CAPUT XVIII.

DE USU CALCULI DIFFERENTIALIS IN RESOLUTIONE FRACTIONUM.

403.

Methodus fractionem quamvis propositam in fractiones simplices resolvendi, quam in Introductione exposuimus, etsi per se satis est facilis; tamen ope calculi differentialis ita perfici potest, ut saepenumero multo minori negotio in usum vocari possit. Praecipue vero si denominator fractionis resolvendae fuerit indefiniti gradus, methodus ante exposita plerumque non mediocriter impeditur, dum loco quantitatis incognitae substitutio valoris, quem ex quopiam factore induit, fieri debet. Imprimis autem his casibus divisio denominatoris per factorem iam inventum nimis fit molesta. Quae operatio, si calculus differentialis in subsidium vocetur, evitari poterit, ita ut non opus sit alterum denominatoris factorem, qui oritur, si denominator per factorem iam cognitum dividatur, nosse. Hunc autem usum praestat methodus determinandi valorem fractionis, cuius numerator ac denominator certo casu ambo evanescunt, cuius beneficio, quemadmodum resolutio fractionum iam supra tradita commodior & tractabilior reddi queat, hoc Capite doceamus, simulque finem huic libro, in quo usum calculi differentialis in Analysi exposuimus, imponamus.

404. Si igitur proposita fuerit fractio quaecunque $\frac{P}{Q}$, cuius numerator ac denominator sint functiones variabilis quantitatis x, rationales & integrae; primum videndum est, utrum x in numeratore P tot pluresve dimensiones habeat, quam in deno-

iusmodi $ff - 2fgx\cos\varphi + ggxx$, ex eo orietur fractio simplex talis formae $\dfrac{\mathfrak{A} + \mathfrak{a}x}{ff - 2fgx\cos\varphi + ggxx}$; &, si duo huiusmodi factores fuerint aequales uti $(ff - 2fgx\cos\varphi + ggxx)^2$, hinc prodibunt duae fractiones

$$\frac{\mathfrak{A} + \mathfrak{a}x}{(ff - 2fgx\cos\varphi + ggxx)^2} + \frac{\mathfrak{B} + \mathfrak{b}x}{ff - 2fgx\cos\varphi + ggxx}.$$

Huiusmodi autem factor cubicus $(ff - 2fgx\cos\varphi + ggxx)^3$ dabit tres fractiones simplices, biquadratus quatuor, & ita porro.

406. Resolutio ergo fractionis cuiuscunque $\dfrac{P}{Q}$ ita instituatur. Quaerantur primo omnes factores tam simplices seu binomiales, quam trinomiales denominatoris Q, & si qui fuerint inter se aequales, ii probe notentur, & instar unius habeantur. Tum ex singulis his denominatoris factoribus eliciantur fractiones simplices, vel modo iam supra ostenso, vel eo, quem hic sumus tradituri, & qui pro lubitu in locum prioris substitui poterit. Quo facto aggregatum omnium istarum fractionum simplicium una cum parte integra, si quam continet fractio proposita $\dfrac{P}{Q}$, huius valorem exhaurient. Inventionem quidem factorum denominatoris Q hic tanquam cognitam assumimus, cum pendeat a resolutione aequationis $Q = 0$; methodumque hic trademus per calculum differentialem pro dato quovis denominatoris factore fractionem simplicem inde ortam definiendi. Quod, cum istarum fractionum simplicium denominatores iam habeantur, praestabitur, si numeratorem cuiusque fractionis investigare doceamus.

407. Ponamus ergo fractionis $\dfrac{P}{Q}$ denominatorem Q factorem habere $f + gx$, ita ut sit $Q = (f + gx)S$ neque vero hic alter factor S insuper eundem factorem $f + gx$ contineat.

Sit

simplex $\frac{\mathfrak{A}}{f+gx}$, existente $\mathfrak{A} = \frac{gPdx}{dQ}$, postquam hic ubique loco x valor $\frac{-f}{g}$ ex aequatione $f+gx=0$ oriundus fuerit substitutus. Hoc ergo modo non necesse est, ut ante quaeratur alter denominatoris Q factor S, qui oritur, si Q per $f+gx$ dividatur. Hinc, si Q non in factoribus exprimatur, hanc divisionem saepe non parum molestam, praecipue si x in denominatore Q habeat exponentes indefinitos omittere poterimus, cum valor ipsius $\mathfrak{A}$ ex formula $\frac{gPdx}{dQ}$ obtineatur. Sin autem denominator Q iam in factoribus fuerit expressus, ita ut inde valor ipsius S sponte pateat, tum praeferenda erit altera expressio, qua invenimus $\mathfrak{A} = \frac{P}{S}$, ponendo pariter ubique $x = \frac{-f}{g}$. Sicque pro inveniendo valore ipsius $\mathfrak{A}$ quovis casu ea formula adhiberi poterit, quae commodior & expeditior videatur. Usum autem novae formulae aliquot exemplis illustrabimus.

EXEMPLUM I.

Sit proposita ista fractio $\frac{x^9}{1+x^{17}}$, *cuius fractionem simplicem ex denominatoris factore* $1+x$ *oriundam definiri oporteat.*

Quoniam hic est $Q = 1 + x^{17}$, cuius etsi factor $1 + x$ constat, tamen si, uti prima methodus postulat, per eum dividere velimus, prodiret

$$S = 1 - x + xx - x^3 + \ldots\ldots + x^{16}.$$

Commodius igitur utemur nova formula $\mathfrak{A} = \frac{gPdx}{dQ}$; quia itaque est $f=1$, $g=1$, & $P=x^9$, ob $dQ = 17x^{16}dx$ fiet

fiet $\mathfrak{A} = \frac{x^9}{17 x^{16}} = \frac{1}{17 x^7}$, posito $x = -1$, unde fit $\mathfrak{A} = -\frac{1}{17}$, & fractio simplex ex denominatoris factore $1 + x$ oriunda erit $\frac{-1}{17(1+x)}$.

EXEMPLUM II.

Proposita fractione $\frac{x^m}{1 - x^{2n}}$ *fractionem simplicem ex denominatoris factore* $1 - x$ *oriundam investigare.*

Ob factorem propositum $1 - x$, erit $f = 1$, & $g = -1$. Tum vero denominator $Q = 1 - x^{2n}$ dat $dQ = -2nx^{2n-1}dx$; unde propter $P = x^m$ obtinebitur $\mathfrak{A} = \frac{-x^m}{-2nx^{2n-1}}$. Positoque ex aequatione $1 - x = 0$, $x = 1$, fiet $\mathfrak{A} = \frac{1}{2n}$; ita ut fractio simplex futura sit haec $\frac{1}{2n(1-x)}$.

EXEMPLUM III.

