

INSTITUTIONUM
CALCULI DIFFERENTIALIS

PARS POSTERIOR

CONTINENS

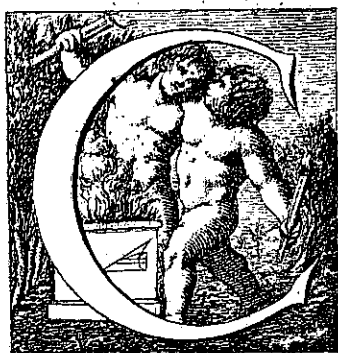
USUM HUIUS CALCULI IN ANALYSI
FINITORUM, NEC NON IN DOCTRINA
SERIERUM.



FI

CAPUT I.

DE TRANSFORMATIONE SERIERUM



I.

Um nobis propositum sit usum Calculi differentialis tam in universa Analyfi, quam in doctrina de seriebus ostendere; nonnulla subsidia ex Algebra comuni, quae vulgo tractari non solent, hic erunt repetenda. Quae quamvis maximam partem iam in introductione sumus complexi, tamen quaedam ibi sunt praetermissa, vel studio quod expediat ea tum demum explicari, quando necessitas id exigat, vel quia cuncta, quibus opus sit futurum, praevideri non poterant. Huc pertinet transformatio serierum, cui hoc Caput destinavimus, qua quaevis series in innumerabiles alias series transmutatur, quae omnes eandem habeant summam communem, ita ut, si seriei propositae summa sit cognita, reliquae series omnes simul summari queant. Hoc autem capite praemisso, eo uberius doctrinam serierum per calculum differentialem & integralem amplificare poterimus.

2. Considerabimus autem potissimum eiusmodi series, quarum singuli termini per potestates successivas quantitatis cuiusdam indeterminatae sunt multiplicati: quoniam hae latius patent, maioremque utilitatem afferent.

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Sit

Sit igitur proposita sequens series generalis, cuius summam, five sit cognita five secus, ponamus = S , sitque

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

Ponatur iam $x = \frac{y}{1+y}$, & cum sit per series infinitas

$$x = y - y^2 + y^3 - y^4 + y^5 - y^6 + \&c.$$

$$x^2 = y^2 - 2y^3 + 3y^4 - 4y^5 + 5y^6 - 6y^7 + \&c.$$

$$x^3 = y^3 - 3y^4 + 6y^5 - 10y^6 + 15y^7 - 21y^8 + \&c.$$

$$x^4 = y^4 - 4y^5 + 10y^6 - 20y^7 + 35y^8 - 56y^9 + \&c.$$

hi valores substituti, serieque secundum potestates ipsius y disposita, dabunt $S = ay - ay^2 + ay^3 - ay^4 + ay^5 \&c.$

$$\begin{array}{r} + b - 2b + 3b - 4b \\ + c - 3c + 6c \\ + d - 4d \\ + e \end{array}$$

3. Quoniam posuimus $x = \frac{y}{1+y}$; erit $y = \frac{x}{1-x}$,

quo valore loco y substituto, series proposita

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

transmutabitur in hanc:

$$S = a \cdot \frac{x}{1-x} + (b-a) \frac{x^2}{(1-x)^2} + (c-2b+a) \frac{x^3}{(1-x)^3} \&c.$$

in qua coefficientis secundi termini $b-a$ est differentia prima ipsius a ex serie $a, b, c, d, e, \&c.$ quam supra per Δa exposuimus; coefficientis tertii termini $c-2b+a$ est differentia secunda $\Delta^2 a$; coefficientis quarti est differentia tertia $\Delta^3 a$, &c. Hinc differentiis ipsius a continuis, quae formantur ex serie $a, b, c, d, e, \&c.$ adhibendis proposita series transmutabitur in hanc

$$S = \frac{x}{1-x} a + \frac{x^2}{(1-x)^2} \Delta a + \frac{x^3}{(1-x)^3} \Delta^2 a + \frac{x^4}{(1-x)^4} \Delta^3 a + \&c.$$

cuius ergo seriei summa habebitur, si propositae summa fuerit cognita.

4. Si igitur series $a, b, c, d, \&c.$ ita fuerit comparata, ut tandem differentias habeat constantes, quod evenit, si eius terminus generalis fuerit functio rationalis integra,

tum series posterior $\frac{x}{1-x} a + \frac{x^2}{(1-x)^2} \Delta a + \&c.$ tandem

habet terminos evanescentes, sicque eius summa per expressionem finitam exhiberi poterit. Ita si seriei $a, b, c, d, \&c.$ differentiae primae iam fuerint constantes, tum seriei huius: $ax + bx^2 + cx^3 + dx^4 + \&c.$ summa erit

$= \frac{x}{1-x} a + \frac{x^2}{(1-x)^2} \Delta a.$ At si illius seriei coefficientium

differentiae secundae fiant constantes, tum ipsius seriei propositae erit

$$= \frac{x}{1-x} a + \frac{x^2}{(1-x)^2} \Delta a + \frac{x^3}{(1-x)^3} \Delta \Delta a.$$

Unde summae huiusmodi serierum ex differentiis coefficientium facile inveniuntur.

