d^9 z}{1.2 \ldots\ldots 11 dx^9} - \frac{\zeta d^{11} z}{1.2 \ldots\ldots 13 dx^{11}} + \&c.$$

iſtae autem litterae α, β, γ, δ, &c. ſequentes valores habere inventae ſunt:

$\alpha =$

$$\alpha = \frac{1}{2} \qquad 1.2\,\alpha = 1$$
$$\beta = \frac{1}{6} \qquad 1.2.3\,\beta = 1$$
$$\gamma = \frac{1}{6} \qquad 1.2.3.4\,\gamma = 4$$
$$\delta = \frac{3}{10} \qquad 1.2.3.4.5\,\delta = 36$$
$$\varepsilon = \frac{5}{6} \qquad 1.2.3\ldots 6\,\varepsilon = 600$$
$$\zeta = \frac{691}{210} \qquad 1.2.3\ldots 7\,\zeta = 24.691$$
$$\eta = \frac{35}{2} \qquad 1.2.3\ldots 8\,\eta = 20160.35$$
$$\theta = \frac{3617}{30} \qquad 1.2.3\ldots 9\,\theta = 12096.3617$$
$$\iota = \frac{43867}{42} \qquad 1.2.3\ldots 10\,\iota = 86400.43867$$
$$\varkappa = \frac{1222277}{110} \qquad 1.2.3\ldots 11\,\varkappa = 362880.1222277$$
$$\lambda = \frac{854513}{6} \qquad 1.2.3\ldots 12\,\lambda = 79833600.854513$$
$$\mu = \frac{1181820455}{546} \qquad 1.2.3\ldots 13\,\mu = 11404800.1181820455$$
$$\nu = \frac{76977927}{2} \qquad 1.2.3\ldots 14\,\nu = 43589145600.76977927$$
$$\xi = \frac{23749461029}{30} \qquad 1.2.3\ldots 15\,\xi = 43589145600.23749461029$$
$$\pi = \frac{8615841276005}{462} \qquad 1.2.3\ldots 16\,\pi = 45287424000.8615841276005$$

&c.

122. Numeri isti per universam serierum doctrinam amplissimum habent usum. Primum enim ex his numeris formari possunt ultimi termini in summis potestatum parium, quos non aeque ac reliquos terminos ex summis praecedentium reperiri posse supra annotavimus. In potestatibus enim paribus postremi summarum termini sunt x per certos numeros multiplicati; qui numeri pro potestatibus II; IV; VI; VIII; &c. sunt $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{30}$, &c. signis alternantibus affecti. Oriuntur autem hi numeri si valores litterarum α, β, γ, δ, &c. supra inventi respective dividantur per numeros impares 3, 5, 7, &c. unde isti numeri, qui ab Inventore *Iacobo Bernoullio* vocari solent Bernoulliani erunt:

$$\frac{\alpha}{3} = \frac{1}{6} = \mathfrak{A}$$
$$\frac{\beta}{5} = \frac{1}{30} = \mathfrak{B}$$
$$\frac{\gamma}{7} = \frac{1}{42} = \mathfrak{C}$$
$$\frac{\delta}{9} = \frac{1}{30} = \mathfrak{D}$$
$$\frac{\varepsilon}{11} = \frac{5}{66} = \mathfrak{E}$$
$$\frac{\zeta}{13} = \frac{691}{2730} = \mathfrak{F}$$
$$\frac{\eta}{15} = \frac{7}{6} = \mathfrak{G}$$
$$\frac{\theta}{17} = \frac{3617}{510} = \mathfrak{H}$$
$$\frac{\iota}{19} = \frac{43867}{798} = \mathfrak{I}$$

$$\frac{\varkappa}{21} = \frac{174611}{330} = \mathfrak{K} = \frac{283.617}{330}$$

$$\frac{\lambda}{23} = \frac{854513}{138} = \mathfrak{L} = \frac{11.131.593}{2.3.23}$$

$$\frac{\mu}{25} = \frac{236364091}{2730} = \mathfrak{M}$$

$$\frac{\nu}{27} = \frac{8553103}{6} = \mathfrak{N} = \frac{13.657931}{6}$$

$$\frac{\xi}{29} = \frac{23749461029}{870} = \mathfrak{O}$$

$$\frac{\pi}{31} = \frac{8615841276005}{14322} = \mathfrak{P}$$

&c.

123. Isti igitur numeri Bernoulliani $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. immediate ex sequentibus aequationibus inveniri poterunt:

$$\mathfrak{A} = \frac{1}{6}$$

$$\mathfrak{B} = \frac{4.3}{1.2}.\frac{1}{5}\mathfrak{A}^2$$

$$\mathfrak{C} = \frac{6.5}{1.2}.\frac{2}{7}\mathfrak{A}\mathfrak{B}$$

$$\mathfrak{D} = \frac{8.7}{1.2}.\frac{2}{9}\mathfrak{A}\mathfrak{C} + \frac{8.7.6.5}{1.2.3.4}.\frac{1}{9}\mathfrak{B}^2$$

$$\mathfrak{E} = \frac{10.9}{1.2}.\frac{2}{11}\mathfrak{A}\mathfrak{D} + \frac{10.9.8.7}{1.2.3.4}.\frac{2}{11}\mathfrak{B}\mathfrak{C}$$

$$\mathfrak{F} = \frac{12.11}{1.2}.\frac{2}{13}\mathfrak{A}\mathfrak{E} + \frac{12.11.10.9}{1.2.3.4}.\frac{2}{13}\mathfrak{B}\mathfrak{D} + \frac{12.11.10.9.8.7}{1.2.3.4.5.6}.\frac{1}{13}\mathfrak{C}^2$$

$$\mathfrak{G} = \frac{14.13}{1.2}.\frac{2}{15}\mathfrak{A}\mathfrak{F} + \frac{14.13.12.11}{1.2.3.4}.\frac{2}{15}\mathfrak{B}\mathfrak{E} + \frac{14.13.12.11.10.9}{1.2.3.4.5.6}.\frac{2}{15}\mathfrak{C}\mathfrak{D}$$