Proposita fractione $\frac{x^m}{1 - 4x^k + 3x^n}$, *eius fractionem simplicem ex denominatoris factore* $1 - x$ *oriundam determinare.*

Hic ergo fit $f = 1$; $g = -1$; $P = x^m$; $Q = 1 - 4x^k + 3x^n$ & $\frac{dQ}{dx} = -4kx^{k-1} + 3nx^{n-1}$; unde fit $\mathfrak{A} = \frac{-x^m}{-4kx^{k-1} + 3nx^{n-1}}$ & posito $x = 1$, erit $\mathfrak{A} = \frac{1}{4k - 3n}$. Fractio ergo simplex ex isto denominatoris factore simplici $1 - x$ oriunda erit $= \frac{1}{(4k-3n)(1-x)}$.

409. Ponamus nunc fractionis $\frac{P}{Q}$ denominatorem Q factorem habere quadratum $(f+gx)^2$, & fractiones simplices hinc oriundas esse $= \frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}$. Sit $Q = (f+gx)^2 S$ & complementum $= \frac{V}{S}$; ita ut sit $\frac{V}{S} = \frac{P}{Q} - \frac{\mathfrak{A}}{(f+gx)^2} - \frac{\mathfrak{B}}{f+gx}$; & $V = \frac{P - \mathfrak{A}S - \mathfrak{B}(f+gx)S}{(f+gx)^2}$. Quia nunc V est functio integra, necesse est ut sit $P - \mathfrak{A}S - \mathfrak{B}S(f+gx)$ divisibile per $(f+gx)^2$; & cum S factorem $f+gx$ amplius non contineat, quoque haec expressio $\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx)$ divisibilis erit per $(f+gx)^2$; ideoque facto $f+gx=0$, seu $x = \frac{-f}{g}$ non solum ipsa, sed etiam eius differentiale $d.\frac{P}{S} - \mathfrak{B}g\,dx$ evanescet. Fiat ergo $x = \frac{-f}{g}$, eritque ex priori aequatione $\mathfrak{A} = \frac{P}{S}$; ex posteriori vero erit $\mathfrak{B} = \frac{1}{g\,dx}\, d.\frac{P}{S}$; quibus valoribus inventis habebuntur fractiones quaesitae: $\frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}$.

EXEMPLUM.

Proposita fractione $\frac{x^m}{1-4x^3+3x^4}$ *cuius denominator factorem habet* $(1-x)^2$, *invenire fractiones simplices hinc oriundas.*

Cum hic sit $f=1$, $g=-1$, $P=x^m$ & $Q=1-4x^3+3x^4$, erit $S = 1+2x+3xx$; $\frac{P}{S} = \frac{x^m}{1+2x+3xx}$; &

d.

Ex prima aequatione ergo erit $\mathfrak{A} = \frac{P}{S}$

Ex secunda vero erit $\mathfrak{B} = \frac{1}{g\,dx}\,d.\frac{P}{S}$

Ex tertia denique definitur $\mathfrak{C} = \frac{1}{2g^2\,dx^2}\,dd.\frac{P}{S}$

411. Generaliter ergo si fractionis $\frac{P}{S}$ denominator Q factorem habeat $(f+gx)^n$, ita ut sit $Q = (f+gx)^n S$; positis fractionibus simplicibus ex hoc factore $(f+gx)^n$ oriundis his:

$$\frac{\mathfrak{A}}{(f+gx)^n} + \frac{\mathfrak{B}}{(f+gx)^{n-1}} + \frac{\mathfrak{C}}{(f+gx)^{n-2}} + \frac{\mathfrak{D}}{(f+gx)^{n-3}} + \frac{\mathfrak{E}}{(f+gx)^{n-4}} + \&c.$$

quoad ad ultimam, cuius denominator est $f+gx$, perveniatur, si ratiocinium ut ante instituatur, reperietur haec expressio:

$$\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx) - \mathfrak{C}(f+gx)^2 - \mathfrak{D}(f+gx)^3 - \mathfrak{E}(f+gx)^4 - \&c.$$

divisibilis esse debere per $(f+gx)^n$, hinc tam ipsa quam singula eius differentialia usque ad gradum $n-1$, casu $x = \frac{-f}{g}$ evanescere debebunt. Ex quibus aequationibus concludetur fore ponendo ubique $x = \frac{-f}{g}$

$$\mathfrak{A} = \frac{P}{S}$$

$$\mathfrak{B} = \frac{1}{1\,g\,dx}\,d.\frac{P}{S}$$

$$\mathfrak{C} = \frac{1}{1.2\,g^2\,dx^2}\,dd.\frac{P}{S}$$

$$\mathfrak{D} = \frac{1}{1.2.3\,g^3\,dx^3}\,d^3.\frac{P}{S}$$

$$\mathfrak{E} = \frac{1}{1.2.3.4\,g^4\,dx^4}\,d^4.\frac{P}{S} \quad \&c.$$

Ubi

Ponamus esse $Q = (ff - 2fgx\cos\varphi + ggxx)S$, atque S praeterea per $ff - 2fgx\cos\varphi + ggxx$ non esse divisibile. Sit fractio ex isto factore denominatoris oriunda:

$$\frac{\mathfrak{A} + \mathfrak{a}x}{ff - 2fgx\cos\varphi + ggxx}$$

& complementum ad propositam $\frac{P}{Q}$ fit $= \frac{V}{S}$, erit

$$V = \frac{P - (\mathfrak{A} + \mathfrak{a}x)S}{ff - 2fgx\cos\varphi + ggxx};$$

unde $P - (\mathfrak{A} + \mathfrak{a}x)S$ ac propterea quoque $\frac{P}{S} - \mathfrak{A} - \mathfrak{a}x$ divisibile erit per $ff - 2fgx\cos\varphi + ggxx$. Evanescet ergo $\frac{P}{S} - \mathfrak{A} - \mathfrak{a}x$ si ponatur $ff - 2fgx\cos\varphi + ggxx = 0$, hoc est si ponatur

$$\text{vel} \quad x = \frac{f}{g}\cos\varphi + \frac{f}{g\sqrt{-1}}\sin\varphi$$

$$\text{vel} \quad x = \frac{f}{g}\cos\varphi - \frac{f}{g\sqrt{-1}}\sin\varphi.$$

414. Quoniam P & S sunt functiones integrae ipsius x, fiat in utroque seorsim utraque substitutio; & quia pro quavis potestate ipsius x, puta x^n binomium hoc $x^n = \frac{f^n}{g^n}\cos n\varphi \pm \frac{f^n}{g^n\sqrt{-1}}\sin n\varphi$ substitui debet. Ponamus primo ubique $\frac{f^n}{g^n}\cos n\varphi$ pro x^n, hocque facto abeat P in $\mathfrak{P}$, & S in $\mathfrak{S}$. Deinde ponatur ubique $\frac{f^n}{g^n}\sin n\varphi$ pro x^n, hocque facto abeat P in $\mathfrak{p}$ & S in $\mathfrak{s}$; ubi notandum est ante has substitutiones utramque functionem P & S penitus debere evolvi, ita ut, si forte factoribus sint implicatae, ii per actualem multiplicationem tollantur. His valoribus $\mathfrak{P}$, $\mathfrak{p}$, $\mathfrak{S}$, $\mathfrak{s}$, inventis, manifestum erit, si ponatur:

$x =$

ctionem $\frac{dQ}{dx}$ abire in $\mathfrak{Q}$; sin autem statuatur $x^n = \frac{f^n}{g^n} \sin n\phi$, eam abire in $\mathfrak{q}$; atque manifestum est si ponatur

$$x = \frac{f}{g}\cos\phi \pm \frac{f}{g\sqrt{-1}}\sin\phi$$

functionem $\frac{dQ}{dx}$ abire in $\mathfrak{Q} \pm \frac{\mathfrak{q}}{\sqrt{-1}}$. Ex quo functio S abibit in $\frac{\mathfrak{Q} \pm \mathfrak{q} : \sqrt{-1}}{\pm 2fg\sin\phi : \sqrt{-1}}$. Cum ergo sit $S = \mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}$ eodem valore pro x posito, habebitur,

$$\mathfrak{Q} \pm \frac{\mathfrak{q}}{\sqrt{-1}} = \pm\frac{2fg\mathfrak{S}}{\sqrt{-1}}\sin\phi - 2fg\mathfrak{s}\sin\phi.$$

Erit ergo $\mathfrak{s} = \frac{-\mathfrak{Q}}{2fg\sin\phi}$ & $\mathfrak{S} = \frac{\mathfrak{q}}{2fg\sin\phi}$.

Hisque valoribus substitutis, fiet $\mathfrak{a} = \frac{2gg(\mathfrak{p}\mathfrak{q} + \mathfrak{P}\mathfrak{Q})}{\mathfrak{Q}^2 + \mathfrak{q}^2}$

$$\& \quad \mathfrak{A} = \frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\sin\phi}{\mathfrak{Q}^2 + \mathfrak{q}^2} - \frac{2fg(\mathfrak{p}\mathfrak{q} + \mathfrak{P}\mathfrak{Q})\cos\phi}{\mathfrak{Q}^2 + \mathfrak{q}^2}.$$

416. Hinc ergo idonea obtinetur ratio ex quovis factore secundae potestatis fractionem simplicem formandi, hicque cum ipse fractionis propositae denominator in computo retineatur, divisionem, qua valor litterae S definiri deberet, & quae saepe non parum est molesta, evitamus. Si igitur fractionis $\frac{P}{Q}$ denominator Q factorem habeat talem $ff - 2fgx\cos\phi + ggxx$, sequenti modo fractio simplex ex hoc factore oriunda, quam fingamus $= \frac{\mathfrak{A} + \mathfrak{a}x}{ff - 2fgx\cos\phi + ggxx}$, definietur. Ponatur $x = \frac{f}{g}\cos\phi$, & pro quavis ipsius x potestate x^n

x^n scribatur $\frac{f^n}{g^n}\cos n\phi$; quo facto abeat P in $\mathfrak{P}$, & functio $\frac{dQ}{dx}$ in $\mathfrak{Q}$. Deinde ibidem ponatur $x = \frac{f}{g}\sin\phi$, & potestas eius quaevis $x^n = \frac{f^n}{g^n}\sin n\phi$; abeatque P in $\mathfrak{p}$, & $\frac{dQ}{dx}$ in $\mathfrak{q}$. Inventisque hoc modo valoribus litterarum $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{p}$ & $\mathfrak{q}$ quantitates $\mathfrak{A}$ & $\mathfrak{a}$ ita definientur, ut sit

$$\mathfrak{A} = \frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\sin\phi}{\mathfrak{Q}^2 + \mathfrak{q}^2} - \frac{2fg(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})\cos\phi}{\mathfrak{Q}^2 + \mathfrak{q}^2}$$

$$\mathfrak{a} = \frac{2gg(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})}{\mathfrak{Q}^2 + \mathfrak{q}^2}.$$

Fractio ergo ex denominatoris Q factore $ff - 2fgx\cos\phi + ggxx$ oriunda erit:

$$\frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\sin\phi + 2g(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})(gx - f\cos\phi)}{(\mathfrak{Q}^2 + \mathfrak{q}^2)(ff - 2fgx\cos\phi + ggxx)}$$

EXEMPLUM I.

Si proposita fuerit haec fractio $\frac{x^m}{a + bx^n}$, *cuius denominator* $a + bx^n$ *factorem habeat hunc:* $ff - 2fgx\cos\phi + ggxx$ *invenire fractionem simplicem huic factori convenientem.*

Quoniam hic est $P = x^m$ & $Q = a + bx^n$, erit $\frac{dQ}{dx} = nbx^{n-1}$, unde fiet:

$$\mathfrak{P} = \frac{f^m}{g^m}\cos m\phi \quad ; \quad \mathfrak{p} = \frac{f^m}{g^m}\sin m\phi$$

$$\mathfrak{Q} = \frac{nbf^{n-1}}{g^{n-1}}\cos(n-1)\phi \; ; \; \mathfrak{q} = \frac{nbf^{n-1}}{g^{n-1}}\sin(n-1)\phi.$$

Ex his erit: $\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{n^2b^2f^{2(n-1)}}{g^{2(n-1)}}$;

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} \sin(n-m-1)\phi;$$

atque $$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} \cos(n-m-1)\phi.$$

Quamobrem erit fractio simplex quaesita:

$$\frac{2g^{n-m}[f\sin\phi.\sin(n-m-1)\phi + gx\cos(n-m-1)\phi - f\cos\phi.\cos(n-m-1)\phi]}{nbf^{n-m-1}(ff-2fgx\cos\phi+ggxx)}$$

seu

$$\frac{2g^{n-m}[gx\cos(n-m-1)\phi - f\cos(n-m)\phi]}{nbf^{n-m-1}(ff-2fgx\cos\phi+ggxx)}.$$

EXEMPLUM II.

Sit proposita haec fractio $\frac{1}{x^m(a+bx^n)}$, *cuius denominator factorem habeat* $ff-2fgx\cos\phi+ggxx$, *invenire fractionem simplicem inde oriundam.*

Cum sit $P=1$, & $Q = ax^m + bx^{m+n}$, erit $\frac{dQ}{dx} = max^{m-1} + (m+n)bx^{m+n-1}$, ideoque posito $x^n = \frac{f^n}{g^n}\cos n\phi$ ob $P = x^0$ & $\mathfrak{P} = 1$.

$$\mathfrak{Q} = \frac{maf^{m-1}}{g^{m-1}}\cos(m-1)\phi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}}\cos(m+n-1)\phi$$

& $\mathfrak{p} = 0$; atque

$$\mathfrak{q} = \frac{maf^{m-1}}{g^{m-1}}\sin(m-1)\phi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}}\sin(m+n-1)\phi$$

