CAPUT II.

DE INVESTIGATIONE SERIERUM SUMMABILIUM.

19.

Si Seriei, in cuius terminis inest quantitas indeterminata x, summa fuerit cognita, quae utique erit functio ipsius x; tum quicunque valor ipsi x tribuatur, seriei summa perpetuo assignari poterit. Quare si loco x ponatur $x + dx$, seriei resultantis summa erit aequalis summae prioris, una cum ipsius differentiali: unde sequitur fore differentiale summae = differentiali seriei. Quia vero hoc modo tam summa, quam singuli seriei termini multiplicati erunt per dx, si ubique per dx dividatur, habebitur nova series, cuius summa erit cognita. Simili modo si haec series cum sua summa denuo differentietur, & ubique per dx dividatur, nova exoritur series cum sua summa: sicque ex una serie summabili quantitatem indeterminatam x involvente, per continuam differentiationem innumerae novae series pariter summabiles elicientur.

20. Quo haec clarius perspiciantur, proposita sit progressio geometrica indeterminata, quippe cuius summa est cognita, haec:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \&c.$$

Si nunc differentiatio instituatur, erit:

$$\frac{dx}{(1-x)^2} = dx + 2xdx + 3x^2dx + 4x^3dx + 5x^4dx + \&c.$$

atque divisione per dx instituta, habebitur:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \&c.$$

Si

Si denuo differentietur, atque per dx dividatur, prodibit:

$$\frac{2}{(1-x)^3} = 2 + 2.3x + 3.4x^2 + 4.5x^3 + 5.6x^4 + \&c. \quad \text{seu}$$

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + \&c.$$

ubi coefficientes ſunt numeri trigonales. Si haec porro differentietur, atque per $3dx$ dividatur, obtinebitur:

$$\frac{1}{(1-x)^4} = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + \&c.$$

cuius coefficientes ſunt numeri pyramidales primi. Sicque ulterius procedendo oriuntur eaedem ſeries quas ex evolutione fractionis $\frac{1}{(1-x)^n}$ naſci conſtat.

21. Latius autem patebit haec ſerierum inveſtigatio, ſi antequam quaevis differentiatio ſuſcipiatur, ipſa ſeries una cum ſumma per quamvis ipſius x poteſtatem ſeu functionem multiplicetur. Sic cum ſit

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \&c.$$

multiplicetur ubique per x^m, eritque

$$\frac{x^m}{1-x} = x^m + x^{m+1} + x^{m+2} + x^{m+3} + x^{m+4} + \&c.$$

nunc differentietur haec ſeries, fietque per dx diviſo:

$$\frac{mx^{m-1} - (m-1)x^m}{(1-x)^2} = mx^{m-1} + (m+1)x^m$$
$$+ (m+2)x^{m+1} + (m+3)x^{m+2} + \&c.$$

Dividatur nunc per x^{m-1}, habebitur:

$$\frac{m-(m-1)x}{(1-x)^2} = \frac{m}{1-x} + \frac{x}{(1-x)^2} = m + (m+1)x$$
$$+ (m+2)x^2 + \&c.$$

multiplicetur haec antequam nova differentiatio ſuſcipiatur per x^n, ut ſit

$$\frac{mx^n}{1-x}+\frac{x^{n+1}}{(1-x)^2}=mx^n+(m+1)x^{n+1}+(m+2)x^{n+2}+ \&c.$$

Nunc instituatur differentiatio, & diviso per dx erit:

$$\frac{mnx^{n-1}}{1-x}+\frac{(m+n+1)x^n}{(1-x)^2}+\frac{2x^{n+1}}{(1-x)^3}=mnx^{n-1}$$
$$+(m+1)(n+1)x^n+(m+2)(n+2)x^{n+1}+ \&c.$$

Divisione autem per x^{n-1} instituta, fiet

$$\frac{mn}{1-x}+\frac{(m+n+1)x}{(1-x)^2}+\frac{2xx}{(1-x)^3}=$$
$$mn+(m+1)(n+1)x+(m+2)(n+2)x^2+ \&c.$$

sicque ulterius progredi licebit: invenientur autem perpetuo eaedem series, quae ex evolutione fractionum summam constituentium nascuntur.

22. Quoniam progressionis geometricae primum assumtae summa ad quemvis terminum usque assignari potest, hoc modo quoque series definito terminorum numero constantes summabuntur. Ita cum sit

$$\frac{x-x^{n+1}}{1-x}=x+x^2+x^3+x^4+ \ldots\ldots\ldots +x^n$$

erit differentiatione instituta & terminis per dx divisis

$$\frac{1}{(1-x)^2}-\frac{(n+1)x^n-nx^{n+1}}{(1-x)^2}=1+2x+3x^2+4x^3$$
$$+\ldots\ldots+nx^{n-1}$$

Hinc summae potestatum numerorum naturalium ad quemvis terminum inveniri poterunt. Multiplicetur enim haec series per x, ut fiat:

$$\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}=x+2x^2+3x^3+\ldots+nx^n$$

quae denuo differentiata ac per dx divisa dabit:

$$\frac{1+x-(n+1)^2x^n+(2nn+2n-1)x^{n+1}-nnx^{n+2}}{(1-x)^3}$$
$$=1+4x+9x^2+\ldots\ldots+n^2x^{n-1}$$

quae

quae per x multiplicata dabit:

$$\frac{x + x^2 - (n+1)^2 x^{n+1} + (2nn + 2n - 1)x^{n+2} - nnx^{n+3}}{(1-x)^3}$$

$$= x + 4x^2 + 9x^3 + \ldots\ldots + n^2 x^n$$

quae differentiata, per dx divisa ac per x multiplicata producet seriem hanc:

$$x + 8x^2 + 27x^3 + \ldots\ldots + n^3 x^n$$

cuius summa propterea invenietur. Ex hacque simili modo summa biquadratorum altiorumque potestatum indefinita eruetur.

23. Methodus igitur haec ad omnes series quantitatem indeterminatam continentes accommodari potest, quarum quidem summae constant. Cum igitur praeter geometricas series recurrentes omnes eadem praerogativa gaudeant, ut non solum in infinitum, sed etiam ad quemvis terminum summari queant; ex iis quoque hac methodo innumerae aliae series summabiles inveniri poterunt. Quod cum opus foret maxime diffusum, si id persequi vellemus, unicum casum perpendamus.

