CAPUT IV.

DE CONVERSIONE FUNCTIONUM IN SERIES.

70.

In Capite superiori iam ex parte ostensus est usus, quem expressiones generales ibi pro differentiis finitis inventae habent in investigatione serierum, quae valorem cuiusque functionis ipsius x exhibeant. Si enim y fuerit functio data ipsius x, eius valor quem induit posito $x = 0$, erit cognitus; hicque si ponatur $= A$, erit uti invenimus:

$$y - \frac{xdy}{dx} + \frac{x^2 ddy}{1.2dx^2} - \frac{x^3 d^3y}{1.2.3dx^3} + \frac{x^4 d^4y}{1.2.3.4dx^4} - \&c. = A.$$

Hinc ergo non solum habemus seriem plerumque in infinitum excurrentem, cuius summa aequetur quantitati constanti A, etiamsi in singulis terminis insit quantitas variabilis x, sed etiam ipsam functionem y per seriem exprimere poterimus, erit enim:

$$y = A + \frac{xdy}{dx} - \frac{xxddy}{1.2dx^2} + \frac{x^3 d^3y}{1.2.3dx^3} - \frac{x^4 d^4y}{1.2.3.4dx^4} + \&c.$$

cuius exempla iam aliquot sunt allata.

71. Quo autem haec investigatio latius pateat, ponamus functionem y abire in z, si loco x ubique scribatur $x + \omega$, ita ut z talis sit functio ipsius $x + \omega$, qualis y est ipsius x, atque ostendimus fore:

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{1.2dx^2} + \frac{\omega^3 d^3y}{1.2.3dx^3} + \frac{\omega^4 d^4y}{1.2.3.4dx^4} + \&c.$$

Cum igitur huius seriei singuli termini per continuam ipsius y differentiationem ponendo dx constans inveniri, simulque valor ipsius z per substitutionem $x + \omega$ in locum ipsius x actu exhiberi queat; hoc modo perpetuo obtinebitur series valori ipsi-

ipsius z aequalis, quae si ω fuerit quantitas vehementer parva, maxime convergit, atque non admodum multis terminis capiendis valorem ipsius z proxime verum praebebit. Ex quo huius formulae in negotio approximationum uberrimus erit usus.

72. Ut igitur in insigni huius formulae usu ostendendo ordine procedamus, substituamus primo in locum ipsius y functiones ipsius x algebraicas. Ac primo quidem sit $y = x^n$; eritque si $x+\omega$ loco x ponatur $z = (x+\omega)^n$. Cum igitur sit:

$$\frac{dy}{dx} = nx^{n-1} \quad ; \quad \frac{ddy}{dx^2} = n(n-1)x^{n-2}$$

$$\frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3} \quad ; \quad \frac{d^4y}{dx^4} = n(n-1)(n-2)(n-3)x^{n-4}$$

&c.

his valoribus substitutis fiet:

$$(x+\omega)^n = x^n + \frac{n}{1}x^{n-1}\omega + \frac{n(n-1)}{1.2}x^{n-2}\omega^2 + \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}\omega^3 + \&c.$$

quae est notissima expressio Neutoniana, qua potestas binomii $(x+\omega)^n$ in seriem convertitur. Huiusque seriei terminorum numerus semper est finitus, si n fuerit numerus integer affirmativus.

73. Poterimus hinc quoque progressionem invenire, quae valorem potestatis binomii ita exprimat, ut ea abrumpatur, quoties exponens potestatis fuerit numerus negativus. Statuamus enim

$$\omega = \frac{-ux}{x+u}; \quad \text{erit} \quad z = (x+\omega)^n = \left(\frac{xx}{x+u}\right)^n$$

ideoque habebitur:

$$\frac{x^{2n}}{(x+u)^n} = x^n - \frac{nx^n u}{1(x+u)} + \frac{n(n-1)x^n u^2}{1.2(x+u)^2} - \frac{n(n-1)(n-2)x^n u^3}{1.2.3(x+u)^3} + \&c.$$

dividatur ubique per x^{2n}, eritque

$$(x+u)^{-n} = x^{-n} - \frac{nx^{-n}u}{1(x+u)} + \frac{n(n-1)x^{-n}u^2}{1.2(x+u)^2} - \frac{n(n-1)(n-2)x^{-n}u^3}{1.2.3(x+u)^3} + \&c.$$

Ponatur nunc $-n = m$; prodibitque

$$(x+u)^m = x^m + \frac{mx^m u}{1(x+u)} + \frac{m(m+1)x^m u^2}{1.2(x+u)^2} + \frac{m(m+1)(m+2)x^m u^3}{1.2.3(x+u)^3} + \&c.$$

quae series, quoties m est numerus integer negativus, finito terminorum numero constabit. Haec igitur series aequalis est primum inventae, si pro ω & n scribantur u & m; erit enim inde

$$(x+u)^m = x^m + \frac{mx^{m-1}u}{1} + \frac{m(m-1)x^{m-2}u^2}{1.\ 2} + \frac{m(m-1)(m-2)x^{m-3}u^3}{1.\ 2.\ 3} + \&c.$$

74. Haec eadem series quoque deduci potest ex expressione initio §. 70. data. Cum enim, si posito $x = 0$, abeat y in A fit:

$$y - \frac{xdy}{dx} + \frac{xxddy}{1.2dx^2} - \frac{x^3 d^3 y}{1.2.3dx^3} + \frac{x^4 d^4 y}{1.2.3.4dx^4} - \&c. = A,$$

ponatur $y = (x+a)^n$; eritque $A = a^n$; & ob

$$\frac{dy}{dx} = n(x+a)^{n-1};\quad \frac{ddy}{dx^2} = n(n-1)(x+a)^{n-2};$$

$$\frac{d^3y}{dx^3} = n(n-1)(n-2)(x+a)^{n-3};\ \&c.\ \text{fiet}$$

$$(x+a)^n - \frac{n}{1}x(x+a)^{n-1} + \frac{n(n-1)}{1.\ 2}x^2(x+a)^{n-2} - \&c. = a^n$$

dividatur per $a^n(x+a)^n$, atque prodibit:

$$(x+a)^{-n} = a^{-n} - \frac{na^{-n}x}{1(x+a)} + \frac{n(n-1)a^{-n}x^2}{1.2(x+a)^2} - \&c.$$

quae positis respective u, x & $-m$ pro x, a & n orietur series ante inventa.

75. Si pro m statuantur numeri fracti, ambae series in infinitum excurrent, interim tamen si u prae x fuerit quantitas valde parva, vehementer ad verum valorem convergent. Sit igitur $m = \frac{\mu}{\nu}$; & $x = a^\nu$, erit ex serie primum inventa:

$$(a^\nu + u)^{\frac{\mu}{\nu}} = a^\mu \left(1 + \frac{\mu u}{\nu . a^\nu} + \frac{\mu(\mu-\nu)}{\nu .\ 2\nu} . \frac{uu}{a^{2\nu}} + \frac{\mu(\mu-\nu)(\mu-2\nu)}{\nu .\ 2\nu .\ 3\nu} . \frac{u^3}{a^{3\nu}} + \&c.\right)$$

Series autem posterius inventa dabit:

$$(a^\nu + u)^{\frac{\mu}{\nu}} = a^\mu \left(1 + \frac{\mu u}{\nu(a^\nu + u)} + \frac{\mu(\mu+\nu)u^2}{\nu . 2\nu(a^\nu + u)^2} + \frac{\mu(\mu+\nu)(\mu+2\nu)u^3}{\nu .\ 2\nu .\ 3\nu\ (a^\nu + u)^3} + \&c.\right)$$

Haec autem posterior series magis convergit quam prior, cum eius termini etiam decrescant, si fuerit $u > a^\nu$, quo casu tamen prior series divergit.

Si igitur sit $\mu = 1$, $\nu = 2$, erit

$$\sqrt{(a^2 + u)} = a\left(1 + \frac{1\ u}{2(a^2 + u)} + \frac{1.\ 3.\ u^2}{2.4(a^2 + u)^2} + \frac{1.\ 3.\ 5\ u^3}{2.4.6(a^2 + u)^3} + \&c.\right)$$

simili modo pro ν ponendo numeros 3, 4, 5, &c. manente $\mu = 1$, erit:

$$\sqrt[3]{(a^3 + u)} = a\left(1 + \frac{1\ u}{3(a^3 + u)} + \frac{1.\ 4.\ u^2}{3.6(a^3 + u)^2} + \frac{1.\ 4.\ 7\ u^3}{3.6.9(a^4 + u)^3} + \&c.\right)$$

$$\sqrt[4]{(a^4 + u)} = a\left(1 + \frac{1\ u}{4(a^4 + u)} + \frac{1.\ 5.\ u^2}{4.8(a^4 + u)^2} + \frac{1.\ 5.\ 9\ u^3}{4.8.12(a^4 + u)^3} + \&c.\right)$$

$$\sqrt[5]{(a^5 + u)} = a\left(1 + \frac{1\ u}{5(a^5 + u)} + \frac{1.\ 6\ u^2}{5.10(a^5 + u)^2} + \frac{1.\ 6.\ 11\ u^3}{5.10.15(a^5 + u)^3} + \&c.\right)$$

&c.

76. Ex his ergo formulis facile cuiusque numeri propositi radix cuiusvis potestatis inveniri poterit. Proposito enim numero c quaeratur potestas ei proxima, sive major sive minor: priori casu u fiet numerus negativus, posteriori affirmativus. Quod si vero series resultans non satis convergere videatur, multiplicetur numerus c per quampiam potestatem puta per f^ν, si radix dignitatis ν extrahi debeat, & quaeratur numeri $f^\nu c$ radix, quae per f divisa dabit radicem numeri c quaesitam. Quo major autem accipitur numerus f, eo magis series converget; idque imprimis, si quaepiam similis potestas a^ν non multum ab $f^\nu c$ discrepet.

EXEMPLUM I.

Quaeratur radix quadrata ex numero 2.