Ergo

$$\mathfrak{Q}^2+\mathfrak{q}^2 = \frac{m^2a^2f^{2(m-1)}}{g^{2(m-1)}} + \frac{2m(m+n)abf^{2m+n-2}}{g^{2m+n-2}}\cos n\phi + \frac{(m+n)^2b^2f^{2(m+n-1)}}{g^{2(m+n-1)}}.$$

Quodsi

Quodsi vero est $ff - 2fgx\cos\phi + ggxx$ divisor ipsius $a + bx^n$, erit $a + \frac{bf^n}{g^n}\cos n\phi = 0$ & $\frac{bf^n}{g^n}\sin n\phi = 0$, unde: $aa = \frac{bbf^{2n}}{g^{2n}}$. Erit ergo:

$$\mathfrak{Q} + \mathfrak{q}^2 = \frac{(m+n)^2 bbf^{2(m+n-1)}}{g^{2(m+n-1)}} - \frac{m(2n+m)aaf^{2(m-1)}}{g^{2(m-1)}}$$

$$= \frac{nnaaf^{2(m-1)}}{g^{2(m-1)}} = \frac{nnbbf^{2(m+n-1)}}{g^{2(m+n-1)}}.$$

Deinde vero erit:

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \frac{maf^{m-1}}{g^{m-1}}\sin(m-1)\phi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}}\sin(m+n-1)\phi$$

$$= \frac{bf^{m+n-1}}{g^{m+n-1}}[(m+n)\sin(m+n-1)\phi - m\cos n\phi \,.\, \sin(m-1)\phi]$$

$$= \frac{bf^{m+n-1}}{g^{m+n-1}}[n\cos n\phi\sin(m-1)\phi + (m+n)\sin n\phi\cos(m-1)\phi]$$

& $\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} =$

$$= \frac{bf^{m+n-1}}{g^{m+n-1}}[(m+n)\cos(m+n-1)\phi - m\cos n\phi \,.\, \cos(m-1)\phi]$$

Vel cum $ff - 2fg\cos\phi + ggxx$ sit quoque divisor ipsius $ax^{m-1} + bx^{m+n-1}$, erit:

$$\frac{af^{m-1}}{g^{m-1}}\cos(m-1)\phi + \frac{bf^{m+n-1}}{g^{m+n-1}}\cos(m+n-1)\phi = 0$$

& $$\frac{af^{m-1}}{g^{m-1}}\sin(m-1)\phi + \frac{bf^{m+n-1}}{g^{m+n-1}}\sin(m+n-1)\phi = 0,$$

unde erit:

$$\mathfrak{Q} = \frac{nbf^{m+n-1}}{g^{m+n-1}}\cos(m+n-1)\phi \;\&\; \mathfrak{q} = \frac{nbf^{m+n-1}}{g^{m+n-1}}\sin(m+n-1)\phi$$

seu

$$\mathfrak{Q} = \frac{-naf^{m-1}}{g^{m-1}}\cos(m-1)\phi \;\&\; \mathfrak{q} = \frac{-naf^{m-1}}{g^{m-1}}\sin(m-1)\phi.$$

Ex quibus resultabit fractio quaesita:

$$+\frac{2g^m[f\cos m\phi - gx\cos(m-1)\phi]}{naf^{n-1}(ff-2fgx\cos\phi+ggxx)}.$$

Quae formula ex priori exemplo sequitur, si ponatur m negativum, unde non opus fuisset hunc casum peculiarem constituisse.

EXEMPLUM III.

Si huius fractionis $\frac{x^m}{a+bx^n+cx^{2n}}$ *denominator habuerit factorem* $ff-2fgx\cos\phi+ggxx$, *fractionem simplicem investigare ex hoc factore oriundam.*

Si $ff-2fgx\cos\phi+ggxx$ est factor denominatoris $a+bx^n+cx^{2n}$, erit ut supra ostendimus:

$$a+\frac{bf^n}{g^n}\cos n\phi+\frac{cf^{2n}}{g^{2n}}\cos 2n\phi=0$$

$$\&\ \frac{bf^n}{g^n}\sin n\phi+\frac{cf^{2n}}{g^{2n}}\sin 2n\phi=0.$$

Cum igitur sit $P=x^m$ & $Q=a+bx^n+cx^{2n}$, erit $\frac{dQ}{dx}=nbx^{n-1}+2ncx^{2n-1}$; unde efficitur:

$$\mathfrak{P}=\frac{f^m}{g^m}\cos m\phi \quad \& \quad \mathfrak{p}=\frac{f^m}{g^m}\sin m\phi:$$

$$\mathfrak{Q}=\frac{nbf^{n-1}}{g^{n-1}}\cos(n-1)\phi+\frac{2ncf^{2n-1}}{g^{2n-1}}\cos(2n-1)\phi$$

$$\mathfrak{q}=\frac{nbf^{n-1}}{g^{n-1}}\sin(n-1)\phi+\frac{2ncf^{2n-1}}{g^{2n-1}}\sin(2n-1)\phi.$$

Quamobrem habebimus:

$$\mathfrak{Q}^2+\mathfrak{q}^2=\frac{n^2f^{2(n-1)}}{g^{2(n-1)}}\left(bb+\frac{4bcf^n}{g^n}\cos n\phi+\frac{4ccf^{2n}}{g^{2n}}\right)$$

At ex duabus prioribus aequationibus est:

$$\frac{f^{2n}}{g^{2n}}\left(bb+\frac{2bcf^n}{g^n}\cos n\phi+\frac{ccf^{2n}}{g^{2n}}\right)=aa;$$

ideo-

ideoque

$$\frac{4bcf^n}{g^n}\cos n\phi = \frac{2g^{2n}aa}{f^{2n}} - 2bb - \frac{2ccf^{2n}}{g^{2n}}$$

quo valore ibi substituto erit:

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{n^2 f^{2n-2}}{g^{2n-2}}\left(\frac{2aag^{2n}}{f^{2n}} - bb + \frac{2ccf^{2n}}{g^{2n}}\right)$$

seu

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{n^2(2aag^{4n} - bbf^{2n}g^{2n} + 2ccf^{4n})}{ff\,g^{4n-2}}$$

Deinde erit: $\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} =$

$$\frac{nbf^{m+n-1}}{g^{m+n-1}}\sin(n-m-1)\phi + \frac{2ncf^{m+2n-1}}{g^{m+2n-1}}\sin(2n-m-1)\phi$$

$$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} =$$

$$\frac{nbf^{m+n-1}}{g^{m+n-1}}\cos(n-m-1)\phi + \frac{2ncf^{m+2n-1}}{g^{m+2n-1}}\cos(2n-m-1)\phi.$$

Quibus valoribus inventis erit fractio simplex quaesita:

$$\frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\sin\phi + 2g(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})(gx - f\cos\phi)}{(\mathfrak{Q}^2 + \mathfrak{q}^2)(ff - 2fgx\cos\phi + ggxx)}$$