&c.

qua-

quarum aequationum lex per se est manifesta, si tantum notetur, ubi quadratum cuiuspiam litterae occurrit, eius coefficientem duplo esse minorem, quam secundum regulam esse debere videatur: Revera autem termini, qui continent producta ex disparibus litteris, bis occurrere censendi sunt, erit enim verbi gratia:

$$13\mathfrak{F} = \frac{12.11}{1.2}\mathfrak{AE} + \frac{12.11.10.9}{1.2.3.4}\mathfrak{BD} + \frac{12.11.10.9.8.7}{1.2.3.4.5.6}\mathfrak{CC} +$$
$$\frac{12.11.10\ldots5}{1.2.3\ldots8}\mathfrak{DB} + \frac{12.11.10\ldots3}{1.2.3\ldots10}\mathfrak{EA}$$

124. Deinde vero etiam iidem numeri $\alpha, \beta, \gamma, \delta$, &c. ingrediuntur in expressiones summarum serierum fractionum in hac forma generali:

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \&c.$$

quoties n est numerus par affirmativus; contentarum. Has enim summas in introductione per potestates semiperipheriae circuli π radio existente $= 1$ expressas dedimus, atque in harum potestatum coefficientibus isti ipsi numeri $\alpha, \beta, \gamma, \delta$, &c. ingredi deprehenduntur. Quo autem haec convenientia non casu evenire, sed necessario locum habere intelligatur, has easdem summas singulari modo investigemus, quo lex summarum illarum facilius patebit. Quoniam supra invenimus esse:

$$\frac{\pi}{n}\cot.\frac{m}{n}\pi = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \&c.$$

binis terminis coniungendis habebimus:

$$\frac{\pi}{n}\cot.\frac{m}{n}\pi = \frac{1}{m} - \frac{2m}{nn-m^2} - \frac{2m}{4n^2-m^2} - \frac{2m}{9n^2-m^2} - \frac{2m}{16n^2-m^2} - \&c.$$

unde colligimus fore:

$$\frac{1}{n^2-m^2} + \frac{1}{4n^2-m^2} + \frac{1}{9n^2-m^2} + \frac{1}{16n^2-m^2} + \&c. = \frac{1}{2mm} - \frac{\pi}{2mn}\cot.\frac{m}{n}\pi$$

 Sta-

Statuamus nunc $n=1$, & pro m ponamus u; ut fit:

$$\frac{1}{1-u^2}+\frac{1}{4-u^2}+\frac{1}{9-u^2}+\frac{1}{16-u^2}+\&c.=\frac{1}{2uu}-\frac{\pi}{2u}\cot.\pi u.$$

Resolvantur fingulae istae fractiones in feries:

$$\frac{1}{1-u^2}=1+u^2+u^4+u^6+u^8+\&c.$$

$$\frac{1}{4-u^2}=\frac{1}{2^2}+\frac{u^2}{2^4}+\frac{u^4}{2^6}+\frac{u^6}{2^8}+\frac{u^8}{2^{10}}+\&c.$$

$$\frac{1}{9-u^2}=\frac{1}{3^2}+\frac{u^2}{3^4}+\frac{u^4}{3^6}+\frac{u^6}{3^8}+\frac{u^8}{3^{10}}+\&c.$$

$$\frac{1}{16-u^2}=\frac{1}{4^2}+\frac{u^2}{4^4}+\frac{u^4}{4^6}+\frac{u^6}{4^8}+\frac{u^8}{4^{10}}+\&c.$$

&c.

125. Quod fi ergo ponatur:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c.=\mathfrak{a}$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\&c.=\mathfrak{b}$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\&c.=\mathfrak{c}$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\&c.=\mathfrak{d}$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\&c.=\mathfrak{e}$$

&c.

fuperior feries transfmutabitur in hanc:

$$\mathfrak{a}+\mathfrak{b}u^2+\mathfrak{c}u^4+\mathfrak{d}u^6+\mathfrak{e}u^8+\mathfrak{f}u^{10}+\&c.=\frac{1}{2uu}-\frac{\pi}{2u}\cot.\pi u.$$

Cum igitur in §. 118. litterae A, B, C, D, &c. ita comparatae fint inventae, ut pofito:

$s=$

$$s = \frac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \&c.$$

fit $s = \frac{1}{2}\cot.\frac{1}{2}u$, erit posito πu loco $\frac{1}{2}u$ seu $2\pi u$ loco u

$$\tfrac{1}{2}\cot\pi u = \frac{1}{2\pi u} - 2A\pi u - 2^3B\pi^3u^3 - 2^5C\pi^5u^5 - 2^7D\pi^7u^7 - \&c.$$

unde per $\frac{\pi}{u}$ multiplicando erit:

$$\frac{\pi}{2u}\cot.\pi u = \frac{1}{2uu} - 2A\pi^2 - 2^3B\pi^4u^2 - 2^5C\pi^6u^4 - \&c.$$

hincque sequitur fore:

$$\frac{1}{2uu} - \frac{\pi}{2u}\cot.\pi u = 2A\pi^2 + 2^3B\pi^4u^2 + 2^5C\pi^6u^4 + 2^7D\pi^8u^6 + \&c.$$