Si sine ulteriori praeparatione ponatur $a = 1$ & $u = 1$ fiet

$$\sqrt{2} = 1 + \frac{1}{2 \cdot 2} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 2^2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 2^3} + \&c.$$

quae etsi iam satis convergit, tamen praestabit numerum 2 ante per quadratum quodpiam uti 25 multiplicare, ut productum 50 ab alio quadrato 49 minime discrepet. Hancobrem quaeratur radix quadrata ex 50, quae per 5 divisa dabit $\sqrt{2}$. Erit autem tum $a = 7$ & $u = 1$, unde fiet:

$$\sqrt{50} = 5\sqrt{2} = 7\left(1 + \frac{1}{2 \cdot 50} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 50^2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 50^3} + \&c.\right)$$

seu

$$\sqrt{2} = \tfrac{7}{5}\left(1 + \frac{1}{100} + \frac{1 \cdot 3}{100 \cdot 200} + \frac{1 \cdot 3 \cdot 5}{100 \cdot 200 \cdot 300} + \&c.\right)$$

quae ad computum in fractionibus decimalibus instituendum est aptissima.

Erit enim

$$\begin{aligned}
\tfrac{7}{5} &= 1{,}4000000000000 \\
\tfrac{7}{5} \cdot \tfrac{1}{100} &= 14000000000 \\
\tfrac{7}{5} \cdot \tfrac{1}{100} \cdot \tfrac{3}{200} &= 210000000 \\
\tfrac{7}{5} \cdot \tfrac{1}{100} \cdot \tfrac{3}{200} \cdot \tfrac{5}{300} &= 35000000 \\
\text{praec. in } \tfrac{7}{400} &= 612500 \\
\text{praec. in } \tfrac{9}{500} &= 11025 \\
\text{praec. in } \tfrac{11}{600} &= 202 \\
\text{praec. in } \tfrac{13}{700} &= 3
\end{aligned}$$

Ergo $\sqrt{2} = 1{,}4142135623730$

EXEMPLUM II.

Quaeratur radix cubica ex 3.

Multiplicetur 3 per cubum 8, & quaeratur radix cubica ex 24, erit enim $\sqrt[3]{24} = 2\sqrt[3]{3}$. Ponatur ergo

$a = 3$ & $u = -3$, eritque

$$\sqrt[3]{24} = 3\left(1 - \frac{1\cdot 3}{3\cdot 24} + \frac{1\cdot 4\cdot 3^2}{3\cdot 6\cdot 24^2} - \frac{1\cdot 4\cdot 7\cdot 3^3}{3\cdot 6\cdot 9\cdot 24^3} + \&c.\right)$$ &

$$\sqrt[3]{3} = \tfrac{3}{2}\left(1 - \frac{1}{3\cdot 8} + \frac{1\cdot 4}{3\cdot 6\cdot 8^2} - \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9\cdot 8^3} + \&c.\right)$$ seu

$$\sqrt[3]{3} = \tfrac{3}{2}\left(1 - \frac{1}{24} + \frac{1}{24}\cdot\frac{4}{48} - \frac{1}{24}\cdot\frac{4}{48}\cdot\frac{7}{72} + \&c.\right)$$

quae series iam vehementer convergit, cum quilibet terminus plusquam octies minor sit praecedente. Sin autem 3 multiplicetur per cubum 729 fiet 2187, &

$$\sqrt[3]{2187} = \sqrt[3]{(13^3 - 10)} = 9\sqrt[3]{3}.$$

Erit ergo ob $a = 13$ & $u = -10$

$$\sqrt[3]{3} = \tfrac{13}{9}\left(1 - \frac{1\cdot 10}{3\cdot 2187} + \frac{1\cdot 4\cdot 10^2}{3\cdot 6\cdot 2187^2} - \frac{1\cdot 4\cdot 7\cdot 10^3}{3\cdot 6\cdot 9\cdot 2187^3} + \&c.\right)$$

cuius quivis terminus plusquam ducenties minor est quam praeced.

77. Evolutio binomii potestatis tam late patet, ut omnes functiones algebraicae in ea comprehendi queant. Si enim ver. gr. quaeratur valor huius functionis $\sqrt{(a + 2bx + cxx)}$ per seriem expressus, hoc per praecedentes formulas, duos terminos tanquam unum considerando fieri poterit. Deinde vero haec explicatio fieri poterit ope expressionis primum traditae: nam si ponatur $\sqrt{(a + 2bx + cxx)} = y$, quia posito $x = 0$ fit $y = \sqrt{a}$, erit $A = \sqrt{a}$, & cum differentialia ipsius y ita se habeant:

$$\frac{dy}{dx} = \frac{b + cx}{\sqrt{(a + 2bx + cxx)}}$$

$$\frac{ddy}{dx^2} = \frac{ac - bb}{(a + 2bx + cxx)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = + \frac{3(bb - ac)(b + cx)}{(a + 2bx + cxx)^{\frac{5}{2}}}$$

$$\frac{d^4y}{dx^4} = \frac{3(bb - ac)(ac - 5bb - 8bcx - 4ccxx)}{(a + 2bx + cxx)^{\frac{7}{2}}}$$

&c.

Ex

Ex his ergo obtinebitur:

$$\sqrt{(a+2bx+cxx)}-\frac{(b+cx)x}{\sqrt{(a+2bx+cxx)}}-\frac{(bb-ac)xx}{2(a+2bx+cxx)^{\frac{3}{2}}}-\frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^{\frac{5}{2}}}-\frac{(bb-ac)(5bb-ac+8bcx+4ccxx)x^4}{8(a+2bx+cxx)^{\frac{7}{2}}}-\&c.=\sqrt{a}.$$

Quodsi ergo ubique per $\sqrt{(a+2bx+cxx)}$ multiplicetur series fiet rationalis, eritque

$$\sqrt{a(a+2bx+cxx)}=a+2bx+cxx-(b+cx)x-\frac{(bb-ac)xx}{2(a+2bx+cxx)}-\frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^2}-\frac{(bb-ac)(5bb-ac+8bcx+4ccxx)x^4}{8(a+2bx+cxx)^3}-\&c.$$

sive

$$\sqrt{(a+2bx+cxx)}=\sqrt{a}+\frac{bx}{\sqrt{a}}-\frac{(bb-ac)xx}{2(a+2bx+cxx)\sqrt{a}}-\frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^2\sqrt{a}}\ \&c.$$

78. Transeamus ergo ad functiones transcendentes, quas loco y substituamus. Sit itaque primum $y=lx$, ac posito $x+\omega$ loco x fiet $z=l(x+\omega)$. Sint autem hi logarithmi quicunque, qui ad hiperbolicos rationem teneant $n:1$, eritque pro logarithmis hyperbolicis $n=1$ & pro tabularibus erit $n=0,4342944819032$. Hinc differentialia ipsius $y=lx$ erunt: $\frac{dy}{dx}=\frac{n}{x}$; $\frac{ddy}{dx^2}=-\frac{n}{x^2}$; $\frac{d^3y}{dx^3}=\frac{2n}{x^3}$; &c. ex quibus conficitur: $l(x+\omega)=lx+\frac{n\omega}{x}-\frac{n\omega^2}{2x^2}+\frac{n\omega^3}{3x^3}-\frac{n\omega^4}{4x^4}+\&c.$

Simili modo si ω statuatur negativum, erit:

$$l(x-\omega)=lx-\frac{n\omega}{x}-\frac{n\omega^2}{2x^2}-\frac{n\omega^3}{3x^3}-\frac{n\omega^4}{4x^4}-\&c.$$

Quodsi ergo haec series a priori subtrahatur, fiet

$$l\frac{x+\omega}{x-\omega}=2n\left(\frac{\omega}{x}+\frac{\omega^3}{3x^3}+\frac{\omega^5}{5x^5}+\frac{\omega^7}{7x^7}+\&c.\right)$$

79. Si in ſerie primum inventa:

$$l(x+\omega)=lx+\frac{n\omega}{x}-\frac{n\omega^2}{2x^2}+\frac{n\omega^3}{3x^3}-\frac{n\omega^4}{4x^4}+\&c.$$

ponatur $\omega=\frac{xx}{u-x}$; erit $x+\omega=\frac{ux}{u-x}$; &

$$l(x+\omega)=lu+lx-l(u-x)=lx+\frac{nx}{u-x}-\frac{nxx}{2(u-x)^2}+\&c.$$

atque

$$l(u-x)=lu-\frac{nx}{u-x}+\frac{nxx}{2(u-x)^2}-\frac{nx^3}{3(u-x)^3}+\&c.$$

ſumtoque x negativo habebitur:

$$l(u+x)=lu+\frac{nx}{u+x}+\frac{nxx}{2(u+x)^2}+\frac{nx^3}{3(u+x)^3}+\frac{nx^4}{4(u+x)^4}.$$

Harum ergo ſerierum ope logarithmi expedite inveniri poterunt, ſi quidem ſeries valde convergant. Huiuſmodi autem erunt ſequentes, quae ex inventis facile deducuntur:

$$l(x+1)=lx+n\left(\frac{1}{x}-\frac{1}{2xx}+\frac{1}{3x^3}-\frac{1}{4x^4}+\&c.\right)$$

$$l(x-1)=lx-n\left(\frac{1}{x}+\frac{1}{2xx}+\frac{1}{3x^3}+\frac{1}{4x^4}+\&c.\right)$$

quae duae ſeries, cum tantum ſignis a ſe invicem diſcrepent, ſi ad calculum revocentur, ex logarithmo numeri x cognito, eadem opera logarithmi amborum numerorum $x+1$ & $x-1$ reperientur. Deinde ex reliquis ſeriebus erit:

$$l(x+1)=l(x-1)+2n\left(\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\frac{1}{7x^7}+\&c.\right)$$

$$l(x-1)=lx-n\left(\frac{1}{x-1}-\frac{1}{2(x-1)^2}+\frac{1}{3(x-1)^3}-\frac{1}{4(x-1)^4}+\&c.\right)$$

$$l(x+1)=lx+n\left(\frac{1}{x+1}+\frac{1}{2(x+1)^2}+\frac{1}{3(x+1)^3}+\frac{1}{4(x+1)^4}+\&c.\right)$$

80. Ex dato ergo logarithmo numeri x, logarithmi nu-

numerorum contiguorum $x+1$ & $x-1$ facile inveniri poterunt; quin etiam ex logarithmo numeri $x-1$ logarithmus numeri binario maioris & viciſſim eruetur. Quod quamvis in Introductione uberius ſit oſtenſum, tamen hic quaedam exempla adiungemus.

EXEMPLUM I.