417. Hae autem fractiones facilius exprimentur, si ipsos denominatorum factores determinemus. Sit igitur denominator fractionis propositae: $a + bx^n$ cuius factor trinomialis si ponatur: $ff - 2fgx\cos\phi + ggxx$ erit uti in Introductione ostendimus:

$$a + \frac{bf^n}{g^n}\cos n\phi = 0 \quad \& \quad \frac{bf^n}{g^n}\sin n\phi = 0,$$

cum igitur sit $\sin n\phi = 0$, erit vel $n\phi = (2k-1)\pi$, vel $n\phi = 2k\pi$, priori casu erit $\cos n\phi = -1$, posteriori $\cos n\phi = +1$. Si ergo a & b sint quantitates affirmativae, prior casus solus locum habebit, quo fit $a = \frac{bf^n}{g^n}$; ac propterea: $f = a^{\frac{1}{n}}$ & $g = b^{\frac{1}{n}}$

reti-

retineamus autem loco harum quantitatum irrationalium litteras f & g, seu ponamus potius $a = f^n$ & $b = g^n$, ita ut factores investigari debeant huius functionis $f^n + g^n x^n$. Cum igitur sit $\varphi = \frac{(2k-1)\pi}{n}$, ubi k numerum quemcunque affirmativum integrum designare potest; at vero maiores numeri pro k non sunt sumendi, quam qui reddant $\frac{2k-1}{n}$ unitate minorem; hinc functionis propositae $f^n + g^n x^n$ factores erunt sequentes:

$$ff - 2fgx \cos\frac{\pi}{n} + ggxx$$

$$ff - 2fgx \cos\frac{3\pi}{n} + ggxx$$

$$ff - 2fgx \cos\frac{5\pi}{n} + ggxx$$

&c.

ubi notandum est si n sit numerus impar, unum factorem haberi binomium hunc: $f + gx$; sin autem n sit numerus par nullus factor aderit binomius.

EXEMPLUM I.

Resolvere hanc fractionem $\frac{x^m}{f^n + g^n x^n}$ *in suas fractiones simplices.*

Cum denominatoris unusquisque factor trinomialis contineatur in hac forma: $ff - 2fgx \cos\frac{(2k-1)\pi}{n} + ggxx$, erit in §. praecedente Exempl. I. $a = f^n$, $b = g^n$; & $\varphi = \frac{(2k-1)\pi}{n}$, unde erit:

$\sin(n-m-1)\varphi = \sin(m+1)\varphi = \sin\frac{(m+1)(2k-1)\pi}{n}$ &

cos

$A x^{m-n} + B x^{m-2n} + C x^{m-3n} + D x^{m-4n} + \&c.$
quamdiu exponentes manent affirmativi, eritque:

$$A g^n = 1 \; ; \quad \text{Ergo} \quad A = \frac{1}{g^n}$$

$$A f^n + B g^n = 0 \; . \; . \; B = -\frac{f^n}{g^{2n}}$$

$$B f^n + C g^n = 0 \; . \; . \; C = +\frac{f^{2n}}{g^{3n}}$$

$$C f^n + D g^n = 0 \; . \; . \; D = -\frac{f^{3n}}{g^{4n}}$$

&c. &c.

EXEMPLUM II.

Resolvere hanc fractionem $\frac{1}{x^m (f^n + g^n x^n)}$ *in suas fractiones simplices.*

Quod ad factores ipsius $f^n + g^n x^n$ attinet, ex iis oriuntur eaedem fractiones; quas Exemplo praecedente eruimus, dummodo ibi sumatur m negative: superest igitur tantum, ut fractiones simplices ex denominatoris altero factore x^m definiamus, quod hoc modo commodissime fit: statuatur fractio proposita $= \frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{N} x^{n-m}}{f^n + g^n x^n}$, eritque

$$\mathfrak{A} f^n = 1 \; ; \; \text{Ergo} \; \mathfrak{A} = \frac{1}{f^n}$$

$$\mathfrak{A} g^n + \mathfrak{N} = 0 \; . \; . \; \mathfrak{N} = -\frac{g^n}{f^n}.$$

Si $n-m$ adhuc fuerit numerus negativus, simili modo erit operandum, ita ut, si m fuerit numerus quantumvis magnus, resultent huiusmodi fractiones simplices

$$\frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{B}}{x^{m-n}} + \frac{\mathfrak{C}}{x^{m-2n}} + \frac{\mathfrak{D}}{x^{m-3n}} + \&c.$$

cuius

cuius seriei tot termini sunt sumendi, quot habentur ipsius x exponentes affirmativi in denominatore. Eritque

$$\mathfrak{A}f^n = 1 \;; \qquad \text{Ergo} \quad \mathfrak{A} = \frac{1}{f^n}$$

$$\mathfrak{A}g^n + \mathfrak{B}f^n = 0 \quad . \quad . \quad \mathfrak{B} = -\frac{g^n}{f^{2n}}$$

$$\mathfrak{B}g^n + \mathfrak{C}f^n = 0 \quad . \quad . \quad \mathfrak{C} = +\frac{g^{2n}}{f^{3n}}$$

$$\mathfrak{C}g^n + \mathfrak{D}f^n = 0 \quad . \quad . \quad \mathfrak{D} = -\frac{g^{3n}}{f^{4n}}$$

&c. &c.

Fractio ergo proposita omnino in has fractiones simplices resolvetur:

$$\frac{1}{f^n x^m} - \frac{g^n}{f^{2n} x^{m-n}} + \frac{g^{2n}}{f^{3n} x^{m-2n}} - \frac{g^{3n}}{f^{4n} x^{m-3n}} + \&c.$$

$$\frac{-2fg^m \sin\frac{\pi}{n} . \sin\frac{(m-1)\pi}{n} - 2g^m \cos\frac{(m-1)\pi}{n}\left(gx - f\cos\frac{\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{\pi}{n} + ggxx\right)}$$

$$\frac{-2fg^m \sin\frac{3\pi}{n} . \sin\frac{3(m-1)\pi}{n} - 2g^m \cos\frac{3(m-1)\pi}{n}\left(gx - f\cos\frac{3\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{3\pi}{n} + ggxx\right)}$$

$$\frac{-2fg^m \sin\frac{5\pi}{n} . \sin\frac{5(m-1)\pi}{n} - 2g^m \cos\frac{5(m-1)\pi}{n}\left(gx - f\cos\frac{5\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{5\pi}{n} + ggxx\right)}$$

&c.

Quibus formulis si n fuerit numerus impar, ob $f + gx$ factoctorem denominatoris, insuper adiici debet:

$$\pm \frac{g^m}{nf^{n+m-1}(f+gx)}$$

ubi signorum ambiguorum $\pm$ superius valet, si m fuerit numerus par, inferius vero si m impar.