Sit scilicet proposita haec series:

$$\frac{x}{1 - x - xx} = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \&c.$$

quae differentiata ac per dx divisa dabit:

$$\frac{1 + xx}{(1 - x - xx)^2} = 1 + 2x + 6x^2 + 12x^3 + 25x^4 + 48x^5 + 91x^6 + \&c.$$

Facile autem patet omnes has series hoc modo resultantes fore quoque recurrentes, quarum adeo summae ex ipsarum natura inveniri poterunt.

24. In genere igitur si seriei cuiuspiam in hac forma contentae: $ax + bx^2 + cx^3 + dx^4 + \&c.$ summa fuerit cognita, quam ponamus $= S$, eiusdem seriei, si singuli termini singulatim per terminos progressionis arithmeticae multiplicentur, summa inveniri poterit. Sit enim

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

multiplicetur per x^m, erit

$$Sx^m = ax^{m+1} + bx^{m+2} + cx^{m+3} + dx^{m+4} + \&c.$$

differentietur haec aequatio, & dividatur per dx:

$$mSx^{m-1} + x^m \frac{dS}{dx} = (m+1)ax^m + (m+2)bx^{m+1}$$
$$+ (m+3)cx^{m+2} + \&c.$$

dividatur per x^{m-1}, eritque

$$mS + \frac{xdS}{dx} = (m+1)ax + (m+2)bx^2 + (m+3)cx^3 + \&c.$$

Quodsi ergo huius sequentis seriei summa desideretur:
$\alpha ax + (\alpha+\beta)bx^2 + (\alpha+2\beta)cx^3 + (\alpha+3\beta)dx^4 + \&c.$
multiplicetur superior per β ac statuatur $m\beta + \beta = \alpha$ ut sit $m = \frac{\alpha-\beta}{\beta}$, eritque huius seriei summa $= (\alpha-\beta)S + \frac{\beta xdS}{dx}$

25. Poterit etiam huius seriei propositae summa inveniri, si singuli eius termini multiplicentur per terminos seriei secundi ordinis singulatim, cuius scilicet differentiae demum secundae sint constantes. Quoniam enim iam invenimus.

$$mS + \frac{xdS}{dx} = (m+1)ax + (m+2)bx^2 + (m+3)cx^3 + \&c.$$

multiplicetur per x^n, ut sit

$$mSx^n + \frac{x^{n+1}dS}{dx} = (m+1)ax^{n+1} + (m+2)bx^{n+2} + \&c.$$

differentietur posito dx constante, & per dx dividatur:

$$mnSx^{n-1} + \frac{(m+n+1)x^n dS}{dx} + \frac{x^{n+1}ddS}{dx^2} =$$
$$(m+1)(n+1)ax^n + (m+2)(n+2)bx^{n+1} + \&c.$$

Dividatur per x^{n-1}, ac multiplicetur per k, ut sit

$$mnkS + \frac{(m+n+1)kxdS}{dx} + \frac{kx^2ddS}{dx^2} =$$
$$(m+1)(n+1)kax + (m+2)(n+2)kbx^2 + (m+3)(n+3)kcx^3 + \&c.$$

Comparetur nunc haec series cum ista:

erit

erit:

	Diff. 1.	Diff. 2.
$kmn + km + kn + k = \alpha$	$km + kn + 3k = \beta$	$2k = \gamma$
$kmn + 2km + 2kn + 4k = \alpha + \beta$	$km + kn + 5k = \beta + \gamma$	
$kmn + 3km + 3kn + 9k = \alpha + 2\beta + \gamma$		

Ergo $k = \frac{1}{2}\gamma$; &. $m + n = \frac{2\beta}{\gamma} - 3$; atque

$$mn = \frac{\alpha}{k} - m - n - 1 = \frac{2\alpha}{\gamma} - \frac{2\beta}{\gamma} + 2 = \frac{2(\alpha - \beta + \gamma)}{\gamma}.$$

Hinc summa seriei quaesita erit:

$$(\alpha - \beta + \gamma)S + \frac{(\beta - \gamma)xdS}{dx} + \frac{\gamma x^2 ddS}{2dx^2}.$$

26. Simili modo summa reperiri poterit seriei huius

$$Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \&c.$$

si quidem cognita fuerit summa S seriei huius:

$$S = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \&c.$$

atque A, B, C, D, &c. constituant seriem, quae ad differentias constantes deducitur. Fingatur enim summa, quoniam eius forma ex antecedentibus colligitur, haec

$$\alpha S + \frac{\beta xdS}{dx} + \frac{\gamma x^2 ddS}{2dx^2} + \frac{\delta x^3 d^3S}{6dx^3} + \frac{\varepsilon x^4 d^4S}{24dx^4} + \&c.$$

Nunc ad litteras α, β, γ, δ, &c. inveniendas evolvantur singulae series, eritque:

$$\alpha S = \alpha a + \alpha bx + \alpha cx^2 + \alpha dx^3 + \alpha ex^4 + \&c.$$

$$\frac{\beta xdS}{dx} = \beta bx + 2\beta cx^2 + 3\beta dx^3 + 4\beta ex^4 + \&c.$$

$$\frac{\gamma x^2 ddS}{2dx^2} = \gamma cx^2 + 3\gamma dx^3 + 6\gamma ex^4 + \&c.$$

$$\frac{\delta x^3 d^3S}{6dx^3} = \delta dx^3 + 4\delta ex^4 + \&c.$$

$$\frac{\varepsilon x^4 d^4S}{24dx^4} = \varepsilon ex^4 + \&c.$$

quae simul sumtae comparentur cum proposita:

$$Z = Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \&c.$$

fietque comparatione singulorum terminorum instituta:

$$\alpha = A$$
$$\beta = B - \alpha = B - A$$
$$\gamma = C - 2\beta - \alpha = C - 2B + A$$
$$\delta = D - 3\gamma - 3\beta - \alpha = D - 3C + 3B - A$$
&c.