Ex dato numeri 10 *logarithmo hyperbolico, qui eſt* 2,302585092994 0, *logarithmos hyperbolicos numerorum* 11 *&* 9 *invenire.*

Quoniam haec quaeſtio logarithmos hyperbolicos ſpectat, erit $n=1$; ideoque habebuntur hae ſeries:

$$l11 = l10 + \frac{1}{10} - \frac{1}{2.\,10^2} + \frac{1}{3.\,10^3} - \frac{1}{4.\,10^4} + \frac{1}{5.\,10^5} - \&c.$$

$$l9 = l10 - \frac{1}{10} - \frac{1}{2.\,10^2} - \frac{1}{3.\,10^3} - \frac{1}{4.\,10^4} - \frac{1}{5.\,10^5} - \&c.$$

Ad quarum ſerierum ſummas inveniendas, colligantur termini pares & impares ſeorſim, eritque

$\frac{1}{10}$	$=0,1000000000000$	$\frac{1}{2.\,10^2}$	$=0,0050000000000$
$\frac{1}{3.\,10^3}$	$=0,0003333333333$	$\frac{1}{4.\,10^4}$	$=0,0000250000000$
$\frac{1}{5.\,10^5}$	$=0,0000020000000$	$\frac{1}{6.\,10^6}$	$=0,0000001666666$
$\frac{1}{7.\,10^7}$	$=0,0000000142857$	$\frac{1}{8.\,10^8}$	$=0,0000000012500$
$\frac{1}{9.\,10^9}$	$=0,0000000001111$	$\frac{1}{10.\,10^{10}}$	$=0,0000000000100$
$\frac{1}{11.\,10^{11}}$	$=0,0000000000009$	$\frac{1}{12.\,10^{12}}$	$=0,0000000000001$
ſumma	$=0,1003353477310$	ſumma	$=0,0050251679267$

Sum-

Summa utriusque erit . . . 0,1053605156577
Differentia ambarum erit 0,0953101798043
Iam est $l10 = 2,3025850929940$
Ergo erit $l11 = 2,3978952727983$
& $l9 = 2,1972245773363$
Hinc porro erit $l3 = 1,0986122886681$
& $l99 = 4,5951198501346$

EXEMPLUM II.

Ex logarithmo hyperbolico numeri 99 nunc invento invenire logarithmum numeri 101.

Adhibeatur ad hoc series supra inventa:

$$l(x+1) = l(x-1) + \frac{2}{x} + \frac{2}{3x^3} + \frac{2}{5x^5} + \frac{2}{7x^7} + \text{\&c.}$$

in qua fiat $x = 100$; eritque:

$$l101 = l99 + \frac{2}{100} + \frac{2}{3.\ 100^3} + \frac{2}{5.\ 100^5} + \frac{2}{7.\ 100^7} + \text{\&c.}$$

cuius seriei summa ex his quatuor terminis colligitur $= 0,0200006667066$, quae ad $l99$ addita dabit $l101 = 4,6151205168412$.

EXEMPLUM III.

Ex dato logarithmo tabulari numeri 10, qui est $= 1$, *invenire logarithmos numerorum* 11 & 9.

Quoniam hic logarithmos communes tabulares quaerimus, erit $n = 0,4342944819032$, posito ergo $x = 10$ erit:

$$l11 = l10 + \frac{n}{10} + \frac{n}{2.\ 10^2} + \frac{n}{3.\ 10^3} + \frac{n}{4.\ 10^4} + \text{\&c.}$$

$$l9 = l10 - \frac{n}{10} - \frac{n}{2.\ 10^2} - \frac{n}{3.\ 10^3} - \frac{n}{4.\ 10^4} - \text{\&c.}$$

Colligantur termini pares & impares seorsim:

$\frac{n}{10} = 0{,}0434294481903$	$\frac{n}{2.\,10^2} = 0{,}0021714724095$
$\frac{n}{3.\,10^3} = 0{,}0001447648273$	$\frac{n}{4.\,10^4} = 0{,}0000108573620$
$\frac{n}{5.\,10^5} = 0{,}0000008685889$	$\frac{n}{6.\,10^6} = 0{,}0000000723824$
$\frac{n}{7.\,10^7} = 0{,}0000000062042$	$\frac{n}{8.\,10^8} = 0{,}0000000005428$
$\frac{n}{9.\,10^9} = 0{,}0000000000482$	$\frac{n}{10.\,10^{10}} = 0{,}0000000000043$
$\frac{n}{11.\,10^{11}} = 0{,}0000000000003$	$\frac{n}{12.\,10^{12}} = 0{,}0000000000000$
summa $= 0{,}0435750878593$	summa $= 0{,}0021824027010$

Aggregatum ambarum est $= 0{,}0457574905603$
Differentia earum est $= 0{,}0413926851583$
Cum ergo sit $l10 = 1{,}0000000000000$
Erit $l11 = 1{,}0413926851583$
& $l9 = 0{,}9542425094396$
Hinc $l3 = 0{,}4771212547198$
& $l99 = 1{,}9956351945979$

EXEMPLUM IV.

Ex logarithmo tabulari numeri 99 hic invento invenire logarithmum tabularem numeri 101.

Adhibendo hic eandem seriem, qua in Exemplo secundo usi sumus, habebimus:

$$l101 = l99 + 2n\left(\frac{1}{100} + \frac{1}{3.\,100^3} + \frac{1}{5.\,100^5} + \&c.\right)$$

cuius seriei posito pro n valore debito, summa mox reperietur $= 0{,}0086861791849$ quae addita ad $l99 = 1{,}9956351945979$ oritur

$l101 = 2{,}0043213737829$

81.

81. Tribuamus nunc in expressione nostra generali y valorem exponentialem, sitque $y = a^x$, posito $x + \omega$ loco x; erit $z = a^{x+\omega}$, cuius valor ob differentialia:
$\frac{dy}{dx} = a^x la$; $\frac{ddy}{dx^2} = a^x (la)^2$; $\frac{d^3y}{dx^3} = a^x (la)^3$; &c. erit

$$a^{x+\omega} = a^x \left(1 + \frac{\omega la}{1} + \frac{\omega^2 (la)^2}{1 \cdot 2} + \frac{\omega^3 (la)^3}{1 \cdot 2 \cdot 3} + \&c. \right)$$

quae si dividatur per a^x prodibit series valores quantitatis exponentialis exprimens, quam supra in Introductione iam elicuimus: nempe

$$a^\omega = 1 + \frac{\omega la}{1} + \frac{\omega^2 (la)^2}{1 \cdot 2} + \frac{\omega^3 (la)^3}{1 \cdot 2 \cdot 3} + \frac{\omega^4 (la)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.$$

Simili modo sumto ω negativo erit:

$$a^{-\omega} = 1 - \frac{\omega la}{1} + \frac{\omega^2 (la)^2}{1 \cdot 2} - \frac{\omega^3 (la)^3}{1 \cdot 2 \cdot 3} + \frac{\omega^4 (la)^4}{1 \cdot 2 \cdot 3 \cdot 4} - \&c.$$

ex quarum combinatione oritur

$$\frac{a^\omega + a^{-\omega}}{2} = 1 + \frac{\omega^2 (la)^2}{1 \cdot 2} + \frac{\omega^4 (la)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{\omega^6 (la)^6}{1 \cdot 2 \ldots 6} + \&c.$$

$$\frac{a^\omega - a^{-\omega}}{2} = \frac{\omega la}{1} + \frac{\omega^3 (la)^3}{1 \cdot 2 \cdot 3} + \frac{\omega^5 (la)^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \&c.$$

Ubi notandum est la denotare logarithmum hyperbolicum numeri a.

82. Huius formulae ope ex dato quovis logarithmo numerus ei conveniens reperiri poterit. Sit enim propositus logarithmus quicunque u ad canonem, in quo numeri a logarithmus $= 1$ statuitur, pertinens. Quaeratur in eodem canone logarithmus x proxime ad u accedens, sitque $u = x + \omega$; numerus autem logarithmo x conveniens sit $= y = a^x$, erit numerus logarithmo $u = x + \omega$ respondens $= a^{x+\omega} = z$; fietque

$$z = y \left(1 + \frac{\omega la}{1} + \frac{\omega^2 (la)^2}{1 \cdot 2} + \frac{\omega^3 (la)^3}{1 \cdot 2 \cdot 3} + \frac{\omega^4 (la)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c. \right)$$

quae series ob ω numerum valde parvum, vehementer converget, cuius usum sequenti exemplo declaremus.

EXEMPLUM.

Quaeratur numerus isti binarii potestati, $2^{2^{24}}$ *aequalis*.

Cum sit $2^{24} = 16777216$, erit $2^{2^{24}} = 2^{16777216}$, sumendisque logarithmis vulgaribus, erit huius numeri logarithmus $= 16777216\, l2$. Cum autem sit:

$$l2 = 0,30102999566398119521373889$$

numeri quaesiti logarithmus erit:

$$5050445,2597336759320390 63$$

cuius characteristica indicat numerum quaesitum exprimi 5050446 figuris, quae cum omnes exhiberi nequeant, sufficiet figuras initiales assignasse, quae ex mantissa

$$,259733675932039063 = u$$

investigari debent. Ex tabulis autem colligitur, numerum cuius logarithmus proxime ad hunc accedat fore $18 \cdot 101 = 1,818$; qui ponatur y; cuius logarithmus

$x = 0,259593878885948644$, unde erit

$\omega = 0,000139797046090419$. Cum iam sit

$a = 10$ erit

$la = 2,302585092994045684017991 4$ &

$\omega la = 0,000321894594372398$ Deinde erit

$$y = 1,818000000000000000$$

$$\frac{\omega la}{1}\, y = \quad 585204372569020$$

$$\frac{\omega^2 (la)^2}{1 \cdot 2}\, y = \quad 94187062064$$

$$\frac{\omega^3 (la)^3}{1 \cdot 2 \cdot 3}\, y = \quad 10106100$$

$$\frac{\omega^4 (la)^4}{1 \cdot 2 \cdot 3 \cdot 4}\, y = \quad 813$$

$$18185852985697 37997$$

haeque sunt figurae initiales numeri quaesiti, cuius omnes figurae excepta forte ultima sunt iustae.

