CAPUT XVI.

DE DIFFERENTIATIONE FUNCTIONUM INEXPLICABILIUM.

367.

Functiones inexplicabiles hic voco, quae neque expressionibus determinatis, neque per aequationum radices explicari possunt; ita ut non solum non sint algebraicae, sed etiam plerumque incertum sit, ad quod genus transcendentium pertineant. Huiusmodi functio inexplicabilis est $1 + \frac{1}{2} + \frac{1}{3} \ldots\ldots + \frac{1}{x}$, quae utique ab x pendet, at nisi x sit numerus integer nullo modo explicari potest. Simili modo haec expressio $1.2.3.4\ldots x$, erit functio inexplicabilis ipsius x, quoniam si x sit numerus quicunque eius valor non solum non algebraice, sed ne quidem per ullum certum quantitatum transcendentium genus exprimi potest. Generatim ergo talium functionum inexplicabilium notio ex seriebus derivari potest. Sit enim proposita series quaecunque

$$\overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \overset{\ldots}{\ldots} + \overset{x}{X}$$

cuius summa si formula finita exprimi nequeat, praebebit functionem inexplicabilem ipsius x, nempe

$$S = A + B + C + D + \ldots + X.$$

Similiter continua producta ex terminis serierum uti

$$P = A\,B\,C\,D \ldots . X$$

exhibebunt functiones inexplicabiles ipsius x, quae autem ope logarithmorum ad formam priorem revocari possunt, erit enim:

$$lP = lA + lB + lC + lD + \ldots + lX$$

368. Hoc igitur capite methodum explicare constitui, huiusmodi functionum inexplicabilium differentialia investigandi.

di. Quod argumentum, quamvis ad primam huius operis partem, ubi praecepta calculi differentialis sunt tradita, pertinere videatur; tamen quoniam uberiorem doctrinae serierum cognitionem postulat, ad quam in hac altera parte pervenire licuit, ordinem naturalem relinquere coacti hoc loco attingamus. Cum autem haec investigatio prorsus sit nova, neque a quoquam adhuc tractata, tantum abest ut hanc calculi differentialis partem absolvere queamus, ut potius prima tantum eius elementa adumbrare conemur. Praeterea vero nonnullas quaestiones proponam, quarum enodatio differentiationem huiusmodi functionum inexplicabilium requirat, quo simul usus huius tractationis, qui autem in posterum sine dubio multo amplior erit, clarius perspiciatur.

369. Ad huiusmodi functiones inexplicabiles differentiandas ante omnia necesse est, ut earum valores investigemus, quos induunt, si pro x ponatur $x+\omega$. Sit igitur

$$S = \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \ldots + \overset{x}{X}$$

atque ponatur Σ valor ipsius S, quem recipit, si pro x ponatur $x+\omega$, sitque Z terminus seriei respondens indicii $x+\omega$. Iam igitur termini, qui respondent indicibus $x+1$, $x+2$, $x+3$, &c. indicentur per X', X'', X''', X^{IV}, &c. atque is, qui convenit indici infinito $x+\infty$ per $X^{|\infty|}$. Similique modo termini competentes indicibus $x+\omega+1$, $x+\omega+2$, $x+\omega+3$ &c. indicentur per Z', Z'', Z''', &c. & sit $Z^{|\infty|}$ terminus respondens indici $x+\omega+\infty$. Quibus positis erit

$$S' = S + X'$$
$$S'' = S + X' + X''$$
$$S''' = S + X' + X'' + X'''$$
&c.
$$S^{|\infty|} = S + X' + X'' + X''' + \ldots + X^{|\infty|}.$$ Simili modo cum etiam Σ successive terminis Z', Z'' &c. augeatur, erit

Σ'

$$\Sigma' = \Sigma + Z'$$
$$\Sigma'' = \Sigma + Z' + Z''$$
$$\Sigma''' = \Sigma + Z' + Z'' + Z'''$$
&c.
$$\Sigma^{|\infty|} = \Sigma + Z' + Z'' + Z''' + \ldots + Z^{|\infty|}$$

370. Nunc natura feriei S, S', S'', S''', &c. eft perpendenda, qualis futura fit, fi in infinitum continuetur: quae fi in infinito cum progreffione arithmetica confundatur; quod fit fi termini feriei X, X', X'', X''', &c. in infinito ad aequalitatem convergant, ita ut differentiae feriei S, S', S'', &c. tandem fiant aequales: hoc cafu quantitates $S^{|\infty|}$, $S^{|\infty+1|}$, $S^{|\infty+2|}$ &c. erunt in arithmetica progreffione, & cum fit $\Sigma^{|\infty|} = S^{|\infty+\omega|}$ ob $S^{|\infty+\omega|} = S^{|\infty|} + \omega(S^{|\infty+1|} - S^{|\infty|}) = \omega S^{|\infty+1|} + (1-\omega) S^{|\infty|}$ erit $\Sigma^{|\infty|} = \omega S^{|\infty+1|} + (1-\omega) S^{|\infty|}$. At eft $S^{|\infty+1|} = S^{|\infty|} + X^{|\infty+1|}$, unde fit $\Sigma^{|\infty|} = S^{|\infty|} + \omega X^{|\infty+1|}$, ex quo obtinebitur haec aequatio

$$\Sigma + Z' + Z'' + Z''' + \ldots + Z^{|\infty|} =$$
$$S + X' + X'' + X''' + \ldots + X^{|\infty|} + \omega X^{|\infty+1|}$$

ex qua definitur valor quaefitus Σ, quem induit functio S, dum in ea $x + \omega$ loco x fubftituitur; eritque

$$\Sigma = S + \omega X^{|\infty+1|} + X' + X'' + X''' + \text{\&c. in infinitum}$$
$$- Z' - Z'' - Z''' - \text{\&c. in infinitum}$$

Quare fi feriei A, B, C, D, &c. termini infinitefimi evanefcant, terminus $\omega X^{|\infty+1|}$ evanefcit, & omitti poteft.

371. Exprimitur ergo valor ipfius Σ per novam feriem infinitam, quae exhiberi poteft, fi feriei $A + B + C +$ &c. habeatur terminus generalis, ex quo valores terminorum Z', Z'', Z''', &c. definiri queant. Pofito ergo ω infinite parvo, cum fit $\Sigma - S$ differentiale functionis S, hoc differentiale dS per feriem infinitam exprimetur. Atque fi nequidem altiores poteftates ipfius ω negligantur, habebitur differentiale completum functionis huius inexplicabilis S, cuius natura, quo clarius ob oculos ponatur, fequentibus exemplis hoc negotium illuftrabimus.

EXEM-

EXEMPLUM I.

Invenire differentiale huius functionis inexplicabilis

$$S = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots + \frac{1}{x}.$$

Quoniam huius feriei terminus generalis X eft $= \frac{1}{x}$ ac propterea

$X' = \frac{1}{x+1}$	$Z' = \frac{1}{x+1+\omega}$
$X'' = \frac{1}{x+2}$	$Z'' = \frac{1}{x+2+\omega}$
$X''' = \frac{1}{x+3}$	$Z''' = \frac{1}{x+3+\omega}$
&c.	&c.

ob $X^{|\infty+1|} = \frac{1}{x+\infty+1} = 0$, fi loco x ponatur $x+\omega$ functio S abibit in Σ, ut fit

$$\Sigma = S + \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \text{\&c.}$$
$$- \frac{1}{x+1+\omega} - \frac{1}{x+2+\omega} - \frac{1}{x+3+\omega} - \text{\&c.}$$

five binis his terminis in fingulos colligendis, erit $\Sigma = S$

$$+ \frac{\omega}{(x+1)(x+1+\omega)} + \frac{\omega}{(x+2)(x+2+\omega)} + \frac{\omega}{(x+3)(x+3+\omega)} + \text{\&c.}$$

feu cum fit

$$\frac{1}{x+1+\omega} = \frac{1}{x+1} - \frac{\omega}{(x+1)^2} + \frac{\omega^2}{(x+1)^3} - \frac{\omega^3}{(x+1)^4} + \text{\&c.}$$
$$\frac{1}{x+2+\omega} = \frac{1}{x+2} - \frac{\omega}{(x+2)^2} + \frac{\omega^2}{(x+2)^3} - \frac{\omega^3}{(x+2)^4} + \text{\&c.}$$
&c.