His igitur valoribus inventis erit summa quaesita:

$$Z = AS + \frac{(B-A)xdS}{1dx} + \frac{(C-2B+A)x^2 ddS}{1.2dx^2} + \frac{(D-3C+3B-A)x^3 d^3S}{1.2.3\,dx^3} + \&c.$$

seu si seriei A, B, C, D, E, &c. differentiae continuae more consueto indicentur, erit

$$Z = AS + \frac{\Delta A.xdS}{1dx} + \frac{\Delta^2 A.x^2 d^2S}{1.2\,dx^2} + \frac{\Delta^3 A.x^3 d^3S}{1.2.3\,dx^3} + \&c.$$

si quidem fuerit uti assumsimus:

$$S = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \&c.$$

Si ergo series A, B, C, D, &c. tandem habeat differentias constantes, summa seriei Z finite exprimi poterit.

27. Quia sumto e pro numero, cuius logarithmus hyperbolicus est $= 1$, est:

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \&c.$$

sumatur haec series pro priori, & cum sit

$$S = e^x \quad \text{erit} \quad \frac{dS}{dx} = e^x; \quad \frac{ddS}{dx^2} = e^x; \quad \&c.$$

Quare huius seriei, quae ex illa & hac A, B, C, D, &c. componitur:

$$A + \frac{Bx}{1} + \frac{Cx^2}{1.2} + \frac{Dx^3}{1.2.3} + \frac{Ex^4}{1.2.3.4} + \&c.$$

sum-

summa hoc modo exprimetur:

$$e^x\left(A+\frac{x\Delta A}{1}+\frac{xx\Delta^2 A}{1.2}+\frac{x^3\Delta^3 A}{1.2.3}+\frac{x^4\Delta^4 A}{1.2.3.4}+\&c.\right)$$

Sic si proponatur haec series:

$$2+\frac{5x}{1}+\frac{10x^2}{1.2}+\frac{17x^3}{1.2.3}+\frac{26x^4}{1.2.3.4}+\frac{37x^5}{1.2.3.4.5}+\&c.$$

Ob seriem A, B, C, D, E, &c.

erit $A = 2,\ 5,\ 10,\ 17,\ 26$ &c.

$\Delta A = 3,\ 5,\ 7,\ 9$ &c.

$\Delta^2 A = 2,\ 2,\ 2$ &c.

erit huius seriei:

$$2+5x+\frac{10x^2}{2}+\frac{17x^3}{6}+\frac{26x^4}{24}+\&c.$$

summa $= e^x(2+3x+xx) = e^x(1+x)(2+x)$: quod quidem sponte patet. Est enim

$$2e^x = 2+\frac{2x}{1}+\frac{2x^2}{2}+\frac{2x^3}{6}+\frac{2x^4}{24}+\&c.$$

$$3xe^x = 3x+\frac{3x^2}{1}+\frac{3x^3}{2}+\frac{3x^4}{6}+\&c.$$

$$xxe^x = xx+\frac{x^3}{1}+\frac{x^4}{2}+\&c.$$

$$e^x(2+3x+xx) = 2+5x+\frac{10xx}{2}+\frac{17x^3}{6}+\frac{26x^4}{24}+\&c.$$

28. Quae hactenus sunt tradita non solum ad series in infinitum excurrentes spectant, sed etiam ad summas quotcunque terminorum: coefficientes enim a, b, c, d, &c. vel in infinitum progredi, vel ubicunque libuerit abrumpi possunt. Verum cum hoc non egeat uberiori explicatione, quae ex hactenus allatis sequuntur, accuratius perpendamus. Proposita ergo quacunque serie, cuius singuli termini duobus constent factoribus, quorum alteri seriem ad differentias constantes de-

ducentem constituant, huius seriei summa poterit assignari; dummodo omissis istis factoribus, series fuerit summabilis.

Scilicet si proposita sit ista series

$$Z = Aa + Bbx + Ccxx + Ddx^3 + Eex^4 + \&c.$$

in qua quantitates A, B, C, D, E, &c. eiusmodi seriem constituant, quae tandem ad differentias constantes perducatur; tum istius seriei summa exhiberi poterit, dummodo habeatur summa S huius seriei

$$S = a + bx + cx^2 + dx^3 + ex^4 + \&c.$$

Sumtis enim ex progressione A, B, C, D, E, &c. differentiis continuis, uti initio huius libri ostendimus:

$$\begin{array}{c} A,\quad B,\quad C,\quad D,\quad E,\quad F,\quad \&c. \\ \Delta A,\quad \Delta B,\quad \Delta C,\quad \Delta D,\quad \Delta E,\quad \&c. \\ \Delta^2 A,\ \Delta^2 B,\ \Delta^2 C,\ \Delta^2 D,\ \&c. \\ \Delta^3 A,\ \Delta^3 B,\ \Delta^3 C,\ \&c. \\ \Delta^4 A,\ \Delta^4 B,\ \&c. \\ \Delta^5 A,\ \&c. \\ \&c. \end{array}$$

erit seriei propositae summa

$$Z = SA + \frac{xdS}{1dx}\Delta A + \frac{x^2 ddS}{1.2dx^2}\Delta^2 A + \frac{x^3 d^3 S}{1.2.3dx^3}\Delta^3 A + \&c.$$

posito in altioribus ipsius S differentialibus dx constante.

29. Si igitur series A, B, C, D, &c. nunquam ad differentias constantes deducat, summa seriei Z per novam seriem infinitam exprimetur, quae interdum magis converget quam proposita; sique ista series in aliam sibi aequalem transformabitur. Sit ad hoc declarandum proposita haec series:

$$Y = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \frac{y^5}{5} + \frac{y^6}{6} + \&c.$$

quam constat exprimere $l\frac{1}{1-y}$, ita ut sit $Y = -l(1-y)$. Dividatur haec series per y, & statuatur $y = x$, & $Y = yZ$, ut sit $Z = -\frac{1}{y}l(1-y) = -\frac{1}{x}l(1-x)$, erit

$$Z = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \&c.$$

quae comparata cum ista:

$$S = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \&c. = \frac{1}{1-x},$$

dabit pro serie A, B, C, D, E, &c. hos valores:

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}$$

$$-\frac{1}{1.2}, -\frac{1}{2.3}, -\frac{1}{3.4}, -\frac{1}{4.5}$$

$$\frac{1.2}{1.2.3}, \frac{1.2}{2.3.4}, \frac{1.2}{3.4.5}$$

$$-\frac{1.2.3}{1.2.3.4}, -\frac{1.2.3}{2.3.4.5}$$

&c.