Quia igitur modo invenimus esse:

$$\frac{1}{2uu} - \frac{\pi}{2u}\cot.\pi u = \mathfrak{a} + \mathfrak{b}u^2 + \mathfrak{c}u^4 + \mathfrak{d}u^6 + \&c.$$

necesse est ut sit:

$$\mathfrak{a} = 2\,A\,\pi^2 = \frac{2\alpha}{1.2.3}.\pi^2 = \frac{2\mathfrak{A}}{1.2}.\pi^2$$

$$\mathfrak{b} = 2^3\,B\,\pi^4 = \frac{2^3\beta}{1.2.3.4.5}.\pi^4 = \frac{2^3\mathfrak{B}}{1.2.3.4}.\pi^4$$

$$\mathfrak{c} = 2^5\,C\,\pi^6 = \frac{2^5\gamma}{1.2.3\ldots7}\pi^6 = \frac{2^5\mathfrak{C}}{1.2\ldots\ldots6}\pi^6$$

$$\mathfrak{d} = 2^7\,D\,\pi^8 = \frac{2^7\delta}{1.2.3\ldots9}\pi^8 = \frac{2^7\mathfrak{D}}{1.2\ldots\ldots8}\pi^8$$

$$\mathfrak{e} = 2^9\,E\,\pi^{10} = \frac{2^9\varepsilon}{1.2.3\ldots11}\pi^{10} = \frac{2^9\mathfrak{E}}{1.2\ldots.10}\pi^{10}$$

$$\mathfrak{f} = 2^{11}F\,\pi^{12} = \frac{2^{11}\zeta}{1.2.3\ldots13}\pi^{12} = \frac{2^{11}\mathfrak{F}}{1.2\ldots.12}\pi^{12}\quad\&c.$$

126. Ex hoc ergo tam facili ratiocinio non solum omnes series potestatum reciprocarum, quas §. praeced. exhibui-

buimus, expedite summantur; sed simul quoque perspicitur, quemadmodum istae summae ex cognitis valoribus litterarum $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. vel etiam ex numeris Bernoullianis $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, &c. formentur. Quare cum istorum numerorum quindecim §. 122. definiverimus, ex iis summae omnium potestatum parium usque ad summam huius seriei inclusive assignari poterunt:

$1 + \frac{1}{2^{30}} + \frac{1}{3^{30}} + \frac{1}{4^{30}} + \frac{1}{5^{30}} + \&c.$ erit enim huius seriei summa $= \frac{2^{29}\pi}{1.2.3\ldots31}\pi^{30} = \frac{2^{29}\mathfrak{P}}{1.2\ldots\ldots30}\pi^{30}$.

Atque si quis voluerit has summas ulterius determinare, id continuandis numeris α, β, γ, &c. vel his $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, &c. facillime praestabitur.

127. Origo ergo horum numerorum $\alpha, \beta, \gamma, \delta$, &c. vel inde formatorum $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, &c. potissimum debetur evolutioni cotangentis cuiusvis anguli in seriem infinitam. Cum enim sit

$$\tfrac{1}{2}\cot.\tfrac{1}{2}u = \frac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \&c.$$

erit:

$$Au^2 + Bu^4 + Cu^6 + Du^8 + \&c. = 1 - \frac{u}{2}\cot.\tfrac{1}{2}u,$$

si igitur loco coefficientium A, B, C, D, &c. valores ipsorum substituantur, reperietur:

$$\frac{\alpha u^2}{1.2.3} + \frac{\beta u^4}{1.2\ldots5} + \frac{\gamma u^6}{1.2\ldots\ldots7} + \frac{\delta u^8}{1.2\ldots9} + \&c. = 1 - \frac{u}{2}\cot\tfrac{1}{2}u$$

atque numeros Bernoullianos adhibendo erit:

$$\frac{\mathfrak{A}u^2}{1.2} + \frac{\mathfrak{B}u^4}{1.2.3.4} + \frac{\mathfrak{C}u^6}{1.2\ldots6} + \frac{\mathfrak{D}u^8}{1.2\ldots\ldots8} + \&c. = 1 - \frac{u}{2}\cot\tfrac{1}{2}u$$

ex quibus seriebus per differentiationem innumerabiles aliae deduci possunt, sicque infinitae series summari, in quas isti numeri notatu tantopere digni ingrediuntur.

128. Sumamus aequationem priorem, quam per u multiplicemus, ut fit:

$$\frac{\alpha u^3}{1.2.3} + \frac{\beta u^5}{1.2\ldots 5} + \frac{\gamma u^7}{1.2\ldots 7} + \frac{\delta u^9}{1.2\ldots 9} + \&c. = u - \frac{uu}{2}\cot\tfrac{1}{2}u$$

quae differentiata ac per du divisa dat:

$$\frac{\alpha u^2}{1.2} + \frac{\beta u^4}{1.2.3.4} + \frac{\gamma u^6}{1.2\ldots 6} + \frac{\delta u^8}{1.2\ldots 8} + \&c. = 1 - u\cot\tfrac{1}{2}u + \frac{uu}{4(\sin\frac{1}{2}u)^2}$$

&, si denuo differentietur erit:

$$\frac{\alpha u}{1} + \frac{\beta u^3}{1.2.3} + \frac{\gamma u^5}{1.2.3.4.5} + \&c. = -\cot\tfrac{1}{2}u + \frac{u}{(\sin\frac{1}{2}u)^2} - \frac{uu\cos\frac{1}{2}u}{4(\sin\frac{1}{2}u)^3}$$

