CAPUT XVII.

DE INTERPOLATIONE SERIERUM.

389.

Series interpolari dicitur, dum eius termini assignantur, qui respondent indicibus fractis, vel etiam surdis. Si igitur seriei terminus generalis fuerit cognitus, interpolatio nullam habet difficultatem, cum quicunque numerus loco indicis x substituatur, ista expressio praebeat terminum respondentem. Verum si series ita fuerit comparata, ut eius terminus generalis nullo modo exhiberi queat; tum interpolatio huiusmodi serierum plerumque est maxime difficilis, neque maximam partem termini indicibus non integris respondentes aliter nisi per series infinitas definiri possunt. Quoniam ergo in Capite praecedente huiusmodi expressionum, quae more consueto finite exprimi non possunt, valores quibuscunque indicibus respondentes determinavimus; ea tractatio maximam afferet utilitatem ad interpolationes perficiendas. Quam ob causam usum, qui ex superiori Capite in hoc negotium redundat, hic diligentius prosequemur.

390. Sit ergo proposita series quaecunque

$$\overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \;.\;.\;.\;.\; \overset{x}{X}$$

cuius terminus generalis X sit cognitus, summatorius autem S lateat. Hinc formetur alia series, cuius terminus generalis aequetur illius seriei termino summatorio, eritque ista nova series:

$$\overset{1}{A};\ (A + \overset{2}{B});\ (A + \overset{3}{B} + C);\ (A + B + \overset{4}{C} + D);\ (A + B + \overset{5}{C} + D + E);\ \&c.$$

eiusque terminus generalis seu indici indefinito x respondens

 erit

erit $= A + B + C + D + \ldots + X = S$, qui cum explicite non ſit cognitus, interpolatio huius novae ſeriei iiſdem difficultatibus erit obnoxia, quas ante meminimus. Ad hanc ergo ſeriem interpolandam inveſtigari oportet valores ipſius S, quos recipit, ſi loco x numeri quicunque non integri ſubſtituantur. Si enim x eſſet numerus integer, tum conveniens ipſius S valor ſine difficultate reperiretur, additione ſcilicet tot terminorum ſeriei $A + B + C + D +$ &c. quot x contineat unitates.

391. Quo igitur ea, quae in Capite praecedente ſunt tradita, in uſum vocari poſſint, ponamus x eſſe numerum integrum, ita ut valor ei reſpondens $S = A + B + C + \ldots + X$ ſit cognitus, & quaeramus valorem Σ, in quem S tranſmutetur, ſi loco x ſcribatur $x + \omega$, exiſtente ω fractione quacunque; eritque Σ terminus ſeriei propoſitae interpolandae, qui reſpondet indici $x + \omega$; quo ergo invento, interpolatio huius ſeriei erit in promtu. Sit Z terminus ſeriei A, B, C, D, E, &c. qui reſpondet indici $x + \omega$, ſintque Z', Z'', Z''', &c. termini eius conſecutivi indices habentes $x + \omega + 1$; $x + \omega + 2$: $x + \omega + 3$; &c. Ac primo quidem ponamus ſeriei A, B, C, D, &c. terminos infiniteſimos evaneſcere. His ergo poſitis ſeries

$$\overset{1}{A};\ \overset{2}{(A+B)};\ \overset{3}{(A+B+C)};\ \overset{4}{(A+B+C+D)};\ \text{\&c.}$$

cuius terminus indici x reſpondens eſt

$$S = A + B + C + \ldots + X$$

interpolabitur quaerendo eius terminum Σ, qui indici fracto $x + \omega$ reſpondeat, erit autem uti invenimus:

$$\Sigma = S \begin{array}{l} + X' + X'' + X''' + X'''' + \text{\&c.} \\ - Z' - Z'' - Z''' - Z'''' - \text{\&c.} \end{array}$$

ſicque habebitur ſeries infinita iſti termino quaeſito Σ aequalis, quae ob

$$Z = X + \frac{\omega dX}{dx} + \frac{\omega^2 ddX}{1.2\, dx^2} + \frac{\omega^3 d^3X}{1.2.3 dx^3} + \text{\&c.}$$

in

in hanc formam transmutatur, ut sit:

$$\Sigma = S - \frac{\omega}{dx}\, d.\,(X' + X'' + X''' + X'''' + \&c.)$$
$$- \frac{\omega^2}{2\,dx^2}\, dd.\,(X' + X'' + X''' + X'''' + \&c.)$$
$$- \frac{\omega^3}{6\,dx^3}\, d^3.\,(X' + X'' + X''' + X'''' + \&c.)$$
&c.

quarum formularum ea, quae quovis casu commodior videatur, adhiberi poterit.

392. Sumamus pro A, B, C, D &c. seriem harmonicam quamcunque $\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \&c.$ cuius terminus generalis seu indici x respondens est $= \frac{1}{a+(x-1)b} = X$. Hinc formata sit ista series:

$$\overset{1}{\frac{1}{a}};\ \overset{2}{\left(\frac{1}{a}+\frac{1}{a+b}\right)};\ \overset{3}{\left(\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}\right)};\ \overset{4}{\left(\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}+\frac{1}{a+3b}\right)}\ \&c.$$

cuius propterea terminus indici x respondens erit:

$$S = \frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \ldots + \frac{1}{a+(x-1)b}.$$

Si iam Σ denotet terminum istius seriei indici $x + \omega$ respondentem, ob $Z = \frac{1}{a+(x+\omega-1)b}$, erit

$$X' = \frac{1}{a+bx};\quad Z' = \frac{1}{a+bx+b\omega}$$
$$X'' = \frac{1}{a+b+bx};\quad Z'' = \frac{1}{a+b+bx+b\omega}$$
$$X''' = \frac{1}{a+2b+bx};\quad Z''' = \frac{1}{a+2b+bx+b\omega}$$
&c. &c.

hinc-

hincque orietur:

$$\Sigma = S + \frac{1}{a + bx} + \frac{1}{a + b + bx} + \frac{1}{a + 2b + bx} + \&c.$$
$$- \frac{1}{a + bx + b\omega} - \frac{1}{a + b + bx + b\omega} - \frac{1}{a + 2b + bx + b\omega} - \&c.$$

altera expressio autem erit huiusmodi:

$$\Sigma = + b\omega \left(\frac{1}{(a + bx)^2} + \frac{1}{(a + b + bx)^2} + \frac{1}{(a + 2b + bx)^2} + \&c. \right)$$
$$- b^2\omega^2 \left(\frac{1}{(a + bx)^3} + \frac{1}{(a + b + bx)^3} + \frac{1}{(a + 2b + bx)^3} + \&c. \right)$$
$$+ b^3\omega^3 \left(\frac{1}{(a + bx)^4} + \frac{1}{(a + b + bx)^4} + \frac{1}{(a + 2b + bx)^4} + \&c. \right)$$
$$\&c.$$

EXEMPLUM I.

Proposita sit ista series

$$\overset{1}{1}\ ;\ \overset{2}{\left(1 + \tfrac{1}{2}\right)}\ ;\ \overset{3}{\left(1 + \tfrac{1}{2} + \tfrac{1}{3}\right)}\ ;\ \overset{4}{\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}\right)}\ ;\ \&c.$$

cuius terminos, qui indicibus fractis respondent, inveniri oporteat.

Erit ergo $a = 1$ & $b = 1$; unde si terminus indici integro x respondens ponatur

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \quad \ldots \quad + \frac{1}{x};$$

terminusque indici fracto $x + \omega$ respondens vocetur $= \Sigma$, erit:

$$\Sigma = S + \frac{1}{1 + x} + \frac{1}{2 + x} + \frac{1}{3 + x} + \frac{1}{4 + x} + \frac{1}{5 + x} + \&c.$$
$$- \frac{1}{1 + x + \omega} - \frac{1}{2 + x + \omega} - \frac{1}{3 + x + \omega} - \frac{1}{4 + x + \omega} - \frac{1}{5 + x + \omega} - \&c.$$

Notandum autem est, si inventus fuerit terminus respondens indici fracto ω, quem ponamus $= T$, ex eo terminum indicis

dicis $x+\omega$ facile inveniri posse; erit enim si T', T'', T''', &c. denotent terminos indicibus $1+\omega$, $2+\omega$, $3+\omega$, &c. respondentes:

$$T' = T + \frac{1}{1+\omega}$$

$$T'' = T + \frac{1}{1+\omega} + \frac{1}{2+\omega}$$

$$T''' = T + \frac{1}{1+\omega} + \frac{1}{2+\omega} + \frac{1}{3+\omega} \quad ; \qquad \&c.$$

unde sufficit eos tantum terminos, qui respondent indicibus ω unitate minoribus, investigasse. Quem in finem ponamus $x=0$, erit quoque $S=0$, atque terminus seriei T indici fracto ω respondens ita exprimetur:

$$\begin{aligned} T = {} & \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c. \\ & - \frac{1}{1+\omega} - \frac{1}{2+\omega} - \frac{1}{3+\omega} - \frac{1}{4+\omega} - \&c. \end{aligned}$$

vel his fractionibus in series infinitas conversis prodibit altera expressio:

$$\begin{aligned} T = & +\omega \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c.\right) \\ & - \omega^2 \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right) \\ & + \omega^3 \left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right) \\ & - \omega^4 \left(1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \&c.\right) \\ & \&c. \end{aligned}$$

quae ad valorem ipsius T proxime inveniendum perquam est apta.