I. *Quaeratur summa huius seriei:*

$$1x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \&c.$$

Diff. I. 2, 2, 2, 2, &c.

Cum ergo differentiae primae sint constantes, ob $a = 1$ & $\Delta a = 2$: erit seriei propositae summa

$$= \frac{x}{1-x} + \frac{2xx}{(1-x)^2} = \frac{x + 2xx}{(1-x)^2}.$$

II. *Quaeratur summa huius seriei*

$$1x + 4xx + 9x^3 + 16x^4 + 25x^5 + \&c.$$

Diff. I. 3, 5, 7, 9, &c.

Diff. II. 2, 2, 2, &c.

Quia itaque est

$$a = 1; \Delta a = 3, \Delta^2 a = 2;$$

erit seriei propositae summa

$$= \frac{x}{1-x} + \frac{3xx}{(1-x)^2} + \frac{2x^3}{(1-x)^3} = \frac{x + 3xx + 2x^3}{(1-x)^3}.$$

III. *Quaeratur summa huius seriei*

$$S = 4x + 15x^2 + 40x^3 + 85x^4 + 156x^5 + 259x^6 + \&c.$$

Diff. I. 11, 25, 45, 71, 103

Diff. II. 14, 20, 26, 32

Diff. III. 6, 6, 6,

Quia est $a = 4$; $\Delta a = 11$; $\Delta^2 a = 14$; $\Delta^3 a = 6$;
erit summa

$$S = \frac{4x}{1-x} + \frac{11xx}{(1-x)^2} + \frac{14x^3}{(1-x)^3} + \frac{6x^4}{(1-x)^4},$$

five $S = \frac{4x - xx + 4x^3 - x^4}{(1-x)^4} = \frac{x(1+xx)(4-x)}{(1-x)^4}.$

5. Quanquam hoc modo istarum serierum in infinitum progredientium summae inveniuntur; tamen ex iisdem principiis hae series quoque ad datum quemvis terminum summari possunt. Proposita enim sit haec series

$$S = ax + bx^2 + cx^3 + dx^4 + \dots + ox^n$$

& quaeratur eius summa, si in infinitum progrediatur, quae

$$\text{erit} = \frac{x}{1-x} a + \frac{xx}{(1-x)^2} \Delta a + \frac{xx^3}{(1-x)^3} \Delta^2 a + \&c.$$

Nunc considerentur eiusdem seriei termini post ultimum ox^n sequentes, qui sint

$$px^{n+1} + qx^{n+2} + rx^{n+3} + sx^{n+4} + \&c.$$

cuius seriei, si per x^n dividatur, summa, ut ante inveniri poterit; quae rursus per x^n multiplicata erit

$$\frac{x^{n+1}}{1-x} p + \frac{x^{n+2}}{(1-x)^2} \Delta p + \frac{x^{n+3}}{(1-x)^3} \Delta^2 p + \&c.$$

quae summa si a totius seriei in infinitum continuatae summa subtrahatur, remanebit summa portionis propositae quaesita: $S =$

$$\frac{x}{1-x} (a - x^n p) + \frac{x^2}{(1-x)^2} (\Delta a - x^n \Delta p) + \frac{x^3}{(1-x)^3} (\Delta^2 a - x^n \Delta^2 p) + \&c.$$

I. *Quaeratur summa huius seriei finitae.*

$$S = 1x + 2x^2 + 3x^3 + 4x^4 + \dots + nx^n$$

Tam

Tam horum coefficientium, quam terminum ultimum frequentium quaerantur differentiae:

$$\begin{array}{l|l} 1, 2, 3, 4, \&c. & n+1, n+2, n+3, \&c. \\ 1, 1, 1, & 1, 1; \\ \text{eritque } a=1, \Delta a=1, p=n+1, \Delta p=1, & \\ & \text{unde summa quaesita est;} \end{array}$$

$$S = \frac{x}{1-x} (1 - (n+1)x^n) + \frac{x^2}{(1-x)^2} (1 - x^n), \text{ feu}$$

$$S = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

II. *Quaeratur summa huius seriei finitae*

$$S = 1x + 4x^2 + 9x^3 + 16x^4 + \dots + n^2 x^n.$$

Investigentur primum differentiae hoc modo:

$$\begin{array}{l|l} 1, 4, 9, 16, \&c. & (n+1)^2, (n+2)^2, (n+3)^2, \&c. \\ 3, 5, 7, & 2n+3, 2n+5 \\ 2, 2, & 2 \end{array}$$

quibus inventis erit summa quaesita $S =$

$$\frac{x}{1-x} (1 - (n+1)^2 x^n) + \frac{x^2}{(1-x)^2} (3 - (2n+3)x^n) + \frac{x^3}{(1-x)^3} (2 - 2x^n),$$

$$\text{feu } S = \frac{x + nx - (n+1)^2 x^{n+1} + (2nn + 2n - 1)x^{n+2} - nx^{n+3}}{(1-x)^3}$$

6. Quodsi autem series proposita non eiusmodi habeat coefficientes, qui tandem ad differentias constantes deducantur, tum transmutatio hic exhibita nihil confert ad eius summam determinandam. Neque vero etiam eius ope summa proxime definiiri poterit commodius, quam per ipsam seriei propositae additionem fieri licet. Si enim in serie

$$ax + bx^2 + cx^3 + dx^4 + \&c.$$

fuerit $x < 1$ quo solo casu summatio proprie sic dicta locum

habere potest, erit $\frac{x}{1-x} > x$, ideoque non series minus con-

vergit quam proposita. Sin autem in serie proposita fuerit

$$x = 1$$

$x = 1$ tum novae seriei omnes plane termini fiunt infiniti, quo ergo casu ista transmutatio nullius prorsus erit usus.

7. Consideremus autem seriem, in qua signa $+$ & $-$ alternatim se excipiant, quae ex praecedente deducetur ponendo x negativum. Si itaque fuerit

$$S = ax - bx^2 + cx^3 - dx^4 + ex^5 - \&c.$$

cuius seriei negativa oritur, si in praecedente statuarur x negativum. Sumantur ergo ut ante differentiae $\Delta a, \Delta^2 a, \Delta^3 a, \&c.$ ex serie coefficientium $a, b, c, d, e, \&c.$ signis ad solas ipsius x potestates relatis, atque series proposita transformabitur in hanc: $S =$

$$\frac{x}{1+x} a - \frac{x^2}{(1+x)^2} \Delta a + \frac{x^3}{(1+x)^3} \Delta^2 a - \frac{x^4}{(1+x)^4} \Delta^3 a + \&c.$$

unde perspicitur aequationem propositam iisdem casibus summi posse quibus praecedens. Scilicet si series $a, b, c, d, \&c.$ tandem ad differentias constantes deducatur.

8. Hoc autem casu ista transformatio commodam praebet approximationem ad valorem seriei propositae:

$$ax - bx^2 + cx^3 - dx^4 + ex^5 - fx^6 + \&c.$$

quantumcunque enim x fuerit numerus, fractio $\frac{x}{1+x}$ secundum

cuius potestates altera series progreditur, fit unitate minor:

atque si fit $x = 1$, erit $\frac{x}{1+x} = \frac{1}{2}$. Sin autem fit $x < 1$

puta $x = \frac{1}{n}$ fiet $\frac{x}{1+x} = \frac{1}{n+1}$, ideoque series per transformationem inventa semper magis convergit quam proposita. Consideremus imprimis casum, quo $x = 1$, qui ad series summendas ingens affert subsidium, sitque

$$S = a - b + c - d + e - f + \&c.$$

ac denotentur differentiae primae, secundae & sequentes ipsius a , quas progressio $a, b, c, d, e, \&c.$ praebet per Δ ,

$\Delta a, \Delta^2 a, \Delta^3 a, \&c.$ quibus inventis erit

$$S = \frac{1}{2} a - \frac{1}{4} \Delta a + \frac{1}{8} \Delta^2 a - \frac{1}{16} \Delta^3 a + \&c.$$

quae nisi actu terminatur, summam vero proximam satis commode exhibet.

9. Usus igitur huius ultimae transmutationis, qua sumimus $n = 1$, in aliquot exemplis ostendamus, ac primo quidem in eiusmodi, quibus vera summa finite exprimi potest. Tales sunt series divergentes, quibus numeri $a, b, c, d, \&c.$ tandem ad differentias constantes deducant, quarum summae, cum recepto huius vocis significato, proprie non exhiberi queant, vocem summae hic eo sensu, quem supra tribuimus, accipimus, ita ut denotet valorem expressionis finitae, ex cuius evolutione proposita series nascatur.

I. Sit igitur proposita haec series Leibnitzii:

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \&c.$$

in qua cum omnes termini sint aequales, fient omnes differentiae $= 0$, ideoque ob $a = 1$, erit $S = \frac{1}{2}$.