Sin autem altera aequatio differentietur erit:

$$\frac{\mathfrak{A}u}{1} + \frac{\mathfrak{B}u^3}{1.2.3} + \frac{\mathfrak{C}u^5}{1.2\ldots 5} + \frac{\mathfrak{D}u^7}{1.2\ldots 7} = -\tfrac{1}{2}\cot\tfrac{1}{2}u + \frac{u}{4(\sin\frac{1}{2}u)^2}$$

Ex his ergo si ponatur $u = \pi$, ob $\cot\frac{1}{2}\pi = 0$, & $\sin\frac{1}{2}u = 1$, sequuntur istae summationes:

$$1 = \frac{\alpha\pi^2}{1.2.3} + \frac{\beta\pi^4}{1.2.3.4.5} + \frac{\gamma\pi^6}{1.2.3\ldots 7} + \frac{\delta\pi^8}{1.2.3\ldots 9} + \&c.$$

$$1 + \frac{\pi^2}{4} = \frac{\alpha\pi^2}{1.2} + \frac{\beta\pi^4}{1.2.3.4} + \frac{\gamma\pi^6}{1.2.3\ldots 6} + \frac{\delta\pi^8}{1.2.3\ldots 8} + \&c.$$

$$\pi = \frac{\alpha\pi}{1} + \frac{\beta\pi^3}{1.2.3} + \frac{\gamma\pi^5}{1.2.3.4.5} + \frac{\delta\pi^7}{1.2.3\ldots 7} + \&c.$$

seu $$1 = \alpha + \frac{\beta\pi^2}{1.2.3} + \frac{\gamma\pi^4}{1.2.3.4.5} + \frac{\delta\pi^6}{1.2.3\ldots 7} + \&c.$$

a qua si prima subtrahatur remanebit:

$$\alpha = \frac{(\alpha-\beta)\pi^2}{1.2.3} + \frac{(\beta-\gamma)\pi^4}{1.2.3.4.5} + \frac{(\gamma-\delta)\pi^6}{1.2.3\ldots 7} + \&c.$$

Tum vero erit:

$$1 = \frac{\mathfrak{A}\pi^2}{1.2} + \frac{\mathfrak{B}\pi^4}{1.2.3.4} + \frac{\mathfrak{C}\pi^6}{1.2.3\ldots 6} + \frac{\mathfrak{D}\pi^8}{1.2.3\ldots 8} + \&c.$$

$$\frac{\pi}{4} = \frac{\mathfrak{A}\pi}{1} + \frac{\mathfrak{B}\pi^3}{1.2.3} + \frac{\mathfrak{C}\pi^5}{1.2.3.4.5} + \frac{\mathfrak{D}\pi^7}{1.2.3\ldots 7} + \&c.$$

seu $\frac{1}{6} = \frac{\mathfrak{A}}{1} + \frac{\mathfrak{B}\pi^2}{1.\,2.\,3} + \frac{\mathfrak{C}\pi^4}{1.2.3.4.5} + \frac{\mathfrak{D}\pi^6}{1.2.3\ldots 7} + \&c.$

129. Ex tabula valorum numerorum α, β, γ, δ, &c. quam supra §. 121. exhibuimus, patet eos primum decrescere tum vero iterum crescere, & quidem in infinitum. Operae igitur pretium erit investigare, in quanam ratione hi numeri, postquam iam vehementer longe fuerint continuati, ulterius progredi pergant. Sit igitur ϕ numerus quicunque huius seriei numerorum α, β, γ, δ, &c. longissime ab initio remotus, & sit ψ numerorum sequens. Quoniam per hos numeros summae potestatum reciprocarum definiuntur, sit $2n$ exponens potestatis, in cuius summa numerus ϕ ingreditur, erit $2n+2$ exponens potestatis numero ψ respondens, atque numerus n iam erit vehementer magnus. Hinc ex §. 125. habebitur:

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \&c. = \frac{2^{2n-1}\,\phi}{1.2.3\ldots(2n+1)}\,\pi^{2n}$$

$$1 + \frac{1}{2^{2n+2}} + \frac{1}{3^{2n+2}} + \frac{1}{4^{2n+2}} + \&c. = \frac{2^{2n+1}\,\psi}{1.2.3\ldots(2n+3)}\,\pi^{2n+2}$$

Quod si ergo haec per istam dividatur, erit:

$$\frac{1 + \frac{1}{2^{2n+2}} + \frac{1}{3^{2n+2}} + \&c.}{1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \&c.} = \frac{4\psi}{(2n+2)\,(2n+3)}\;\frac{\pi^2}{\phi}$$

Quia vero n est numerus vehementer magnus, ob seriem utramque proxime $= 1$, erit:

$$\frac{\psi}{\phi} = \frac{(2n+2)(2n+3)}{4\pi^2} = \frac{nn}{\pi\pi}$$

Cum igitur n designet, quotus sit numerus ϕ a primo α computatus, se habebit hic numerus ϕ ad suum sequentem ψ ut π^2 ad n^2, quae ratio, si n fuerit numerus infinitus, veritati penitus sit consentanea. Quoniam est fere $\pi\pi = 10$, si ſi ponatur $n = 100$; erit terminus centeſimus circiter millies minor ſuo ſequente. Conſtituunt ergo numeri $\alpha, \beta, \gamma, \delta$, &c. pariter ac Bernoulliani $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, &c. ſeriem maxime divergentem, quae etiam magis increſcat, quam ulla ſeries geometrica terminis creſcentibus procedens.