Quaeratur ergo propositae seriei terminus respondens indici $\frac{1}{2}$, qui si ponatur $= T$, erit: $T =$

$$T = 1 - \frac{2}{3} + \frac{1}{2} - \frac{2}{5} + \frac{1}{3} - \frac{2}{7} + \frac{1}{4} - \frac{2}{9} + \&c.$$

seu $T = 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \&c.\right)$

cuius seriei valor est $= 2 - 2\,l\,2$, sicque terminus indicis $= \frac{1}{2}$ finite exprimi potest. Erunt ergo termini sequentes, quorum indices sunt $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, &c. ita expressi:

Ind.	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	
Term.	$2 - 2l2$;	$2 + \frac{2}{3} - 2l2$;	$2 + \frac{2}{3} + \frac{2}{5} - 2l2$;	$2 + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} - 2l2$;	&c.

EXEMPLUM II.

Proposita sit ista series:

$$\overset{1}{1};\ \overset{2}{\left(1 + \frac{1}{3}\right)};\ \overset{3}{\left(1 + \frac{1}{3} + \frac{1}{5}\right)};\ \overset{4}{\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}\right)};\ \&c.$$

cuius terminos indicibus fractis respondentes exprimere oporteat.

Erit ergo $a = 1$, $b = 2$, unde si terminus indici integro x respondens ponatur

$$S = 1 + \frac{1}{3} + \frac{1}{5} + \ \ldots\ + \frac{1}{2x - 1}$$

terminusque indici fracto $x + \omega$ vocetur $= \Sigma$, erit

$$\Sigma = S + \frac{1}{1 + 2x} + \frac{1}{3 + 2x} + \frac{1}{5 + 2x} + \frac{1}{7 + 2x} + \&c.$$
$$- \frac{1}{1 + 2(x + \omega)} - \frac{1}{3 + 2(x + \omega)} - \frac{1}{5 + 2(x + \omega)} - \frac{1}{7 + 2(x + \omega)} - \&c.$$

Cum igitur sufficiat terminos indicibus unitate minoribus assignasse, sit $x = 0$, & $S = 0$: quocirca si terminus indici ω conveniens ponatur $= T$, erit:

$$T = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \&c.$$
$$- \frac{1}{1 + 2\omega} - \frac{1}{3 + 2\omega} - \frac{1}{5 + 2\omega} - \frac{1}{7 + 2\omega} - \frac{1}{9 + 2\omega} - \&c.$$

& si ω numerum quemcunque denotare ponatur, quoniam T est terminus indici ω respondens, erit T terminus generalis seriei propositae, qui etiam hoc modo exprimetur:

$$T = \frac{2\omega}{1(1 + 2\omega)} + \frac{2\omega}{3(3 + 2\omega)} + \frac{2\omega}{5(5 + 2\omega)} + \frac{2\omega}{7(7 + 2\omega)} + \&c. \text{ vel ita}$$

$T =$

$$T = 2\omega\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.\right)$$
$$-4\omega^2\left(1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{9^3} + \&c.\right)$$
$$+8\omega^3\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.\right)$$
$$-16\omega^4\left(1 + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{9^5} + \&c.\right)$$
$$\&c.$$

Ponamus esse $\omega = \frac{1}{2}$, erit terminus huic indici respondens $T = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{5} - \&c. = l\,2$, eruntque

Ind.	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	
Term.	$l\,2$;	$\frac{1}{2} + l\,2$;	$\frac{1}{2} + \frac{1}{4} + l\,2$;	$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + l\,2$;	&c.

Si sit $\omega = \frac{1}{4}$; erit

$$T = \begin{array}{l} +\,1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \&c. \\ -\,\frac{2}{3} - \frac{2}{7} - \frac{2}{11} - \frac{2}{15} - \&c. \end{array}$$

sive $\quad T = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c. = \frac{\pi}{4}$

393. Quod si ergo huius seriei generalis:

$$\frac{1}{a}\,;\ \left(\frac{1}{a} + \frac{1}{a+b}\right);\ \left(\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b}\right);\ \&c.$$

quaeratur terminus respondens indici $= \frac{1}{2}$, ponatur in expressionibus §. praeced. $x = 0$, & $\omega = \frac{1}{2}$; fietque $S = 0$, & terminus indici $\frac{1}{2}$ respondens quaesitus erit

$$\Sigma = \frac{1}{a} - \frac{2}{2a+b} + \frac{1}{a+b} - \frac{2}{2a+3b} + \frac{1}{a+2b} - \frac{2}{2a+5b} + \&c.$$

sive terminis ad maiorem uniformitatem perductis erit

$$\tfrac{1}{2}\Sigma = \frac{1}{2a} - \frac{1}{2a+b} + \frac{1}{2a+2b} - \frac{1}{2a+3b} + \frac{1}{2a+4b} - \&c.$$

in qua serie cum signa + & − alternentur, sumendis continuis differentiis per methodum supra expositam valor ipsius $\frac{1}{2}\Sigma$ per seriem magis convergentem exprimetur. Erunt autem differentiarum series:

$$\frac{b}{2a(2a+b)};\quad \frac{b}{(2a+b)(2a+2b)};\quad \frac{b}{(2a+2b)(2a+3b)};\quad \&c.$$

$$\frac{2bb}{2a(2a+b)(2a+2b)};\quad \frac{2bb}{(2a+b)(2a+2b)(2a+3b)};\quad \&c.$$

$$\frac{6b^3}{2a(2a+b)(2a+2b)(2a+3b)};\quad \&c.$$

&c.

Ex quibus concluditur fore:

$$\tfrac{1}{2}\Sigma = \frac{1}{4a} + \frac{1b}{8a(2a+b)} + \frac{1.\,2bb}{16a(2a+b)(2a+2b)} + \frac{1.\,2.\,3b^3}{32a(2a+b)(2a+2b)(2a+3b)} + \&c.$$

Hincque ergo habebitur:

$$\Sigma = \frac{1}{2a} + \frac{\frac{1}{2}.b}{2a(2a+b)} + \frac{\frac{1}{2}.\frac{2}{2}bb}{2a(2a+b)(2a+2b)} + \frac{\frac{1}{2}.\frac{2}{2}.\frac{3}{2}b^3}{2a(2a+b)(2a+2b)(2a+3b)} + \&c.$$

quae series maxime convergit, atque valorem termini Σ facili labore proxime exhibet.

394. Quod si autem in genere seriei A, B, C, D, E, &c. termini infinitesimi evanescant, terminusque indici ω respondens fuerit $=Z$, eiusque sequentes, qui indicibus $\omega+1$, $\omega+2$, $\omega+3$, &c. respondeant, sint Z', Z'', Z''', Z^{IV}, &c. Si in superioribus (391) ponatur $x=0$, ut sit $S=0$ & $X'=A$, $X''=B$, $X'''=C$, &c. sequetur, si formetur huiusmodi series:

$$\overset{1}{A},\ (A\overset{2}{+}B),\ (A+\overset{3}{B}+C),\ (A+B\overset{4}{+}C+D),\ \&c.$$

eiusque terminus indici ω respondens ponatur $=\Sigma$, fore

$$\Sigma = (A-Z') + (B-Z'') + (C-Z''') + (D-Z^{IV}) + \&c.$$

ex qua expressione termini quicunque intermedii definiri poterunt. Sufficiet autem ad interpolationem perficiendam eos terminos investigasse, qui respondeant indicibus ω unitate minorinoribus. Si enim terminus Σ indici huiusmodi cuicunque ω respondens fuerit repertus, iique qui conveniant indicibus $\omega+1$, $\omega+2$, $\omega+3$, &c. ponantur Σ', Σ'', Σ''', Σ^{IV} &c. erit

$$\Sigma' = \Sigma + Z'$$
$$\Sigma'' = \Sigma + Z' + Z''$$
$$\Sigma''' = \Sigma + Z' + Z'' + Z'''$$
&c.