Erit ergo $A = 1$; $\Delta A = -\frac{1}{2}$, $\Delta^2 A = \frac{1}{3}$, $\Delta^3 A = -\frac{1}{4}$ &c.

Porro cum sit $S = \frac{1}{1-x}$, erit

$$\frac{dS}{dx} = \frac{1}{(1-x)^2}$$

$$\frac{ddS}{1.2\,dx^2} = \frac{1}{(1-x)^3}$$

$$\frac{d^3S}{1.2.3\,dx^3} = \frac{1}{(1-x)^4}$$

&c.

Quibus valoribus substitutis orietur summa: $Z =$

$$\frac{1}{1-x} - \frac{x}{2(1-x)^2} + \frac{x^2}{3(1-x)^3} - \frac{x^3}{4(1-x)^4} + \frac{x^4}{5(1-x)^5} - \&c.$$

Cum ergo sit $x = y$, & $Y = -l(1-y) = yZ$; erit

$$-l(1-y) = \frac{y}{1-y} - \frac{y^2}{2(1-y)^2} + \frac{y^3}{3(1-y)^3} - \frac{y^4}{4(1-y)^4} + \&c.$$

quae series utique exprimit

$$l\left(1+\frac{y}{1-y}\right)=l\frac{1}{1-y}=-l(1-y),$$

cuius adeo veritas per ante demonstrata constat.

30. Proposita nunc sit ista series, ut etiam usus pateat, si potestates tantum impares occurrant, & signa alternentur.

$$Y=y-\frac{y^3}{3}+\frac{y^5}{5}-\frac{y^7}{7}+\frac{y^9}{9}-\frac{y^{11}}{11}+\&c.$$

ex qua constat esse $Y=A\ \text{tang}\ y$.

Dividatur haec series per y, & ponatur $\frac{Y}{y}=Z$, & $yy=x$; erit:

$$Z=1-\frac{x}{3}+\frac{xx}{5}-\frac{x^3}{7}+\frac{x^4}{9}-\frac{x^5}{11}+\&c.$$

quae si comparetur cum ista:

$S=1-x+xx-x^3+x^4-\&c.$ fiet $S=\frac{1}{1+x}$, & series coefficientium A, B, C, D, &c. fiet:

$$A=1,\ \frac{1}{3},\ \frac{1}{5},\ \frac{1}{7},\ \frac{1}{9}\ \&c.$$

$$\Delta A=-\frac{2}{3};\ -\frac{2}{3.5};\ -\frac{2}{5.7};\ -\frac{2}{7.9}$$

$$\Delta^2 A=\frac{2.4}{3.5};\ \frac{2.4}{3.5.7};\ \frac{2.4}{5.7.9}$$

$$\Delta^3 A=-\frac{2.4.6}{3.5.7};\ -\frac{2.4.6}{3.5.7.9}$$

$$\Delta^4 A=\frac{2.4.6.8}{3.5.7.9}$$

&c.

At

At cum sit $S = \frac{1}{1+x}$; erit

$$\frac{dS}{1dx} = \frac{-1}{(1+x)^2};\ \frac{ddS}{1.2dx^2} = \frac{1}{(1+x)^3};\ \frac{d^3S}{1.2.3dx^3} = \frac{-1}{(1+x)^4}$$
&c.

Quare substitutis his valoribus fiet forma $Z =$

$$\frac{1}{1+x} + \frac{2x}{3(1+x)^2} + \frac{2.4x^2}{3.5(1+x)^3} + \frac{2.4.6x^3}{3.5.7(1+x)^4} + \&c.$$

Restituto ergo $x = yy$; & per y multiplicato fiet.

$Y = A$ tang $y =$

$$\frac{y}{1+yy} + \frac{2y^3}{3(1+yy)^2} + \frac{2.4.y^5}{3.5(1+yy)^3} + \frac{2.4.6.y^7}{3.5.7(1+yy)^4} + \&c.$$

31. Potest quoque superior series, qua arcus circuli per tangentem exprimitur, alio modo transmutari, eam comparando cum serie logarithmica.

Consideremus nempe seriem

$$Z = 1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{x^5}{11} + \&c.$$

quam comparemus cum hac;

$$S = \frac{1}{0} - \frac{x}{2} + \frac{xx}{4} - \frac{x^3}{6} + \frac{x^4}{8} - \&c. = \frac{1}{0} - \frac{1}{2}\, l(1+x),$$

atque valores litterarum A, B, C, D, &c. erunt

$$A = \frac{0}{1};\ \frac{2}{3};\ \frac{4}{5};\ \frac{6}{7};\ \frac{8}{9};\ \&c.$$

$$\Delta A = \frac{2}{3};\ \frac{+2}{3.5};\ \frac{+2}{5.7};\ \frac{+2}{7.9};\ \&c.$$

$$\Delta^2 A = \frac{-2.4}{3.5};\ \frac{-2.4}{3.5.7};\ \frac{-2.4}{5.7.9};\ \&c.$$

$$\Delta^3 A = \frac{2.4.6}{3.5.7};\ \&c.$$

Dein-

Deinde cum sit $S = \frac{1}{0} - \frac{1}{2} l(1+x)$; erit

$$\frac{dS}{1.dx} = -\frac{1}{2(1+x)}$$

$$\frac{ddS}{1.2dx^2} = +\frac{1}{4(1+x)^2}$$

$$\frac{d^3S}{1.2.3dx^3} = -\frac{1}{6(1+x)^3}$$

$$\frac{d^4S}{1.2.3.4dx^4} = \frac{1}{8(1+x)^4}$$

&c.

erit igitur $SA = S.\frac{0}{1} = 1$: & ex reliquis fiet:

$$Z = 1 - \frac{x}{3(1+x)} - \frac{2xx}{3.5(1+x)^2} - \frac{2.4.x^3}{3.5.7(1+x)^3} - \&c.$$

Ponatur nunc $x = yy$, & multiplicetur per y fiet:

$$Y = A \text{ tang } y =$$

$$y - \frac{y^3}{3(1+yy)} - \frac{2y^5}{3.5(1+yy)^2} - \frac{2.4y^7}{3.5.7(1+yy)^3} - \&c.$$

Haec ergo transmutatio non impediebatur termino infinito $\frac{1}{0}$ qui in seriem S ingrediebatur. Sin autem cui supersit dubium, is tantum singulos terminos praeter primum secundum potestates ipsius y in series resolvat, atque deprehendet actu seriem primum propositam resultare.