83. Consideremus quantitates transcendentes a circulo pendentes, sitque uti perpetuo ponimus, radius circuli $= 1$, atque y denotet arcum circuli cuius sinus $= x$ seu sit $y = A \sin x$. Ponatur $x + \omega$ loco x, eritque $z = A \sin (x + \omega)$: ad quem valorem exprimendum quaerantur differentialia ipsius y:

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-xx)}}\,; \quad \frac{ddy}{dx^2} = \frac{+x}{(1-xx)^{\frac{3}{2}}}\,; \quad \frac{d^3y}{dx^3} = \frac{1+2xx}{(1-xx)^{\frac{5}{2}}}$$

$$\frac{d^4y}{dx^4} = \frac{9x+6x^3}{(1-xx)^{\frac{7}{2}}}\,; \quad \frac{d^5y}{dx^5} = \frac{9+72x^2+24x^4}{(1-xx)^{\frac{9}{2}}}$$

$$\frac{d^6y}{dx^6} = \frac{225x+600x^3+120x^5}{(1-xx)^{\frac{11}{2}}}\,; \quad \&c.$$

Ex his ergo invenitur:

$$A \sin (x+\omega) = A \sin x + \frac{\omega}{\sqrt{(1-xx)}} + \frac{\omega^2 x}{2(1-xx)^{\frac{3}{2}}} + \frac{\omega^3(1+2xx)}{6(1-xx)^{\frac{5}{2}}}$$
$$+ \frac{\omega^4(9x+6x^3)}{24(1-xx)^{\frac{7}{2}}} + \frac{\omega^5(9+72x^2+24x^4)}{120(1-xx)^{\frac{9}{2}}} + \&c.$$

84. Si ergo cognitus fuerit arcus, cuius sinus est $= x$, huius formulae beneficio inveniri poterit arcus, cuius sinus est $x + \omega$, si fuerit ω quantitas valde parva. Series autem cuius summa addi debet, exprimetur in partibus radii, quae ad arcum facile reducentur: uti ex hoc exemplo intelligetur.

EXEMPLUM.

Quaeratur arcus circuli, cuius sinus est $= \frac{1}{3} = 0{,}3333333333$.

Quaeratur ex tabulis sinuum arcus, cuius sinus sit proxime minor, quam $\frac{1}{3}$, qui erit $19^\circ, 28'$, cuius sinus est $= 0{,}3332584$. Statuatur ergo $19^\circ, 28', = A \sin x = y$ erit $x = 0{,}3332584$, & $\omega = 0{,}0000749$, atque ex tabulis $\sqrt{(1-xx)} = \cos y = 0{,}9428356$. Erit ergo arcus quaesitus z, cuius sinus $= \frac{1}{3}$ proponitur $= 19^\circ, 28' + \frac{\omega}{\cos y} + \frac{\omega\omega \sin y}{2 \cos y^3}$, quae expressio iam sufficit; erit ergo per logarithmos calculum instituendo:

$l\omega = 5,8744818$

$l\cos y = 9,9744359$

$l\dfrac{\omega}{\cos y} = 5,9000459 \; ; \quad \dfrac{\omega}{\cos y} = 0,000079441 2$

$l\dfrac{\omega^2}{\cos y^2} = 1,8000918$

$l\dfrac{\sin y}{\cos y} = 9,5483452$

$1,3484370$

$l2 = 0,3010300$

$l\dfrac{\omega^2 \sin y}{2\cos y^3} = 1,0474070 \; ; \quad \dfrac{\omega^2 \sin y}{2\cos y^3} = 0,00000000 11$

Summa $= 0,0000794423$

qui eſt valor arcus ad $19^\circ, 28^{I}$ addendi, ad quem in minutis ſecundis exprimendum, ſumamus eius logarithmum

qui eſt	5,9000518
a quo ſubtrahatur	4,6855749
	1,2144769
cui log. reſpondet num. $=$	16,38615

qui eſt numerus minutorum ſecundorum; fractionem vero in tertiis & quartis exprimendo fiet arcus quaeſitus

$= 19^\circ, 28^{I}, 16^{II}, 23^{III}, 10^{IV}, 8^{V}, 24^{VI}.$

85. Simili modo expreſſio pro coſinibus eruetur; poſito enim $y = A\cos x$; quia eſt $dy = \dfrac{-dx}{\sqrt{(1-xx)}}$, ſeries ante inventa invariata manebit, dummodo eius ſigna permutentur. Erit itaque

$$A\cos(x+\omega) = A\cos x - \frac{\omega}{\sqrt{(1-xx)}} - \frac{\omega^2 x}{2(1-xx)^{\frac{3}{2}}} - \frac{\omega^3(1+2xx)}{6(1-xx)^{\frac{5}{2}}}$$

$$- \frac{\omega^4(9x+6x^3)}{24(1-xx)^{\frac{7}{2}}} - \frac{\omega^5(9+72x^2+24x^4)}{120(1-xx)^{\frac{9}{2}}} - \&c.$$

quae series pariter ac praecedens vehementer semper converget, si ex tabulis sinuum proxime veri anguli excerpantur, ita ut plerumque unicus terminus primus $\frac{\omega}{\sqrt{(1-xx)}}$ sufficiat. Interim tamen si x fuerit ipsi 1 seu sinui toti proxime aequalis, tum ob denominatores admodum parvos illa series convergentiam amittit. His igitur casibus, quibus x non multum ab 1 deficit, quoniam tum differentiae fiunt minimae, commodius utemur solita interpolatione.

86. Ponamus quoque pro y arcum cuius tangens datur, sitque $y = \text{A tang}\, x$ & $z = \text{A tang}(x+\omega)$ ita ut sit

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \&c.$$

Ad quos terminos indagandos quaerantur ipsius y singula differentialia:

$$\frac{dy}{dx} = \frac{1}{1+xx}; \quad \frac{ddy}{dx^2} = \frac{-2x}{(1+xx)^2}; \quad \frac{d^3y}{dx^3} = \frac{-2+6xx}{(1+xx)^3}$$

$$\frac{d^4y}{dx^4} = \frac{24x-24x^3}{(1+xx)^4}; \quad \frac{d^5y}{dx^5} = \frac{24-240x^2+120x^4}{(1+xx)^5};$$

$$\frac{d^6y}{dx^6} = \frac{-720x+2400x^3-720x^5}{(1+xx)^6}; \quad \&c.$$

unde colligitur fore: $\text{A tang}(x+\omega) = \text{A tang}\, x +$

$$\frac{\omega}{1(1+xx)} - \frac{\omega^2 x}{(1+xx)^2} + \frac{\omega^3}{(1+xx)^3}\left(xx-\tfrac{1}{3}\right) - \frac{\omega^4}{(1+xx)^4}(x^3-x) +$$

$$\frac{\omega^5}{(1+xx)^5}\left(x^4-2x^2+\tfrac{1}{5}\right) - \frac{\omega^6}{(1+xx)^6}\left(x^5-\tfrac{10}{3}x^3+x\right) + \&c.$$

87. Haec series, cuius lex progressionis non adeo manifesta est, transmutari potest in aliam formam, cuius progressio statim in oculos incurrit. Ponatur in hunc finem $\text{A tang}\, x = 90° - u$, ut sit $x = \cot u = \frac{\cos u}{\sin u}$; erit

1 +

$1 + xx = \frac{1}{\sin u^2}$, unde fit $\frac{dy}{dx} = \frac{1}{1+xx} = \sin u^2$. Cum deinde fit $dx = \frac{-du}{\sin u^2}$, seu $du = -dx \sin u^2$, fiet ulteriora differentialia sumendo:

$$\frac{ddy}{dx} = 2du \sin u \cos u = du \sin 2u = -dx \sin u^2 . \sin 2u$$

ideoque $\frac{ddy}{1dx^2} = -\sin u^2 . \sin 2u.$

$$\frac{d^3y}{2dx^2} = -du \sin u . \cos u . \sin 2u - du \sin u^2 \cos 2u = -du \sin u . \sin 3u$$
$$= dx \sin u^3 \sin 3u$$

ideoque $\frac{d^3y}{1.2dx^3} = + \sin u^3 . \sin 3u$

$$\frac{d^4y}{1.2.3dx^3} = du \sin u^2 . (\cos u . \sin 3u + \sin u . \cos 3u) = du \sin u^2 . \sin 4u$$
$$= -dx \sin u^4 . \sin 4u$$

ideoque $\frac{d^4y}{1.2.3dx^4} = -\sin u^4 . \sin 4u$

$$\frac{d^5y}{1.2.3.4dx^4} = -du \sin u^3 (\cos u \sin 4u + \sin u . \cos 4u) = -du . \sin u^3 . \sin 5u$$
$$= + dx \sin u^5 . \sin 5u$$

ideoque $\frac{d^5y}{1.2.3.4dx^5} = + \sin u^5 . \sin 5u$

&c.

Ex quibus colligitur fore:

$$A\,tg(x+\omega) = A\,tg\,x + \frac{\omega}{1} \sin u . \sin u - \frac{\omega^2}{2} \sin u^2 . \sin 2u + \frac{\omega^3}{3} \sin u^3 . \sin 3u$$
$$- \frac{\omega^4}{4} \sin u^4 \sin 4u + \frac{\omega^5}{5} \sin u^5 . \sin 5u - \frac{\omega^6}{6} \sin u^6 \sin 6u + \&c.$$

ubi

ubi cum sit $A\,tg\,x = y$ & $A\,tg\,x = 90^\circ - u$, erit $y = 90^\circ - x$.

88. Si ponatur $A \cot x = y$ & $A \cot (x + \omega) = z$; erit

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{1.2 dx^2} + \frac{\omega^3 d^3 y}{1.2.3 dx^3} + \frac{\omega^4 d^4 y}{1.2.3.4 dx^4} + \&c.$$

Cum autem sit $dy = \frac{-dx}{1 + xx}$, termini huius seriei congruent praeter primum cum ante inventis, exceptis tantum signis. Quare si ponatur, ut ante A tang. $x = 90^\circ - u$, seu $A \cot x = u$, ut sit $u = y$; erit:

$$A \cot(x + \omega) = A \cot x - \frac{\omega}{1} \sin u . \sin u + \frac{\omega^2}{2} \sin u^2 . \sin 2u - \frac{\omega^3}{3} \sin u^3 \sin 3u$$

$$+ \frac{\omega^4}{4} \sin u^4 . \sin 4u - \frac{\omega^5}{5} \sin u^5 . \sin 5u + \&c.$$

quae expressio immediate ex praecedente sequitur: quia enim est $A \cot(x + \omega) = 90 - A\,\text{tang}(x + \omega)$

& $A \cot x = 90 - A\,\text{tang}\,x$; erit

$A \cot(x + \omega) - A \cot x = - A\,\text{tang}(x + \omega) + A\,\text{tang}\,x$.