130. Inventis ergo his valoribus numerorum $\alpha, \beta, \gamma, \delta$, &c. ſeu $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, &c. ſi proponatur ſeries, cuius terminus generalis z fuerit functio quaecunque ipſius indicis x, terminus ſummatorius Sz huius ſeriei ſequenti modo exprimetur, ut ſit:

$$
\begin{aligned}
Sz = \int z\,dx + \tfrac{1}{2}z &+ \frac{1}{6}\cdot\frac{dz}{1.2\,dx} - \frac{1}{30}\cdot\frac{d^3z}{1.2.3.4\,dx^3} \\
&+ \frac{1}{42}\cdot\frac{d^5z}{1.2.3\ldots 6\,dx^5} - \frac{1}{30}\cdot\frac{d^7z}{1.2.3\ldots 8\,dx^7} \\
&+ \frac{5}{66}\cdot\frac{d^9z}{1.2.3\ldots 10\,dx^9} - \frac{691}{2730}\cdot\frac{d^{11}z}{1.2.3\ldots 12\,dx^{11}} \\
&+ \frac{7}{6}\cdot\frac{d^{13}z}{1.2.3\ldots 14\,dx^{13}} - \frac{3617}{510}\cdot\frac{d^{15}z}{1.2.3\ldots 16\,dx^{15}} \\
&+ \frac{43867}{798}\cdot\frac{d^{17}z}{1.2.3\ldots 18\,dx^{17}} - \frac{174611}{330}\cdot\frac{d^{19}z}{1.2.3\ldots 20\,dx^{19}} \\
&+ \frac{854513}{138}\cdot\frac{d^{21}z}{1.2.3\ldots 22\,dx^{21}} - \frac{236364091}{2730}\cdot\frac{d^{23}z}{1.2.3\ldots 24\,dx^{23}} \\
&+ \frac{8553103}{6}\cdot\frac{d^{25}z}{1.2.3\ldots 26\,dx^{25}} - \frac{23749461029}{870}\cdot\frac{d^{27}z}{1.2.3\ldots 28\,dx^{27}} \\
&+ \frac{8615841276005}{14322}\cdot\frac{d^{29}z}{1.2.3\ldots 30\,dx^{29}} - \&c.
\end{aligned}
$$

Si igitur innotuerit integrale $\int z\,dx$, ſeu quantitas illa cuius differentiale ſit $= z\,dx$, terminus ſummatorius ope continuae differentiationis invenietur. Perpetuo autem notandum eſt ad hanc expreſſionem ſemper eiuſmodi conſtantem addi oportere, ut ſumma fiat $= 0$, ſi index x ponatur in nihilum abire:

131. Si igitur z fuerit functio rationalis integra ipſius

x,

x, quia eius differentialia tandem evanescunt, terminus summatorius per expressionem finitam exprimetur; id quod sequentibus exemplis illustrabimus.

EXEMPLUM I.

Quaeratur terminus summatorius huius seriei:

$$\overset{1}{1}+\overset{2}{9}+\overset{3}{25}+\overset{4}{49}+\overset{5}{81}+\;\ldots\ldots\;+\overset{x}{(2x-1)^2}$$

Quia hic est $z=(2x-1)^2=4xx-4x+1$;

erit $\int zdx=\frac{4}{3}x^3-2x^2+x$,

ex huius enim differentiatione oritur:

$$4xxdx-4xdx+dx=zdx.$$

Deinde vero per differentiationem erit:

$$\frac{dz}{dx}=8x-4$$

$$\frac{ddz}{dx^2}=8$$

$$\frac{d^3z}{dx^3}=0 \quad \&c.$$

Hinc erit terminus summatorius quaesitus: $=$
$\frac{4}{3}x^3-2x^2+x+2xx-2x+\frac{1}{2}+\frac{2}{3}x-\frac{1}{3}\pm$ Const.
qua constante tolli debent termini $\frac{1}{2}-\frac{1}{3}$, unde erit

$$S(2x-1)^2=\frac{4}{3}x^3-\frac{1}{3}x=\frac{x}{3}(2x-1)(2x+1).$$

Sic erit posito $x=4$ summa 4 primorum terminorum

$$1+9+25+49=\frac{4}{3}\cdot 7\cdot 9=84.$$

EXEMPLUM II.

Quaeratur terminus summatorius huius seriei:

$$\overset{1}{1}+\overset{2}{27}+\overset{3}{125}+\overset{4}{343}+\;\ldots\ldots\;\overset{x}{(2x-1)^3}$$

Quia est $z=(2x-1)^3=8x^3-12x^2+6x-1$; erit $\int zdx$

$\int z dx = 2x^4 - 4x^3 + 3x^2 - x$; $\frac{dz}{dx} = 24x^2 - 24x + 6$;

$\frac{ddz}{dx^2} = 48x - 24$; $\frac{d^3z}{dx^3} = 48$; sequentia evanescunt.

Quare erit $S(2x-1)^3 = 2x^4 - 4x^3 + 3x^2 - x$
$\qquad + 4x^3 - 6x^2 + 3x - \frac{1}{2}$
$\qquad + 2x^2 - 2x + \frac{1}{2}$
$\qquad - \frac{1}{15}$

hoc est $S(2x-1)^3 = 2x^4 - x^2 = x^2(2xx-1)$. Sic erit posito $x = 4$ $\quad 1 + 27 + 125 + 343 = 16.31 = 496$.

132. Ex hac inventa generali expressione pro termino summatorio sponte sequitur ille terminus summatorius, quem superiori parte pro potestatibus numerorum naturalium dedimus, cuiusque demonstrationem ibi tradere non licuerat. Quod si enim ponamus $z = x^n$, erit utique

$\int z dx = \frac{1}{n+1} x^{n+1}$; differentialia vero ita se habebunt:

$\frac{dz}{dx} = nx^{n-1}$.