EXEMPLUM I.

Interpolare hanc seriem:

$$\overset{1}{1};\ \overset{2}{\left(1+\tfrac{1}{4}\right)};\ \overset{3}{\left(1+\tfrac{1}{4}+\tfrac{1}{9}\right)};\ \overset{4}{\left(1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}\right)};$$
&c.

Sit Σ huius seriei terminus respondens indici ω, & cum haec series formata sit ex summatione huius:

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \&c.$$

cuius terminus indici ω respondens est $= \frac{1}{\omega^2}$ erit

$$\Sigma = +1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \&c.$$
$$- \frac{1}{(1+\omega)^2} - \frac{1}{(2+\omega)^2} - \frac{1}{(3+\omega)^2} - \frac{1}{(4+\omega)^2} - \&c.$$

Quod si ergo seriei propositae quaeratur terminus indici $\frac{1}{2}$ respondens, poni debebit $\omega = \frac{1}{2}$, fietque:

$$\Sigma = 1 - \tfrac{4}{9} + \tfrac{1}{4} - \tfrac{4}{25} + \tfrac{1}{9} - \tfrac{4}{49} + \&c. \quad \text{sive}$$
$$\Sigma = 4\left(\tfrac{1}{4} - \tfrac{1}{9} + \tfrac{1}{16} - \tfrac{1}{25} + \tfrac{1}{36} - \tfrac{1}{49} + \&c.\right)$$

Cum igitur sit $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \&c. = \frac{\pi^2}{12}$, erit

$$\Sigma = 4\left(1 - \frac{\pi\pi}{12}\right) = 4 - \tfrac{1}{3}\pi^2,$$

qui est terminus indici $\frac{1}{2}$ respondens. Hinc ergo respondebunt

Indicibus $\frac{1}{2}$; $\frac{3}{2}$; $\frac{5}{2}$; &c.

Termini $4 - \frac{1}{3}\pi^2$; $\frac{4}{1} + \frac{4}{9} - \frac{1}{3}\pi^2$; $\frac{4}{1} + \frac{4}{9} + \frac{4}{25} - \frac{1}{3}\pi^2$; &c.

EXEMPLUM II.

Interpolare hanc seriem:

$$\overset{1}{1};\ \overset{2}{(1+\tfrac{1}{9})};\ \overset{3}{(1+\tfrac{1}{9}+\tfrac{1}{25})};\ \overset{4}{(1+\tfrac{1}{9}+\tfrac{1}{25}+\tfrac{1}{49})}\ \&c.$$

Sit Σ terminus respondens indici cuicunque ω, & cum haec series formata sit ex summatione huius:

$$1+\tfrac{1}{9}+\tfrac{1}{25}+\tfrac{1}{49}+\tfrac{1}{81}+\&c.$$

ex qua sit terminus indici ω respondens $Z=\frac{1}{(2\omega-1)^2}$

erit $Z'=\frac{1}{(2\omega+1)^2};\ Z''=\frac{1}{(2\omega+3)^2};\ Z'''=\frac{1}{(2\omega+5)^2}$ &c.

Quamobrem habebitur:

$$\Sigma = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \&c.$$
$$-\frac{1}{(1+2\omega)^2}-\frac{1}{(3+2\omega)^2}-\frac{1}{(5+2\omega)^2}-\frac{1}{(7+2\omega)^2}-\&c.$$

Ponamus $\omega=\frac{1}{2}$, ut inveniamus terminum seriei propositae respondentem indici $=\frac{1}{2}$, qui erit:

$$\Sigma = 1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\&c.=\frac{\pi\pi}{12},$$

ex quo termini, qui medium interiacent inter binos quosvis datos, sequenti modo exprimentur. Respondebunt

Ind. $\frac{1}{2}$; $\frac{3}{2}$; $\frac{5}{2}$; $\frac{7}{2}$; &c.

Term. $\frac{\pi\pi}{12};\ \frac{1}{4}+\frac{\pi\pi}{12};\ \frac{1}{4}+\frac{1}{16}+\frac{\pi\pi}{12};\ \frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\frac{\pi\pi}{12};$ &c.

EXEMPLUM III.

Interpolare hanc seriem:

$$\overset{1}{1};\ \overset{2}{\left(1+\frac{1}{2^n}\right)};\ \overset{3}{\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)};\ \overset{4}{\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right)};\ \&c.$$

Sit ut ante Σ terminus indici ω respondens, erit $Z=$

$Z = \frac{1}{\omega^n}$; & $Z' = \frac{1}{(1+\omega)^n}$; $Z'' = \frac{1}{(2+\omega)^n}$; $Z''' = \frac{1}{(3+\omega)^n}$; &c.

hincque habebitur:

$$\Sigma = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c.$$
$$- \frac{1}{(1+\omega)^n} - \frac{1}{(2+\omega)^n} - \frac{1}{(3+\omega)^n} - \frac{1}{(4+\omega)^n} - \&c.$$

Si igitur desideretur terminus indici $\frac{1}{2}$ respondens: erit is

$$1 - \frac{2^n}{3^n} + \frac{1}{2^n} - \frac{2^n}{5^n} + \frac{1}{3^n} - \frac{2^n}{7^n} + \&c.$$

$$\text{seu} = 2^n \left(\frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \&c. \right)$$

Quare si ponatur:

$$\mathfrak{N} = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \&c.$$

erit seriei propositae terminus qui indici $\frac{1}{2}$ respondet $= 2^n(1 - \mathfrak{N})$; hincque respondebunt

Indic.	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	&c.
Term.	$2^n - 2^n\mathfrak{N}$;	$2^n + \frac{2^n}{3^n} - 2^n\mathfrak{N}$;	$2^n + \frac{2^n}{3^n} + \frac{2^n}{5^n} - 2^n\mathfrak{N}$;	&c.

EXEMPLUM IV.

Interpolare hanc seriem:

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 1; & \left(1 + \frac{1}{3^n}\right); & \left(1 + \frac{1}{3^n} + \frac{1}{5^n}\right); & \left(1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n}\right); \&c. \end{matrix}$$

Sit Σ terminus qui indici cuicunque ω respondeat, & cum sit $Z = \frac{1}{(2\omega - 1)^n}$, erit:

$$Z' = \frac{1}{(2\omega+1)^n}; \; Z'' = \frac{1}{(2\omega+3)^n}; \; Z''' = \frac{1}{(2\omega+5)^n}; \; \&c.$$

 at-

atque $\mathfrak{M} = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \&c.$

$$- \frac{1}{(1+2\omega)^n} - \frac{1}{(3+2\omega)^n} - \frac{1}{(5+2\omega)^n} - \frac{1}{(7+2\omega)^n} - \&c.$$

Ponatur $\omega = \frac{1}{2}$, & prodibit terminus indici $\frac{1}{2}$ respondens

$$= 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \&c. = \mathfrak{N}$$

ex quo porro erunt reliqui termini inter binos datos medii.

Indices: $\frac{1}{2}$; $\frac{3}{2}$; $\frac{5}{2}$; &c.

Termini: $\mathfrak{N}$; $\frac{1}{2^n} + \mathfrak{N}$; $\frac{1}{2^n} + \frac{1}{4^n} + \mathfrak{N}$; &c.