418. Consideremus nunc quoque formulam $a+bx^n$, si b sit numerus negativus, sitque proposita haec functio:

$$f^n - g^n x^n$$

cuius primo semper erit factor $f-gx$; atque si n sit numerus par, quoque $f+gx$ eius erit factor. Reliqui vero erunt trinomiales, quorum forma generalis si ponatur

$$ff - 2fgx\cos\phi + ggxx$$

erit $f^n - f^n \cos n\phi = 0$ & $f^n \sin n\phi = 0$ sive $\sin n\phi = 0$ & $\cos n\phi = 1$. Quibus ut satisfiat, oportet esse $n\phi = 2k\pi$ existente k numero quocunque integro, atque propterea erit $\phi = \frac{2k\pi}{n}$. Factor ergo generalis erit:

$$ff - 2fgx\cos\frac{2k\pi}{n} + ggxx$$

sumendo ergo pro $2k$ omnes numeros pares exponente n minores, prodibunt factores trinomiales omnes:

$$ff - 2fgx\cos\frac{2\pi}{n} + ggxx$$

$$ff - 2fgx\cos\frac{4\pi}{n} + ggxx$$

$$ff - 2fgx\cos\frac{6\pi}{n} + ggxx$$

&c.

EXEMPLUM I.

Resolvere hanc fractionem $\frac{x^m}{f^n - g^n x^n}$ *in suas fractiones simplices.*

Quoniam denominatoris factor est $f-gx$, inde orietur fra-

fractio huiusmodi $\frac{\mathfrak{A}}{f-gx}$, ad cuius numeratorem inveniendum, ponatur $x^m = P$ & $f^n - g^n x^n = Q$, erit $\frac{dQ}{dx} = -ng^n x^{n-1}$, fietque $\mathfrak{A} = \frac{-gx^m}{-ng^n x^{n-1}} = \frac{x^m}{ng^{n-1}x^{n-1}}$, posito $x = \frac{f}{g}$. Ergo erit $\mathfrak{A} = \frac{1}{nf^{n-m-1}g^m}$, hincque fractio simplex ex factore $f-gx$ orta erit:

$$\frac{1}{nf^{n-m-1}g^m(f-gx)}.$$

Si n sit numerus par, quia tum denominatoris factor quoque est $f+gx$, ponatur fractio simplex inde oriunda

$$= \frac{\mathfrak{A}}{f+gx}, \text{ erit } \mathfrak{A} = \frac{-gx^m}{ng^n x^{n-1}} = \frac{-x^m}{ng^{n-1}x^{n-1}},$$

posito $x = \frac{-f}{g}$. Fiet ergo ob $n-1$ numerum imparem $g^{n-1}x^{n-1} = -f^{n-1}$: at erit $x^m = \frac{\pm f^m}{g^m}$, ubi signum superius valet, si m fuerit numerus par, inferius si m fit numerus impar. Quare cum sit $\mathfrak{A} = \frac{\mp 1}{nf^{n-m-1}g^m}$, erit fractio simplex ex factore $f+gx$ oriunda haec:

$$\frac{\mp 1}{nf^{n-m-1}g^m(f+gx)}.$$

Deinde cum factorum trinomialium forma generalis sit:

$$ff - 2fgx\cos\frac{2k\pi}{n} + ggxx,$$

si comparatio cum Exemplo I. §. 416. instituatur, erit $a = f^n$, $b = -g^n$ & $\phi = \frac{2k\pi}{n}$; unde sin $n\phi = 0$ & cos $n\phi = 1$;

 at-

atque $\sin(n-m-1)\phi = -\sin(m+1)\phi = -\sin\frac{2k(m+1)\pi}{n}$;

& $\cos(n-m-1)\phi = \cos(m+1)\phi = \cos\frac{2k(m+1)\pi}{n}$.

Ex quibus erit fractio simplex hinc oriunda:

$$\frac{2f\sin\frac{2k\pi}{n}.\sin\frac{2k(m+1)\pi}{n} - 2\cos\frac{2k(m+1)\pi}{n}\left(gx - f\cos\frac{2k\pi}{n}\right)}{nf^{n-m-1}g^{m}\left(ff - 2fgx\cos\frac{2k\pi}{n} + ggxx\right)}$$

Hancobrem fractiones simplices quaesitae erunt:

$$\frac{1}{nf^{n-m-1}g^{m}(f-gx)}$$

$$+\frac{2f\sin\frac{2\pi}{n}.\sin\frac{2(m+1)\pi}{n} - 2\cos\frac{2(m+1)\pi}{n}\left(gx - f\cos\frac{2\pi}{n}\right)}{nf^{n-m-1}g^{m}\left(ff - 2fgx\cos\frac{2\pi}{n} + ggxx\right)}$$

$$+\frac{2f\sin\frac{4\pi}{n}.\sin\frac{4(m+1)\pi}{n} - 2\cos\frac{4(m+1)\pi}{n}\left(gx - f\cos\frac{4\pi}{n}\right)}{nf^{n-m-1}g^{m}\left(ff - 2fgx\cos\frac{4\pi}{n} + ggxx\right)}$$

$$+\frac{2f\sin\frac{6\pi}{n}.\sin\frac{6(m+1)\pi}{n} - 2\cos\frac{6(m+1)\pi}{n}\left(gx - f\cos\frac{6\pi}{n}\right)}{nf^{n-m-1}g^{m}\left(ff - 2fgx\cos\frac{6\pi}{n} + ggxx\right)}$$

&c.

quibus si n fuerit numerus par, insuper addi debet haec fractio: $\frac{\mp 1}{nf^{n-m-1}g^{m}(f+gx)}$

cu-

cuius signum superius — est sumendum, si m fuerit numerus par, inferius si impar. Praeterea vero si m sit numerus non minor quam n, adiiciendae sunt partes integrae:

$$Ax^{m-n} + Bx^{m-2n} + Cx^{m-3n} + Dx^{m-4n} + \&c.$$

quamdiu exponentes non fuerint negativi, eritque:

$$-Ag^n = 1 \quad \text{seu} \quad A = -\frac{1}{g^n}$$

$$Af^n - Bg^n = 0 \quad \ldots \quad B = -\frac{f^n}{g^{2n}}$$

$$Bf^n - Cg^n = 0 \quad \ldots \quad C = -\frac{f^{2n}}{g^{3n}}$$

$$Cf^n - Dg^n = 0 \quad \ldots \quad D = -\frac{f^{3n}}{g^{4n}}$$

&c. &c.

EXEMPLUM II.

Resolvere hanc fractionem $\frac{1}{x^m(f^n - g^n x^n)}$ *in suas fractiones simplices*

Fractiones quae ex denominatoris factore $f^n - g^n x^n$ oriuntur, eaedem erunt quae ante, dummodo in illis formulis m negative accipiatur. Quare ad alterum factorem x^m est respiciendum, ex quo si ponamus has fractiones resultare:

$$\frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{B}}{x^{m-n}} + \frac{\mathfrak{C}}{x^{m-2n}} + \frac{\mathfrak{D}}{x^{m-3n}} + \&c.$$

quae series eousque est continuanda, donec exponentes ipsius x fiant negativi. Erit vero

$$\mathfrak{A}f^n = 1 \; ; \quad \text{Ergo} \quad \mathfrak{A} = \frac{1}{f^n}$$

$$\mathfrak{B}f^n - \mathfrak{A}g^n = 0 \quad \ldots \quad \mathfrak{B} = \frac{g^n}{f^{2n}}$$

$$\mathfrak{C}f^n - \mathfrak{B}g^n = 0 \quad \ldots \quad \mathfrak{C} = \frac{g^{2n}}{f^{3n}}$$

$$\mathfrak{D}f^n - \mathfrak{C}g^n = 0 \quad \ldots \quad \mathfrak{D} = \frac{g^{3n}}{f^{4n}}$$

&c. &c.