$\frac{ddz}{dx^2} = n(n-1)x^{n-2}$

$\frac{d^3z}{dx^3} = n(n-1)(n-2)x^{n-3}$

$\frac{d^5z}{dx^5} = n(n-1)(n-2)n-3)(n-4)x^{n-5}$

$\frac{d^7z}{dx^7} = n(n-1) \; : \; : \; : \; : \; (n-6)x^{n-7}$ &c.

Ex his ergo deducetur sequens terminus summatorius respondens termino generali x^n; scilicet

$$Sx^n = \frac{1}{n+1} x^{n+1} + \frac{1}{2} x^n + \frac{1}{6} \cdot \frac{n}{2} x^{n-1}$$
$$- \frac{1}{30} \cdot \frac{n(n-1)(n-2)}{2. \ 3. \ 4.} x^{n-3}$$
$$+ \frac{1}{42} \cdot \frac{n(n-1)(n-2)(n-3)(n-4)}{2. \ 3. \ 4. \ 5. \ 6} x^{n-5}$$
$$- \frac{1}{30} \cdot \frac{n(n-1) \ldots (n-6)}{2.3 \ldots 8} x^{n-7}$$
$$+ \frac{5}{66} \cdot \frac{n(n-1) \ldots (n-8)}{2.3 \ldots 10} x^{n-9}$$
$$- \frac{691}{2730} \cdot \frac{n(n-1) \ldots (n-10)}{2.3 \ldots 12} x^{n-11}$$
$$+ \frac{7}{6} \cdot \frac{n(n-1) \ldots (n-12)}{2.3 \ldots 14} x^{n-13}$$
$$- \frac{3617}{510} \cdot \frac{n(n-1) \ldots (n-14)}{2.3 \ldots 16} x^{n-15}$$
$$+ \frac{43867}{798} \cdot \frac{n(n-1) \ldots (n-16)}{2.3 \ldots 18} x^{n-17}$$
$$- \frac{174611}{330} \cdot \frac{n(n-1) \ldots (n-18)}{2.3 \ldots 20} x^{n-19}$$
$$+ \frac{854513}{138} \cdot \frac{n(n-1) \ldots (n-20)}{2.3 \ldots 22} x^{n-21}$$
$$- \frac{236364091}{2730} \cdot \frac{n(n-1) \ldots (n-22)}{2.3 \ldots 24} x^{n-23}$$
$$+ \frac{8553103}{6} \cdot \frac{n(n-1) \ldots (n-24)}{2.3 \ldots 26} x^{n-25}$$
$$- \frac{23749461029}{870} \cdot \frac{n(n-1) \ldots (n-26)}{2.3 \ldots 28} x^{n-27}$$
$$+ \frac{8615841276005}{14322} \cdot \frac{n(n-1) \ldots (n-28)}{2.3 \ldots 30} x^{n-29} \ \&c.$$

quae

quae expreſſio non differt ab ea, quam ſupra dedimus, niſi quod hic numeros Bernoullianos $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. introduximus, cum ſupra uſi eſſemus numeris α, β, γ, δ, &c. interim tamen conſenſus ſponte elucet. Hinc ergo terminos ſummatorios omnium poteſtatum uſque ad poteſtatem trigeſimam incluſive exhibere licuit; quae inveſtigatio, ſi alia via fuiſſet ſuſcepta, ſine longiſſimis & taedioſiſſimis calculis abſolvi non potuiſſet.

133. Iam ſupra §. 59. ſimilem fere expreſſionem pro termino ſummatorio dedimus ex termino generali definiendo. Ea enim pariter ſecundum differentialia termini generalis procedebat; ab iſta autem in hoc potiſſimum erat diverſa, quod illa non integrale $\int z dx$ requirebat, ſingula vero termini generalis differentialia per certas ipſius x functiones habebat multiplicata. Eandem igitur expreſſionem ſequenti modo ad naturam ſerierum magis accommodato denuo eliciamus, ex quo ſimul lex clarius patebit, ſecundum quam coefficientes illi differentialium progrediuntur. Sit igitur ſeriei terminus generalis z, functio quaecunque ipſius indicis x, terminus vero ſummatorius quaeſitus ſit s: qui quoniam uti vidimus eiuſmodi erit functio ipſius x, ut evaneſcat poſito $x = 0$, erit per ea, quae ſupra de natura huiuſmodi functionum demonſtravimus:

$$s - \frac{x ds}{1 dx} + \frac{x^2 dds}{1.2 dx^2} - \frac{x^3 d^3 s}{1.2.3 dx^3} + \frac{x^4 d^4 s}{1.2.3.4 dx^4} - \&c. = 0.$$

134. Quia s denotat ſummam omnium terminorum ſeriei a primo uſque ad ultimum z, perſpicuum eſt ſi in s loco x ponatur $x - 1$, tum priorem ſummam ultimo termino z mulctari: erit ſcilicet

$$s - z = s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3 s}{6dx^3} + \frac{d^4 s}{24dx^4} - \&c.$$

ideoque $$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3 s}{6dx^3} - \frac{d^4 s}{24dx^4} + \&c.$$

 quae

quae aequatio modum suppeditat ex dato termino summatorio s definiendi terminum generalem, quod quidem per se est facillimum. Ex idonea autem combinatione huius aequationis cum ea, quam §. praeced. invenimus, valor ipsius s per x & z determinari poterit. Ponamus in hunc finem esse:

$$s - Az + \frac{Bdz}{dx} - \frac{Cddz}{dx^2} + \frac{Dd^3z}{dx^3} - \frac{Ed^4z}{dx^4} + \&c. = 0.$$