395. Ponamus nunc feriei A, B, C, D, E, &c. ex cuius fummatione feries interpolanda formatur, terminos infinitefimos non evanefcere, fed ita effe comparatos, ut eorum differentiae evanefcant; fitque X huius feriei terminus refpondens indici x, & Z terminus refpondens exponenti $x+\omega$, tum vero fint X', X'', X''', X'''', &c. termini ipfum X fequentes, & Z', Z'', Z''', &c. termini ipfum Z fequentes. Quibus pofitis proponatur haec feries interpolanda:

$$\overset{1}{A}\,;\ (A+\overset{2}{B})\,;\ (A+\overset{3}{B}+C)\,;\ (A+B+\overset{4}{C}+D)\,;\ \&c.$$

cuius terminus indici x refpondens fit $= S$, at terminus indici $x+\omega$ refpondens fit $= \mathfrak{S}$; eritque ex iis, quae Capite praecedente funt tradita:

$$\mathfrak{S} = S + X' + X'' + X''' + \&c. - Z' - Z'' - Z''' - \&c. + \omega X' + \omega \left\{ \begin{array}{l} X'' + X''' + X'''' + \&c. \\ -X' - X'' - X''' - \&c. \end{array} \right\}$$

Quia autem ut ante fufficit terminos indicibus unitate minoribus refpondentes inveftigaffe, ponamus $x = 0$, ut fit $S = 0$, $X' = A$, $X'' = B$, &c. eritque terminus indici ω refpondens:

$$\mathfrak{S} = (A - Z') + (B - Z'') + (C - Z''') + (D - Z'''')\ \&c. + \omega A + \omega[(B-A) + (C-B) + (D-C) + (E-D) + \&c.]$$

Vel fi differentias has more fupra recepto exprimere velimus quo eft $\Delta A = B - A$; $\Delta B = C - B$; &c. habebitur:

$$\Sigma = (A - Z') + (B - Z'') + (C - Z''') + (D - Z'''') + \&c. + \omega(A + \Delta A + \Delta B + \Delta C + \Delta D + \&c.)$$

396. Sin autem seriei A, B, C, D, E, &c. ex cuius summatione series interpolanda formatur, termini infinitesimi neque ipsi evanescant, neque differentias primas habeant evanescentes; tum plures series ad valorem ipsius Σ exprimendum adiici debebunt, quoad scilicet ad differentias terminorum infinitesimorum evanescentes perveniatur. Sit enim ut ante seriei A, B, C, D, E, &c. terminus indici x respondens $= X$, eumque sequentes X', X'', X''', &c. indici autem $x + \omega$ respondeat terminus Z, quem sequantur Z', Z'', &c. atque proponatur haec series:

$$\overset{1}{A}\,;\ (A + \overset{2}{B})\,;\ (A + \overset{3}{B} + C)\,;\ (A + B \overset{4}{+} C + D)\,;\ \&c.$$

cuius terminus indici x respondens sit

$$S = A + B + C + D + \ldots\ldots + X$$

indici vero $x + \omega$ respondeat terminus Σ; ita ut

indicibus	respondeant termini
$x + \omega + 1$	$\Sigma' = \Sigma + Z'$
$x + \omega + 2$	$\Sigma'' = \Sigma + Z' + Z''$
$x + \omega + 3$	$\Sigma''' = \Sigma + Z' + Z'' + Z'''$
&c.	&c.

Si iam differentiae terminorum ita exprimantur, ut sit

$$\Delta X' = X'' - X';\ \Delta X'' = X''' - X'';\ \Delta X''' = X'''' - X''';\ \&c.$$
$$\Delta^2 X' = \Delta X'' - \Delta X';\ \Delta^2 X'' = \Delta X''' - \Delta X'';\ \Delta^2 X''' = \Delta X'''' - \Delta X''';\ \&c.$$
$$\Delta^3 X' = \Delta^2 X'' - \Delta^2 X';\ \Delta^3 X'' = \Delta^2 X''' - \Delta^2 X'';\ \&c.$$

ex §. 377. terminus Σ sequenti modo exprimetur:

$$\begin{aligned}\Sigma = S &+ X' + X'' + X''' + X'''' + \&c.\\ &- Z' - Z'' - Z''' - Z'''' - \&c.\\ &+ \omega[X' + \Delta X' + \Delta X'' + \Delta X''' + \Delta X'''' + \&c.]\\ &+ \frac{\omega(\omega - 1)}{1 \cdot 2}[\Delta X' + \Delta^2 X' + \Delta^2 X'' + \Delta^2 X''' + \Delta^2 X'''' + \&c.]\\ &+ \frac{\omega(\omega - 1)(\omega - 2)}{1 \cdot 2 \cdot 3}[\Delta^2 X' + \Delta^3 X' + \Delta^3 X'' + \Delta^3 X''' + \Delta^3 X'''' + \&c.]\end{aligned}$$

397.

397. Sufficit, uti iam notavimus, tot huiusmodi series adiecisse, donec ad terminorum infinitesimorum differentias evanescentes perveniatur: si enim has ipsas series quoque in infinitum continuare velimus, vel eo usque saltem, donec terminorum finitorum differentiae evanescant; tum ob

$$Z' = X' + \omega\Delta X' + \frac{\omega(\omega-1)}{1.\,2}\Delta^2 X' + \frac{\omega(\omega-1)(\omega-2)}{1.\,2.\,3}\Delta^3 X' + \&c.$$

tota expressio inventa contrahetur in hanc:

$$\Sigma = S + \omega X' + \frac{\omega(\omega-1)}{1.\,2}\Delta X' + \frac{\omega(\omega-1)(\omega-2)}{1.\,2.\,3}\Delta^2 X' + \&c.$$

quae terminum summatorium seriei $A + B + C + D +$ &c. involvit; qui autem si esset cognitus, interpolatio nullam haberet difficultatem. Interim tamen & hac formula uti licebit, quippe quae, quoties abrumpitur, quemvis terminum interpolandum finite & algebraice expressum exhibet: sin autem in infinitum progrediatur, plerumque praestat priorem formulam adhibere, in qua ratio terminorum infinitesimorum habetur. Haec vero, si ponatur $x = 0$, ut Σ denotet terminum indici ω respondentem, ob $S = 0$ hanc formam induet:

$$\begin{array}{rl}
\Sigma = & + A + B + C + D + \&c. \\
& - Z' - Z'' - Z''' - Z'''' - \&c. \\
& + \omega[A + \Delta A + \Delta B + \Delta C + \Delta D + \&c.] \\
& + \frac{\omega(\omega-1)}{1.\,2}[\Delta A + \Delta^2 A + \Delta^2 B + \Delta^2 C + \Delta^2 D + \&c.] \\
& + \frac{\omega(\omega-1)(\omega-2)}{1.\,2.\,3}[\Delta^2 A + \Delta^3 A + \Delta^3 B + \Delta^3 C + \Delta^3 D + \&c.] \\
& \&c.
\end{array}$$

Vel si ponatur brevitatis gratia:

$$\omega = \alpha;\quad \frac{\omega(\omega-1)}{1.\,2} = \beta;\quad \frac{\omega(\omega-1)(\omega-2)}{1.\,2.\,3} = \gamma;\ \&c.$$

erit

$$\begin{array}{rl}
\Sigma = & \alpha A + \beta\Delta A + \gamma\Delta^2 A + \delta\Delta^3 A + \&c. \\
& + A + \alpha\Delta A + \beta\Delta^2 A + \gamma\Delta^3 A + \&c. - Z' \\
& + B + \alpha\Delta B + \beta\Delta^2 B + \gamma\Delta^3 B + \&c. - Z'' \\
& + C + \alpha\Delta C + \beta\Delta^2 C + \gamma\Delta^3 C + \&c. - Z''' \\
& \&c.
\end{array}$$

qua-

quarum serierum horizontalium numerus in infinitum quidem progreditur, at quaelibet finito terminorum numero constat.

EXEMPLUM.

Interpolare hanc seriem:

$$\overset{1}{\tfrac{1}{2}};\ \overset{2}{\tfrac{1}{2}+\tfrac{2}{3}};\ \overset{3}{\tfrac{1}{2}+\tfrac{2}{3}+\tfrac{3}{4}};\ \overset{4}{\tfrac{1}{2}+\tfrac{2}{3}+\tfrac{3}{4}+\tfrac{4}{5}};\ \&c.$$

Sit huius seriei terminus indici ω respondens $= \Sigma$, & cum ea oriatur ex summatione huius seriei:

$$\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \&c. \quad \text{erit} \quad Z = \frac{\omega}{\omega+1};$$

& quia termini infinitesimi differentias suas primas iam habent evanescentes, differentiae tantum primae sunt accipiendae; quae erunt:

$$\text{ob}\ A = \frac{1}{2};\ B = \frac{2}{3};\ C = \frac{3}{4};\ D = \frac{4}{5};\ \&c.$$

$$\Delta A = \frac{1}{2.3};\ \Delta B = \frac{1}{3.4};\ \Delta C = \frac{1}{4.5};\ \&c.$$