erit feriebus fecundum poteftates ipfius ω difpofitis

$$\Sigma =$$

$$\Sigma = S + \omega\left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \&c.\right)$$
$$- \omega^2\left(\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \frac{1}{(x+4)^3} + \&c.\right)$$
$$+ \omega^3\left(\frac{1}{(x+1)^4} + \frac{1}{(x+2)^4} + \frac{1}{(x+3)^4} + \frac{1}{(x+4)^4} + \&c.\right)$$
$$- \omega^4\left(\frac{1}{(x+1)^5} + \frac{1}{(x+2)^5} + \frac{1}{(x+3)^5} + \frac{1}{(x+4)^5} + \&c.\right) \quad \&c.$$

Posito ergo dx pro ω obtinebimus functionis propositae S differentiale completum

$$dS = dx\left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \&c.\right)$$
$$- dx^2\left(\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \frac{1}{(x+4)^3} + \&c.\right)$$
$$+ dx^3\left(\frac{1}{(x+1)^4} + \frac{1}{(x+2)^4} + \frac{1}{(x+3)^4} + \frac{1}{(x+4)^4} + \&c.\right)$$
$$- dx^4\left(\frac{1}{(x+1)^5} + \frac{1}{(x+2)^5} + \frac{1}{(x+3)^5} + \frac{1}{(x+4)^5} + \&c.\right) \quad \&c.$$

EXEMPLUM II.

Invenire differentiale huius functionis inexplicabilis ipsius S

$$S = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \;.\;.\; + \frac{1}{2x-1}.$$

Quia huius seriei terminus generalis est $X = \frac{1}{2x-1}$; erit:

$$X' = \frac{1}{2x+1} \qquad Z' = \frac{1}{2x+1+2\omega}$$
$$X'' = \frac{1}{2x+3} \qquad Z'' = \frac{1}{2x+3+2\omega}$$
$$X''' = \frac{1}{2x+5} \qquad Z''' = \frac{1}{2x+5+2\omega}$$
$$\&c. \qquad\qquad \&c.$$

ob

ob terminos huius seriei infinitesimos evanescentes & aequales, prodibit valor ipsius S, si loco x ponatur $x+1$:

$$\Sigma = S + \frac{1}{2x+1} + \frac{1}{2x+3} + \frac{1}{2x+5} + \&c.$$
$$- \frac{1}{2x+1+2\omega} - \frac{1}{2x+3+2\omega} - \frac{1}{2x+5+2\omega} - \&c.$$

seu

$$\Sigma = S + \frac{2\omega}{(2x+1)(2x+1+2\omega)} + \frac{2\omega}{(2x+3)(2x+3+2\omega)} + \&c.$$

Verum si singuli termini in series secundum dimensiones ipsius ω resolvantur, erit:

$$\begin{aligned}\Sigma = S &+ 2\omega \left(\frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \&c.\right)\\ &- 4\omega^2\left(\frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \&c.\right)\\ &+ 8\omega^3\left(\frac{1}{(2x+1)^4} + \frac{1}{(2x+3)^4} + \frac{1}{(2x+5)^4} + \&c.\right)\\ &- 16\omega^4\left(\frac{1}{(2x+1)^5} + \frac{1}{(2x+3)^5} + \frac{1}{(2x+5)^5} + \&c.\right)\\ &\&c.\end{aligned}$$

Ponatur nunc dx pro ω, atque prodibit differentiale completum functionis inexplicabilis S propositae:

$$\begin{aligned}dS = &\; 2dx \left(\frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \&c.\right)\\ &- 4dx^2\left(\frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \&c.\right)\\ &+ 8dx^3\left(\frac{1}{(2x+1)^4} + \frac{1}{(2x+3)^4} + \frac{1}{(2x+5)^4} + \&c.\right)\\ &- 16dx^4\left(\frac{1}{(2x+1)^5} + \frac{1}{(2x+3)^5} + \frac{1}{(2x+5)^5} + \&c.\right)\\ &\&c.\end{aligned}$$

EXEM-

EXEMPLUM III.

Invenire differentiale completum functionis huius inexplicabilis ipsius S:

$$S = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots\ldots + \frac{1}{x^n}.$$

Cum huius seriei terminus generalis sit $= \frac{1}{x^n}$, erunt termini infinitesimi evanescentes & inter se aequales. Hincque ob

$$X' = \frac{1}{(x+1)^n} \qquad Z' = \frac{1}{(x+1+\omega)^n}$$
$$X'' = \frac{1}{(x+2)^n} \qquad Z'' = \frac{1}{(x+2+\omega)^n}$$
$$X''' = \frac{1}{(x+3)^n} \qquad Z''' = \frac{1}{(x+3+\omega)^n}$$
&c. &c.

erit:

$$X' - Z' = \frac{n\omega}{(x+1)^{n+1}} - \frac{n(n+1)\omega^2}{2(x+1)^{n+2}} + \frac{n(n+1)(n+2)\omega^3}{6(x+1)^{n+3}} - \&c.$$
$$X'' - Z'' = \frac{n\omega}{(x+2)^{n+1}} - \frac{n(n+1)\omega^2}{2(x+2)^{n+2}} + \frac{n(n+1)(n+2)\omega^3}{6(x+2)^{n+3}} - \&c.$$
&c.

ex quibus invenitur:

$$\Sigma - S = n\omega\left(\frac{1}{(x+1)^{n+1}} + \frac{1}{(x+2)^{n+1}} + \frac{1}{(x+3)^{n+1}} + \&c.\right)$$
$$- \frac{n(n+1)}{1.\ 2}\omega^2\left(\frac{1}{(x+1)^{n+2}} + \frac{1}{(x+2)^{n+2}} + \frac{1}{(x+3)^{n+2}} + \&c.\right)$$
$$+ \frac{n(n+1)(n+2)}{1.\ 2.\ 3}\omega^3\left(\frac{1}{(x+1)^{n+3}} + \frac{1}{(x+2)^{n+3}} + \frac{1}{(x+3)^{n+3}} + \&c.\right)$$
&c.

Quare posito $\omega = dx$ prodibit differentiale completum functionis S quaesitum:

$$dS = +\, n\,dx\left(\frac{1}{(x+1)^{n+1}} + \frac{1}{(x+2)^{n+1}} + \frac{1}{(x+3)^{n+1}} + \&c.\right)$$
$$-\frac{n(n+1)}{1.\ 2}\,dx^2\left(\frac{1}{(x+1)^{n+2}} + \frac{1}{(x+2)^{n+2}} + \frac{1}{(x+3)^{n+2}} + \&c.\right)$$
$$+\frac{n(n+1)(n+2)}{1.\ 2.\ 3}\,dx^3\left(\frac{1}{(x+1)^{n+3}} + \frac{1}{(x+2)^{n+3}} + \frac{1}{(x+3)^{n+3}} + \&c.\right)$$
&c.

372. Ex his quoque summae istarum serierum interpolari, seu valores terminorum summatoriorum exhiberi possunt, quando numerus terminorum non est numerus integer. Si enim ponatur $x = 0$, erit quoque $S = 0$, atque Σ exprimet summam tot terminorum, quot numerus ω continet unitates, etiamsi iste numerus ω non sit integer. Ita in exemplo primo si ponatur

$$\Sigma = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{\omega}$$

erit:

$$\Sigma = \frac{\omega}{1(1+\omega)} + \frac{\omega}{2(2+\omega)} + \frac{\omega}{3(3+\omega)} + \frac{\omega}{4(4+\omega)} + \&c.$$

sive

$$\Sigma = \omega\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \&c.\right)$$
$$-\omega^2\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right)$$
$$+\omega^3\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right)$$
&c.