89. Ex his expressionibus multa egregia corollaria consequuntur, prout loco x & ω dati valores substituantur. Sit igitur primum $x = 0$; & cum sit $u = 90^\circ - A\,\text{tang}\,x$ fiet $u = 90^\circ$; atque $\sin u = 1$; $\sin 2u = 0$; $\sin 3u = -1$; $\sin 4u = 0$; $\sin 5u = 1$; $\sin 6u = 0$; $\sin 7u = -1$; &c. unde fiet

$$A\,\text{tang}\,\omega = \frac{\omega}{1} - \frac{\omega^3}{3} + \frac{\omega^5}{5} - \frac{\omega^7}{7} + \frac{\omega^9}{9} - \frac{\omega^{11}}{11} + \&c.$$

quae est notissima series exprimens arcum, cuius tangens est $= \omega$. Sit $x = 1$, erit $A\,\text{tang}\,x = 45^\circ$, ideoque $u = 45^\circ$, hinc $\sin u = \frac{1}{\sqrt{2}}$; $\sin 2u = 1$; $\sin 3u = \frac{1}{\sqrt{2}}$; $\sin 4u = 0$; $\sin 5u = -\frac{1}{\sqrt{2}}$; $\sin 6u = -1$; $\sin 7u = -\frac{1}{\sqrt{2}}$; $\sin 8u = 0$ $\sin 9u = \frac{1}{\sqrt{2}}$ &c. Ex quibus fit:

$$A\,tg(1+\omega) = 45^\circ + \frac{\omega}{2} - \frac{\omega^2}{2.2} + \frac{\omega^3}{3.4} - \frac{\omega^5}{5.8} + \frac{\omega^6}{6.8} - \frac{\omega^7}{7.16} + \frac{\omega^9}{9.32} - \frac{\omega^{10}}{10.32} + \frac{\omega^{11}}{11.64} - \frac{\omega^{13}}{13.128} + \frac{\omega^{14}}{14.128} - \&c.$$

Si igitur fit $\omega = -1$; ob $A\,tang(1+\omega) = 0$, & $45^\circ = \frac{\pi}{4}$ fiet:

$$\frac{\pi}{4} = \frac{1}{1.2} + \frac{1}{2.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{6.2^3} - \frac{1}{7.2^4} + \frac{1}{9.2^5} + \frac{1}{10.2^5} + \frac{1}{11.2^6} - \&c.$$

qui valor si loco arcus 45° substituatur in illa expressione erit:

$$A\,tang(1+\omega) = \frac{\omega+1}{1.2} - \frac{\omega^2+1}{2.2} + \frac{\omega^3+1}{3.2^2} - \frac{\omega^5-1}{5.2^3} + \frac{\omega^6-1}{6.2^3} - \frac{\omega^7-1}{7.2^4} + \&c.$$

Illa autem series maxime est idonea ad valorem ipsius $\frac{\pi}{4}$ proxime inveniendum Cum fit

$$\frac{\pi}{4} = \frac{1}{1.2} + \frac{1}{2.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{6.2^3} - \frac{1}{7.2^4} + \&c.$$

termini autem in denominatoribus habentes 2, 6, 10, &c.

$\frac{1}{2.2} - \frac{1}{6.2^3} + \frac{1}{10.2^5} - \frac{1}{14.2^7} + \&c.$ exprimunt $\frac{1}{2}\,A\,tg\,\frac{1}{2}$; erit

$$\frac{\pi}{4} = \frac{1}{2}\,A\,tang\,\frac{1}{2} + \frac{1}{1.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{7.2^4} + \frac{1}{9.2^5} + \frac{1}{11.2^6}\ \&c.$$

In altera autem formula posito ω negativo, cum fit

$$A\,tang(1-\omega) = \begin{cases} +\frac{1}{1.2} + \frac{1}{2.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{6.2^3} - \frac{1}{7.2^4} + \&c. \\ -\frac{\omega}{1.2} - \frac{\omega^2}{2.2} - \frac{\omega^3}{3.2^2} + \frac{\omega^5}{5.2^3} + \frac{\omega^6}{6.2^3} + \frac{\omega^7}{7.2^4} - \&c. \end{cases}$$

si fiat $\omega = \frac{1}{2}$; erit:

A

$$\text{A tang}\,\tfrac{1}{2} = \left\{\begin{array}{l} \dfrac{1}{1.2} + \dfrac{1}{2.2} + \dfrac{1}{3.2^2} + \dfrac{1}{5.2^3} - \dfrac{1}{6.2^3} - \dfrac{1}{7.2^4} + \&c. \\ -\dfrac{1}{1.2^2} - \dfrac{1}{2.2^3} - \dfrac{1}{3.2^5} + \dfrac{1}{5.2^8} + \dfrac{1}{6.2^9} + \dfrac{1}{7.2^{11}} - \&c. \end{array}\right.$$

& terminis per 2, 6, 10, &c. divisis seorsim sumtis erit

$$\text{A tg}\,\tfrac{1}{2} = \tfrac{1}{2}\text{A tg}\,\tfrac{1}{2} + \frac{1}{1.2} + \frac{1}{3.2^2} - \frac{1}{5.2^3} - \frac{1}{7.2^4} + \frac{1}{9.2^5} + \&c.$$

$$-\tfrac{1}{2}\,\text{A tang}\,\tfrac{1}{8} - \frac{1}{1.2^2} - \frac{1}{3.2^5} + \frac{1}{5.2^8} + \frac{1}{7.2^{11}} - \frac{1}{9.2^{14}} - \&c.$$

ideoque $\tfrac{1}{2}\,\text{A tang}\,\tfrac{1}{2} = \dfrac{1}{1.2} + \dfrac{1}{3.2^2} - \dfrac{1}{5.2^3} - \dfrac{1}{7.2^4} + \&c.$

$$-\tfrac{1}{2}\,\text{A tang}\,\tfrac{1}{8} - \frac{1}{1.2^2} - \frac{1}{3.2^5} + \frac{1}{5.2^8} + \frac{1}{7.2^{11}} - \&c.$$

qui valor si in superiore serie substituatur, atque A tang $\tfrac{1}{8}$ ipse in seriem convertatur, reperietur

$$\frac{\pi}{4} = \left\{\begin{array}{l} 1 + \dfrac{1}{3.2^1} - \dfrac{1}{5.2^2} - \dfrac{1}{7.2^3} + \dfrac{1}{9.2^4} + \&c. \\ -\dfrac{1}{1.2^2} - \dfrac{1}{3.2^5} + \dfrac{1}{5.2^8} + \dfrac{1}{7.2^{11}} - \dfrac{1}{9.2^{14}} - \&c. \\ -\dfrac{1}{1.2^4} + \dfrac{1}{3.2^{10}} - \dfrac{1}{5.2^{16}} + \dfrac{1}{7.2^{22}} - \dfrac{1}{9.2^{28}} + \&c. \end{array}\right.$$

90. Sequuntur hae multaeque aliae series ex positione $x = 1$: sin autem ponamus $x = \sqrt{3}$, ut sit A tang $x = 60°$, fiet $u = 30$, & sin $u = \tfrac{1}{2}$; sin $2u = \dfrac{\sqrt{3}}{2}$;

sin $3u = 1$; sin $4u = \dfrac{\sqrt{3}}{2}$; sin $5u = \tfrac{1}{2}$; sin $6u = 0$; sin $7u = -\tfrac{1}{2}$; &c. unde erit:

$$\text{A tang}(\sqrt{3} + \omega) = 60° + \frac{\omega}{1.2^2} - \frac{\omega^2\sqrt{3}}{2.2^3} + \frac{\omega^3}{3.2^3} - \frac{\omega^4\sqrt{3}}{4.2^5}$$

+

$$+\frac{\omega^5}{5.2^6}-\frac{\omega^7}{7.2^8}+\frac{\omega^8\sqrt{3}}{8.2^9}-\frac{\omega^9}{9.2^9}+\frac{\omega^{10}\sqrt{3}}{10.2^{11}}-\frac{\omega^{11}}{11.2^{12}}+\&c.$$

Sin autem ponatur $x=\frac{1}{\sqrt{3}}$, ut fit A tang $x=30^\circ$; erit $u=60^\circ$; atque fin $u=\frac{\sqrt{3}}{2}$; fin $2u=\frac{\sqrt{3}}{2}$; fin $3u=0$; fin $4u=-\frac{\sqrt{3}}{2}$; fin $5u=-\frac{\sqrt{3}}{2}$; fin $6u=0$; fin $7u=\frac{\sqrt{3}}{2}$; &c.

quibus valoribus fubftitutis erit:

$$\text{A tg}\left(\frac{1}{\sqrt{3}}+\omega\right)=30^\circ+\frac{3\omega}{1.2^2}-\frac{3\omega^2\sqrt{3}}{2.2^3}+\frac{3^2\omega^4\sqrt{3}}{4.2^5}-\frac{3^3\omega^5}{5.2^6}+\&c.$$

fi igitur fit $\omega=-\frac{1}{\sqrt{3}}$, ob $30^\circ=\frac{\pi}{6}$; erit:

$$\frac{\pi}{6\sqrt{3}}=\frac{1}{1.2^2}+\frac{1}{2.2.^3}-\frac{1}{4.2^5}-\frac{1}{5.2^6}+\frac{1}{7.2^8}+\frac{1}{8.2^9}-\&c.$$

91. Refumamus expreffionem generalem inventam:

$$\text{A tang.}(x+\omega)=\text{A tang } x$$
$$+\frac{\omega}{1}\text{fin } u.\text{ fin } u-\frac{\omega^2}{2}\text{fin } u^2.\text{ fin } 2u+\frac{\omega^3}{3}\text{fin } u^3.\text{ fin } 3u-\&c.$$

ac ponamus $\omega=-x$, ut fit A tang $(x+\omega)=0$, eritque

$$\text{A tang } x=$$
$$\frac{x}{1}\text{fin } u.\text{ fin } u+\frac{x^2}{2}\text{fin } u^2.\text{ fin } 2u+\frac{x^3}{3}\text{fin } u^3.\text{ fin } 3u+\&c.$$