Fra-

Fractio ergo propoſita reſolvetur in has fractiones ſimplices:

$$\frac{1}{f^n x^m} + \frac{g^n}{f^{2n} x^{m-n}} + \frac{g^{2n}}{f^{3n} x^{m-2n}} + \frac{g^{3n}}{f^{4n} x^{m-3n}} + \&c.$$

$$+ \frac{g^m}{n f^{n+m-1} (f - gx)}$$

$$\frac{-2fg^m \sin\frac{2\pi}{n}\sin\frac{2(m-1)\pi}{n} - 2g^m \cos\frac{2(m-1)\pi}{n}\left(gx - f\cos\frac{2\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{2\pi}{n} + ggxx\right)}$$

$$\frac{-2fg^m \sin\frac{4\pi}{n}\sin\frac{4(m-1)\pi}{n} - 2g^m \cos\frac{4(m-1)\pi}{n}\left(gx - f\cos\frac{4\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{4\pi}{n} + ggxx\right)}$$

$$\frac{-2fg^m \sin\frac{6\pi}{n}\sin\frac{6(m-1)\pi}{n} - 2g^m \cos\frac{6(m-1)\pi}{n}\left(gx - f\cos\frac{6\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{6\pi}{n} + ggxx\right)}$$

&c.

quibus ſi n fuerit numerus par, inſuper addi debet haec fractio:

$$\frac{\mp g^m}{nf^{n+m-1}(f + gx)}$$

quae autem praetermittitur, ſi n fuerit numerus impar. Signorum ambiguorum vero ſuperius — valet, ſi m ſit numerus par, inferius vero +, ſi m ſit numerus impar.

419. Hoc ergo modo omnes fractiones, quarum denominator ex duobus conſtat membris huiuſmodi $a + bx^n$, in fractiones ſimplices reſolvuntur. At ſi denominator conſtet tribus huiuſmodi membris $a + bx^n + cx^{2n}$, tum primum videndum eſt, utrum is in duos factores reales prioris formae reſolvi poſſit. Hoc enim ſi eveniat, reſolutio in fractiones ſimpli-

plices modo ante expoſito inſtitui poterit. Si enim proponatur huiuſmodi fractio

$$\frac{x^m}{(f^n+g^nx^n)(f^n+h^nx^n)}$$

ea primum in duas fractiones transformabitur huiuſmodi:

$$\frac{\alpha x^m}{f^n+g^nx^n}+\frac{\beta x^m}{f^n+h^nx^n}$$

eritque $\alpha f^n+\beta f^n=1$ & $\alpha h^n+\beta g^n=0$, unde fit

$$\alpha=\frac{1}{f^n}-\beta=-\frac{\beta g^n}{h^n};\ \text{ideoque habebitur}$$

$$\beta=\frac{h^n}{f^n(h^n-g^n)}\ \&\ \alpha=\frac{g^n}{f^n(g^n-h^n)}.$$

Si exponens m fuerit maior quam n, tranſmutatio in ſequentes fractiones erit commodior:

$$\frac{\alpha x^{m-n}}{f^n+g^nx^n}+\frac{\beta x^{m-n}}{f^n+h^nx^n}$$

qua fit $\alpha+\beta=0$ & $\alpha h^n+\beta g^n=1$, ideoque $\alpha=\frac{1}{h^n-g^n}$ & $\beta=\frac{1}{g^n-h^n}$. Utra autem transformatio adhibeatur, utraque fractio hoc modo oriunda methodo ante expoſita reſolvetur in ſuas fractiones ſimplices, quae iunctim ſumtae fractioni propoſitae erunt aequales.

420. Simili modo methodus hactenus tradita ſufficiet, ſi denominator ex pluribus membris conſtet huiuſmodi $a+bx^n+cx^{2n}+dx^{3n}+ex^{4n}+$ &c. dummodo is in factores formae $f^n\pm g^nx^n$ reſolvi queat. Ponamus enim occurrere hanc fractionem in ſuas fractiones ſimplices reſolvendam:

x^m

$$\frac{x^m}{(a-x^n)(b-x^n)(c-x^n)(d-x^n)}$$

Haec primum resolvetur in has:

$$\frac{Ax^m}{a-x^n}+\frac{Bx^m}{b-x^n}+\frac{Cx^m}{c-x^n}+\frac{Dx^m}{d-x^n}$$

quarum numeratores sequenti modo determinabuntur, ut sit

$$A=\frac{1}{(b-a)(c-a)(d-a)}$$

$$B=\frac{1}{(a-b)(c-b)(d-b)}$$

$$C=\frac{1}{(a-c)(b-c)(d-c)} \quad \&c.$$

Hac ergo praeparatione facta, singulae istae fractiones methodo ante exposita in suas fractiones simplices resolventur; quae cunctae in unam summam erunt colligendae.

421. Quodsi vero huiusmodi denominator $a+bx^n+cx^{2n}+dx^{3n}+$ &c. non omnes factores formae $f^n+g^n x^n$ habeant reales, bini imaginarii erunt coniungendi. Ponamus ergo huiusmodi binorum factorum productum esse:

$$f^{2n}-2f^n g^n x^n \cos\omega + g^{2n}x^{2n}$$

& cum haec expressio nullos habeat factores simplices reales, ponamus factores trinomiales in hac forma generali contineri:

$$ff-2fgx\cos\varphi+ggxx$$

quorum numerus erit $=n$. Posito ergo $x^n=\frac{f^n}{g^n}\cos n\varphi$; orietur haec aequatio:

$$1-2\cos\omega.\cos n\varphi+\cos 2n\varphi=0.$$

Deinde posito $x^n=\frac{f^n}{g^n}\sin n\varphi$; erit quoque:

$$-2\cos\omega.\sin n\varphi+\sin 2n\varphi=0$$

quae divisa per $\sin n\varphi$ dat $\cos n\varphi=\cos\omega$, sicque simul priori ri aequationi satisfit. Erit ergo $n\phi = 2k\pi \pm \omega$ denotante k numerum quemvis integrum, ideoque erit $\phi = \frac{2k\pi \pm \omega}{n}$, & factores omnes continebuntur in hac forma:

$ff - 2fgx\cos\frac{2k\pi \pm \omega}{n} + ggxx$ unde sequentes habebuntur factores:

$$ff - 2fgx\cos\frac{\omega}{n} + ggxx$$

$$ff - 2fgx\cos\frac{2\pi - \omega}{n} + ggxx$$

$$ff - 2fgx\cos\frac{2\pi + \omega}{n} + ggxx$$

$$ff - 2fgx\cos\frac{4\pi - \omega}{n} + ggxx$$

$$ff - 2fgx\cos\frac{4\pi + \omega}{n} + ggxx \quad \&c.$$

quorum tot sunt sumendi, donec eorum numerus fiat $= n$.