32. Hactenus eiusmodi tantum series sumus contemplati, in quibus omnes potestates variabilis occurrunt. Nunc igitur ad alias series progrediamur, quae in singulis terminis eandem potestatem ipsius variabilis complectantur, cuiusmodi est haec:

$$S = \frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} + \frac{1}{d+x} + \&c.$$

Hu-

Huius enim feriei fi fumma S fuerit cognita, ac per functionem quampiam ipfius x exprimatur, erit differentiando ac per $-dx$ dividendo:

$$\frac{-dS}{dx} = \frac{1}{(a+x)^2} + \frac{1}{(b+x)^2} + \frac{1}{(c+x)^2} + \frac{1}{(d+x)^2} + \&c.$$

Si haec ulterius differentietur, atque per $-2dx$ dividatur, cognofcetur feries cuborum:

$$\frac{ddS}{2dx^2} = \frac{1}{(a+x)^3} + \frac{1}{(b+x)^3} + \frac{1}{(c+x)^3} + \frac{1}{(d+x)^3} + \&c.$$

Haecque denuo differentiata, atque per $-3dx$ divifa dabit:

$$\frac{-d^3S}{6dx^3} = \frac{1}{(a+x)^4} + \frac{1}{(b+x)^4} + \frac{1}{(c+x)^4} + \frac{1}{(d+x)^4} + \&c.$$

Similique modo omnium fequentium poteftatum fummae reperientur, dummodo fumma feriei primae fuerit cognita.

33. Huiufmodi autem feries fractionum quantitatem indeterminatam involventes fupra in introductione elicuimus, ubi oftendimus, fi circuli, cuius radius $=1$, femiperipheria ftatuatur $=\pi$, fore

$$\frac{\pi}{n \sin \frac{m}{n}\pi} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \&c.$$

$$\frac{\pi \cos \frac{m}{n}\pi}{n \sin \frac{m}{n}\pi} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \&c.$$

Cum igitur pro m & n numeros quofcunque affumere liceat, ftatuamus $n=1$, & $m=x$; ut obtineamus feries illi quam in §. praec. propofueramus fimiles; hoc facto erit:

$$\frac{\pi}{\sin \pi x} = \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \&c.$$

$$\frac{\pi \cos \pi x}{\sin \pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \&c.$$

Per differentiationes ergo summae quarumvis potestatum ex his fractionibus oriundarum exhiberi poterunt.

34. Consideremus seriem priorem, sitque brevitatis gratia $\frac{\pi}{\sin \pi x} = S$, cuius differentialia altiora capiantur posito dx constante: eritque

$$S = \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \&c.$$

$$\frac{-dS}{dx} = \frac{1}{xx} - \frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(2+x)^2} - \frac{1}{(3-x)^2} - \&c.$$

$$\frac{ddS}{2dx^2} = \frac{1}{x^3} + \frac{1}{(1-x)^3} - \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(2+x)^3} + \frac{1}{(3-x)^3} - \&c.$$

$$\frac{-d^3S}{6dx^3} = \frac{1}{x^4} - \frac{1}{(1-x)^4} - \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(2+x)^4} - \frac{1}{(3-x)^4} - \&c.$$

$$\frac{d^4S}{24dx^4} = \frac{1}{x^5} + \frac{1}{(1-x)^5} - \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(2+x)^5} + \frac{1}{(3-x)^5} - \&c.$$

$$\frac{-d^5S}{120dx^5} = \frac{1}{x^6} - \frac{1}{(1-x)^6} - \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(2+x)^6} - \frac{1}{(3-x)^6} - \&c.$$

&c.

ubi notandum est in potestatibus paribus signa eandem sequi legem, pariterque in imparibus eandem legem signorum observari. Omnium ergo istarum serierum summae invenientur ex differentialibus expressionis $S = \frac{\pi}{\sin \pi x}$.

35. Ad Differentialia haec simplicius exprimenda ponamus

$\sin \pi x = p$ & $\cos \pi x = q$, erit

$dp = \pi dx \cos \pi x = \pi q dx$, & $dq = -\pi p dx$. Cum ergo sit

$S = \frac{\pi}{p}$ erit

$-dS$

$$\frac{-dS}{dx} = \frac{\pi^2 q}{pp}$$

$$\frac{ddS}{dx^2} = \frac{\pi^3(pp+2qq)}{p^3} = \frac{\pi^3(qq+1)}{p^3} \text{ ob } pp+qq=1$$

$$\frac{-d^3S}{dx^3} = \pi^4\left(\frac{5q}{pp}+\frac{6q^3}{p^4}\right) = \frac{\pi^4(q^3+5q)}{p^4}$$

$$\frac{d^4S}{dx^4} = \pi^5\left(\frac{24q^4}{p^5}+\frac{28q^2}{p^3}+\frac{5}{p}\right) = \frac{\pi^5(q^4+18q^2+5)}{p^5}$$

$$\frac{-d^5S}{dx^5} = \pi^6\left(\frac{120q^5}{p^6}+\frac{180q^3}{p^4}+\frac{61q}{pp}\right) = \frac{\pi^6(q^5+58q^3+61q)}{p^6}$$

$$\frac{d^6S}{dx^6} = \pi^7\left(\frac{720q^6}{p^7}+\frac{1320q^4}{p^5}+\frac{662q^2}{p^3}+\frac{61}{p}\right)$$

$$\text{vel} = \frac{\pi^7(q^6+179q^4+479q^2+61)}{p^7}$$

$$\frac{-d^7S}{dx^7} = \pi^8\left(\frac{5040q^7}{p^8}+\frac{10920q^5}{p^6}+\frac{7266q^3}{p^4}+\frac{1385q}{p^2}\right)$$

$$\text{vel} = \frac{\pi^8}{p^8}(q^7+543q^5+3111q^3+1385q)$$

$$\frac{d^8S}{dx^8} = \pi^9\left(\frac{40320q^8}{p^9}+\frac{1008000q^6}{p^7}+\frac{83664q^4}{p^5}+\frac{24568q^2}{p^3}+\frac{1385}{p}\right)$$