Cum autem fit A tang $x=90-u=\frac{\pi}{2}-u$;

erit: $x=\cot u=\frac{\text{cof } u}{\text{fin } u}$. Quamobrem erit:

$$\frac{\pi}{2}=u+\text{cof } u.\text{ fin } u+\tfrac{1}{2}\text{cof } u^2.\text{ fin } 2u+\tfrac{1}{3}\text{cof } u^3.\text{ fin } 3u+\tfrac{1}{4}\text{cof } u^4.\text{ fin } 4u+\&c.$$

quae

quae series eo magis est notatu digna, quod quicunque arcus loco u accipiatur, valor seriei semper prodeat idem $= \frac{\pi}{2}$. Sin autem sit $\omega = -2x$, ob A tang $(-x) = -$ A tg x;

fiet: $$2 \text{ A tang } x =$$

$$\frac{2x}{1} \sin u . \sin u + \frac{4x^2}{2} \sin u^2 . \sin 2u + \frac{8x^3}{3} \sin u^3 . \sin 3u + \&c.$$

Cum autem sit A tang $x = \frac{\pi}{2} - u$ & $x = \frac{\cos u}{\sin u}$, erit:

$$\pi = 2u + \frac{2}{1} \cos u . \sin u + \frac{2^2}{2} \cos u^2 . \sin 2u + \frac{2^3}{3} \cos u^3 . \sin 3u + \&c.$$

Sit $u = 45° = \frac{\pi}{4}$; erit $\cos u = \frac{1}{\sqrt{2}}$; $\sin u = \frac{1}{\sqrt{2}}$; $\sin 2u = 1$;

$\sin 3u = \frac{1}{\sqrt{2}}$; $\sin 4u = 0$; $\sin 5u = \frac{-1}{\sqrt{2}}$; $\sin 6u = -1$;

$\sin 7u = \frac{-1}{\sqrt{2}}$; $\sin 8u = 0$; $\sin 9u = \frac{1}{\sqrt{2}}$; eritque

$$\frac{\pi}{2} = \frac{1}{1} + \frac{2}{2} + \frac{2}{3} - \frac{2^2}{5} - \frac{2^3}{6} - \frac{2^3}{7} + \frac{2^4}{9} + \frac{2^5}{10} + \frac{2^5}{11} - \&c.$$

quae series etsi divergit, tamen ob simplicitatem est notatu digna.

92. Ponatur in expressione generali inventa:

$$\omega = -x - \frac{1}{x} = \frac{-1}{\sin u . \cos u}, \text{ ob } x = \frac{\cos u}{\sin u}; \text{ erit:}$$

$$\text{A tang}(x + \omega) = \text{A tang} - \frac{1}{x} = - \text{A tang} \frac{1}{x} = - \frac{\pi}{2} + \text{A tang } x.$$

Hinc ergo obtinebitur sequens expressio:

$$\frac{\pi}{2} = \frac{\sin u}{1 \cos u} + \frac{\sin 2u}{2 \cos u^2} + \frac{\sin 3u}{3 \cos u^3} + \frac{\sin 4u}{4 \cos u^4} + \frac{\sin 5u}{5 \cos u^5} + \&c.$$

quae posito $u = 45°$ dat eandem seriem, quam ultimo loco invenimus. Sin autem ponamus $\omega = -\sqrt{(1 + xx)}$

ob

ob $x = \frac{\cos u}{\sin u}$, fiet $\omega = -\frac{1}{\sin u}$, &

$$\text{A tang}\,[x - \surd(1+xx)] - \text{A tang}\,[\surd(1+xx) - x]$$

$$= -\tfrac{1}{2}\,\text{A tang}\,\frac{1}{x} = -\tfrac{1}{2}\left(\frac{\pi}{2} - \text{A tang}\,x\right) = -\tfrac{1}{2}u,$$

& $\text{A tang}\,x = \frac{\pi}{2} - u$. Hancobrem erit:

$$\frac{\pi}{2} = \tfrac{1}{2}u + \tfrac{1}{1}\sin u + \tfrac{1}{2}\sin 2u + \tfrac{1}{3}\sin 3u + \tfrac{1}{4}\sin 4u + \&c.$$

Quodſi haec aequatio differentietur erit:

$$0 = \tfrac{1}{2} + \cos u + \cos 2u + \cos 3u + \cos 4u + \cos 5u + \&c.$$

cuius ratio ex natura ſerierum recurrentium intelligitur.

93. Si ſimili modo ſeries ante inventae differentientur, novae ſeries ſummabiles reperientur. Ac primo quidem ex ſerie:

$$\text{A tang}\,(1+\omega) = \frac{\pi}{4} + \frac{\omega}{2} - \frac{\omega^2}{2.2} + \frac{\omega^3}{3.4} - \frac{\omega^5}{5.8} + \frac{\omega^6}{6.8} - \&c.$$

ſequitur

$$\frac{1}{2+2\omega+\omega^2} = \tfrac{1}{2} - \frac{\omega}{2} + \frac{\omega^2}{4} - \frac{\omega^4}{8} + \frac{\omega^5}{8} - \frac{\omega^6}{16} + \frac{\omega^8}{32} - \&c.$$

quae oritur ex evolutione fractionis $\frac{2-2\omega+\omega^2}{4+\omega^4} = \frac{1}{2+2\omega+\omega^2}$,

Deinde iſta ſeries:

$$\frac{\pi}{2} = u + \cos u \sin u + \tfrac{1}{2}\cos u^2 . \sin 2u + \tfrac{1}{3}\cos u^3 \sin 3u + \tfrac{1}{4}\cos u^4 \sin 4u + \&c.$$

per differentiationem dabit:

$$0 = 1 + \cos 2u + \cos u . \cos 3u + \cos u^2 . \cos 4u + \cos u^3 \cos 5u + \&c.$$

Denique ſeries $\frac{\pi}{2} = \frac{\sin u}{\cos u} + \frac{\sin 2u}{2\cos u^2} + \frac{\sin 3u}{3\cos u^3} + \frac{\sin 4u}{4\cos u^4} + \&c.$

dat $0 = \frac{1}{\cos u^2} + \frac{\cos u}{\cos u^3} + \frac{\cos 2u}{\cos u^4} + \frac{\cos 3u}{\cos u^5} + \frac{\cos 4u}{\cos u^6} + \&c.$

ſeu

seu $0 = 1 + \frac{\cos u}{\cos u} + \frac{\cos 2u}{\cos u^2} + \frac{\cos 3u}{\cos u^3} + \frac{\cos 4u}{\cos u^4} + \&c.$

94. Imprimis autem expressio inventa:

$$A\,\mathrm{tang}\,(x+\omega) =$$

$$A\,\mathrm{tang}\,x + \frac{\omega}{1}\sin u.\sin u - \frac{\omega^2}{2}\sin u^2.\sin 2u + \frac{\omega^3}{3}\sin u^3.\sin 3u - \&c.$$

existente $x = \cot u$ seu $u = A\cot x = 90° - A\,\mathrm{tang}\,x$ inserviet ad angulum seu arcum datae cuique tangenti respondentem inveniendum. Sit enim proposita tangens $= t$, quaeraturque in tabulis tangens ad hanc proxime accedens $= x$, cui respondeat arcus $= y$; eritque $u = 90° - y$. Tum ponatur $x + \omega = t$, seu $\omega = t - x$; eritque arcus quaesitus:

$$= y + \frac{\omega}{1}\sin u.\sin u - \frac{\omega^2}{2}\sin u^2.\sin 2u + \&c.$$

quae regula tum praecipue est utilis, cum tangens proposita fuerit admodum magna, ac propterea arcus quaesitus parum a 90° discrepet. His enim casibus ob tangentes vehementer increscentes, solita methodus interpolationum nimium a veritate abducit. Sit ergo propositum hoc exemplum.

EXEMPLUM.

Quaeratur arcus, cuius tangens sit $= 100$, posito radio $= 1$. Arcus proxime quaesito aequalis est $89°, 25'$, cuius tangens est $x = 98,217943$ secund.

quae subtrahatur a $t = 100,000000$

remanebit $\omega = 1,782057$

Deinde cum sit $y = 89°, 25'$, erit $u = 0°, 35'$, $2u = 1°, 10'$, $3u = 1°, 45'$, &c. Iam singuli termini per logarithmos investigentur.

Ad

Ad $l\omega = 0,2509215$
add. $l \sin u = 8,0077867$
$l \sin u = 8,0077867$

$l\omega \sin u . \sin u = 6,2664949$
$4,6855749$

subtr. $= 1,5809200$

Ergo $\omega \sin u . \sin u = 38,09956$ secund.

Ad $l \omega \sin u^2 = 6,2664949$
add. $l\omega = 0,2509215$
$l \sin 2u = 8,3087941$

$4,8262105$

subtr. $l\, 2 = 0,3010300$

$l \tfrac{1}{2} \omega^2 \sin u^2 . \sin 2u = 4,5251805$
subtr. $4,6855749$

Remanet $9,8396056$

Ergo $\tfrac{1}{2} \omega^2 \sin u^2 . \sin 2u = 0,69120$ secund.

Porro ad $l\omega^3 = 0,7527645$
add. $l \sin u^3 = 4,0233601$
$l \sin 3u = 8,4848479$

$3,2609725$

subtr. $l\, 3 = 0,4771213$

$2,7838512$

subtr. $4,6855749$

$8,0982763$

Ergo $\tfrac{1}{3} \omega^3 \sin u^3 \sin 3u = 0,01254$ secund.

Denique ad $l \omega^4 = 1,0036860$
add. $l \sin u^4 = 2,0311468$
$l \sin 4u = 8,6097341$

$1,6445669$

subtr. $l\, 4 = 0,6020600$

$1,0425069$

subtr. $4,6855749$

$6,3569320$

Ergo $\tfrac{1}{4} \omega^4 \sin u^4 \sin 4u = 0,00023$ secund.

