THEORIA MOTUS LUNAE

EXHIBENS

OMNES EIUS INAEQUALITATES

IN

ADDITAMENTO

HOC IDEM ARGUMENTUM ALITER TRACTATUR

SIMULQUE OSTENDITUR

QUEMADMODUM MOTUS LUNAE CUM OMNIBUS

INAEQUALITATIBUS

INNUMERIS ALIIS MODIS

REPRAESENTARI

ATQUE AD CALCULUM REUOCARI POSSIT

AUCTORE

L. EULERO

IMPENSIS
ACADEMIAE IMPERIALIS SCIENTIARUM
PETROPOLITANAE
ANNO 1753.

Academiam Scientiarum Petropolitanam triennio abhinc omnes, qui ingenii viribus confisi, ad examinandam Neutonianam de motu Lunae Theoriam, animum applicare vellent, inuitasse, atque ei, qui in hac parte tenuisset primas, praemii loco proposuisse nummos aureos centum; postmodum Celeberrimum Clairautum huius certaminis exstitisse victorem, publico Academicorum, qui Petropoli agunt, iudicio diuulgatum.

Celeberrimus Eulerus, Academiae Petropolitanae membrum honorarium, officii ſui eſſe exiſtimauit, ferre vna cum ceteris de illa Clairauti diſſertatione iudicium. Transmiſit ergo huc ad nos una cum ſua ſententia ampliſſimam de eodem argumento diſſertationem; quae quo celerius innoteſceret, viſum eſt Academiae Praeſidi, minoris Ruſſiae Hetmano Illuſtriſſimo Cyrillo Gregoridae, Comiti Raſumouio, eam tradere Academicis in ſolenni conuentu examinandam, ea fini, vt ſi ſuffragio Academicorum comprobata, dignaque iudicata foret, quae orbis eruditi proponeretur theatro, ea praelo quam maturrime ſubiiceretur; quandoquidem ille mos iam indè a principio obtinuit, vt, quae in publico coetu praeleguntur diſſertationes, eae vel ante ſolennem actum, vel paruulo intermiſſo ſpatio typis diuulgentur poſtea. Cete-

Ceterum diſſertatio illa conſtat magnam partem calculis, veris quidem illis et omnibus numeris abſolutis, ſed propter nimiam ſui molem atque difficultatem auditu moleſtiſſimis: quae ſi recitarentur publice, periculum erat, ne auditorum animi auſcultandis iis deficerent, neue Academia Summorum patientia Virorum abvti, caeterosque enicare odio videretur. Praeuidit hoc incommodum, prouiditque Illuſtriſſimi Praeſidis ſapientia. Mandavit Aſtronomiae Profeſſori V. C. Nicetae Popouio, cuius id temporis dicendi erat prouincia, vt prolixam illam Euleri disſertationem, omiſſis calculis, redigeret in compendium, et quae inde excepiſſet, auditorum cauſa recitaret publice. Quo quidem facto et auribus hominum conſuluit, et tamen rerum capitum participes eos facere aequabili temperamento inſtituit: atque

que dissertationem, ne quid forte naeuorum obreperet, ipso auctore coram typis excudi aequum censuit.

Iam qui illo tempore interfuerant conuentui Academicorum solenni auditores, aequis nimis auribusque auscultarunt excerpta recitantem Euleriana Popouium: est ergo quod sperare liceat, et iis, quorum interest, vt qui hoc genere studiorum maxime delectantur, factum iri satis ipsa dissertatione. Datum Petropoli Nov. 1752.

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INTRODUCTIO.

Cum nullum sit dubium, quin summi Newtoni Theoria, qua motum Planetarum felicissimo cum successu certis legibus adstrinxit, plurimum ad motum Lunae accuratius determinandum contulerit, maximi sane in Astronomia momenti est quaestio: vtrum haec Newtoni Theoria omnino sit sufficiens omnibus motus inaequalitatibus, quae in Luna obseruantur, exactissime explicandis nec ne? Quanquam autem a Newtoni asseclis plerumque affirmari solet, nullam in motu Lunae obseruari inaequalitatem, cuius ratio in ista Theoria non contineatur; tamen tantum abest, vt hic consensus a quoquam perspicue sit monstratus, vt potius applicatio huius Theoriae ad Lunam tantis implicetur calculi difficultatibus, quibus penitus euoluendis vires ingenii humani vix sufficere videntur. Plurimae quidem adhuc prodierunt Tabulae Lunares, quae ex Theoria Newtoniana deductae perhibentur, sed praeterquam quod saepius ultra 5′ ab obseruationibus discrepent, earum conuenientia cum Theoria ipsa neutiquam est euicta; quin potius pleraeque Tabulae inaequalitatum non tam Theoriae quam obseruationibus sunt superstructae. Huiusmodi ergo tabularum siue consensus siue dissensus cum obseruationibus neque ad Theoriam Newtonianam plenissime confirmandam, neque ad eam infringendam allegari poteſt: nam quatenus iſtae tabulae obſeruationibus ſatisfaciunt, hoc non ſoli Theoriae eſt tribuendum; quatenus autem cum obſeruationibus minus conueniunt, hoc ne Theoriae quidem imputari poterit, propterea quod iſtae Tabulae non ſoli Theoriae innituntur.