Hinc ergo habebitur:

$$\begin{aligned}\Sigma = \frac{\omega}{2} &+ \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \&c.\\ &+ \frac{\omega}{2.3} + \frac{\omega}{3.4} + \frac{\omega}{4.5} + \frac{\omega}{5.6} + \&c.\\ &- \frac{\omega+1}{\omega+2} - \frac{\omega+2}{\omega+3} - \frac{\omega+3}{\omega+4} - \frac{\omega+4}{\omega+5} - \&c.\end{aligned}$$

seu ob $\frac{\omega}{2} + \frac{\omega}{2.3} + \frac{\omega}{3.4} + \frac{\omega}{4.5} + \&c. = \omega$; erit

$$\begin{aligned}\Sigma = \omega &+ \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \&c.\\ &- \frac{\omega+1}{\omega+2} - \frac{\omega+2}{\omega+3} - \frac{\omega+3}{\omega+4} - \frac{\omega+4}{\omega+5} - \&c.\end{aligned}$$

Si ergo quaeratur terminus indici $\frac{1}{2}$ respondens, erit is

$\Sigma =$

$$\Sigma = \frac{1}{2} + \frac{1}{2} - \frac{3}{5} + \frac{2}{3} - \frac{5}{7} + \frac{3}{4} - \frac{7}{9} + \frac{4}{5} - \frac{9}{11} + \&c.$$

seu $\Sigma = \frac{1}{2} - \frac{1}{2.5} - \frac{1}{3.7} - \frac{1}{4.9} - \frac{1}{5.11} - \frac{1}{6.13} - \&c.$

ideoque $\frac{1}{2}\Sigma = \frac{1}{4} - \frac{1}{4.5} - \frac{1}{6.7} - \frac{1}{8.9} - \frac{1}{10.11} - \frac{1}{12.13} - \&c.$

$$\text{seu } \tfrac{1}{2}\Sigma = \frac{1}{4} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} - \&c.$$
$$+ \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \&c.$$

Quare cum sit $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c. = l2$

erit $\frac{1}{2}\Sigma = l2 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = l2 - \frac{7}{12}$,

ideoque $\Sigma = 2l2 - \frac{7}{6}$.

398. Pergamus nunc ad ſeries interpolandas, quarum termini ex factoribus ſunt conflati, ſitque propoſita haec ſeries generaliſſima:

$$\overset{1}{A};\ \overset{2}{AB};\ \overset{3}{ABC};\ \overset{4}{ABCD};\ \overset{5}{ABCDE};\ \&c.$$

cuius terminus indici ω reſpondens ſit $= \Sigma$. Erit ergo $l\Sigma$ terminus reſpondens indici ω in hac ſerie:

$$\overset{1}{lA};\ \overset{2}{(lA + lB)};\ \overset{3}{(lA + lB + lC)};\ \overset{4}{(lA + lB + lC + lD)};\ \&c.$$

Quodſi ergo ponamus huius ſeriei terminos infiniteſimos evaneſcere; atque ſeriei A, B, C, D, E, &c. terminum indici ω reſpondentem eſſe Z, eiuſque ſequentes indicibus $\omega+1, \omega+2, \omega+3$, &c. reſpondentes eſſe Z', Z'', Z''', Z'''', &c. erit ex ſupra demonſtratis: $l\Sigma = \begin{matrix} + lA + lB + lC + lD + \&c. \\ - lZ' - lZ'' - lZ''' - lZ'''' - \&c. \end{matrix}$

Hinc

Hinc igitur ad numeros progrediendo habebitur:

$$\Sigma = \frac{A}{Z'} \cdot \frac{B}{Z''} \cdot \frac{C}{Z'''} \cdot \frac{D}{Z''''} \cdot \&c.$$

399. Quodsi autem terminorum infinitesimorum seriei A, B, C, D, &c. logarithmi non evanescant, sed habeant differentias evanescentes, erit uti vidimus:

$$\begin{aligned} l\Sigma = &+ lA + lB + lC + \&c. \\ &- lZ' - lZ'' - lZ''' - \&c. \\ &+ \omega lA + \omega \left(l\frac{B}{A} + l\frac{C}{B} + l\frac{D}{C} + \&c. \right) \end{aligned}$$

hincque ad numeros a logarithmis procedendo fiet

$$\Sigma = A^{\omega} \cdot \frac{A^{1-\omega} B^{\omega}}{Z'} \cdot \frac{B^{1-\omega} C^{\omega}}{Z''} \cdot \frac{C^{1-\omega} D^{\omega}}{Z'''} \cdot \frac{D^{1-\omega} E^{\omega}}{Z''''} \cdot \&c.$$

At si illorum logarithmorum infinitesimorum differentiae demum secundae evanescant, erit:

$$\begin{aligned} l\Sigma = \; & lA + lB + lC + lD + \&c. \\ & - lZ' - lZ'' - lZ''' - lZ'''' - \&c. \\ & + \omega \left(lA + l\frac{B}{A} + l\frac{C}{B} + l\frac{D}{C} + l\frac{E}{D} + \&c. \right) \\ & + \frac{\omega(\omega-1)}{1 \cdot 2} \left(l\frac{B}{A} + l\frac{AC}{B^2} + l\frac{BD}{C^2} + l\frac{CE}{D^2} + l\frac{DE}{E^2} + \&c. \right) \end{aligned}$$

Ex his itaque obtinebitur:

$$\Sigma = A^{\frac{\omega(3-\omega)}{2}} \cdot B^{\frac{\omega(\omega-1)}{1 \cdot 2}} \cdot$$

$$\frac{A^{\frac{(\omega-1)(\omega-2)}{1 \cdot 2}} B^{\omega(2-\omega)} C^{\frac{\omega(\omega-1)}{1 \cdot 2}}}{Z'} \cdot \frac{B^{\frac{(\omega-1)(\omega-2)}{1 \cdot 2}} C^{\omega(2-\omega)} D^{\frac{\omega(\omega-1)}{1 \cdot 2}}}{Z''} \; \&c.$$

quae si $\omega < 1$ commodius ita exprimetur:

$$\Sigma = \frac{A^{\frac{\omega(3-\omega)}{1 \cdot 2}}}{B^{\frac{\omega(1-\omega)}{1 \cdot 2}}} \cdot \frac{A^{\frac{(1-\omega)(2-\omega)}{1 \cdot 2}} B^{\omega(2-\omega)}}{C^{\frac{\omega(1-\omega)}{1 \cdot 2}} Z'} \cdot \frac{B^{\frac{(1-\omega)(2-\omega)}{1 \cdot 2}} C^{\omega(2-\omega)}}{D^{\frac{\omega(1-\omega)}{1 \cdot 2}} Z''} \cdot \&c.$$

400. Accommodemus hanc interpolationem ad iſtam ſeriem:

$$\overset{1}{\frac{a}{b}};\quad \overset{2}{\frac{a(a+c)}{b(b+c)}};\quad \overset{3}{\frac{a(a+c)(a+2c)}{b(b+c)(b+2c)}};\quad \overset{4}{\frac{a(a+c)(a+2c)(a+3c)}{b(b+c)(b+2c)(b+3c)}};\ \&c.$$

cuius factores deſumti ſunt ex hac ſerie:

$$\overset{1}{\frac{a}{b}};\quad \overset{2}{\frac{a+c}{b+c}};\quad \overset{3}{\frac{a+2c}{b+2c}};\quad \overset{4}{\frac{a+3c}{b+3c}};\ \&c.$$

cuius terminorum infiniteſimorum logarithmi ſunt $=0$.

Erit ergo $Z=\frac{a-c+c\omega}{b-c+c\omega}$; $Z'=\frac{a+c\omega}{b+c\omega}$; &c.

Hinc ſi illius ſeriei terminus indici ω reſpondens ponatur $=\Sigma$, erit ex §. 398:

$$\Sigma=\frac{a(b+c\omega)}{b(a+c\omega)}\cdot\frac{(a+c)(b+c+c\omega)}{(b+c)(a+c+c\omega)}\cdot\frac{(a+2c)(b+2c+c\omega)}{(b+2c)(b+2c+c\omega)}\cdot\ \&c.$$

Quare ſi deſideretur terminus indici $\frac{1}{2}$ reſpondens, facto $\omega=\frac{1}{2}$, erit:

$$\Sigma=\frac{a(2b+c)}{b(2a+c)}\cdot\frac{(a+c)(2b+3c)}{(b+c)(2a+3c)}\cdot\frac{(a+2c)(2b+5c)}{(b+2c)(2a+5c)}\cdot\ \&c.$$

EXEMPLUM.