Hinc:

Termini addendi	Termini subtrahendi
38,09956	0,69120
0,01254	0,00023
38,11210	0,69143
subtr. 0,69143	
37,42067 $= 37^{II}, 25^{III}, 14^{IV}, 24^{V}, 36^{VI}$.	

Quocirca arcus, cuius tangens centies superat radium erit:

$$89^{0},\ 25^{I},\ 37^{II},\ 25^{III},\ 14^{IV},\ 24^{V},\ 36^{VI},$$

neque error ad minuta quarta ascendit; sed in minutis tantum quintis inesse potest, ex quo vere hunc angulum pronunciare poterimus $= 89^{0}, 25^{I}, 37^{II}, 25^{III}, 14^{IV}$. Si tangens adhuc maior proponatur, etiamsi fortasse ω maius prodeat, tamen ob u angulum adhuc minorem, aeque expedite arcus definiri poterit.

95. Cum hic pro y arcum circuli substituerimus, nunc functiones reciprocas in locum y ponamus, cuiusmodi sunt $\sin x$, $\cos x$, $\tang x$, $\cot x$, &c. Sit igitur $y = \sin x$, positoque $x+\omega$ loco x, fiet: $z = \sin(x+\omega)$, atque aequatio

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \&c.$$

ob $\frac{dy}{dx} = \cos x$; $\frac{ddy}{dx^2} = -\sin x$; $\frac{d^3y}{dx^3} = -\cos x$; &c. dabit

$$\sin(x+\omega) = \sin x + \omega\cos x - \tfrac{1}{2}\omega^2 \sin x - \tfrac{1}{6}\omega^3 \cos x + \tfrac{1}{24}\omega^4 \sin x + \&c.$$

& sumto ω negativo erit:

$$\sin(x-\omega) = \sin x - \omega\cos x - \tfrac{1}{2}\omega^2\sin x + \tfrac{1}{6}\omega^3\cos x + \tfrac{1}{24}\omega^4\sin x - \&c.$$

Quod si vero statuatur $y = \cos x$,

ob $\frac{dy}{dx} = -\sin x$; $\frac{ddy}{dx^2} = -\cos x$; $\frac{d^3y}{dx^3} = \sin x$; $\frac{d^4y}{dx^4} = \cos x$; &c.

erit:

$$\cos(x+\omega) = \cos x - \omega\sin x - \tfrac{1}{2}\omega^2\cos x + \tfrac{1}{6}\omega^3\sin x + \tfrac{1}{24}\omega^4\cos x - \&c.$$

& facto ω negativo erit:

$$\cos(x-\omega)=\cos x+\omega\sin x-\tfrac{1}{2}\omega^2\cos x-\tfrac{1}{6}\omega^3\sin x+\tfrac{1}{24}\omega^4\cos x+\&c.$$

96. Usus harum formularum eximius est cum in condendis, tum interpolandis tabulis sinuum & cosinuum. Si enim cogniti fuerint sinus & cosinus cuiuspiam arcus x, ex iis facili negotio sinus & cosinus angulorum $x+\omega$ & $x-\omega$ inveniri possunt, si quidem differentia ω fuerit satis exigua: hoc enim casu series inventae vehementer convergunt. Ad hoc vero necesse est, ut arcus ω in partibus radii exprimatur; quod cum arcus 180° sit: 3,14159265358979323846 facile fiet: erit enim divisione per 180 instituta

arcus $1^{\circ} = 0,01745329251994329576\,9$
arcus $1^{I} = 0,00029088820866572159\,6$
arcus $10^{II} = 0,00004848136811095359\,9$

EXEMPLUM I.

Invenire sinus & cosinus angulorum 45°, 1^{I}, *&* 44°, 59^{I}, *ex datis sinu & cosinu anguli* 45°, *quorum uterque est*

$$=\frac{1}{\sqrt{2}}=0,7071067811865.$$

Cum igitur sit:

$$\sin x=\cos x=0,7071067811865$$
$$\text{atque}\quad \omega=0,0002908882086$$

erit ad multiplicationes facilius instituendas:

$$2\,\omega=0,0005817764173$$
$$3\,\omega=0,0008726646259$$
$$4\,\omega=0,0011635528346$$
$$5\,\omega=0,0014544410432$$
$$6\,\omega=0,0017453292519$$
$$7\,\omega=0,0020362174605$$
$$8\,\omega=0,0023271056692$$
$$9\,\omega=0,0026179938779$$

Ergo $\omega\sin x$ & $\omega\cos x$ hoc modo invenietur:

7 .	0,00020362174605
0 .	. . . .
7 .	0,00000203621746
1 .	. 2908882
0 .	. . .
6 .	. 174532
7 .	. 20362
8 .	. 2327
1 .	. 29
1 .	. . 2
8 .	. . 2
6 .	. . 0
$\omega \sin x = \omega \cos x =$	0,00020568902490
Ergo $\frac{1}{2}\omega \cos x =$	0,00010284451245
per ω . 1 .	0,00000002908882
0 .	. . .
2 .	. 58177
8 .	. 23271
4 .	. 1163
4 .	. 116
5 .	. . 14
$\frac{1}{2}\omega^2 \cos x =$	0,00000002991625
$\frac{1}{6}\omega^2 \cos x =$	0,00000000997208
per ω . 9 .	0,00000000000261
9 .	. . 26
7 .	. . 2
$\frac{1}{6}\omega^3 \cos x =$	0,00000000000290

Ergo ad sin 45°, 1′, inveniendum:

Ad $\sin x =$	0,70710678811865
add. $\omega \cos x =$	2056890249
	0,70731247702114
subtr. $\frac{1}{2}\omega^2 \sin x =$	299162
	0,70731244402952

$$0{,}7073124402952$$

subtr. $\frac{1}{6}\omega^3 \cos x = 29$

$\sin 45°, 1', = 0{,}7073124402923 = \cos 44°, 59'.$

At ad $\cos 45°, 1'$, inveniendum:

$A \cos x = 0{,}7071067811865$

subtr. $\omega \sin x = 2056890249$

$0{,}7069010921616$

subtr. $\frac{1}{2}\omega^2 \cos x = 299162$

$0{,}7069010622454$

add. $\frac{1}{6}\omega^3 \sin x = 29$

$\cos 45°, 1', = 0{,}7069010622483 = \sin 44°, 59'.$

EXEMPLUM II.

Ex datis sinu & cosinu arcus $67°, 30'$, invenire sinus & cosinus arcuum $67°, 31'$, & $67°, 29'$.

Absolvamus hunc calculum in fractionibus decimalibus, tantum ad 7 notas, uti tabulae vulgares construi solent, sicque negotium facile per logarithmos conficietur. Cum sit $x = 67°, 30'$, & $\omega = 0{,}000290888$; erit: $l\omega = 6{,}4637259$ &

$l \sin x = 9{,}9656153$; $l \cos x = 9{,}5828397$

$l\omega = 6{,}4637259$; $l\omega = 6{,}4637259$

$l\omega \sin x = 6{,}4293412$; $l\omega \cos x = 6{,}0465656$

$l\frac{1}{2}\omega = 6{,}1626959$; $l\frac{1}{2}\omega = 6{,}1626959$

$l\frac{1}{2}\omega^2 \sin x = 2{,}5920371$; $l\frac{1}{2}\omega^2 \cos x = 2{,}2092615$

ergo:

$\omega \sin x = 0{,}00026874$; $\omega \cos x = 0{,}00011132$

$\frac{1}{2}\omega^2 \sin x = 0{,}00000004$; $\frac{1}{2}\omega^2 \cos x = 0{,}00000001$

unde fit:

$\sin 67°, 31' = 0{,}9239908$; $\cos 67°, 31' = 0{,}3824147$

$\sin 67°, 29' = 0{,}9237681$; $\cos 67°, 29' = 0{,}3829522$

ubi ne quidem terminis $\frac{1}{2}\omega^2 \sin x$ & $\frac{1}{2}\omega^2 \cos x$ erat opus.

97. Ex seriebus quas supra invenimus:

$\sin(x+\omega) = \sin x + \omega \cos x - \frac{1}{2}\omega^2 \sin x - \frac{1}{6}\omega^3 \cos x + \frac{1}{24}\omega^4 \sin x + \&c.$

cos

$$\cos(x+\omega)=\cos x-\omega\sin x-\tfrac{1}{2}\omega^2\cos x+\tfrac{1}{6}\omega^3\sin x+\tfrac{1}{24}\omega^4\cos x-\&c.$$
$$\sin(x-\omega)=\sin x-\omega\cos x-\tfrac{1}{2}\omega^2\sin x+\tfrac{1}{6}\omega^3\cos x+\tfrac{1}{24}\omega^4\sin x-\&c.$$
$$\cos(x-\omega)=\cos x+\omega\sin x-\tfrac{1}{2}\omega^2\cos x-\tfrac{1}{6}\omega^3\sin x+\tfrac{1}{24}\omega^4\cos x+\&c.$$

sequitur per combinationem fore:

$$\frac{\sin(x+\omega)+\sin(x-\omega)}{2}=$$
$$\sin x-\tfrac{1}{2}\omega^2\sin x+\tfrac{1}{24}\omega^4\sin x-\tfrac{1}{720}\omega^6\sin x+\&c.=\sin x\cos\omega$$

Et
$$\frac{\sin(x+\omega)-\sin(x-\omega)}{2}=$$
$$\omega\cos x-\tfrac{1}{6}\omega^3\cos x+\tfrac{1}{120}\omega^5\cos x-\&c.=\cos x\sin\omega$$

unde prodeunt series pro sinibus & cosinibus iam supra inventae:

$$\cos\omega=1-\tfrac{1}{2}\omega^2+\tfrac{1}{24}\omega^4-\tfrac{1}{720}\omega^6+\&c.$$
$$\sin\omega=\omega-\tfrac{1}{6}\omega^3+\tfrac{1}{120}\omega^5-\tfrac{1}{5040}\omega^7+\&c.$$

quae eaedem series ex primis ponendo $x=0$ consequuntur; cum enim sit $\cos x=1$ & $\sin x=0$ prima series *sin* ω, secunda vero *cos* ω exhibebit.