Quaeſtio itaque, cujus mentionem feci, recte enodari nequit, niſi ante eiusmodi Tabulae exhibeantur, quae ex ſola Theoria, nullis obſeruationibus in ſubſidium vocatis, ſint formatae; tum enim demum ex huiusmodi Tabularum collatione cum ingenti obſeruationum ſummo ſtudio inſtitutarum copia diiudicare licebit, vtrum Theoria omnibus obſeruationibus reſpondeat, an vero correctione quapiam indigeat. Non difficile quidem eſt ex principiis mechanicis motum Lunae aequationibus analyticis complecti; quoniam autem hae aequationes plures variabiles inter ſe permixtas continent, atque adeo differentialibus ſecundi ordinis implicantur, earum reſolutio maximis difficultatibus eſt obnoxia; et quoniam alio modo niſi per approximationem ſuſcipi non poteſt, vtcunque inſtituatur, ſemper non leue dubium remanet, vtrum partes, quae in calculo ſunt neglectae et praetermiſſae, nihil, quod in motu Lunae eſſet notabile, efficere potuerint. Hoc modo explicatio motuum Lunae tota ad ſolam Analyſin transfertur, ac difficultates, quibus premitur, inde oriuntur, quod Analyſis nondum ſatis eſt exculta.

Cum igitur Theoria Newtoniana hoc principio latiſſime patente innitatur, quod omnia corpora coeleſtia ſe mutuo attrahant in ratione reciproca duplicata diſtantiarum,

rum,

rum, ſi motum Lunae ſecundum hanc Theoriam definire velimus, vires erunt ſpectandae omnes, quibus Luna ſollicitatur. Atque inter has vires primaria eſt ea, qua ad terram vrgetur, quae ſi ſola adeſſet, terraque quieſceret, Luna in ellipſi perfecta ſecundum regulas Keplerianas motum ſuum circa terram abſolueret. At cum Luna praeterea aeque ac terra ipſa etiam ad ſolem trahatur, hac vi motus ille regularis non mediocriter perturbabitur: atque haec vis a ſole profecta omnium difficultatum, quae in determinatione motus Lunae offenduntur, cauſa eſt exiſtimanda. Reliquae enim vires, quibus forte Luna ſecundum Theoriam Newtoni ad reliquos planetas vrgeri deberet, tam ſunt exiguae, vt effectus inde oriundus merito pro nihilo haberi queat.

Solas ergo vires ſolis ac terrae in computum duci oportet, ſi motum Lunae ſecundum Theoriam definire velimus, atque cum ex his viribus formulae analyticae fuerint erutae, quae motum Lunae complectantur, omne ſtudium in his formulis ita euoluendis erit impendendum, vt inde ad quoduis tempus propoſitum locus Lunae aſſignari, ac more apud Aſtronomos ſolito ſecundum longitudinem et latitudinem definiri queat. Hinc porro Tabulae Aſtronomicae pro motu Lunae erunt condendae, quibus omnes inaequalitates tam in longitudine quam in latitudine exhibeantur, ex quibus ſi pro cujusuis obſeruationis momento locus Lunae computetur, conſenſus vel diſſenſus calculi ab obſeruationibus Theoriam vel confirmabit, vel ejus defectum declarabit. Neque tamen Theoria ſola hujusmodi Tabulis conſtruendis ſufficit, ſed quaedam ele-

menta

menta extrinfecus ab obferuationibus affumi oportet, quae funt 1°. Excentricitas orbitae Lunaris, quae falua theoria vel major vel minor effe potuiffet; pendet enim a motu Lunae primitus impreffo, quem Theoria non determinat, fed tanquam cognitum affumit. 2°. Locus Lunae medius pro quapiam Epocha propofita, qui pariter ex obferuationibus eft concludendus. 3°. Locus Apogei orbitae Lunaris pro Epocha quadam data. 4°. Tempus periodicum Lunae fecundum motum medium, quod pendet a diftantia Lunae media a Terra, ideoque ex fola Theoria definiri nequit. 5°. Locus nodorum Lunae pro Epocha quadam data: et 6to denique inclinatio media orbitae Lunaris ad planum Eclipticae.