In exemplo vero tertio erit:

$$\Sigma = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots + \frac{1}{\omega^n}$$

Valorque ipsius Σ, sive ω sit numerus integer sive fractus, per series sequenti modo exprimetur:

$\Sigma =$

$$\begin{aligned}\Sigma = {} & n\omega\left(1+\frac{1}{2^{n+1}}+\frac{1}{3^{n+1}}+\frac{1}{4^{n+1}}+\&c.\right)\\ & -\frac{n(n+1)}{1.\ 2}\omega^2\left(1+\frac{1}{2^{n+2}}+\frac{1}{3^{n+2}}+\frac{1}{4^{n+2}}+\&c.\right)\\ & +\frac{n(n+1)(n+2)}{1.\ 2.\ 3}\omega^3\left(1+\frac{1}{2^{n+3}}+\frac{1}{3^{n+3}}+\frac{1}{4^{n+3}}+\&c.\right)\\ & \&c.\end{aligned}$$

373. Haec eadem quoque ad feriem generalem accommodari poffunt, cum enim fit

$$S = \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \ldots \ldots + \overset{x}{X}$$

atque pofito $x+\omega$ loco x, abeat X in Z, & S in Σ, erit:

$$Z = X + \frac{\omega dX}{dx} + \frac{\omega^2 ddX}{1.\ 2\, dx^2} + \frac{\omega^3 d^3X}{1.2.3\, dx^3} + \&c.$$

& quia fimili modo Z', Z'', Z''', &c. per X', X'', X''', &c. exprimuntur, erit:

$$\begin{aligned}\Sigma = S + \omega X|\infty+1| & - \frac{\omega}{dx}\, d.\,[X'+X''+X'''+X''''+\&c.]\\ & - \frac{\omega^2}{1.\ 2\, dx^2}\, dd.\,[X'+X''+X'''+X''''+\&c.]\\ & - \frac{\omega^3}{1.\ 2.\ 3\, dx^3}\, d^3.\,[X'+X''+X'''+X''''+\&c.]\\ & \&c.\end{aligned}$$

& nifi $X|\infty+1|$ fit $=0$, hoc modo exprimi poterit, ut confideratio infiniti tollatur:

$$X|\infty+1| = X' + (X''-X') + (X'''-X'') + (X''''-X''') + \&c.$$

eritque ergo:

$$\begin{aligned}\Sigma = S + \omega X' & + \omega[(X''-X') + (X'''-X'') + (X''''-X''') + \&c.]\\ & - \frac{\omega}{dx}\, d.\,[X'+X''+X'''+X''''+X'''''+\&c.]\\ & - \frac{\omega^2}{2dx^2}\, dd.\,[X'+X''+X'''+X''''+\&c.]\\ & - \frac{\omega^3}{6dx^3}\, d^3.\,[X'+X''+X'''+X''''+\&c.] \qquad \&c.\end{aligned}$$

Si

Si ergo ponatur $\omega = dx$, orietur differentiale completum ipsius $S = A + B + C + \ldots + X$, ita expressum:

$$dS = X'dx + dx[(X'' - X') + (X''' - X'') + (X'''' - X''') + \&c.]$$
$$- d.[X' + X'' + X''' + X'''' + \&c.]$$
$$- \tfrac{1}{2}dd.[X' + X'' + X''' + X'''' + \&c.]$$
$$- \tfrac{1}{6}d^3.[X' + X'' + X''' + X'''' + \&c.] \quad \&c.$$

374. Ponamus esse $x = 0$, fiet $X' = A$, $X'' = B$, &c. ideoque $X' + X'' + X''' + \&c.$ erit series infinita cuius terminus generalis est $= X$. Formentur deinde series ex his terminis generalibus:

$$\frac{dX}{dx};\quad \frac{ddX}{2dx^2};\quad \frac{d^3X}{6dx^3};\quad \frac{d^4X}{24dx^4};\quad \&c.$$

quarum serierum in infinitum continuatarum summae sint:

$$\int. X = \mathfrak{A}$$
$$\int. \frac{dX}{dx} = \mathfrak{B}$$
$$\int. \frac{ddX}{2dx^2} = \mathfrak{C}$$
$$\int. \frac{d^3X}{6dx^3} = \mathfrak{D} \quad \&c.$$

& quia posito $x = 0$, fit quoque $S = 0$, & Σ erit summa seriei $A + B + C + D + \ldots + Z$ continentis ω terminos; est enim Z terminus indicis ω, sive ω sit numerus integer sive fractus. Quare habebitur

$$\Sigma = \omega A + \omega[(B - A) + (C - B) + (D - C) + \&c.]$$
$$- \omega\mathfrak{B} - \omega^2\mathfrak{C} - \omega^3\mathfrak{D} - \omega^4\mathfrak{E} - \&c.$$

ubi prima series praetermitti potest, si seriei propositae termini tandem evanescant.

375. Scribamus nunc x loco ω, abibitque Σ in S ita ut sit

$$\begin{matrix} & 1 & & 2 & & 3 & & 4 & & \ldots & & x \\ S = & A & + & B & + & C & + & D & + & \ldots & + & X \end{matrix}$$

atque idem ipsius S valor iam per seriem infinitam exprimetur hoc modo:

S =

$$S = Ax + x[(B-A)+(C-B)+(D-C)+\&c.]$$
$$-\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \mathfrak{F}x^5 - \&c.$$

cuius valor cum aeque diſtincte exprimatur, ſive x ſit numerus integer ſive fractus, differentialia ipſius S cuiuſque ordinis hinc facile exhiberi poſſunt:

$$\frac{dS}{dx} = A + (B-A) + (C-B) + (D-C) + \&c.$$
$$-\mathfrak{B} - 2\mathfrak{C}x - 3\mathfrak{D}x^2 - 4\mathfrak{E}x^3 - \&c.$$
$$\frac{ddS}{2dx^2} = -\mathfrak{C} - 3\mathfrak{D}x - 6\mathfrak{E}x^2 - 10\mathfrak{F}x^3 - \&c.$$
$$\frac{d^3S}{6dx^3} = -\mathfrak{D} - 4\mathfrak{E}x - 10\mathfrak{F}x^2 - 20\mathfrak{G}x^3 - \&c.$$
$$\frac{d^4S}{24dx^4} = -\mathfrak{E} - 5\mathfrak{F}x - 15\mathfrak{G}x^2 - \&c.$$

quare cum differentiale completum ſit

$$= dS + \tfrac{1}{2}ddS + \tfrac{1}{6}d^3S + \tfrac{1}{24}d^4S + \&c.$$

erit functionis propoſitae S differentiale completum:

$$dS = A\,dx + (B-A)\,dx + (C-B)\,dx + (D-C)\,dx + \&c.$$
$$-\mathfrak{B}\,dx - \mathfrak{C}(2x\,dx + dx^2) - \mathfrak{D}(3x^2dx + 3x\,dx^2 + dx^3)$$
$$-\mathfrak{E}(4x^3dx + 6x^2dx^2 + 4x\,dx^3 + dx^4) - \&c.$$

376. Hoc ergo modo functionis cuiuſque inexplicabilis S differentiale aſſignari poteſt, ſi ſeriei $A + B + C + \&c.$ termini infiniteſimi vel evaneſcant vel inter ſe ſint aequales. Quodſi enim huius termini infiniteſimi non fuerint $=0$, tum ſumma ſeriei $\mathfrak{B}$, quae ex termino generali $\frac{dX}{dx}$ formatur, fiet infinita; at vero cum ſerie $A+(B-A)+(C-B)+(D-C)+$ &c. coniuncta ſummam finitam conſtituet. At fieri poteſt, ut termini ſeriei $A + B + C + D + \&c.$ ita in infinitum augeantur, ut non ſolum ſeriei $\mathfrak{B}$, ſed etiam ſeriei $\mathfrak{C}$ ſumma fiat

fiat infinite magna, quo casu non sufficit seriem $A+(B-A)+(C-B)+$ &c. adiecisse: sed quoniam hoc casu valores infinitesimi §. 370. considerati, nempe $S|\infty|$, $S|\infty+1|$, $S|\infty+2|$, non amplius in arithmetica sunt progressione, uti assumseramus, huius progressionis ratio erit habenda. Quemadmodum ergo assumsimus, horum terminorum differentias primas esse aequales; ita methodum amplius extendemus, si horum valorum differentias demum secundas, vel tertias, vel ulteriores constantes statuamus.