II. Sit proposita ista series:

$$S = 1 - 2 + 3 - 4 + 5 - 6 + \&c.$$

Diff. I. $= 1, 1, 1, 1, 1, \&c.$

Cum ergo sit $a = 1, \Delta a = 1$, erit $S = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

III. Sit proposita haec series:

$$S = 1 - 3 + 5 - 7 + 9 - \&c.$$

Diff. I. $= 2, 2, 2, 2, \&c.$

Ob $a = 1 \& \Delta a = 2$ fit $S = \frac{1}{2} - \frac{2}{4} = 0$.

IV. Sit proposita haec series trigonalium numerorum.

$$S = 1 - 3 + 6 - 10 + 15 - 21 + \&c.$$

Diff. I. $= 2, 3, 4, 5, 6, \&c.$

Diff. II. $= 1, 1, 1, 1, \&c.$

Hic ergo ob $a = 1, \Delta a = 2, \& \Delta \Delta a = 1$; erit

$$S = \frac{1}{2} - \frac{2}{4} + \frac{1}{8} = \frac{1}{8}.$$

V. Sit proposita series quadratorum:

$$S = 1 - 4 + 9 - 16 + 25 - 36 + \&c.$$

Diff. I. $= 3, 5, 7, 9, 11, \&c.$

Diff. II. $= 2, 2, 2, 2, \&c.$

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Ob

Ob $a = 1$; $\Delta a = 3$; $\Delta\Delta a = 2$; erit $S = \frac{1}{2} - \frac{3}{4} + \frac{2}{8} = 0$.

VI. *Sit proposita series biquadratorum:*

$$S = 1 - 16 + 81 - 256 + 625 - 1296 + \&c.$$

$$\text{Diff. I.} = 15, 65, 175, 369, 671$$

$$\text{Diff. II.} = 50, 110, 194, 302$$

$$\text{Diff. III.} = 60, 84, 108$$

$$\text{Diff. IV.} = 24, 24$$

$$\text{Erit ergo } S = \frac{1}{2} - \frac{15}{4} + \frac{10}{8} - \frac{60}{16} + \frac{24}{32} = 0.$$

10. Si series magis divergant uti geometricae aliaeque similes, eae hoc modo statim in feriem magis convergentem transmutantur, quae nisi adhuc fatis convergat, eodem modo in aliam magis convergentem convertetur.

I. *Sit proposita haec series geometrica:*

$$S = 1 - 2 + 4 - 8 + 16 - 32 + \&c.$$

$$\text{Diff. I.} = 1, 2, 4, 8, 16, \&c.$$

$$\text{Diff. II.} = 1, 2, 4, 8, \&c.$$

$$\text{Diff. III.} = 1, 2, 4, \&c.$$

Cum igitur in omnibus differentiis primus terminus fit $= 1$.
Summa seriei exprimetur hoc modo

$$S = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \&c.$$

cuius summa est $= \frac{1}{3}$, oritur enim ex evolutione fractionis

$$\frac{1}{2+1}, \text{ dum proposita oritur ex } \frac{1}{1+2}.$$

II. *Sit proposita haec series recurrens:*

$$S = 1 - 2 + 5 - 12 + 29 - 70 + 169 - \&c.$$

$$\text{Diff. I.} = 1, 3, 7, 17, 41, 99 \quad \&c.$$

$$\text{Diff. II.} = 2, 4, 10, 24, 58 \quad \&c.$$

$$\text{Diff. III.} = 2, 6, 14, 34 \quad \&c.$$

$$\text{Diff. IV.} = 4, 8, 20 \quad \&c.$$

$$\text{Diff. V.} = 4, 12 \quad \&c.$$

$$\text{Diff. VI.} = 8 \quad \&c.$$

&c.

Continuarum ergo differentiarum termini primi constituunt hanc progressionem geometricam geminatam:

1,

$r, 1, 2, 2, 4, 4, 8, 8, 16, 16, \&c.$ unde erit

$$S = \frac{1}{2} - \frac{1}{4} + \frac{2}{8} - \frac{2}{16} + \frac{4}{32} - \frac{4}{64} + \frac{8}{128} \&c.$$

cum igitur praeter primum terminum reliqui bini se continuo destruant, erit $S = \frac{1}{2}$. Oritur autem series proposita ex

evolutione fractionis $\frac{r}{1+2-r} = \frac{1}{2}$, uti in expressione natu-

rae serierum recurrentium ostendimus.