Interpolare hanc ſeriem:

$$\overset{1}{\frac{1}{2}};\quad \overset{2}{\frac{1\cdot3}{2\cdot4}};\quad \overset{3}{\frac{1\cdot3\cdot5}{2\cdot4\cdot6}};\quad \overset{4}{\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}};\quad \overset{5}{\frac{1\cdot3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10}};\ \&c.$$

Cum hic ſit $a=1$, $b=2$, & $c=2$; ſit terminus indici cuicunque ω reſpondens $=\Sigma$, erit

$$\Sigma=\frac{1(2+2\omega)}{2(1+2\omega)}\cdot\frac{3(4+2\omega)}{4(3+2\omega)}\cdot\frac{5(6+2\omega)}{6(5+2\omega)}\cdot\frac{7(8+2\omega)}{8(7+2\omega)}\cdot\ \&c.$$

Hinc ſi termini, qui indicibus $\omega+1$, $\omega+2$, $\omega+3$, &c. reſpondent, ponantur Σ', Σ'', Σ''', &c. erit: $\Sigma'=$

$$\Sigma' = \frac{1+2\omega}{2+2\omega} \cdot \Sigma$$

$$\Sigma'' = \frac{1+2\omega}{2+2\omega} \cdot \frac{3+2\omega}{4+2\omega} \cdot \Sigma$$

$$\Sigma''' = \frac{1+2\omega}{2+2\omega} \cdot \frac{3+2\omega}{4+2\omega} \cdot \frac{5+2\omega}{6+2\omega} \cdot \Sigma \qquad \&c.$$

Si itaque desideretur terminus indici $\frac{1}{2}$ respondens, facto $\omega = \frac{1}{2}$,

erit: $\Sigma = \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} \cdot \frac{7 \cdot 9}{8 \cdot 8} \cdot \frac{9 \cdot 11}{10 \cdot 10} \cdot$ &c.

Verum posito $\pi =$ semicircumferentiae circuli, cuius radius est $= 1$, supra ostendimus esse:

$$\pi = 2 \cdot \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdot \frac{8 \cdot 8}{7 \cdot 9} \cdot \&c.$$

Hancobrem termini intermedii indicibus $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, &c. per peripheriam circuli exprimi poterunt, hoc modo:

Indices: $\frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{2}$ $\frac{7}{2}$

Termini: $\frac{2}{\pi}$; $\frac{2}{3} \cdot \frac{2}{\pi}$; $\frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{2}{\pi}$; $\frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \cdot \frac{2}{\pi}$; &c.

Quam eandem interpolationem *Wallisius* in arithmetica infinitorum invenit:

401. Consideremus nunc istam seriem:

$$\overset{1}{a};\ \overset{2}{a(a+b)};\ \overset{3}{a(a+b)(a+2b)};\ \overset{4}{a(a+b)(a+2b)(a+3b)};\ \&c.$$

cuius factores hanc progressionem arithmeticam constituunt:

$$a,\ (a+b),\ (a+2b),\ (a+3b),\ (a+4b),\ \&c.$$

huiusque termini infinitesimi ita sunt comparati, ut eorum logarithmorum differentiae evanescant. Cum igitur sit $Z = a - b + b\omega$, & $Z' = a + b\omega$; $Z'' = a + b + b\omega$; $Z''' = a + 2b + b\omega$; &c. si Σ denotet terminum seriei propositae, cuius index est $= \omega$, erit:

$$\Sigma = a^{\omega} \cdot \frac{a^{1-\omega}(a+b)^{\omega}}{a+b\omega} \cdot \frac{(a+b)^{1-\omega}(a+2b)^{\omega}}{a+b+b\omega} \cdot \frac{(a+2b)^{1-\omega}(a+3b)^{\omega}}{a+2b+b\omega} \cdot \&c.$$

 Hoc-

Hocque valore invento, si ω denotet numerum quemvis fractum unitate minorem, termini sequentes indicibus $1+\omega$, $2+\omega$, $3+\omega$, &c. respondentes ita determinabuntur, ut sit

$$\Sigma' = (a+b\omega)\,\Sigma$$
$$\Sigma'' = (a+b\omega)\,(a+b+b\omega)\,\Sigma$$
$$\Sigma''' = (a+b\omega)\,(a+b+b\omega)\,(a+2b+b\omega)\,\Sigma$$
&c.

Quare si desideretur terminus indici $\frac{1}{2}$ respondens, facto $\omega = \frac{1}{2}$, erit:

$$\Sigma = a^{\frac{1}{2}} \cdot \frac{a^{\frac{1}{2}}(a+b)^{\frac{1}{2}}}{a+\frac{1}{2}b} \cdot \frac{(a+b)^{\frac{1}{2}}(a+2b)^{\frac{1}{2}}}{a+\frac{3}{2}b} \cdot \frac{(a+2b)^{\frac{1}{2}}(a+3b)^{\frac{1}{2}}}{a+\frac{5}{2}b} \cdot \&c.$$

ideoque sumtis quadratis:

$$\Sigma^2 = a \cdot \frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{1}{2}b)} \cdot \frac{(a+b)(a+2b)}{(a+\frac{3}{2}b)(a+\frac{3}{2}b)} \cdot \frac{(a+2b)(a+3b)}{(a+\frac{5}{2}b)(a+\frac{5}{2}b)} \cdot \&c.$$

402. Ponatur in serie quam supra tractavimus:

$$\overset{1}{\frac{f}{g}};\ \overset{2}{\frac{f(f+b)}{g(g+b)}};\ \overset{3}{\frac{f(f+b)(f+2b)}{g(g+b)(g+2b)}};\ \overset{4}{\frac{f(f+b)(f+2b)(f+3b)}{g(g+b)(g+2b)(g+3b)}};\ \&c.$$

terminus indici $\frac{1}{2}$ respondens $= \Theta$, erit:

$$\Theta = \frac{f(g+\frac{1}{2}b)}{g(f+\frac{1}{2}b)} \cdot \frac{(f+b)(g+\frac{3}{2}b)}{(g+b)(f+\frac{3}{2}b)} \cdot \frac{(f+2b)(g+\frac{5}{2}b)}{(g+2b)(f+\frac{5}{2}b)} \cdot \&c.$$

statuatur nunc: $f = a$; $g = a+\frac{1}{2}b$; & $b = b$; erit

$$\Theta = \frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{1}{2}b)} \cdot \frac{(a+b)(a+2b)}{(a+\frac{3}{2}b)(a+\frac{3}{2}b)} \cdot \&c.$$

ideoque fiet $\Sigma^2 = a\Theta$, & $\Sigma = \sqrt{a\Theta}$. Quocirca si huius seriei:

$$\overset{1}{a};\ \overset{2}{a(a+b)};\ \overset{3}{a(a+b)(a+2b)};\ \overset{4}{a(a+b)(a+2b)(a+3b)};\ \&c.$$

terminus indici $\frac{1}{2}$ respondens statuatur $= \Sigma$; atque huius seriei:

$$\overset{1}{\frac{a}{a+\frac{1}{2}b}};\ \overset{2}{\frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{3}{2}b)}};\ \overset{3}{\frac{a(a+b)(a+2b)}{(a+\frac{1}{2}b)(a+\frac{3}{2}b)(a+\frac{5}{2}b)}};\ \&c.$$

terminus indici $\frac{1}{2}$ reſpondens ponatur $=\Theta$; erit $\Sigma=\sqrt{a\Theta}$.

Cum igitur hic ſeriei ſolorum numeratorum terminus indici $\frac{1}{2}$ reſpondens ſit $=\Sigma$, ſi in ſerie denominatorum terminus indici $\frac{1}{2}$ reſpondens ponatur $=\Lambda$; erit $\Theta=\frac{\Sigma}{\Lambda}$; at eſt $\Theta=\frac{\Sigma^2}{a}$, unde fiet $\Sigma=\frac{a}{\Lambda}$, ſeu $\Sigma\Lambda=a$, quibus theorematibus interpolatio huiuſmodi ſerierum non mediocriter illuſtratur.

EXEMPLUM I.