377. Retento ergo eodem ratiocinio, quo §. 369. sumus usi, ponamus memoratorum valorum differentias demum secundas esse constantes:

$$S|\infty|,\quad S|\infty+1|,\quad S|\infty+2|\;;$$

DIFF. I. $\qquad X|\infty+1|,\quad X|\infty+2|$

DIFF. II. $\qquad X|\infty+2|-X|\infty+1|$

Hinc erit $\Sigma|\infty| = S|\infty+\omega| = S|\infty| + \omega X|\infty+1| + \frac{\omega(\omega-1)}{1.\,2}$

$$(X|\infty+2|-X|\infty+1|) = S|\infty| - \frac{\omega(\omega-3)}{1.\,2}X|\infty+1| + \frac{\omega(\omega-1)}{1.\,2}X|\infty+2|$$

Quamobrem habebimus hanc aequationem:

$$\Sigma+Z'+Z''+Z'''+ \;.\;.\;.\; +Z|\infty| =$$
$$S+X'+X''+X'''+ \;.\;.\;.\; +X|\infty| -$$
$$\frac{\omega(\omega-3)}{1.\,2}X|\infty+1| + \frac{\omega(\omega-1)}{1.\,2}X|\infty+2|,$$

ex qua elicitur:

$$\Sigma = S+X'+X''+X'''+X''''+ \text{\&c. in infinitum}$$
$$-Z'-Z''-Z'''-Z''''- \text{\&c. in infinitum}$$
$$+\omega X|\infty+1| + \frac{\omega(\omega-1)}{1.\,2}(X|\infty+2|-X|\infty+1|).$$

Termini autem isti infinitesimi ita repraesentari poterunt, ut sit

$\Sigma =$

$$\begin{aligned}
\Sigma = {} & S + X' + X'' + X''' + X'''' + \&c. \\
& - Z' - Z'' - Z''' - Z'''' - \&c. \\
& + \omega X' + \omega \left\{ \begin{array}{l} + X'' + X''' + X'''' + X''''' + \&c. \\ - X' - X'' - X''' - X'''' - \&c. \end{array} \right. \\
& + \frac{\omega(\omega-1)}{1.\,2} X'' \qquad + \frac{\omega(\omega-1)}{1.\,2} \left\{ \begin{array}{l} + X''' + X^{iv} + X^{v} + \&c. \\ - 2X'' - 2X''' - 2X^{iv} - \&c. \\ + X' + X'' + X''' + \&c. \end{array} \right\} \\
& - \frac{\omega(\omega-1)}{1.\,2} X'
\end{aligned}$$

unde simul lex patet, qua haec expressio erit comparata, si differentiae demum tertiae vel quartae vel ulteriores fuerint constantes.

378. Cum igitur sit, ut supra demonstravimus:

$$Z = X + \frac{\omega dX}{1 dx} + \frac{\omega^2 ddX}{1.\,2 dx^2} + \frac{\omega^3 d^3 X}{1.2.3 dx^3} + \&c.$$

si loco Z', Z'', Z''', &c. valores hinc oriundos substituamus, erit valor ipsius S, si loco x scribatur $x + \omega$, sequens:

$$\begin{aligned}
\Sigma = {} & S + \omega X' + \omega \left\{ \begin{array}{l} + X'' + X''' + X^{iv} + X^{v} + \&c. \\ - X' - X'' - X''' - X^{iv} - \&c. \end{array} \right\} \\
& + \frac{\omega(\omega-1)}{1.\,2} X'' \qquad + \frac{\omega(\omega-1)}{1.\,2} \left\{ \begin{array}{l} + X''' + X^{iv} + X^{v} + X^{vi} + \&c. \\ - 2X'' - 2X''' - 2X^{iv} - 2X^{v} - \&c. \\ + X' + X'' + X''' + X^{iv} + \&c. \end{array} \right\} \\
& - \frac{\omega(\omega-1)}{1.\,2} X' \\
& \qquad - \frac{\omega}{dx} d.\,(X' + X'' + X''' + X'''' + \&c.) \\
& \qquad - \frac{\omega^2}{2 dx^2} d^2.\,(X' + X'' + X''' + X'''' + \&c.) \\
& \qquad - \frac{\omega^3}{6 dx^3} d^3.\,(X' + X'' + X''' + X'''' + \&c.) \\
& \qquad\qquad \&c.
\end{aligned}$$

Si ergo loco ω ponatur dx, prodibit differentiale completum functionis inexplicabilis propositae S; scilicet

dS

$$dS = X'dx + dx\left\{\begin{array}{l} + X'' + X''' + X^{IV} + X^{V} + \&c. \\ - X' - X'' - X''' - X^{IV} - \&c. \end{array}\right\}$$

$$\begin{array}{l} - X''\dfrac{dx(1-dx)}{1.\ 2} \\ + X'\dfrac{dx(1-dx)}{1.\ 2} \end{array} - \frac{dx(1-dx)}{1.\ 2}\left\{\begin{array}{l} + X''' + X^{IV} + X^{V} + X^{VI} + \&c. \\ - 2X'' - 2X''' - 2X^{IV} - 2X^{V} - \&c. \\ + X' + X'' + X''' + X^{IV} + \&c. \end{array}\right\}$$

$$\begin{array}{l} + X'''\dfrac{dx(1-dx)(2-dx)}{1.\ 2.\ 3} \\ - 2X''\dfrac{dx(1-dx)(2-dx)}{1.\ 2.\ 3} \\ + X'\dfrac{dx(1-dx)(2-dx)}{1.\ 2.\ 3} \end{array} + \frac{dx(1-dx)(2-dx)}{1.\ 2.\ 3}\left\{\begin{array}{l} + X^{IV} + X^{V} + \&c. \\ - 3X''' - 3X^{IV} - \&c. \\ + 3X'' + 3X''' + \&c. \\ - X' - X'' - \&c. \end{array}\right\}$$

&c.

$$\begin{array}{l} - d.\ (X' + X'' + X''' + X^{IV} + X^{V} + \&c.) \\ - \tfrac{1}{2}dd.\ (X' + X'' + X''' + X^{IV} + X^{V} + \&c.) \\ - \tfrac{1}{6}d^{3}.\ (X' + X'' + X''' + X^{IV} + X^{V} + \&c.) \\ - \tfrac{1}{24}d^{4}.\ (X' + X'' + X''' + X^{IV} + X^{V} + \&c.) \end{array}$$

&c.

quae expressio latissime patet, & quotaecunque demum differentiae fuerint constantes, differentiale quaesitum exhibet. Accommodata enim est haec formula ad differentias constantes, & simul lex patet, si forte ulterius progredi necesse sit.

379. Quod si series $A + B + C + D +$ &c. ex qua formatur functio inexplicabilis

$$S = \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \ldots + \overset{x}{X}$$

ita fuerit comparata, ut eius termini infinitesimi evanescant, tum uti iam notavimus erit:

$$\begin{array}{rl} dS = & - d.\ (X' + X'' + X''' + X^{IV} + \&c.) \\ & - \tfrac{1}{2}dd.\ (X' + X'' + X''' + X^{IV} + \&c.) \\ & - \tfrac{1}{6}d^{3}.\ (X' + X'' + X''' + X^{IV} + \&c.) \\ & - \tfrac{1}{24}d^{4}.\ (X' + X'' + X''' + X^{IV} + \&c.) \end{array}$$

&c.

Sin

Sin autem illius feriei termini infinitefimi non fint $=0$, fed tamen differentias habeant evanefcentes, tum ad iftam expreffionem infuper addi debet

$$dx\left\{\begin{array}{llllll} X' & +X'' & +X''' & +X^{IV} & +X^{V} & +\&c. \\ & -X' & -X'' & -X''' & -X^{IV} & -\&c. \end{array}\right\}$$

Verum fi terminorum infinitefimorum huius feriei $A+B+C+D+$ &c. differentiae demum fecundae evanefcant, tum praeterea adiici oportet:

$$\frac{dx(dx-1)}{1\cdot 2}\left\{\begin{array}{llllll} +X'' & +X''' & +X^{IV} & +X^{V} & +\&c. \\ -X' & -2X'' & -2X''' & -2X^{IV} & -\&c. \\ & +X' & +X'' & +X''' & +\&c. \end{array}\right\}$$

Atque fi memoratorum terminorum infinitefimorum differentiae demum tertiae fuerint evanefcentes, tum praeter has iam exhibitas expreffiones infuper addi debet.

$$\frac{dx(dx-1)(dx-2)}{1\cdot 2\cdot 3}\left\{\begin{array}{llllll} X''' & +X^{IV} & +X^{V} & +X^{VI} & +\&c. \\ -2X'' & -3X''' & -3X^{IV} & -3X^{V} & -\&c. \\ +X' & +3X'' & +3X''' & +3X^{IV} & +\&c. \\ & -X' & -X'' & -X''' & -\&c. \end{array}\right\}$$

Sicque porro expreffiones infuper addendae erunt comparatae, fi ulteriores demum differentiae terminorum infinitefimorum feriei $A+B+C+D+$ &c. evanefcant. Hincque adeo quaecunque feries affumatur, dummodo eius termini infinitefimi tandem ad differentias evanefcentes perducantur, functionis inexplicabilis ex ea formatae differentiale definiri poterit.