422. Si igitur proponatur ista fractio in suas fractiones simplices resolvenda:

$$\frac{x^{m-1}}{f^{2n} - 2f^n g^n x^n \cos\omega + g^{2n} x^{2n}}$$

quoniam denominatoris factor trinomialis quicunque continetur in hac forma: $ff - 2fgx\cos\phi + ggxx$ existente $\phi = \frac{2k\pi \pm \omega}{n}$, consideretur ista fractio:

$$\frac{x^m}{f^{2n}x - 2f^n g^n x^{n+1}\cos\omega + g^{2n} x^{2n+1}}$$

illi aequalis, ac ponatur numerator $x^m = P$ ac denominator $f^{2n}x - 2f^n g^n x^{n+1}\cos\omega + g^{2n} x^{2n+1} = Q$: erit

 dQ

$\frac{dQ}{dx} = f^{2n} - 2(n+1) f^n g^n x^n \cos\omega + (2n+1) g^{2n} x^{2n}$.

Hinc ponendo $x^n = \frac{f^n}{g^n}\cos n\phi$; erit: $\mathfrak{P} = \frac{f^m}{g^m}\cos m\phi$

seu $\mathfrak{P} = \frac{f^m}{g^m}\cos\frac{m(2k\pi\pm\omega)}{n}$ &

$\mathfrak{Q} = f^{2n}[1-2(n+1)\cos\omega\cos n\phi + (2n+1)\cos 2n\phi]$. Cum autem sit $\cos n\phi = \cos\omega$, erit $\cos 2n\phi = 2\cos\omega^2 - 1$; ideoque $\mathfrak{Q} = f^{2n}(-2n + 2n\cos\omega^2) = -2nf^{2n}\sin\omega^2$.

Deinde posito $x^n = \frac{f^n}{g^n}\sin n\phi$, fiet:

$$\mathfrak{p} = \frac{f^m}{g^m}\sin m\phi = \frac{f^m}{g^m}\sin\frac{m(2k\pi\pm\omega)}{n} \quad \&$$

$$\mathfrak{q} = -f^{2n}[2(n+1)\cos\omega\sin n\phi - (2n+1)\sin 2n\phi]$$

ob $\sin 2n\phi = 2\sin n\phi\cos n\phi = 2\cos\omega\sin n\phi$; erit $\mathfrak{q} = 2nf^{2n}\cos\omega\sin n\phi$. Cum autem sit $n\phi = 2k\pi\pm\omega$, erit $\sin n\phi = \pm\sin\omega$ & $\mathfrak{q} = \pm 2nf^{2n}\sin\omega.\cos\omega$.
His inventis erit: $\mathfrak{Q}^2 + \mathfrak{q}^2 = 4n^2f^{4n}\sin\omega^2$

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \frac{2nf^{m+2n}}{g^m}(\pm\cos m\phi.\sin\omega.\cos\omega + \sin m\phi.\sin\omega^2)$$

sive $\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \pm\frac{2nf^{m+2n}}{g^m}\sin\omega.\cos(m\phi\mp\omega)$ seu

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \pm\frac{2nf^{m+2n}}{g^m}\sin\omega.\cos\frac{2km\pi\pm(m-n)\omega}{n}$$

$$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} = \frac{2nf^{m+2n}}{g^m}(-\cos m\phi\sin\omega^2 \pm \sin m\phi.\sin\omega\cos\omega)$$

$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} = \pm\frac{2nf^{m+2n}}{g^m}\sin\omega.\sin(m\phi\mp\omega)$ seu

$$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} = \pm\frac{2nf^{m+2n}}{g^m}\sin\omega.\sin\frac{2km\pi\pm(m-n)\omega}{n}.$$

Hinc

Hinc ex denominatoris factore: $ff-2fgx\cos\frac{2k\pi\pm\omega}{n}+ggxx$ nascitur ista fractio simplex:

$$\frac{\pm f\sin\frac{2k\pi\pm\omega}{n}\cos\frac{2km\pi\pm(m-n)\omega}{n}\pm\sin\frac{2km\pi\pm(m-n)\omega}{n}\left(gx-f\cos\frac{2k\pi\pm\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{2k\pi\pm\omega}{n}+ggxx\right)}$$

seu

$$\frac{\pm gx\sin\frac{2km\pi\pm(m-n)\omega}{n}\mp f\sin\frac{2k(m-1)\pi\pm(m-n-1)\omega}{n}}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{2k\pi\pm\omega}{n}+ggxx\right)}$$

EXEMPLUM.

Resolvere hanc fractionem $\frac{x^{m-1}}{f^{2n}-2f^ng^nx^n\cos\omega+g^{2n}x^{2n}}$ *in suas fractiones simplices.*

Istae fractiones simplices quaesitae ergo erunt:

$$\frac{+f\sin\frac{\omega}{n}\cos\frac{(m-n)\omega}{n}+\sin\frac{(m-n)\omega}{n}\left(gx-f\cos\frac{\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{\omega}{n}+ggxx\right)}$$

$$\frac{-f\sin\frac{2\pi-\omega}{n}\cos\frac{2m\pi-(m-n)\omega}{n}-\sin\frac{2m\pi-(m-n)\omega}{n}\left(gx-f\cos\frac{2\pi-\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{2\pi-\omega}{n}+ggxx\right)}$$

$$\frac{\mp f\sin\frac{2\pi\pm\omega}{n}\cos\frac{2m\pi\pm(m-n)\omega}{n}\pm\sin\frac{2m\pi\pm(m-n)\omega}{n}\left(gx-f\cos\frac{2\pi\pm\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{2\pi\pm\omega}{n}+ggxx\right)}$$

$$-\frac{f\sin\frac{4\pi-\omega}{n}\cos\frac{4m\pi-(m-n)\omega}{n}-\sin\frac{4m\pi-(m-n)\omega}{n}\left(gx-f\cos\frac{4\pi-\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{4\pi-\omega}{n}+ggxx\right)}$$

$$+\frac{f\sin\frac{4\pi+\omega}{n}\cos\frac{4m\pi+(m-n)\omega}{n}+\sin\frac{4m\pi+(m-n)\omega}{n}\left(gx-f\cos\frac{4\pi+\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{4\pi+\omega}{n}+ggxx\right)}$$

&c.

ſicque eouſque erit progrediendum, quoad harum fractionum numerus fuerit n. Si m fuerit numerus vel maior quam $2n-1$ vel numerus negativus, priori caſu partes integrae, poſteriori vero fractiones inſuper ſunt adiiciendae, quae modo ante expoſito facile inveniuntur.

FINIS.