Sit propoſita haec ſeries interpolanda:

$$1,\ 1\cdot 2,\ 1\cdot 2\cdot 3,\ 1\cdot 2\cdot 3\cdot 4,\ \&c.$$

Quia hic eſt $a=1$, & $b=1$, ſi terminus indici ω reſpondens ponatur $=\Sigma$, erit:

$$\Sigma=\frac{1^{1-\omega}2^{\omega}}{1+\omega}\cdot\frac{2^{1-\omega}3^{\omega}}{2+\omega}\cdot\frac{3^{1-\omega}4^{\omega}}{3+\omega}\cdot\frac{4^{1-\omega}5^{\omega}}{4+\omega}\ \&c.$$

Hic pro ω ſemper fractio unitate minor accipi poteſt nihilominus enim interpolatio per totam ſeriem extendetur. Nam ſi termini indicibus $1+\omega$, $2+\omega$, $3+\omega$, &c. reſpondentes ponantur Σ', Σ'', Σ''', &c. erit:

$$\Sigma'=(1+\omega)\Sigma$$
$$\Sigma''=(1+\omega)(2+\omega)\Sigma$$
$$\Sigma'''=(1+\omega)(2+\omega)(3+\omega)\Sigma$$
$$\&c.$$

Seriei ergo propoſitae terminus indici $\frac{1}{2}$ reſpondens erit:

$$\Sigma=\frac{1^{\frac{1}{2}}\cdot 2^{\frac{1}{2}}}{\frac{3}{2}};\ \frac{2^{\frac{1}{2}}\cdot 3^{\frac{1}{2}}}{2\frac{1}{2}};\ \frac{3^{\frac{1}{2}}\cdot 4^{\frac{1}{2}}}{3\frac{1}{2}};\ \&c.\quad \text{ſive}$$

$$\Sigma^2=\frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{8\cdot 10}{9\cdot 9}\cdot\ \&c.$$

Unde cum sit $\pi = 2 . \frac{2.2}{1.3} . \frac{4.4}{3.5} . \frac{6.6}{5.7} . \&c.$

erit $\Sigma^2 = \frac{\pi}{4}$ & $\Sigma = \frac{\sqrt{\pi}}{2}$: hincque respondebunt

Indicibus : $\frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{2}$ $\frac{7}{2}$

Termini : $\frac{\sqrt{\pi}}{2}$; $\frac{3}{2} . \frac{\sqrt{\pi}}{2}$; $\frac{3.5}{2.2} . \frac{\sqrt{\pi}}{2}$; $\frac{3.5.7}{2.2.2} . \frac{\sqrt{\pi}}{2}$; &c.

EXEMPLUM II.

Sit proposita haec series interpolanda :

$$\overset{1}{1} ; \overset{2}{1 . 3} ; \overset{3}{1 . 3 . 5} ; \overset{4}{1 . 3 . 5 . 7} ; \&c.$$

Quia hic est $a = 1$; $b = 2$; si terminus indici ω respondens ponatur $= \Sigma$ erit :

$$\Sigma = \frac{1^{1-\omega} 3^{\omega}}{1 + 2\omega} . \frac{3^{1-\omega} 5^{\omega}}{3 + 2\omega} . \frac{5^{1-\omega} 7^{\omega}}{5 + 2\omega} . \&c.$$

terminique ordine sequentes ita erunt comparati :

$$\Sigma' = (1 + 2\omega) \Sigma$$
$$\Sigma'' = (1 + 2\omega)(3 + 2\omega) \Sigma$$
$$\Sigma''' = (1 + 2\omega)(3 + 2\omega)(5 + 2\omega) \Sigma$$
&c.

Si igitur seriei propositae desideretur terminus indici $\frac{1}{2}$ respondens, isque vocetur $= \Sigma$, erit :

$$\Sigma = \frac{\sqrt{1.3}}{2} . \frac{\sqrt{3.5}}{4} . \frac{\sqrt{5.7}}{6} . \frac{\sqrt{7.9}}{8} . \&c. \quad \text{ergo}$$

$$\Sigma^2 = \frac{1.3}{2.2} . \frac{3.5}{4.4} . \frac{5.7}{6.6} . \frac{7.9}{8.8} . \&c. = \frac{2}{\pi} ,$$

ideoque habebitur $\Sigma = \sqrt{\frac{2}{\pi}}$. At respondebunt

Indicibus : $\frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{2}$ $\frac{7}{2}$ &c.

Termini : $\sqrt{\frac{2}{\pi}}$; $2 . \sqrt{\frac{2}{\pi}}$; $2 . 4 \sqrt{\frac{2}{\pi}}$; $2 . 4 . 6 \sqrt{\frac{2}{\pi}}$; &c.

Quodsi ergo prior series & haec invicem multiplicentur ut habeatur haec series:

1

$$\overset{1}{1^2}\,;\quad \overset{2}{1^2.2.3}\,;\quad \overset{3}{1^2.2.3^2.5}\,;\quad \overset{4}{1^2.2.3^2.4.5.7}\,;\quad \overset{5}{1^2.2.3^2.4.5^2.7.9}\,;\ \text{\&c.}$$

cuius terminus indici $\frac{1}{2}$ reſpondens erit $= \frac{\sqrt{\pi}}{2}\cdot\sqrt{\frac{2}{\pi}} = \frac{1}{\sqrt{2}}$; quod facile perſpicitur, ſi iſti ſeriei haec forma tribuatur:

$$\overset{1}{\frac{1.2}{2}}\,;\ \overset{2}{\frac{1.2.3.4}{2^2}}\,;\ \overset{3}{\frac{1.2.3.4.5.6}{2^3}}\,;\ \overset{4}{\frac{1.2.3.4.5.6.7.8}{2^4}}\,;\ \text{\&c.}$$

cuius terminus indici $\frac{1}{2}$ reſpondens manifeſto eſt $= \frac{1}{\sqrt{2}}$.

EXEMPLUM III.

Sit iſta ſeries propoſita interpolanda:

$$\overset{1}{\frac{n}{1}}\,;\ \overset{2}{\frac{n(n-1)}{1.2}}\,;\ \overset{3}{\frac{n(n-1)(n-2)}{1.2.3}}\,;\ \overset{4}{\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}}\,;\ \text{\&c.}$$

Conſiderentur huius ſeriei numeratores ac denominatores ſeorſim, & cum numeratores ſint:

$$\overset{1}{n}\,;\ \overset{2}{n(n-1)}\,;\ \overset{3}{n(n-1)(n-2)}\,;\ \overset{4}{n(n-1)(n-2)(n-3)}\,;\ \text{\&c.}$$

fiet applicatione facta, $a = n$, & $b = -1$, unde huius ſeriei terminus indici ω reſpondens erit =

$$n^{\omega}.\frac{n^{1-\omega}(n-1)^{\omega}}{n-\omega}.\frac{(n-1)^{1-\omega}(n-2)^{\omega}}{n-1-\omega}.\frac{(n-2)^{1-\omega}(n-3)^{\omega}}{n-2-\omega}.\ \text{\&c.}$$

quae autem expreſſio ob factores in negativos abeuntes nihil certi monſtrat. Transformetur ergo ſeries propoſita, ponendo brevitatis gratia $1.2.3\ .\ .\ .\ n = N$, in hanc:

$$N\left(\overset{1}{\frac{1}{1.1.2.3\ldots(n-1)}}\,;\ \overset{2}{\frac{1}{1.2.1.2.3\ldots(n-2)}}\,;\ \overset{3}{\frac{1}{1.2.3.1.2.3\ldots(n-3)}}\,;\ \text{\&c.}\right)$$

cuius denominatores cum conſtent duobus factoribus, alteri conſtituent hanc ſeriem:

$$\overset{1}{1.2.3\ldots(n-1)}\,;\ \overset{2}{1.2.3\ldots(n-2)}\,;\ \overset{3}{1.2.3\ldots(n-3)}\,;\ \text{\&c.}$$

cuius

cuius terminus indici ω respondens, convenit cum termino huius seriei:

$$\overset{1}{1}\,;\ \overset{2}{1\cdot 2}\,;\ \overset{3}{1\cdot 2\cdot 3}\,;\ \overset{4}{1\cdot 2\cdot 3\cdot 4}\,;\ \overset{5}{1\cdot 2\cdot 3\cdot 4\cdot 5}\,;\ \&c.$$

indici $n-\omega$ respondente: qui est

$$\frac{1^{1-n+\omega}2^{n-\omega}}{1+n-\omega}\cdot\frac{2^{1-n+\omega}3^{n-\omega}}{2+n-\omega}\cdot\frac{3^{1-n+\omega}4^{n-\omega}}{3+n-\omega}\cdot\&c.$$