380. Si ponatur $x=0$, fiet $X'=A$, $X''=B$, $X'''=C$ &c. Quare uti $A+B+C+D+$ &c. eft feries, cuius terminus generalis eft X, fi ex terminis generalibus $\frac{dX}{1dx}$; $\frac{ddX}{2dx^2}$; $\frac{d^3X}{6dx^3}$; $\frac{d^4X}{24dx^4}$; &c. fimili modo formentur feries infinitae, earumque fummae denotentur per litteras: $\mathfrak{B}$; $\mathfrak{C}$; $\mathfrak{D}$; $\mathfrak{E}$; &c. refpective. Summa ω terminorum feriei $A+B+C+D+$ &c. ita exprimetur, ut perinde fit, five ω fit numerus integer five fecus. Scribamus ergo x pro ω, ut fit:

$$S = \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} \ldots + \overset{x}{X}$$

atque si huius seriei termini infinitesimi evanescant, erit

$$S = -\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \&c.$$

At si termini infinitesimi differentias saltem primas habeant constantes, tum ad hunc valorem insuper addi debet hic:

$$x \left\{ \begin{matrix} A & + B + C + D + E + \&c. \\ & - A - B - C - D - \&c. \end{matrix} \right\}$$

sin autem illorum terminorum infinitesimorum differentiae demum secundae evanescant, tum praeterea addi debet:

$$\frac{x(x-1)}{1.\ 2} \left\{ \begin{matrix} + B & + C + D + E + F + \&c. \\ - A & - 2B - 2C - 2D - 2E - \&c. \\ & + A + B + C + D + \&c. \end{matrix} \right\}$$

Si differentiae demum tertiae fuerint evanescentes, tum insuper adiici debet haec series infinita:

$$\frac{x(x-1)(x-2)}{1.\ 2.\ 3} \left\{ \begin{matrix} + C & + D + E + F + G + \&c. \\ - 2B & - 3C - 3D - 3E - 3F - \&c. \\ + A & + 3B + 3C + 3D + 3E + \&c. \\ & - A - B - C - D - \&c. \end{matrix} \right\}$$

&c.

381. Accommodemus haec quoque ad alterum functionum inexplicabilium genus, quae constant continuo producto terminorum aliquot seriei popositae $A + B + C + D +$ &c. sitque

$$S = \overset{1}{A}\ \overset{2}{B}\ \overset{3}{C}\ \overset{4}{D} \ldots \overset{x}{X}$$

& quaeratur primo valor Σ, in quem S transmutatur, si loco x scribatur $x + \omega$; ponamus autem ut ante esse Z terminum seriei $A + B + C + D +$ &c. cuius index sit $= x + \omega$, uti X respondet indici x. Quo ergo hunc casum ad praecedentem reducamus sumamus logarithmos, eritque

$$lS = lA + lB + lC + lD + \ldots + lX$$

Quod si iam huius seriei termini infinitesimi evanescant, erit eandem methodum, qua ante usi sumus, adhibendo: $l\Sigma$

$$l\Sigma = lS + lX' + lX'' + lX''' + \&c.$$
$$\qquad - lZ' - lZ'' - lZ''' - \&c.$$

hincque ad numeros regrediendo erit

$$\Sigma = S.\frac{X'}{Z'}.\frac{X''}{Z''}.\frac{X'''}{Z'''}.\frac{X''''}{Z''''}, \&c.$$

quae ergo expressio valet, si seriei A, B, C, D, &c. termini infinitesimi unitati aequentur. Sin autem logarithmi terminorum infinitesimorum huius seriei non evanescant, at tamen differentias habeant evanescentes; tum ad illam seriem, quam pro $l\Sigma$ invenimus, insuper addi debet haec series

$$\omega lX' + \omega\left(l\frac{X''}{X'} + l\frac{X'''}{X''} + l\frac{X''''}{X'''} + \&c.\right)$$

sicque numeris sumendis habebitur

$$\Sigma = SX^{I\omega}.\frac{X^{II\omega}.X^{I(1-\omega)}}{Z'}.\frac{X^{III\omega}.X^{II(1-\omega)}}{Z''}.\frac{X^{IV\omega}.X^{III(1-\omega)}}{Z'''}. \&c.$$

382. Quodsi ergo ponamus $x = 0$, quo casu fit $S = 1$ & $X' = A$, $X'' = B$, $X''' = C$, &c. Σ denotabit productum ω terminorum huius seriei A, B, C, D, &c. Si igitur pro ω scribamus x, ut Σ obtineat valorem, quem ante ipsi S tribueramus, ita ut sit

$$S = \overset{1}{A}.\overset{2}{B}.\overset{3}{C}.\overset{4}{D}.\;\ldots\;\overset{x}{X}$$

quia nunc Z', Z'', Z''', &c. abeunt in X', X'', X''' &c. si logarithmi terminorum infinitesimorum istius seriei A, B, C, D, E, &c. evanescant, exprimetur S hoc modo

$$S = \frac{A}{X'}.\frac{B}{X''}.\frac{C}{X'''}.\frac{D}{X^{IV}}.\frac{E}{X^{V}}, \&c.$$

Sin autem differentiae demum logarithmorum terminorum infinitesimorum seriei A, B, C, D, &c. evanescant, tum ista functio S sequenti modo exprimetur, ut sit:

$$S = A^{x}.\frac{B^{x}A^{1-x}}{X'}.\frac{C^{x}B^{1-x}}{X''}.\frac{D^{x}C^{1-x}}{X'''}.\frac{E^{x}D^{1-x}}{X^{IV}}. \&c.$$

ſi illorum logarithmorum differentiae ſecundae demum ſint evaneſcentes, ex praecedentibus facile colligitur, cuiuſmodi factores inſuper addi debeant; quem caſum, cum vix occurrere ſoleat, hic praetermittamus. Ceterum uſum harum expreſſionum in interpolationis negotio capite ſequente oſtendam.

383. Hic igitur cum differentiatio huiuſmodi functionum inexplicabilium potiſſimum ſit propoſita: inveſtigemus differentiale huius functionis

$$S = A.\,B.\,C.\,D.\,\ldots\ldots X$$

Ad hoc reſumamus aequationem ante inventam

$$l\Sigma = lS + lX' + lX'' + lX''' + \&c.$$
$$- lZ' - lZ'' - lZ''' - \&c.$$

& cum lZ oriatur ex lX, ſi loco x ponatur $x+\omega$, erit

$$lZ = lX + \frac{\omega}{dx}\,d.\,lX + \frac{\omega^2}{2dx^2}\,dd.\,lX + \frac{\omega^3}{6dx^3}\,d^3.\,lX + \&c.$$

quibus valoribus pro lZ', lZ'', lZ''', &c. ſubſtitutis habebitur

$$l\Sigma = lS - \frac{\omega}{dx}\,d.\,(lX' + lX'' + lX''' + lX^{\text{IV}} + \&c.)$$
$$- \frac{\omega^2}{2dx^2}\,dd.(lX' + lX'' + lX''' + lX^{\text{IV}} + \&c.)$$
$$- \frac{\omega^3}{6dx^3}\,d^3.(lX' + lX'' + lX''' + lX^{\text{IV}} + \&c.)$$
$$\&c.$$

Ponatur nunc $\omega = dx$, fietque $l\Sigma = lS + d.\,lS$, ideoque erit

$$\frac{dS}{S} = -\,d.\,(lX' + lX'' + lX''' + lX^{\text{IV}} + \&c.)$$
$$-\tfrac{1}{2}\,dd.(lX' + lX'' + lX''' + lX^{\text{IV}} + \&c.)$$
$$-\tfrac{1}{6}\,d^3.(lX' + lX'' + lX''' + lX^{\text{IV}} + \&c.)$$
$$\&c.$$

quae formula valet, ſi logarithmi terminorum infiniteſimorum ſeriei A, B, C, D, &c. evaneſcant; ſin autem ipſi non evaneſcant, attamen differentias habeant evaneſcentes, tum ad praecedentem differentialis completi expreſſionem inſuper addi debet haec ſeries:

$$dx\,lX' + dx\left(l\frac{X''}{X'} + l\frac{X'''}{X''} + l\frac{X''''}{X'''} + \&c.\right)$$

ut obtineatur differentiale completum.