III. Sit proposita series hypergeometrica :

$$S = 1 - 2 + 6 - 24 + 120 - 720 + 5040 - \&c.$$

cuius differentias continuas hoc modo commodius investigabimus :

Diff. I. Diff. II. Diff. III.

r	r		
2.	4	3.	r
6.	18	14.	64
24.	96.	78.	426
120.	600.	504.	3216 &c.
720.	4320.	3720.	27240
5040.	35280.	30960.	256320
40320.	322560.	287280.	2656080.
362880.	3265920.	2943360.	
3628800.			

Quibus differentiis ulterius continuatis erit :

$$S = \frac{1}{2} - \frac{1}{4} + \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \frac{2119}{128} - \frac{16687}{256} \\ + \frac{148329}{512} - \frac{1468457}{1024} + \frac{16019531}{2048} - \frac{190899411}{4096} + \&c.$$

Colligantur duo termini initiales, eritque $S = \frac{1}{4} + A$

existente $A = \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \frac{2119}{128} - \&c.$

Si nunc eodem modo differentiae capiantur, erit

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A

$$A = \frac{3}{2^4} - \frac{5}{2^6} + \frac{21}{2^8} - \frac{99}{2^{10}} + \frac{615}{2^{12}} - \frac{4401}{2^{14}} + \frac{36585}{2^{16}} \\ - \frac{342207}{2^{18}} + \frac{3565323}{2^{20}} - \frac{40866525}{2^{22}} + \&c.$$

Colligantur duo termini initiales, quia convergunt, fietque

$$A = \frac{7}{2^6} + B \quad \text{existente} \quad B = \frac{21}{2^8} - \frac{99}{2^{10}} + \&c.$$

cuius seriei differentiis denuo sumendis fiet:

$$B = \frac{21}{2^9} - \frac{15}{2^{12}} + \frac{159}{2^{15}} - \frac{429}{2^{18}} + \frac{5241}{2^{21}} - \frac{26283}{2^{24}} + \frac{338835}{2^{27}} \\ - \frac{2771097}{2^{30}} + \&c.$$

Colligantur quatuor termini initiales in unum & statuatur

$$B = \frac{153}{2^{12}} + \frac{843}{2^{18}} + C \quad \text{existente} \quad C = \frac{5241}{2^{21}} - \frac{26283}{2^{24}} + \&c.$$

fietque aliquot terminis actu colligendis proxime:

$$C = \frac{15645}{2^{24}} - \frac{60417}{2^{30}}. \quad \text{Ex his ergo tandem concludetur}$$

summa seriei propositae: $S = 0, 40082038$, quae tamen vix ultra tres quatuorve figuras pro accurata haberi potest ob nimiam seriei divergentiam, est tamen certe iusto minor. Aliunde enim inveni hanc summam esse $= 0,4036524077$, ubi ne ultima quidem nota a vero aberrat.

II. Inprimis autem haec transmutatio ingentem affert utilitatem ad series iam quidem, sed lente convergentes in alias, quae multo promptius convergant, transmutandas. Quoniam vero termini sequentes minores sunt quam praecedentes, differentiae primae fiunt negativae; unde in sequentibus signorum ratio probe est habenda.

I. *Sit proposita haec series:*

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c.$$

Diff.

$$\text{Diff. I.} = -\frac{1}{2}; -\frac{1}{2 \cdot 3}; -\frac{1}{3 \cdot 4}; -\frac{1}{4 \cdot 5}; -\frac{1}{5 \cdot 6}$$

$$\text{Diff. II.} = +\frac{1}{3}; \frac{2}{2 \cdot 3 \cdot 4}; \frac{2}{3 \cdot 4 \cdot 5}; \frac{2}{4 \cdot 5 \cdot 6}$$

$$\text{Diff. III.} = -\frac{1}{4}; -\frac{2 \cdot 3}{2 \cdot 3 \cdot 4 \cdot 5}; -\frac{2 \cdot 3}{3 \cdot 4 \cdot 5 \cdot 6}$$

$$\text{Diff. IV.} = +\frac{1}{5}; \&c.$$

Hinc ergo erit

$$S = \frac{1}{2} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 8} + \frac{1}{4 \cdot 16} + \frac{1}{5 \cdot 32} + \&c.$$

utramque autem hanc seriem logarithmum hyperbolicum binarii exhibere, iam in Introductione ostendimus.