Sit autem huius seriei terminus indici $1-\omega$ respondens $=\Theta$: erit:

$$\Theta=\frac{1^{\omega}.2^{1-\omega}}{2-\omega}\cdot\frac{2^{\omega}.3^{1-\omega}}{3-\omega}\cdot\frac{3^{\omega}.4^{1-\omega}}{4-\omega}\cdot\&c.$$

atque cum respondeant:

Indicibus:	$1-\omega$;	$2-\omega$;	$3-\omega$	
Termini:	Θ;	$(2-\omega)\Theta$;	$(2-\omega)(3-\omega)\Theta$;	&c.

indici $n-\omega$ respondebit hic terminus:

$$(2-\omega)(3-\omega)(4-\omega)\ \ldots\ (n-\omega)\Theta.$$

Deinde illorum denominatorum alteri factores constituent hanc seriem:

$$\overset{1}{1}\,;\ \overset{2}{1\cdot 2}\,;\ \overset{3}{1\cdot 2\cdot 3}\,;\ \overset{4}{1\cdot 2\cdot 3\cdot 4}\,;\ \overset{5}{1\cdot 2\cdot 3\cdot 4\cdot 5}\,;\ \&c.$$

si terminus indici ω respondens ponatur $=\Lambda$, erit:

$$\Lambda=\frac{1^{1-\omega}2^{\omega}}{1+\omega}\cdot\frac{2^{1-\omega}3^{\omega}}{2+\omega}\cdot\frac{3^{1-\omega}4^{\omega}}{3+\omega}\cdot\&c.$$

Quibus inventis si ipsius seriei propositae:

$$\overset{1}{\frac{n}{1}}\,;\ \overset{2}{\frac{n(n-1)}{1\,.\,2}}\,;\ \overset{3}{\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}}\,;\ \overset{4}{\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}}\,;\ \&c.$$

terminus indici ω respondens ponatur Σ, erit:

$$\Sigma=\frac{N}{\Lambda\cdot(2-\omega)(3-\omega)(4-\omega)\ \ldots\ (n-\omega)\Theta}.$$

At vero est:

$$\frac{N}{(2-\omega)(3-\omega)(4-\omega)\ \ldots\ (n-\omega)}=$$

$$\frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\cdot\frac{4}{4-\omega}\cdot\;\cdot\;\cdot\;\cdot\;\cdot\frac{n}{n-\omega};$$

atque

$$\Lambda\Theta = \frac{1\cdot 2}{(1+\omega)(2-\omega)}\cdot\frac{2\cdot 3}{(2+\omega)(3-\omega)}\cdot\frac{3\cdot 4}{(3+\omega)(4-\omega)}\cdot \&c.$$

Ex quibus terminus indici ω respondens quaesitus erit:

$$\Sigma = \frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\cdot\frac{4}{4-\omega}\cdot\frac{5}{5-\omega}\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\frac{n}{n-\omega}\cdot$$
$$\frac{(1+\omega)(2-\omega)}{1\cdot 2}\cdot\frac{(2+\omega)(3-\omega)}{2\cdot 3}\cdot\frac{(3+\omega)(4-\omega)}{3\cdot 4}\cdot\&c. \text{ in infinitum.}$$

Indici ergo $\frac{1}{2}$ respondebit iste terminus.

$$\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot\frac{10}{9}\cdot\frac{12}{11}\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\frac{2n}{2n-1}\cdot$$
$$\frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{9\cdot 9}{8\cdot 10}\cdot\&c.$$

qui reducitur ad $\frac{4\cdot 6\cdot 8\cdot 10\;\cdot\;\cdot\;\cdot\;\cdot\;2n}{3\cdot 5\cdot 7\cdot 9\;\cdot\;\cdot\;\cdot\;(2n-1)}\cdot\frac{4}{\pi}$, seu

erit $= \frac{2}{\pi}\cdot\frac{2\cdot 4\cdot 6\cdot 8\cdot 10\;\cdot\;\cdot\;\cdot\;\cdot\;2n}{1\cdot 3\cdot 5\cdot 7\cdot 9\;\cdot\;\cdot\;\cdot\;(2n-1)}$

Si fuerit $n = 2$, prodibit ista series interpolanda:

0, 1, 2, 3, 4, 5, 6, &c.
1, 2, 1, 0, 0, 0, 0, &c.

cuius propterea terminus indici $\frac{1}{2}$ respondens est $= \frac{16}{3\pi}$.

EXEMPLUM IV.

Quaeratur terminus respondens indici $= \frac{1}{2}$ in hac serie:

$$\overset{0}{1} + \overset{1}{\frac{1}{2}} - \overset{2}{\frac{1\cdot 1}{2\cdot 4}} + \overset{3}{\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}} - \overset{4}{\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}} + \&c.$$

Oritur haec series ex praecedente si ponatur $n = \frac{1}{2}$, eritque propterea terminus quaesitus, qui sit $= \Sigma$:

$$\Sigma = \frac{2}{\pi}\cdot\frac{2.4.6.8 \quad . \quad . \quad . \quad . \quad 2n}{1.3.5.7 \quad . \quad . \quad . \quad (2n-1)} \text{ posito } n = \tfrac{1}{2}.$$

Ponatur $\frac{2.4.6.8 \quad . \quad . \quad . \quad . \quad 2n}{1.3.5.7 \quad . \quad . \quad . \quad (2n-1)} = \Theta$ si sit $n = \frac{1}{2}$, eritque Θ terminus respondens indici $\frac{1}{2}$ in hac serie:

$$\frac{2}{1};\ \frac{2.4}{1.3};\ \frac{2.4.6}{1.3.5};\ \frac{2.4.6.8}{1.3.5.7};\ \&c.$$

qui ex superioribus prodit $= \frac{\pi}{2}$. Quocirca seriei propositae terminus indici $\frac{1}{2}$ respondens, qui quaeritur, erit $= 1$. Quoniam autem in ista serie, si terminus indici cuicunque ω respondens ponatur $= \Sigma$, sequens eum erit $\Sigma' = \frac{1-2\omega}{2+2\omega}\Sigma$; series proposita ita mediis terminis interiiciendis interpolabitur:

Indices:	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	
Termini:	1;	1;	$\frac{1}{2}$;	0;	$\frac{-1.1}{2.4}$;	0;	$\frac{1.1.3}{2.4.6}$;	0;	&c.

EXEMPLUM V.

Si n *fuerit numerus quicunque fractus, invenire terminum indici* ω *respondentem in serie:*

$$\overset{0}{1};\ \overset{1}{\frac{n}{1}};\ \overset{2}{\frac{n(n-1)}{1.\ 2}};\ \overset{3}{\frac{n(n-1)(n-2)}{1.\ 2.\ 3}};\ \overset{4}{\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}}$$

&c.

Si expressionem $\frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\cdot\frac{4}{4-\omega}\ .\ .\ .\ .\ .\ .\ .\ \frac{n}{n-\omega}$ cum §. 400. comparemus, fiat $a = 1$, $c = 1$, $b = 1-\omega$, ibique loco ω posito n, erit:

$$\frac{1}{1-\omega}\cdot\frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\ .\ .\ .\ .\ \frac{n}{n-\omega} = \frac{1(1-\omega+n)}{(1-\omega)(1+n)}\cdot\frac{2(2-\omega+n)}{(2-\omega)(2+n)}\cdot \&c.$$

unde

unde terminus quaesitus indici ω respondens si ponatur $= \Sigma$, erit:

$$\Sigma = \frac{(1-\omega+n)2}{(1+n)(2-\omega)} \cdot \frac{(2-\omega+n)3}{(2+n)(3-\omega)} \cdot \&c. \frac{(1+\omega)(2-\omega)}{1 \cdot 2} \cdot \frac{(2+\omega)(3-\omega)}{2 \cdot 3} \cdot \&c.$$

ideoque

$$\Sigma = \frac{(1+\omega)(1+n-\omega)}{1(1+n)} \cdot \frac{(2+\omega)(2+n-\omega)}{2(2+n)} \cdot \frac{(3+\omega)(3+n-\omega)}{3(3+n)} \cdot \&c.$$

quoties ergo $n - \omega$ fuerit numerus integer valor ipsius Σ rationaliter exprimi potest.

Sic si sit $n = \omega$ erit $\Sigma = 1$

si $n = 1 + \omega$ erit $\Sigma = n$

si $n = 2 + \omega$ erit $\Sigma = \frac{n(n-1)}{1 \cdot 2}$

si $n = 3 + \omega$ erit $\Sigma = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$

&c.

At si fuerit $\omega - n$ numerus integer affirmativus, erit semper $\Sigma = 0$.