Quae expreſſiones facile ulterius quouſque libuerit continuari poſſunt, ſi enim fuerit:

$$\pm\frac{d^nS}{dx^n} = \pi^{n+1}\left(\frac{\alpha q^n}{p^{n+1}}+\frac{\beta q^{n-2}}{p^{n-1}}+\frac{\gamma q^{n-4}}{p^{n-3}}+\frac{\delta q^{n-6}}{p^{n-5}}+\&c.\right)$$

erit differentiale ſequens ſignis mutatis:

$$\mp\frac{d^{n+1}S}{dx^{n+1}} = \pi^{n+2}\left(\frac{(n+1)\alpha q^{n+1}}{p^{n+2}} + \left.\begin{matrix} n\alpha \\ +(n-1)\beta \end{matrix}\right\}\frac{q^{n-1}}{p^n} + \left.\begin{matrix}(n-2)\beta \\ +(n-3)\gamma\end{matrix}\right\}\frac{q^{n-3}}{p^{n-2}} + \left.\begin{matrix}(n-4)\gamma \\ +(n-5)\delta\end{matrix}\right\}\frac{q^{n-5}}{p^{n-4}}\ \&c.\right)$$

36. Ex his ergo obtinebuntur ſummae ſerierum ſuperiorum §. 34. exhibitarum ſequentes:

S

$$S = \pi \cdot \frac{1}{p}$$

$$\frac{-dS}{dx} = \frac{\pi^2}{1} \cdot \frac{q}{p^2}$$

$$\frac{ddS}{2dx^2} = \frac{\pi^3}{2}\left(\frac{2q^2}{p^3} + \frac{1}{p}\right)$$

$$\frac{-d^3S}{6dx^3} = \frac{\pi^4}{6}\left(\frac{6q^3}{p^4} + \frac{5q}{p^2}\right)$$

$$\frac{d^4S}{24dx^4} = \frac{\pi^5}{24}\left(\frac{24q^4}{p^5} + \frac{28q^2}{p^3} + \frac{5}{p}\right)$$

$$\frac{-d^5S}{120dx^5} = \frac{\pi^6}{120}\left(\frac{120q^5}{p^6} + \frac{180q^3}{p^4} + \frac{61q}{p^2}\right)$$

$$\frac{d^6S}{720dx^6} = \frac{\pi^7}{720}\left(\frac{720q^6}{p^7} + \frac{1320q^4}{p^5} + \frac{662q^2}{p^3} + \frac{61}{p}\right)$$

$$\frac{-d^7S}{5040dx^7} = \frac{\pi^8}{5040}\left(\frac{5040q^7}{p^8} + \frac{10920q^5}{p^6} + \frac{7266q^3}{p^4} + \frac{1385q}{p^2}\right)$$

$$\frac{d^8S}{40320dx^8} = \frac{\pi^9}{40320}\left(\frac{40320q^8}{p^9} + \frac{100800q^6}{p^7} + \frac{83664q^4}{p^5} + \frac{24568q^2}{p^3} + \frac{1385}{p}\right)$$

&c.

37. Tractemus simili modo alteram seriem supra inventam:

$$\frac{\pi \cos \pi x}{\sin \pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \&c.$$

atque posito brevitatis ergo $\frac{\pi \cos \pi x}{\sin \pi x} = T$, orientur sequentes summationes:

$$T = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \&c.$$

$$\frac{-dT}{dx} = \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(2+x)^2} + \&c.$$

ddT

$$\frac{ddT}{2dx^2} = \frac{1}{x^3} - \frac{1}{(1-x)^3} + \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(2+x)^3} - \&c.$$

$$\frac{-d^3T}{6dx^3} = \frac{1}{x^4} + \frac{1}{(1-x)^4} + \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(2+x)^4} + \&c.$$

$$\frac{d^4T}{24dx^4} = \frac{1}{x^5} - \frac{1}{(1-x)^5} + \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(2+x)^5} - \&c.$$

$$\frac{-d^5T}{120dx^5} = \frac{1}{x^6} + \frac{1}{(1-x)^6} + \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(2+x)^6} + \&c.$$

&c.

ubi in potestatibus paribus omnes termini sunt affirmativi, in imparibus autem signa + & − alternatim se excipiunt.

38. Quo differentialium horum valores innotescant, ponamus ut ante $\sin \pi x = p$ & $\cos \pi x = q$, ut fit $pp + qq = 1$; erit $dp = \pi q dx$ & $dq = -\pi p dx$. Quibus valoribus adhibitis erit:

$$T = \pi . \frac{q}{p}$$

$$\frac{-dT}{dx} = \pi^2 \left(\frac{qq}{pp} + 1\right) = \frac{\pi^2}{pp}$$

$$\frac{ddT}{dx^2} = \pi^3 \left(\frac{2q^3}{p^3} + \frac{2q}{p}\right) = \frac{2\pi^3 q}{p^3}$$

$$\frac{-d^3T}{dx^3} = \pi^4 \left(\frac{6q^4}{p^4} + \frac{8qq}{pp} + 2\right) = \pi^4 \left(\frac{6qq}{p^4} + \frac{2}{pp}\right)$$

$$\frac{d^4T}{dx^4} = \pi^5 \left(\frac{24q^3}{p^5} + \frac{16q}{p^3}\right)$$

$$\frac{-d^5T}{dx^5} = \pi^6 \left(\frac{120q^4}{p^6} + \frac{120qq}{p^4} + \frac{16}{pp}\right)$$

$$\frac{d^6T}{dx^6} = \pi^7 \left(\frac{720q^5}{p^7} + \frac{960q^3}{p^5} + \frac{272q}{p^3}\right)$$

$$\frac{-d^7T}{dx^7} = \pi^8\left(\frac{5040q^6}{p^8} + \frac{8400q^4}{p^6} + \frac{3696q^2}{p^4} + \frac{272}{p^2}\right)$$

$$\frac{d^8T}{dx^8} = \pi^9\left(\frac{40320q^7}{p^9} + \frac{80640q^5}{p^7} + \frac{48384q^3}{p^5} + \frac{7936q}{p^3}\right) \quad \&c.$$