II. *Sit proposita ista series pro circulo:*

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \&c.$$

$$\text{Diff. I.} = -\frac{2}{1 \cdot 3}; -\frac{2}{3 \cdot 5}; -\frac{2}{5 \cdot 7}; -\frac{2}{7 \cdot 9}; -\frac{2}{9 \cdot 11} \&c.$$

$$\text{Diff. II.} = +\frac{2 \cdot 4}{1 \cdot 3 \cdot 5}; \frac{2 \cdot 4}{3 \cdot 5 \cdot 7}; \frac{2 \cdot 4}{5 \cdot 7 \cdot 9}; \frac{2 \cdot 4}{7 \cdot 9 \cdot 11} \&c.$$

$$\text{Diff. III.} = -\frac{2 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5 \cdot 7}; -\frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7 \cdot 9}; -\&c.$$

Hinc ergo concluditur fore summam seriei:

$$S = \frac{1}{2} + \frac{1}{3 \cdot 2} + \frac{1 \cdot 2}{3 \cdot 5 \cdot 2} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7 \cdot 2} + \&c.$$

$$\text{feu } 2S = 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{3 \cdot 5 \cdot 7 \cdot 9} + \&c.$$

III. *Quaeratur valor huius seriei infinitae*

$$S = 12 - 13 + 14 - 15 + 16 - 17 + 18 - 19 \&c.$$

Quia differentiae ab initio nimis fiunt inaequales, colligantur actu termini usque ad 110 ex tabulis, quorum valor reperietur

$$= -0,3911005; \quad \text{eritque}$$

$$S = -0,3911005 + l_{10} - l_{11} + l_{12} - l_{13} + l_{14} - l_{15} + \&c.$$

in infinitum.

Defumantur hi logarithmi ex tabulis, eorumque differentiae quaerantur hoc modo

	Diff. 1.	Diff. 2.	Diff. 3.	Diff. 4.	Diff. 5.
$l_{10} = 1,0000000$	+	-	+	-	+
$l_{11} = 1,0413927$	413927	36042	5779	1292	368
$l_{12} = 1,0791812$	377885	30263	4487	924	
$l_{13} = 1,1139434$	347622	25776	3563		
$l_{14} = 1,1461280$	321846	22213			
$l_{15} = 1,1760913$	299633				

Ex quibus reperitur

$$l_{10} - l_{11} + l_{12} - l_{13} + \&c. =$$

1,0000000	413927	36042	5779	1292	368
2	4	8	16	32	64

$$= 0,4891606.$$

Hinc valor seriei propositae erit

$$S = l_2 - l_3 + l_4 - l_5 + \&c. = 0,0980601;$$

cui logarithmo respondet numerus 1,253315.

12. Quemadmodum has transmutationes obtinuimus ponendo in serie loco x hanc fractionem $\frac{y}{1+y}$, ita innumerabiles aliae transmutationes orientur, si loco x aliae functiones ipsius y substituuntur. Sit iterum proposita ista series:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \&c.$$

atque ponatur $x = y(1-y)$, quo facto series orietur sequens

$$\begin{aligned}
 S = & ay - ayy \\
 & + byy - 2by^3 + by^4 \\
 & + cy^3 - 3cy^4 + 3cy^5 - cy^6 \\
 & + dy^4 - 4dy^5 + 6dy^6 \\
 & + ey^5 - 5ey^6 \quad \&c. \\
 & + fy^6
 \end{aligned}$$

Quodsi ergo altera harum serierum fuerit summabilis, simul alterius summa erit cognita. Ita si statuatur

$$S = x + x^2 + x^3 + x^4 + x^5 + \&c. = \frac{x}{1-x}, \text{ erit}$$

$$S = y - y^3 - y^4 + y^6 + y^7 - y^9 - y^{10} + \&c.$$

$$\text{Cuius ergo seriei summa erit} = \frac{y - yy}{1 - y + yy}.$$

13. Si altera series alicubi abrumpatur, tum summa prioris absolute exhiberi poterit. Ponamus esse $a = 1$, & in serie inventa omnes terminos post primum evanescere, ut sit $S = y$; ideoque ob $x = y - yy$, erit summa prioris $= \frac{1}{2} - \sqrt{\left(\frac{1}{4} - x\right)}$. Fiet autem ob $a = 1$; ut sequitur:

$$b = 1 = \frac{1}{4} \cdot 2^2$$

$$c = 2 = \frac{1 \cdot 3}{4 \cdot 6} \cdot 2^4$$

$$d = 5 = \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 8} \cdot 2^6$$

$$e = 14 = \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8 \cdot 10} \cdot 2^8$$

$$f = 42 = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} \cdot 2^{10}$$

$$g = 132 = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14} \cdot 2^{12}$$

&c.

unde prior series abibit in hanc: $S = \frac{1}{2} - \sqrt{\left(\frac{1}{4} - x\right)} = x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + 132x^7 + \&c.$ quae eadem invenitur, si quantitas furda $\sqrt{\left(\frac{1}{4} - x\right)}$ in seriem evolvatur, atque ab $\frac{1}{2}$ subtrahatur.