384. Idem adhuc alio modo praestari potest. Ponatur $x = 0$; quo casu abit lS in 0. Tum formentur series, quarum termini generales sint:

$$lX;\ \frac{d.lX}{dx};\ \frac{dd.lX}{2dx^2};\ \frac{d^3.lX}{6dx^3};\ \&c.$$

harumque serierum infinitarum summae sint respective: $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, &c. Scribatur x pro ω, ut sit $\Sigma = S$, eritque

$$lS = -\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \&c.$$

si quidem logarithmi terminorum infinitesimorum seriei A, B, C, D, &c. cuius terminus generalis est X, evanescant: at si horum logarithmorum differentiae demum evanescant erit:

$$lS = xlA + x\left(l\frac{B}{A} + l\frac{C}{B} + l\frac{D}{C} + l\frac{E}{D} + \&c.\right)$$
$$- \mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \&c.$$

Hincque adeo differentiale ipsius lS erit:

$$\frac{dS}{S} = dxlA + dx\left(l\frac{B}{A} + l\frac{C}{B} + l\frac{D}{C} + l\frac{E}{D} + \&c.\right)$$
$$- \mathfrak{B}dx - 2\mathfrak{C}xdx - 3\mathfrak{D}x^2dx - 4\mathfrak{E}x^3dx - \&c.$$

At si differentiale completum desideretur, erit id:

$$\frac{dS}{S} = dx\,lA + dx\left(l\frac{B}{A} + l\frac{C}{B} + l\frac{D}{C} + l\frac{E}{D} + \&c.\right)$$
$$- \mathfrak{B}dx - \mathfrak{C}(2xdx + dx^2) - \mathfrak{D}(3xxdx + 3xdx^2 + dx^3) - \&c.$$

Ad quarum formularum usum ostendendum sequentia exempla adiicimus, quae utroque modo resolvemus.

EXEMPLUM I.

Invenire differentiale huius functionis inexplicabilis:

$$S = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdots\cdots\frac{2x-1}{2x}.$$

Hic ante omnia notandum est, terminos infinitesimos horum

rum factorum abire in unitates, ideoque eorum logarithmos evanescere. Cum igitur sit $X = \frac{2x-1}{2x}$, erit

$$X' = \frac{2x+1}{2x+2};\ X'' = \frac{2x+3}{2x+4};\ X''' = \frac{2x+5}{2x+6};\ \&c.$$

& generaliter $X^{|n|} = \frac{2x+2n-1}{2x+2n}$; unde erit:

$$lX^{|n|} = l(2x+2n-1) - l(2x+2n)$$

$$d.lX^{|n|} = \frac{2dx}{2x+2n-1} - \frac{2dx}{2x+2n}$$

$$dd.lX^{|n|} = -\frac{4dx^2}{(2x+2n-1)^2} + \frac{4dx^2}{(2x+2n)^2}$$

$$d^3.lX^{|n|} = +\frac{2.2.4dx^3}{(2x+2n-1)^3} - \frac{2.2.4dx^3}{(2x+2n)^3}$$

$$d^4.lX^{|n|} = -\frac{2.2.4.6dx^4}{(2x+2n-1)^4} + \frac{2.2.4.6dx^4}{(2x+2n)^4}$$

&c.

unde erit differentiale completum:

$$\frac{dS}{S} = -2dx\left\{\begin{array}{l}\frac{1}{2x+1} + \frac{1}{2x+3} + \frac{1}{2x+5} + \&c.\\ -\frac{1}{2x+2} - \frac{1}{2x+4} - \frac{1}{2x+6} - \&c.\end{array}\right\}$$

$$+\tfrac{4}{2}dx^2\left\{\begin{array}{l}\frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \&c.\\ -\frac{1}{(2x+2)^2} - \frac{1}{(2x+4)^2} - \frac{1}{(2x+6)^2} - \&c.\end{array}\right\}$$

$$-\tfrac{8}{3}dx^3\left\{\begin{array}{l}\frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \&c.\\ -\frac{1}{(2x+2)^3} - \frac{1}{(2x+4)^3} - \frac{1}{(2x+6)^3} - \&c.\end{array}\right\}$$

&c.

Quod

Quod si autem tantum differentiale primum quaeratur erit id:

$$\frac{dS}{S} = -2dx \text{ in}$$

$$\left(\frac{1}{(2x+1)(2x+2)} + \frac{1}{(2x+3)(2x+4)} + \frac{1}{(2x+5)(2x+6)} + \&c.\right)$$

quod idem altera methodo §. 384. tradita ita investigatur. Cum sit $lX = l\frac{2x-1}{2x}$, erit $\frac{d.\,lX}{dx} = \frac{2}{2x-1} - \frac{1}{x}$;

$$\frac{dd.\,lX}{2dx^2} = -\frac{2}{(2x-1)^2} + \frac{1}{2xx};\ \frac{d^3.\,lX}{6dx^3} = +\frac{8}{3(2x-1)^3} - \frac{1}{3x^3} \ \&c.$$

ideoque fiet

$$\mathfrak{A} = l\frac{1}{2} + l\frac{3}{4} + l\frac{5}{6} + l\frac{7}{8} + \&c.$$

$$\mathfrak{B} = \begin{Bmatrix} +\frac{2}{1} + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9} + \&c. \\ -\frac{2}{2} - \frac{2}{4} - \frac{2}{6} - \frac{2}{8} - \frac{2}{10} - \&c. \end{Bmatrix} = 2\,l\,2$$

$$\mathfrak{C} = -\frac{4}{2}\begin{Bmatrix} \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c. \\ -\frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2} - \frac{1}{8^2} - \&c. \end{Bmatrix}$$

$$\mathfrak{D} = \frac{8}{3}\begin{Bmatrix} \frac{1}{1} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \&c. \\ -\frac{1}{2^3} - \frac{1}{4^3} - \frac{1}{6^3} - \frac{1}{8^3} - \&c. \end{Bmatrix} \quad \&c.$$

sive erit:

$$\mathfrak{B} = +\frac{2}{1}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c.\right)$$

$$\mathfrak{C} = -\frac{4}{2}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \&c.\right)$$

$\mathfrak{D} =$

$$\mathfrak{D} = +\frac{8}{3}\left(1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \&c.\right)$$

$$\mathfrak{E} = -\frac{16}{4}\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \&c.\right)$$

&c.

Quibus valoribus inventis ſubſtitutis erit:

$$\frac{dS}{S} = -2\ \ dx\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c.\right)$$
$$+4x\,dx\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \&c.\right)$$
$$-8x^2dx\left(1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \&c.\right)$$
$$+16x^3dx\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \&c.\right)$$

&c.

Si igitur ſit $x = 0$, quo caſu ſit $lS = 0$ & $S = 1$, erit $dS = -2\,dx\,l2$.

EXEMPLUM II.

Invenire differentiale huius functionis inexplicabilis:

$$S = 1.\ 2.\ 3.\ 4.\ \ldots\ldots\ x$$

Huius ſeriei 1, 2, 3, 4, &c. termini in infinitum ita creſcunt, ut logarithmorum differentiae evaneſcant: eſt enim $l(\infty + 1) - l\infty = l\left(1 + \frac{1}{\infty}\right) = \frac{1}{\infty} = 0$. Cum igitur ſit $X = x$ erit $X' = x + 1$; $X'' = x + 2$; $X''' = x + 3$; &c. porro autem ob $lX = lx$ fiet $d.lX = \frac{dx}{x}$; $dd.lX = -\frac{dx^2}{x^2}$; $d^3.lX = \frac{2dx^3}{x^3}$; $d^4.lX = -\frac{2.\,3dx^4}{x^4}$; &c. unde ſi logarithmi ultimi evaneſcerent, foret

dS

$$\frac{dS}{S} = -dx\left(\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\frac{1}{x+4}+\&c.\right)$$
$$+\frac{dx^2}{2}\left(\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\frac{1}{(x+4)^2}+\&c.\right)$$
$$-\frac{dx^3}{3}\left(\frac{1}{(x+1)^3}+\frac{1}{(x+2)^3}+\frac{1}{(x+3)^3}+\frac{1}{(x+4)^3}+\&c.\right)$$
&c.