Quae formulae facile ulterius quousque libuerit continuari possunt. Si enim sit

$$\pm\frac{d^nT}{dx^n} = \pi^{n+1}\left(\frac{\alpha q^{n-1}}{p^{n+1}} + \frac{\beta q^{n-3}}{p^{n-1}} + \frac{\gamma q^{n-5}}{p^{n-3}} + \frac{\delta q^{n-7}}{p^{n-5}} + \&c.\right)$$

erit expressio sequens

$$\mp\frac{d^{n+1}T}{dx^{n+1}} = \pi^{n+2}\left(\frac{(n+1)\alpha q^n}{p^{n+2}} + \frac{(n-1)(\alpha+\beta)q^{n-2}}{p^n} + \frac{(n-3)(\beta+\gamma)q^{n-4}}{p^{n-2}} + \&c.\right)$$

39. Series ergo potestatum §. 37. datae sequentes habebunt summmas posito $\sin\pi x = p$ & $\cos\pi x = q$.

$$T = \pi.\frac{q}{p}$$

$$\frac{-dT}{dx} = \pi^2\frac{1}{pp}$$

$$\frac{ddT}{2dx^2} = \pi^3\frac{q}{p^3}$$

$$\frac{-d^3T}{6dx^3} = \pi^4\left(\frac{qq}{p^4} + \frac{1}{3pp}\right)$$

$$\frac{d^4T}{24dx^4} = \pi^5\left(\frac{q^3}{p^5} + \frac{2q}{3p^3}\right)$$

$$\frac{-d^5T}{120dx^5} = \pi^6\left(\frac{q^4}{p^6} + \frac{3qq}{3p^4} + \frac{2}{15pp}\right)$$

$$\frac{d^6T}{720dx^6} = \pi^7\left(\frac{q^5}{p^7} + \frac{4q^3}{3p^5} + \frac{17q}{45p^3}\right)$$

$$\frac{-d^7T}{5040dx^7} = \pi^8\left(\frac{q^6}{p^8} + \frac{5q^4}{3p^6} + \frac{11q^2}{15p^4} + \frac{17}{315pp}\right)$$

$$\frac{d^8T}{40320dx^8} = \pi^9\left(\frac{q^7}{p^9} + \frac{6q^5}{3p^7} + \frac{6q^3}{5p^5} + \frac{62q}{315p^3}\right) \quad \&c.$$

40. Praeter has ſeries invenimus in introductione nonnullas alias, ex quibus ſimili modo per differentiationes novae elici poſſunt. Oſtendimus enim eſſe: $\frac{1}{2x} - \frac{\pi\sqrt{x}}{2x \operatorname{tang} \pi\sqrt{x}} =$

$$\frac{1}{1-x} + \frac{1}{4-x} + \frac{1}{9-x} + \frac{1}{16-x} + \frac{1}{25-x} + \&c.$$

Ponamus ſummam huius ſeriei eſſe $= S$,

ut ſit $S = \frac{1}{2x} - \frac{\pi}{2\sqrt{x}} \cdot \frac{\operatorname{cof} \pi\sqrt{x}}{\operatorname{ſin} \pi\sqrt{x}}$, erit

$$\frac{dS}{dx} = -\frac{1}{2xx} + \frac{\pi}{4x\sqrt{x}} \cdot \frac{\operatorname{cof} \pi\sqrt{x}}{\operatorname{ſin} \pi\sqrt{x}} + \frac{\pi\pi}{4x(\operatorname{ſin} \pi\sqrt{x})^2},$$

quae ergo expreſſio praebet ſummam huius ſeriei:

$$\frac{1}{(1-x)^2} + \frac{1}{(4-x)^2} + \frac{1}{(9-x)^2} + \frac{1}{(16-x)^2} + \frac{1}{(25-x)^2} + \&c.$$

Deinde quoque oſtendimus eſſe:

$$\frac{\pi}{2\sqrt{x}} \cdot \frac{e^{2\pi\sqrt{x}} + 1}{e^{2\pi\sqrt{x}} - 1} - \frac{1}{2x} =$$

$$\frac{1}{1+x} + \frac{1}{4+x} + \frac{1}{9+x} + \frac{1}{16+x} + \&c.$$

Quodſi ergo haec ſumma ponatur $= S$, erit:

$$\frac{-dS}{dx} = \frac{1}{(1+x)^2} + \frac{1}{(4+x)^2} + \frac{1}{(9+x)^2} + \frac{1}{(16+x)^2} + \&c.$$

At eſt

$$\frac{dS}{dx} = \frac{-\pi}{4x\sqrt{x}} \cdot \frac{e^{2\pi\sqrt{x}} + 1}{e^{2\pi\sqrt{x}} - 1} - \frac{\pi\pi}{x} \cdot \frac{e^{2\pi\sqrt{x}}}{(e^{2\pi\sqrt{x}} - 1)^2} + \frac{1}{2xx}$$

Ergo ſumma huius ſeriei erit:

$$\frac{-dS}{dx} = \frac{\pi}{4x\sqrt{x}} \cdot \frac{e^{2\pi\sqrt{x}} + 1}{e^{2\pi\sqrt{x}} - 1} + \frac{\pi\pi}{x} \cdot \frac{e^{2\pi\sqrt{x}}}{(e^{2\pi\sqrt{x}} - 1)^2} - \frac{1}{2xx}.$$

Similique modo ulterioribus differentiationibus ſummae ſequentium poteſtatum invenientur.