14. Statuamus, quo transmutatio latius pateat $x = y(1 + ny)^r$, atque series proposita:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

transmutabitur in sequentem:

$$S =$$

$$\begin{aligned}
S = & ay + \frac{v}{1} nay^2 + \frac{v(v-1)}{1 \cdot 2} n^2 ay^3 + \frac{v(v-1)(v-2)}{1 \cdot 2 \cdot 3} n^3 ay^4 + \frac{v(v-1)(v-2)(v-3)}{1 \cdot 2 \cdot 3 \cdot 4} n^4 ay^5 \\
& + by^2 + \frac{2v}{1} nby^3 + \frac{2v(2v-1)}{1 \cdot 2} n^2 by^4 + \frac{2v(2v-1)(2v-2)}{1 \cdot 2 \cdot 3} n^3 by^5 \\
& + cy^3 + \frac{3v}{1} ncy^4 + \frac{3v(3v-1)}{1 \cdot 2} n^2 cy^5 \\
& + dy^4 + \frac{4v}{1} ndy^5 \\
& + ey^5 \\
& \&c.
\end{aligned}$$

Si ergo illius seriei summa fuerit cognita, & huius simul summa habebitur, ac vicissim. Quoniam vero n & v pro lubitu accipi possunt, hinc ex una serie summabili innumerae aliae summabiles inveniri possunt.

15. Possunt etiam eiusmodi transmutationes fieri, ut seriei inventae summa fiat irrationalis, hoc modo.

Sit proposita ista series:

$$\begin{aligned}
S &= ax + bx^3 + cx^5 + dx^7 + ex^9 + fx^{11} + \&c. \quad \text{erit} \\
Sx &= ax^2 + bx^4 + cx^6 + dx^8 + ex^{10} + fx^{12} + \&c.
\end{aligned}$$

Iam statuatur $x = \frac{y}{\sqrt{1-nyy}}$; erit $xx = \frac{yy}{1-nyy}$,

atque series proposita transmutabitur in hanc: $\frac{Sy}{\sqrt{1-nyy}} =$

$$\begin{aligned}
& ay^2 + nay^4 + n^2 ay^6 + n^3 ay^8 + n^4 ay^{10} + \&c. \\
& + by^4 + 2nby^6 + 3n^2 by^8 + 4n^3 by^{10} + \&c. \\
& + cy^6 + 3ncy^8 + 6n^2 cy^{10} + \&c. \\
& + dy^8 + 4ndy^{10} + \&c. \\
& + ey^{10} + \&c.
\end{aligned}$$

Si igitur summa S ex priori serie fuerit cognita, habebitur

simul summa sequentis seriei: $\frac{S}{\sqrt{1-nyy}} =$

$$ay + (na+b)y^3 + (n^2a+2nb+c)y^5 + (n^3a+3n^2b+3nc+d)y^7 + \&c.$$

16. Si sumatur $n = -1$; erunt coefficientes huius seriei differentiae continuae ipsius a , ex serie $a, b, c, d, \&c.$ sin autem signa in serie proposita alternentur, tum posito $n = 1$ coefficientes erunt istae differentiae. Denotent ergo $\Delta a, \Delta^2 a, \Delta^3 a, \Delta^4 a, \&c.$ differentias primas, secundas, tertias, $\&c.$ ipsius a ex serie numerorum $a, b, c, d, e, \&c.$

Ac si fuerit

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + \&c.$$

posito $x = \frac{y}{\sqrt{(1+yy)}}$; erit

$$\frac{S}{\sqrt{(1+yy)}} = ay + \Delta a \cdot y^3 + \Delta^2 a \cdot y^5 + \Delta^3 a \cdot y^7 + \&c.$$

Sin autem fuerit:

$$S = ax - bx^3 + cx^5 - dx^7 + ex^9 - \&c.$$

ponaturque $x = \frac{y}{\sqrt{(1-yy)}}$; erit

$$\frac{S}{\sqrt{(1-yy)}} = ay - \Delta a \cdot y^3 + \Delta^2 a \cdot y^5 - \Delta^3 a \cdot y^7 + \&c.$$

Quodsi ergo series $a, b, c, d, e, \&c.$ tandem ad differentias constantes deducatur, tum utraque series absolute summari poterit; quae summatio autem quoque ex superioribus sequitur.