At cum differentiae demum logarithmorum evanescant, insuper addi debet haec expressio:

$$dx\,l(x+1)+dx\left(l\frac{x+2}{x+1}+l\frac{x+3}{x+2}+l\frac{x+4}{x+3}+l\frac{x+5}{x+4}+\&c.\right)$$

Quia vero est:

$$l\frac{x+2}{x+1}=\frac{1}{x+1}-\frac{1}{2(x+1)^2}+\frac{1}{3(x+1)^3}-\frac{1}{4(x+1)^4}+\&c.$$
$$l\frac{x+3}{x+2}=\frac{1}{x+2}-\frac{1}{2(x+2)^2}+\frac{1}{3(x+2)^3}-\frac{1}{4(x+2)^4}+\&c.$$
&c.

erit verum differentiale completum:

$$\frac{dS}{S}=dx\,l(x+1)$$
$$-\tfrac{1}{2}(dx-dx^2)\left(\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\&c.\right)$$
$$+\tfrac{1}{3}(dx-dx^3)\left(\frac{1}{(x+1)^3}+\frac{1}{(x+2)^3}+\frac{1}{(x+3)^3}+\&c.\right)$$
$$-\tfrac{1}{4}(dx-dx^4)\left(\frac{1}{(x+1)^4}+\frac{1}{(x+2)^4}+\frac{1}{(x+3)^4}+\&c.\right)$$
$$+\tfrac{1}{5}(dx-dx^5)\left(\frac{1}{(x+1)^5}+\frac{1}{(x+2)^5}+\frac{1}{(x+3)^5}+\&c.\right)$$
&c.

Sin autem altero modo differentiale hoc exprimere velimus, quia est

$$lX = lx;\ \frac{d.lX}{dx} = \frac{1}{x};\ \frac{dd.lX}{2dx^2} = -\frac{1}{2x^2};$$

$$\frac{d^3.lX}{6dx^3} = \frac{1}{3x^3};\ \frac{d^4.lX}{24dx^4} = -\frac{1}{4x^4};\ \&c.$$

habebuntur sequentes series:

$$\mathfrak{A} = l\,1 + l\,2 + l\,3 + l\,4 + l\,5 + \&c.$$

$$\mathfrak{B} = 1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.\right)$$

$$\mathfrak{C} = -\tfrac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c.\right)$$

$$\mathfrak{D} = \tfrac{1}{3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right)$$

$$\mathfrak{E} = -\tfrac{1}{4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right)$$

&c.

Hinc ob $lA = l\,1 = 0$, fiet ex §. 384:

$$lS = x\left(l\frac{2}{1} + l\frac{3}{2} + l\frac{4}{3} + l\frac{5}{4} + \&c.\right)$$

$$-\,x\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.\right)$$

$$+\,\tfrac{1}{2}x^2\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\right)$$

$$-\,\tfrac{1}{3}x^3\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \&c.\right)$$

$$+\,\tfrac{1}{4}x^4\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \&c.\right)$$

&c.

Binae autem primae series, per quas x est multiplicatum, etiamsi utraque habeat summam infinitam, tamen ambae simul summam habent finitam. Si enim utriusque n termini capiantur, prodibit:

l(n

$$l(n+1) - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \dots - \frac{1}{n}.$$

At supra §. 142. invenimus esse

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} = \text{Const.} + l\,n$$
$$+ \frac{1}{2n} - \frac{\mathfrak{A}}{2n^2} + \frac{\mathfrak{B}}{4n^4} - \&c.$$

haecque constans prodit $= 0,5772156649015325$. Quodsi ergo ponatur $n = \infty$, erit:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{\infty} = \text{Const.} + l\,\infty,$$ unde binarum illarum serierum in infinitum continuatarum valor erit $= l(\infty + 1) - \text{Const.} - l\,\infty = - \text{Const.}$ Ex quo erit:

$$lS = - x \,.\, 0,5772156649015325$$
$$+ \tfrac{1}{2}xx\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c.\right)$$
$$- \tfrac{1}{3}x^3\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right)$$
$$+ \tfrac{1}{4}x^4\left(1 + \frac{1}{2.4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right)$$
$$\&c.$$

unde differentialia cuiusque ordinis facile reperiuntur. Erit enim:

$$\frac{dS}{S} = - dx \,.\, 0,5772156649015325$$
$$+ x\,dx\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c.\right)$$
$$- x^2\,dx\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.\right)$$
$$+ x^3\,dx\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.\right)$$
$$\&c.$$

At

At si hae series in unam colligantur erit:

$$\frac{dS}{S} = -dx \cdot 0,5772156649015325 + \frac{x\,dx}{1(1+x)} + \frac{x\,dx}{2(2+x)} + \frac{x\,dx}{3(3+x)} + \frac{x\,dx}{4(4+x)} + \&c.$$

Quare si sit $x = 0$, fiet:

$$\frac{dS}{S} = -dx \cdot 0,5772156649015325$$

Ex priori vero expressione hoc casu erit:

$$\begin{aligned}\frac{dS}{S} = &-\tfrac{1}{2}dx\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c.\right)\\ &+\tfrac{1}{3}dx\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\&c.\right)\\ &-\tfrac{1}{4}dx\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\&c.\right)\\ &+\tfrac{1}{5}dx\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\&c.\right)\\ &\&c.\end{aligned}$$

385. Hinc ergo etiam huiusmodi functionum inexplicabilium differentialia quovis casu speciali exhiberi possunt, propterea quod hic differentialia completa eruimus. Quamobrem si tales functiones ingrediantur in expressiones, quae indeterminatae videntur, cuiusmodi capite praecedente tractavimus; valores eadem methodo definiri poterunt, uti ex adiunctis exemplis intelligetur.

EXEMPLUM I.

Determinare valorem huius expressionis:

$$\frac{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{x}}{x(x-1)} - \frac{1}{(x-1)(2x-1)}$$

eo casu, quando ponitur $x = 1$.

Ponamus $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{x} = S$, erit ex §. 372.

$S =$

$$S = x\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c.\right)$$
$$-x^2\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\&c.\right)$$
$$+x^3\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\&c.\right)$$
$$\&c.$$

seu cum sit quoque
$$S = +1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\&c.$$
$$-\frac{1}{1+x}-\frac{1}{2+x}-\frac{1}{3+x}-\frac{1}{4+x}-\frac{1}{5+x}-\&c.$$
si quivis terminus superioris seriei cum praecedente inferioris combinetur, prodibit:

$$S = 1+\frac{x-1}{2(1+x)}+\frac{x-1}{3(2+x)}+\frac{x-1}{4(3+x)}+\&c.$$

quae expressio, quoniam poni debet $x=1$ est commodior. Sit ergo $x = 1+\omega$, fietque

$$S = 1+\frac{\omega}{2(2+\omega)}+\frac{\omega}{3(3+\omega)}+\frac{\omega}{4(4+\omega)}+\&c.$$

sive
$$S = 1+\omega\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\&c.\right) = 1+\mathfrak{B}\omega$$
$$-\omega^2\left(\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\&c.\right) \quad -\mathfrak{C}\omega^2$$
$$+\omega^3\left(\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\&c.\right) \quad +\mathfrak{D}\omega^3$$
$$\&c. \qquad \&c.$$

Tota ergo expressio posito $x = 1+\omega$ abibit in hanc:

$$\frac{1+\mathfrak{B}\omega-\mathfrak{C}\omega^2+\mathfrak{D}\omega^3-\&c.}{\omega(1+\omega)}-\frac{1}{\omega(1+2\omega)} \quad \text{seu}$$

$$\frac{\omega+\mathfrak{B}\omega+2\mathfrak{B}\omega^2-\mathfrak{C}\omega^2-\&c.}{\omega(1+\omega)(1+2\omega)} = \frac{1+\mathfrak{B}+2\mathfrak{B}\omega-\mathfrak{C}\omega-\&c.}{(1+\omega)(1+2\omega)}$$

Ponatur nunc $\omega = 0$, atque expressionis propositae valor casu $x = 1$, erit $= 1 + \mathfrak{B} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.$ quae series cum sit $= \frac{1}{6}\pi^2$, sequitur valorem quaesitum esse $= \frac{1}{6}\pi^2$.