98. Ponamus nunc quoque $y=\tang x$, ut sit $z=\tang(x+\omega)$, erit ob

$$y=\frac{\sin x}{\cos x};\quad \frac{dy}{dx}=\frac{1}{\cos x^2};\quad \frac{ddy}{2dx^2}=\frac{\sin x}{\cos x^3};$$
$$\frac{d^3y}{2dx^3}=\frac{1}{\cos x^2}+\frac{3\sin x^2}{\cos x^4}=\frac{3}{\cos x^4}-\frac{2}{\cos x^2};$$
$$\frac{d^4y}{2.4dx^4}=\frac{3\sin x}{\cos x^5}-\frac{\sin x}{\cos x^3};$$
$$\frac{d^5y}{2.4dx^5}=\frac{15}{\cos x^6}-\frac{15}{\cos x^4}+\frac{2}{\cos x^2};$$

unde sequitur fore:

$$\tang(x+\omega)=\tang x+\frac{\omega}{\cos x^2}+\frac{\omega^2\sin x}{\cos x^3}+\frac{\omega^3}{\cos x^4}+\frac{\omega^4\sin x}{\cos x^5}$$
$$-\frac{2\omega^3}{3\cos x^2}-\frac{\omega^4\sin x}{3\cos x^3};$$

cu-

cuius formulae ope ex data cuiusvis anguli tangente inveniri possunt tangentes angulorum proximorum. Quia vero superior series est geometrica, ea in unam summam collecta erit:

$$\mathrm{tang}(x+\omega) = \mathrm{tang}\,x + \frac{\omega + \omega^2 \,\mathrm{tang}\,x}{\mathrm{cos}\,x^2 - \omega^2} - \frac{2\omega^3}{3\,\mathrm{cos}\,x^2} - \frac{\omega^4 \,\mathrm{sin}\,x}{3\,\mathrm{cos}\,x^3} \text{ \&c.} \quad \text{seu}$$

$$\mathrm{tang}(x+\omega) = \frac{\mathrm{sin}\,x\,\mathrm{cos}\,x + \omega}{\mathrm{cos}\,x^2 - \omega^2} - \frac{2\omega^3}{3\,\mathrm{cos}\,x^2} - \frac{\omega^4 \,\mathrm{sin}\,x}{3\,\mathrm{cos}\,x^3} \text{ \&c.}$$

quae formula in hunc finem commodius adhibetur.

99. Similes expressiones quoque pro logarithmis sinuum, cosinuum & tangentium inveniri possunt. Sit enim $y =$ logarithmo sinus anguli x, quod ita exprimamus $y = l\,\mathrm{sin}\,x$, & $z = l\,\mathrm{sin}(x+\omega)$, ob $\frac{dy}{dx} = \frac{n\,\mathrm{cos}\,x}{\mathrm{sin}\,x}$; erit: $\frac{ddy}{dx^2} = \frac{-n}{\mathrm{sin}\,x^2}$; $\frac{d^3y}{dx^3} = \frac{+2n\,\mathrm{cos}\,x}{\mathrm{sin}\,x^3}$ &c. unde fiet:

$$z = l\,\mathrm{sin}(x+\omega) = l\,\mathrm{sin}\,x + \frac{n\omega\,\mathrm{cos}\,x}{\mathrm{sin}\,x} - \frac{n\omega^2}{2\,\mathrm{sin}\,x^2} + \frac{n\omega^3\,\mathrm{cos}\,x}{3\,\mathrm{sin}\,x^3} \text{ \&c.}$$

ubi n denotat numerum, per quem logarithmi hyperbolici multiplicari debent, ut prodeant logarithmi propositi. Sin autem fit

$$y = l\,\mathrm{tang}\,x \quad \& \quad z = l\,\mathrm{tang}(x+\omega) \quad \text{fiet:}$$

$$\frac{dy}{dx} = \frac{n}{\mathrm{sin}\,x\,\mathrm{cos}\,x} = \frac{2n}{\mathrm{sin}\,2x}; \quad \frac{ddy}{2dx^2} = \frac{-2n\,\mathrm{cos}\,2x}{(\mathrm{sin}\,2x)^2}; \quad \text{ideoque}$$

$$l\,\mathrm{tang}(x+\omega) = l\,\mathrm{tang}\,x + \frac{2n\omega}{\mathrm{sin}\,2x} - \frac{n\omega^2\,\mathrm{cos}\,2x}{(\mathrm{sin}\,2x)^2} \text{ \&c.}$$

quarum formularum ope logarithmi sinuum & tangentium interpolari possunt.

100. Ponamus denotare y arcum cuius sinus logarithmus sit $= x$, seu ut sit $y = \mathrm{A}.\,l\,\mathrm{sin}\,x$, & z esse arcum, cuius sinus logarithmus sit $= x + \omega$, seu $z = \mathrm{A}.\,l\,\mathrm{sin}(x+\omega)$; erit $x = l\,\mathrm{sin}\,y$, & $\frac{dx}{dy} = \frac{n\,\mathrm{cos}\,y}{\mathrm{sin}\,y}$, unde $\frac{dy}{dx} = \frac{\mathrm{sin}\,y}{n\,\mathrm{cos}\,y}$; erit:

$$\frac{ddy}{dx} = \frac{dy}{n\,\mathrm{cos}\,y^2} = \frac{dx\,\mathrm{sin}\,y}{n^2\,\mathrm{cos}\,y^3}; \quad \text{ergo} \quad \frac{ddy}{dx^2} = \frac{\mathrm{sin}\,y}{n^2\,\mathrm{cos}\,y^3}:$$

Consequenter $z = y + \frac{\omega \sin y}{n \cos y} + \frac{\omega^2 \sin y}{2n^2 \cos y^3} + \&c.$

Simili modo si logarithmus cosinus detur, expressio reperietur. Sin autem sit

$$y = A.\,l\,\text{tang}\,x \quad \& \quad z = A.\,l\,\text{tang}\,(x+\omega).$$

Cum sit $x = l\,\text{tang}\,y$; fiet:

$$\frac{dx}{dy} = \frac{n}{\sin y \cos y}, \quad \& \quad \frac{dy}{dx} = \frac{\sin y \cos y}{n} = \frac{\sin 2y}{2n};$$

quare $\frac{ddy}{dx} = \frac{2dy\cos 2y}{2n} = \frac{dx \sin 2y \cos 2y}{2nn}$ &

$\frac{ddy}{dx^2} = \frac{\sin 2y \cos 2y}{2nn} = \frac{\sin 4y}{4nn}$; $\frac{d^3y}{dx^3} = \frac{\sin 2y\,.\cos 4y}{2n^3}$ &c. hinc

$$z = y + \frac{\omega \sin 2y}{2n} + \frac{\omega^2 \sin 2y \cos 2y}{4nn} + \frac{\omega^3 \sin 2y\,.\cos 4y}{12n^3} + \&c.$$

101. Quoniam usus harum expressionum in condendis tabulis logarithmorum sinuum & tangentium ex antecedentibus facile perspici potest, his diutius non immorabimur. Consideremus ergo adhuc huiusmodi valorem:

$y = e^x \sin nx$; sitque $z = e^{x+\omega} \sin n(x+\omega)$: quia est

$$\frac{dy}{dx} = e^x(\sin nx + n \cos nx)$$

$$\frac{ddy}{dx^2} = e^x[(1-nn)\sin nx + 2n \cos nx]$$

$$\frac{d^3y}{dx^3} = e^x[(1-3nn)\sin nx + n(3-nn)\cos nx]$$

$$\frac{d^4y}{dx^4} = e^x[(1-6nn+n^4)\sin nx + n(4-4nn)\cos nx]$$

$$\frac{d^5y}{dx^5} = e^x[(1-10nn+5n^4)\sin nx + n(5-10nn+n^4)\cos nx]$$

His substitutis & divisione per e^x instituta erit: $e^\omega \sin n(x+\omega) =$

$$\sin nx + \omega \sin nx + \frac{(1-nn)}{2}\omega^2 \sin nx + n\omega \cos nx + \frac{2n\omega^2}{2} \cos nx$$

$$+\frac{(1-3nn)}{6}\omega^3 \sin nx + \frac{(1-6nn+n^4)}{24}\omega^4 \sin nx + \&c.$$

$$+\frac{n(3-nn)}{6}\omega^3 \cos nx + \frac{n(4-4nn)}{24}\omega^4 \cos nx + \&c.$$

102. Hinc plurima egregia corollaria deduci possunt; sufficiat autem nobis haec annotasse:

Si fuerit $x=0$ erit:

$$e^\omega \sin n\omega = n\omega + \frac{2n\omega^2}{2} + \frac{n(3-nn)}{6}\omega^3 + \frac{n(4-4nn)}{24}\omega^4 + \frac{n(5-10n^2+n^4)}{120}\omega^5 + \&c.$$

Si sit $\omega=-x$, ob $\sin n(x+\omega)=0$; erit:

$$\tan nx = \frac{nx - \frac{2n}{2}x^2 + \frac{n(3-nn)}{6}x^3 - \frac{n(4-4nn)}{24}x^4 + \frac{n(5-10n^2+n^4)}{120}x^5}{1-x+\frac{(1-nn)}{2}x^2 - \frac{(1-3nn)}{6}x^3 + \frac{(1-6nn+n^4)}{24}x^4 - \&c.}$$

Generaliter vero si sit $n=1$ habebitur:

$$e^\omega \sin(x+\omega) = \sin x\left(1+\omega-\tfrac{1}{3}\omega^3-\tfrac{1}{6}\omega^4-\tfrac{1}{30}\omega^5+\tfrac{1}{630}\omega^7+\&c.\right)$$
$$+\omega\cos x\left(1+\omega+\tfrac{1}{3}\omega^2-\tfrac{1}{30}\omega^4-\tfrac{1}{90}\omega^5-\tfrac{1}{630}\omega^6+\&c.\right)$$

Sin autem sit $n=0$, ob $\sin n(x+\omega)=n(x+\omega)$, & $\sin nx = nx$ atque $\cos nx = 1$, si ubique per n dividatur, prodibit:

$$e^\omega(x+\omega) = x + \omega x + \tfrac{1}{2}\omega^2 x + \tfrac{1}{6}\omega^3 x + \tfrac{1}{24}\omega^4 x + \&c.$$
$$+\omega + \omega^2 + \tfrac{1}{2}\omega^3 + \tfrac{1}{6}\omega^4 + \tfrac{1}{24}\omega^5 + \&c.$$

cuius seriei ratio est manifesta.

CA-