41. Si cognitus fuerit valor producti cuiuspiam ex factoribus indeterminatam litteram involventibus compositi, ex eo per eandem methodum innumerabiles series summabiles inveniri poterunt. Sit enim huius producti

$$(1+\alpha x)(1+\beta x)(1+\gamma x)(1+\delta x)(1+\varepsilon x)\ \&c.$$

valor $= S$, functioni scilicet cuipiam ipsius x, erit logarithmis sumendis:

$$lS = l(1+\alpha x) + l(1+\beta x) + l(1+\gamma x) + l(1+\delta x) + \&c.$$

Sumantur iam differentialia, erit divisione per dx instituta:

$$\frac{dS}{Sdx} = \frac{\alpha}{1+\alpha x} + \frac{\beta}{1+\beta x} + \frac{\gamma}{1+\gamma x} + \frac{\delta}{1+\delta x} + \&c.$$

ex cuius ulteriori differentiatione summae potestatum quarumvis istarum fractionum reperietur; plane uti in exemplis praecedentibus fusius exposuimus.

42. Exhibuimus autem in introductione nonnullas istiusmodi expressiones, ad quas hanc methodum accommodemus. Scilicet si sit π arcus $180°$ circuli, cuius radius $= 1$, ostendimus esse:

$$\sin\frac{m\pi}{2n} = \frac{m\pi}{2n}\cdot\frac{4nn-mm}{4nn}\cdot\frac{16nn-mm}{16nn}\cdot\frac{36nn-mm}{36nn}\ \&c.$$

$$\cos\frac{m\pi}{2n} = \frac{nn-mm}{nn}\cdot\frac{9nn-mm}{9nn}\cdot\frac{25nn-mm}{25nn}\cdot\frac{49nn-mm}{49nn}\ \&c.$$

Ponamus $n = 1$ & $m = 2x$; ut sit

$$\sin\pi x = \pi x\cdot\frac{1-xx}{1}\cdot\frac{4-xx}{4}\cdot\frac{9-xx}{9}\cdot\frac{16-xx}{16}\cdot\&c. \text{ vel}$$

$$\sin\pi x = \pi x\cdot\frac{1-x}{1}\cdot\frac{1+x}{1}\cdot\frac{2-x}{2}\cdot\frac{2+x}{2}\cdot\frac{3-x}{3}\cdot\frac{3+x}{3}\cdot\frac{4-x}{4}\cdot\&c.\ \&$$

$$\cos\pi x = \frac{1-4xx}{1}\cdot\frac{9-4xx}{9}\cdot\frac{25-4xx}{25}\cdot\frac{49-4xx}{49}\cdot\&c. \text{ seu}$$

$$\cos\pi x = \frac{1-2x}{1}\cdot\frac{1+2x}{1}\cdot\frac{3-2x}{3}\cdot\frac{3+2x}{3}\cdot\frac{5-2x}{5}\cdot\frac{5+2x}{5}\cdot\&c.$$

Ex

Ex his ergo expressionibus, si logarithmi sumantur, erit:

$$l\sin\pi x = l\pi x + l\frac{1-x}{1} + l\frac{1+x}{1} + l\frac{2-x}{2} + l\frac{2+x}{2} + l\frac{3-x}{3} + \&c.$$

$$l\cos\pi x = l\frac{1-2x}{1} + l\frac{1+2x}{1} + l\frac{3-2x}{3} + l\frac{3+2x}{3} + l\frac{5-2x}{5} + \&c.$$

43. Sumamus nunc harum serierum logarithmicarum differentialia, & divisione ubique per dx facta prior series dabit:

$$\frac{\pi\cos\pi x}{\sin\pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \&c.$$

quae est ea ipsa series, quam §. 37. tractavimus. Altera vero series dabit:

$$\frac{-\pi\sin\pi x}{\cos\pi x} = \frac{-2}{1-2x} + \frac{2}{1+2x} - \frac{2}{3-2x} + \frac{2}{3+2x} - \frac{2}{5-2x} + \&c.$$

Ponamus $2x = z$, ut sit $x = \frac{z}{2}$, & dividamus per -2 erit:

$$\frac{\pi\sin\frac{1}{2}\pi z}{2\cos\frac{1}{2}\pi z} = \frac{1}{1-z} - \frac{1}{1+z} + \frac{1}{3-z} - \frac{1}{3+z} + \frac{1}{5-z} - \&c.$$

Cum autem sit

$$\sin\tfrac{1}{2}\pi z = \sqrt{\frac{1-\cos\pi z}{2}} \quad \& \quad \cos\tfrac{1}{2}\pi z = \sqrt{\frac{1+\cos\pi z}{2}}$$

erit:

$$\frac{\pi\sqrt{(1-\cos\pi z)}}{\sqrt{(1+\cos\pi z)}} = \frac{2}{1-z} - \frac{2}{1+z} + \frac{2}{3-z} - \frac{2}{3+z} + \frac{2}{5-z} - \&c.$$

seu loco z scribendo x erit:

$$\frac{\pi\sqrt{(1-\cos\pi x)}}{\sqrt{(1+\cos\pi x)}} = \frac{2}{1-x} - \frac{2}{1+x} + \frac{2}{3-x} - \frac{2}{3+x} + \frac{2}{5-x} - \&c.$$

Addatur haec series ad primum inventam:

$$\frac{\pi\cos\pi x}{\sin\pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \&c.$$

Atque reperietur huius seriei:

$$\frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \frac{1}{3+x} - \&c.$$

Summa $= \frac{\pi\sqrt{(1-\cos\pi x)}}{\sqrt{(1+\cos\pi x)}} + \frac{\pi\cos\pi x}{\sin\pi x}$. At fractio haec $\frac{\sqrt{(1-\cos\pi x)}}{\sqrt{(1+\cos\pi x)}}$, si numerator & denominator multiplicetur per $\sqrt{(1-\cos\pi x)}$ abit in $\frac{1-\cos\pi x}{\sin\pi x}$. Quocirca summa seriei erit $= \frac{\pi}{\sin\pi x}$, quae est ea ipsa, quam §. 34. habuimus: unde eam ulterius non persequemur.

CA-