EXEMPLUM II.

Invenire valorem huius expressionis:

$$\frac{2x - xx}{(x-1)^2} + \frac{\pi\pi x}{6(x-1)} - \frac{(2x-1)\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{x}\right)}{x(x-1)^2}$$

casu quo ponitur $x = 1$.

Ponatur $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{x} = S$, statuaturque $x = 1 + \omega$, fiet ut in exemplo praecedente invenimus;

$$S = 1 + \mathfrak{B}\omega - \mathfrak{C}\omega^2 + \mathfrak{D}\omega^3 - \&c. \quad \text{existente}$$

$$\mathfrak{B} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c. = \tfrac{1}{6}\pi\pi - 1$$

$$\mathfrak{C} = \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \&c.$$

$$\mathfrak{D} = \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c.$$

&c.

Posito ergo $x = 1 + \omega$ expressio proposita induet hanc formam:

$$\frac{1 - \omega\omega}{\omega\omega} + \frac{(1+\mathfrak{B})(1+\omega)}{\omega} - \frac{(1+2\omega)(1 + \mathfrak{B}\omega - \mathfrak{C}\omega^2 + \mathfrak{D}\omega^3 - \&c.)}{(1+\omega)\omega^2}$$

quae ad eandem denominationem $\omega^2(1+\omega)$ perducta fit:

$$\frac{\begin{matrix} 1 + \omega - \omega^2 - \omega^3 + \omega + 2\omega^2 + \omega^3 + \mathfrak{B}\omega(1 + 2\omega + \omega\omega) \\ - 1 - \mathfrak{B}\omega + \mathfrak{C}\omega^2 - \mathfrak{D}\omega^3 - 2\omega - 2\mathfrak{B}\omega^2 + 2\mathfrak{C}\omega^3\ \&c. \end{matrix}}{\omega^2(1+\omega)}$$

quae reducitur ad hanc formam:

$$\frac{\omega^2 + \mathfrak{C}\omega^2 + \mathfrak{B}\omega^3 + 2\mathfrak{C}\omega^3 - \mathfrak{D}\omega^3\ \&c.}{\omega^2(1+\omega)}$$

Fiat nunc $\omega = 0$, atque prodibit $1 + \mathfrak{C}$. Quocirca expressionis propositae valor casu $x = 1$, erit $= 1 + \mathfrak{C}$, ideoque per hanc ſeriem exprimetur: $1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} +$ &c. cuius ſumma cum neque per logarithmos, neque per peripheriam circuli π exhiberi poſſit, valor quaeſitus etiamnum alio modo finite aſſignari non poteſt. Ex his ergo duobus exemplis uſus, quem differentiatio functionum inexplicabilium in doctrina ſerierum habere poteſt, ſatis luculenter perſpicitur.

386. In methodo hic tradita functiones inexplicabiles differentiandi aſſumſimus ſeriei A, B, C, D, E, &c. terminos infiniteſimos vel eſſe $=0$, vel differentias tandem evaneſcentes habere; quorum ſi neutrum contingat, iſta methodo uti non licebit. Hancobrem aliam exponam methodum huic conditioni non adſtrictam, quam ſummatio generalis ſerierum ex termino generali petita & ſupra fuſius explicata ſuppeditat. Denotent igitur litterae $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, &c. numeros Bernoullianos §. 122. exhibitos, ſitque functio inexplicabilis propoſita haec: $S = \overset{1}{A} + \overset{2}{B} + \overset{3}{C} + \overset{4}{D} + \ldots + \overset{x}{X}$ & quia ſupra (130.) oſtendimus fore:

$$S = \int X dx + \tfrac{1}{2} X + \frac{\mathfrak{A}\, dX}{1.2\, dx} - \frac{\mathfrak{B}\, d^3 X}{1.2.3.4\, dx^3} + \frac{\mathfrak{C}\, d^5 X}{1.2.3.4.5.6\, dx^5} - \text{\&c.}$$

hinc facile erit iſtius functionis S differentiale exhibere erit enim:

$$dS = X dx + \tfrac{1}{2} dX + \frac{\mathfrak{A}\, ddX}{1.2\, dx} - \frac{\mathfrak{B}\, d^4 X}{1.2.3.4\, dx^3} + \frac{\mathfrak{C}\, d^6 X}{1.2.3.4.5.6\, dx^5} - \text{\&c.}$$

387. Sin autem progreſſio propoſita coniuncta ſit cum geometrica, quo caſu termini eius infiniteſimi nunquam ad differentias conſtantes reducuntur, ac propterea methodus prior locum invenit nullum; tum methodus §. 174. tradita medelam afferet. Si enim propoſita ſit haec functio:

$S = Ap + Bp^2 + Cp^3 + Dp^4 + \quad . \quad . \quad . \quad + Xp^x$,

quaerantur valores litterarum α, β, γ, δ, &c. ut ſit

$$\frac{p-1}{p-e^u} = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \text{\&c.}$$

quibus inventis, uti eos §. 170. exhibuimus, erit:

$$S = \frac{p}{p-1}\cdot p^x\left(X - \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} - \frac{\gamma d^3X}{dx^3} + \frac{\delta d^4X}{dx^4} - \&c.\right)$$

$\pm$ Constante, quae summam reddat $= 0$, si ponatur $x = 0$, seu quae cuiquam alii casui satisfaciat. Sumto ergo differentiali haec constans ex computo abibit, eritque:

$$dS = \frac{p}{p-1}\cdot p^x dx lp\left(X - \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} - \frac{\gamma d^3X}{dx^3} + \&c.\right)$$

$$+ \frac{p}{p-1}\cdot p^x\left(dX - \frac{\alpha ddX}{dx} + \frac{\beta d^3X}{dx^2} - \frac{\gamma d^4X}{dx^3} + \&c.\right) \quad \text{sive}$$

$$dS = \frac{p^{x+1}}{p-1}\left(X dx lp - (\alpha lp - 1)dX + (\beta lp - \alpha)\frac{ddX}{dx} - (\gamma lp - \beta)\frac{d^3X}{dx^2} + \&c.\right)$$

quod est differentiale quaesitum functionis propositae S.

388. Sin autem functio inexplicabilis proposita ex factoribus constet, eorumque logarithmi infinitesimi differentias habeant constantes sive minus; tum hac quoque methodo differentiale functionis perpetuo exhiberi poterit. Sit enim

$$S = \overset{1}{A}.\ \overset{2}{B}.\ \overset{3}{C}.\ \overset{4}{D}.\ \ldots\ \overset{x}{X}.$$

Quia hinc fit $lS = lA + lB + lC + lD + \ldots + lX$ methodo superiori, numeros Bernoullianos in subsidium vocando erit:

$$lS = \int dx\, lX + \tfrac{1}{2} lX + \frac{\mathfrak{A} d.lX}{1.2\, dx} - \frac{\mathfrak{B} d^3.lX}{1.2.3.4 dx^3} + \&c.$$

qua expressione differentiata fit:

$$\frac{dS}{S} = dx\, lX + \tfrac{1}{2} d.lX + \frac{\mathfrak{A} dd.lX}{1.2\, dx} - \frac{\mathfrak{B} d^4.lX}{1.2.3.4 dx^3} + \frac{\mathfrak{C} d^6.lX}{1.2.3.4.5.6\, dx^5} - \frac{\mathfrak{D} d^8.lX}{1.2.3\ldots 8\, dx^7} + \&c.$$

Hinc si fuerit $X = x$, ut sit:

$$S = 1.\ 2.\ 3.\ 4.\ \ldots\ x$$

fiet applicatione facta

$$\frac{dS}{S} = dx\, lx + \frac{dx}{2x} - \frac{\mathfrak{A} dx}{2xx} + \frac{\mathfrak{B} dx}{4x^4} - \frac{\mathfrak{C} dx}{6x^6} + \&c.$$

quae forma, si x sit numerus valde magnus, commodius usurpatur, quam eae, quas ante invenimus. CA-