ubi A, B, C, D, &c. denotent coefficientes necessario sive constantes sive variabiles: nam cum sit

$$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} - \&c.$$

si hinc valores pro z, $\frac{dz}{dx}$, $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ &c. in superiori aequatione substituantur, prodibit:

$$\begin{array}{rcl}
s & = & s \\
-Az & = & -\frac{Ads}{dx} + \frac{Adds}{2dx^2} - \frac{Ad^3s}{6dx^3} + \frac{Ad^4s}{24dx^4} - \frac{Ad^5s}{120dx^5} + \&c. \\
+\frac{Bdz}{dx} & = & + \frac{Bdds}{dx^2} - \frac{Bd^3s}{2dx^3} + \frac{Bd^4s}{6dx^4} - \frac{Bd^5s}{24dx^5} + \&c. \\
-\frac{Cddz}{dx^2} & = & - \frac{Cd^3s}{dx^3} + \frac{Cd^4s}{2dx^4} - \frac{Cd^5s}{6dx^5} + \&c. \\
+\frac{Dd^3z}{dx^3} & = & + \frac{Dd^4s}{dx^4} - \frac{Dd^5s}{2dx^5} + \&c. \\
-\frac{Ed^4z}{dx^4} & = & - \frac{Ed^5s}{dx^5} + \&c. \\
 & \&c. &
\end{array}$$

quae igitur series iunctim sumtae aequales erunt nihilo.

135. Cum ergo ante invenimus esse:

$$0 = s - \frac{xds}{dx} + \frac{x^2dds}{2dx^2} - \frac{x^3d^3s}{6dx^3} + \frac{x^4d^4s}{24dx^4} - \frac{x^5d^5s}{120dx^5} + \&c.$$

si

si superior aequatio huic aequalis statuatur, prodibunt sequentes litterarum A, B, C, D, &c. denominationes:

$$A = x$$

$$B = \frac{x^2}{2} - \frac{A}{2}$$

$$C = \frac{x^3}{6} - \frac{B}{2} - \frac{A}{6}$$

$$D = \frac{x^4}{24} - \frac{C}{2} - \frac{B}{6} - \frac{A}{24}$$

$$E = \frac{x^5}{120} - \frac{D}{2} - \frac{C}{6} - \frac{B}{24} - \frac{A}{120} \quad \&c.$$

His igitur litterarum A, B, C, D, &c. valoribus inventis, ex termino generali z terminus summatorius $s = Sz$ ita determinabitur, ut sit:

$$Sz = Az - \frac{B\,dz}{dx} + \frac{C\,ddz}{dx^2} - \frac{D\,d^3z}{dx^3} + \frac{E\,d^4z}{dx^4} - \frac{F\,d^5z}{dx^5} + \&c.$$

136. Cum autem fiat:

$$A = x$$
$$B = \tfrac{1}{2}x^2 - \tfrac{1}{2}x$$
$$C = \tfrac{1}{6}x^3 - \tfrac{1}{4}x^2 + \tfrac{1}{12}x$$
$$D = \tfrac{1}{24}x^4 - \tfrac{1}{12}x^3 + \tfrac{1}{24}xx \quad \&c.$$

patet hos coefficientes esse eosdem, quos supra §. 59. habuimus, unde ista termini summatorii expressio eadem est, quam ibi invenimus; eritque propterea:

$$A = Sx^0 = S1$$
$$B = \tfrac{1}{1}Sx^1 - \tfrac{1}{1}x$$
$$C = \tfrac{1}{2}Sx^2 - \tfrac{1}{2}x^2$$
$$D = \tfrac{1}{6}Sx^3 - \tfrac{1}{6}x^3$$
$$E = \tfrac{1}{24}Sx^4 - \tfrac{1}{24}x^4 \quad \&c.$$

Hinc ergo erit:

$$Sz = xz - \frac{dz}{dx}Sx + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3}Sx^3 + \frac{d^4z}{24dx^4}Sx^4 - \&c.$$

+

$$+\frac{xdz}{dx}-\frac{x^2ddz}{2dx^2}+\frac{x^3d^3z}{6dx^3}-\frac{x^4d^4z}{24dx^4}+\&c.$$

Quodsi autem in termino generali z ponatur $x=0$, prodibit terminus indici $=0$ respondens; qui si ponatur $=a$, erit: $a=z-\frac{xdz}{dx}+\frac{x^2ddz}{2dx^2}-\frac{x^3d^3z}{6dx^3}+\&c.$ ideoque

$$\frac{xdz}{dx}-\frac{x^2ddz}{2dx^2}+\frac{x^3d^3z}{6dx^3}-\frac{x^4d^4z}{24dx^4}+\&c.=z-a,$$

quo valore substituto habebitur:

$$Sz=(x+1)z-a-\frac{dz}{dx}Sx+\frac{ddz}{2dx^2}Sx^2-\frac{d^3z}{6dx^3}Sx^3+\frac{d^4z}{24dx^4}Sx^4-\&c.$$

Cognitis ergo summis potestatum, hinc pro quovis termino generali ei conveniens terminus summatorius exhiberi potest.