17. Ponamus coefficientes $a, b, c, d, \&c.$ constituere hanc seriem $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \&c.$ eritque, uti supra iam vidimus:

$$\begin{aligned} a &= 1 \\ \Delta a &= -\frac{2}{3} \\ \Delta^2 a &= \frac{2 \cdot 4}{3 \cdot 5} \\ \Delta^3 a &= -\frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \quad \&c. \end{aligned}$$

H h

unde

unde sequentes duas series summabimus:

I. Sit $S = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \&c.$

Erit $S = \frac{1}{2}l \frac{1+x}{1-x}$. Pofito iam $x = \frac{y}{\sqrt{(1+yy)}}$,
fiet

$$S = \frac{1}{2}l \frac{\sqrt{(1+yy)} + y}{\sqrt{(1+yy)} - y} = l(\sqrt{(1+yy)} + y):$$

unde erit

$$\frac{l(\sqrt{(1+yy)} + y)}{\sqrt{(1+yy)}} = y - \frac{2}{3}y^3 + \frac{2 \cdot 4}{3 \cdot 5}y^5 - \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}y^7 + \&c.$$

II. Sit $S = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \&c.$

Erit $S = A \text{ tang } x$. Pofito iam $x = \frac{y}{\sqrt{(1-yy)}}$,
fiet

$$S = A \text{ tang } \frac{y}{\sqrt{(1-yy)}} = A \text{ fin } y = A \text{ cof } \sqrt{(1-yy)}.$$

Hancobrem obtinebitur ifta fummatio:

$$\frac{A \text{ fin } y}{\sqrt{(1-yy)}} = y + \frac{2}{3}y^3 + \frac{2 \cdot 4}{3 \cdot 5}y^5 + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}y^7 + \&c.$$

18. Poffunt quoque loco x functiones transcendentēs ipfius y fubftitui, ficque fummatioes aliae inventu difficiliores erui; verumtamen ne series novae fiant nimis perplexae, eiusmodi functiones eligi debent, quarum potestates facile exhiberi queant; quales funt quantitates exponentiales e^y . Propofita igitur hac ferie:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \&c.$$

ponatur $x = e^{ny}$, denotante e numerum, cuius logarithmus hyperbolicus = 1,

$$\text{erit } x^2 = e^{2ny}y^2; \quad x^3 = e^{3ny}y^3; \quad \&c.$$

Generaliter vero est, uti conftat:

$$e^z = 1 + z + \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.$$

Quare ferie propofita in hanc tranfmutabitur:

$$\begin{aligned}
 S = & ay + n ay^2 + \frac{1}{2} n^2 ay^3 + \frac{1}{6} n^3 ay^4 + \frac{1}{24} n^4 ay^5 + \&c. \\
 & + by^2 + \frac{2}{1} n by^3 + \frac{4}{2} n^2 by^4 + \frac{8}{6} n^3 by^5 + \&c. \\
 & + cy^3 + \frac{3}{1} n cy^4 + \frac{9}{2} n^2 cy^5 + \&c. \\
 & + dy^4 + \frac{4}{1} n dy^5 + \&c. \\
 & + ey^5 + \&c.
 \end{aligned}$$

I. Sit series proposita geometrica:

$$S = x + x^2 + x^3 + x^4 + x^5 + \&c. \text{ erit } S = \frac{x}{1-x}.$$

Ponatur iam:

$$n = -1; \text{ ut fit } x = e^{-y}; \text{ \& } S = \frac{e^{-y}}{1-e^{-y}} = \frac{y}{e^y - y},$$

reperietur summa haec

$$\frac{y}{e^y - y} = y - \frac{1}{2} y^3 - \frac{1}{6} y^4 + \frac{5}{24} y^5 + \frac{19}{120} y^6 - \&c.$$

cuius autem seriei lex non perspicitur.

II. Sint in altera serie omnes termini praeter primum

$$= 0; \text{ erit } b = -na; \quad c = \frac{3}{2} n^2 a; \quad d = -\frac{8}{3} n^3 a;$$

$$e = \frac{125}{24} n^4 a; \quad f = -\frac{29}{30} n^5 a; \quad \&c.$$

Cum ergo fit summa $S = ay$; & $x = e^{-y}$; fiet:

$$y = x - nx^2 + \frac{3}{2} n^2 x^3 - \frac{8}{3} n^3 x^4 + \frac{125}{24} n^4 x^5 - \frac{29}{30} n^5 x^6 + \&c.$$

Quoniam vero in his seriebus lex progressionis non est manifesta, summationes ex hac substitutione deductae parum habent utilitatis. Praecipue autem notari merentur transforma-

tiones ex substitutione $x = \frac{y}{1 \pm y}$ derivatae, quippe quae non

solum eximias summationes, sed etiam idoneos modos ad summas serierum appropinquandi suppeditant. His ergo, quae sine calculi differentialis ope sunt expedita, praemissis, ad ipsum huius calculi usum in doctrina serierum ostendendum progrediamur.