137. Quoniam ergo geminam invenimus expressionem termini summatorii Sz pro termino generali z, earumque altera formulam integralem $\int zdx$ continet, si istae duae expressiones sibi aequales ponantur, obtinebitur valor ipsius $\int zdx$ per seriem expressus. Cum enim sit:

$$\int zdx+\tfrac{1}{2}z+\frac{\mathfrak{A}dz}{1.2dx}-\frac{\mathfrak{B}d^3z}{1.2.3.4dx^3}+\frac{\mathfrak{C}d^5z}{1.2\ldots6dx^5}-\&c.$$

$$=(x+1)z-a-\frac{dz}{1dx}Sx+\frac{ddz}{1.2dx^2}Sx^2-\frac{d^3z}{1.2.3dx^3}Sx^3+\&c.$$

erit:

$$\int zdx=(x+\tfrac{1}{2})z-a-\frac{dz}{dx}(Sx+\tfrac{1}{2}\mathfrak{A})+\frac{ddz}{2dx^2}Sx^2-\frac{d^3z}{6dx^3}(Sx^3-\tfrac{1}{4}\mathfrak{B})$$

$$+\frac{d^4z}{24dx^4}Sx^4-\frac{d^5z}{120dx^5}(Sx^5+\tfrac{1}{6}\mathfrak{C})+\frac{d^6z}{720dx^6}Sx^6$$

$$-\frac{d^7z}{5040dx^7}(Sx^7-\tfrac{1}{8}\mathfrak{D})+\&c.$$

ubi $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. denotant numeros Bernoullianos supra §. 122. exhibitos.

Sit

Sit v. gr. $z = xx$, fiet $a = 0$; $\frac{dz}{dx} = 2x$; & $\frac{ddz}{2dx^2} = 1$, hinc erit:
$\int xxdx = (x+\frac{1}{2})xx - 2x(\frac{1}{2}xx + \frac{1}{2}x + \frac{1}{12}) + 1(\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x)$
seu $\int xxdx = \frac{1}{3}x^3$; dat autem $\frac{1}{3}x^3$ differentiatum utique $xxdx$.

138. Nova ergo hinc patet via ad terminos summatorios serierum potestatum inveniendos; quoniam enim ex coefficientibus ante assumtis A, B, C, D, &c. hi termini summatorii facillime formantur, horum autem coefficientium quilibet ex praecedentibus conflatur; si in formulis §. 135. datis loco istarum litterarum valores in §. 136. traditi substituantur, erit:

$$Sx^1 - x = \frac{1}{2}xx - \frac{1}{2}x$$
$$Sx^2 - x^2 = \frac{1}{3}x^3 - \frac{1}{3}x - \frac{2}{2}(Sx - x)$$
$$Sx^3 - x^3 = \frac{1}{4}x^4 - \frac{1}{4}x - \frac{3}{2}(Sx^2 - x^2) - \frac{3 \cdot 2}{2 \cdot 3}(Sx^1 - x^1)$$
$$Sx^4 - x^4 = \frac{1}{5}x^5 - \frac{1}{5}x - \frac{4}{2}(Sx^3 - x^3) - \frac{4 \cdot 3}{2 \cdot 3}(Sx^2 - x^2) - \frac{4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 4}(Sx - x)$$
&c.

Hinc ergo summae potestatum superiorum ex summis inferiorum formari poterunt.

139. Quod si vero legem, qua coefficientes A, B, C, D, &c. supra §. 135. progredi inventi sunt, attentius intueamur, eos seriem recurrentem constituere deprehendemus. Si enim evolvamus hanc fractionem:

$$\frac{x + \frac{1}{2}xxu + \frac{1}{6}x^3u^2 + \frac{1}{24}x^4u^3 + \frac{1}{120}x^5u^4 + \&c.}{1 + \frac{1}{2}u + \frac{1}{6}u^2 + \frac{1}{24}u^3 + \frac{1}{120}u^4 + \&c.}$$

secundum potestates ipsius u, hancque seriem resultare sumamus.

$$A + Bu + Cu^2 + Du^3 + Eu^4 + \&c.$$

erit uti ante invenimus $A = x$; $B = \frac{1}{2}xx - \frac{1}{2}A$; &c. sicque inventa hac serie, obtinebuntur termini summatorii serierum potestatum. Illa autem fractio, ex cuius evolutione ista series nascitur, transit in hac formam: $\frac{e^{xu} - 1}{e^u - 1}$, quae si x fuerit

rit numerus integer affirmativus, abit in
$1 + e^u + e^{2u} + e^{3u} + \ldots + e^{(x-1)u}$; cum ergo fit:

$$1 = 1$$

$$e^u = 1 + \frac{u}{1} + \frac{u^2}{1.2} + \frac{u^3}{1.2.3} + \frac{u^4}{1.2.3.4} + \&c.$$

$$e^{2u} = 1 + \frac{2u}{1} + \frac{4u^2}{1.2} + \frac{8u^3}{1.2.3} + \frac{16u^4}{1.2.3.4} + \&c.$$

$$e^{3u} = 1 + \frac{3u}{1} + \frac{9u^2}{1.2} + \frac{27u^3}{1.2.3} + \frac{81u^4}{1.2.3.4} + \&c.$$

$$e^{(x-1)u} = 1 + \frac{(x-1)u}{1} + \frac{(x-1)^2u^2}{1.2} + \frac{(x-1)^3u^3}{1.2.3} + \frac{(x-1)^4u^4}{1.2.3.4} + \&c.$$

ideoque erit

$$A = x$$

$$B = S(x-1) = Sx - x$$

$$C = \tfrac{1}{2}S(x-1)^2 = \tfrac{1}{2}Sx^2 - \tfrac{1}{2}x^2$$

$$D = \tfrac{1}{6}S(x-1)^3 = \tfrac{1}{6}Sx^3 - \tfrac{1}{6}x^3$$

&c.

Unde nexus horum coefficientium cum fummis poteftatum, ante iam obfervatus, penitus confirmatur ac demonftratur.