His autem fex elementis per obferuationes definitis reliqua omnia, quibus ad locum Lunae pro quouis tempore affignandum opus eft, ex fola Theoria funt petenda; quae primo ad quoduis tempus locum Apogei eiusque ideo motum verum praebere debet, vt inde ex loco Lunae medio eius anomalia media colligi queat. Deinde Theoria quoque omnes correctiones feu Proftaphaerefes, quae loco Lunae medio vel additae vel fublatae eius locum verum exhibeant, fuppeditare debet; atque iftae correctiones, quae motus inaequalitates appellari folent, partim ab Anomalia media Lunae, partim ab eius Phafi feu elongatione a fole, partimque ab Anomalia folis media pendent, ex quo triplici fonte numerus inaequalitatum in immenfum augetur. Porro etiam Theoria motum nodi eiusque omnes inaequalitates indicare tenetur, ac denique etiam pro quouis tempore orbitae Lunaris veram inclinationem

da

ad eclipticam, vt inde eius latitudinem veram eruere liceat.

Cum autem, vt iam innui, nemo adhuc omnes inaequalitates, quae in motu Lunae reperiuntur, ex Theoria elicuerit, vt ex iis iudicium ferri poffit, quantum haec Theoria cum obferuationibus conueniat; etfi nullum eft dubium quin difcrimen, fi quod deprehenderetur, admodum paruum fit futurum: iam pridem haec quaeftio ex folo motu Apogei dirimi eft coepta, dum aliis motus Apogei ex obferuationibus cognitus magnopere a Theoria difcrepare eft vifus, alii autem etiam hoc loco pulcerrimum confenfum Theoriae et veritatis iactauerunt. Mirum autem eft, ipfum Newtonum nihil circa motum Apogei ex Theoria ftatuiffe, fed eum ex folis obferuationibus in calculum tranftuliffe, cum tamen motum nodorum fumma fagacitate ex Theoria elicuiffet, atque veritati confentaneum oftendiffet. Cur igitur motum Apogei plane filentio praeterierit, nulla alia ratio fubeffe videtur, nifi quod animaduerterit hunc motum, prouti ex Theoria prodiret, obferuationibus parum fore conformem. Ex iis enim, quae Newtonus in fuo immortali opere de motu abfidum in genere tradidit, non admodum difficile videtur motum apogei Lunae definire: verum hic praeter expectationem euenit, vt motus apogei annuus vix 20° fuperans reperiatur, cum tamen ex obferuationibus conftet, Apogeum Lunae interuallo unius anni vltra 40° promoueri.

Siue autem ifta motus Apogei quantitas 20° legitime fit ex Theoria deriuata, fiue minus; confideratio Apogei tutiffimum praebet remedium quaeftionem de fufficientia

 Theo-

Theoriae Newtonianae decidendi. Quamuis enim ex Theoria inaequalitas quaepiam in ipso motu Lunae aliquot minutis secundis vel etiam primis maior minorue prodiret, quam experientia monstraret, tamen tantilla differentia merito vel leui cuipiam errori in obseruationibus, vel vitio in approximatione commisso tribueretur; quandoquidem aliunde certum est Theoriam Newtonianam non admodum a veritate recedere. At longe aliter est comparata ratio motus Apogei: quodsi enim vires Lunam sollicitantes tantillum a Theoria Newtoniana discrepent, vt ex iis in ipso motu Lunae vix perceptibile discrimen nasceretur, tamen inde in motu Apogei annuo differentia plurium graduum oriri poterit. Quae tanta differentia cum nulli errori vel obseruationum vel ipsius calculi, siquidem omni cura instituatur, tribui queat, inuestigatio motus apogei certissimum suppeditat criterium iudicandi, vtrum quaepiam Theoria veritati sit consentanea nec ne?

Quodsi ergo calculo rite administrato Theoria Newtoni reperiatur tantum Apogei Lunaris motum exhibere, quantus per obseruationes deprehenditur, scilicet ultra 40° quotannis; fortius certe argumentum, quo veritas hujus Theoriae indubie demonstretur, desiderari nequit. Sin autem contra eueniat, vt progressio Apogei annua ex Theoria rite deriuata notabiliter a 40° deficiat, hinc certo erit concludendum Theoriam Newtonianam correctione quapiam indigere, neque vires, quibus Luna reuera sollicitatur, exactissime huic Theoriae esse conformes.

Verum haec ipsa quaestio, vtrum Theoria Newtoniana ad verum apogei Lunae motum perducat nec ne? est profun-

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fundissimae indaginis, atque summam in calculo circumspectionem ac sollertiam requirit. Quanquam enim ex principiis generalibus, vnde vulgo motus absidum definiri solet, satis luculenter semissis tantum pro motu Apogei Lunae elicitur; tamen quoniam in calculo plures termini, qui in determinationem motus Lunae ingrediuntur, ob paruitatem sunt reiecti, merito dubitatio suboritur, num hi ipsi termini, si eorum ratio esset habita, non istum defectum compensare valuerint? Quin etiam non defuere Geometrae, qui consensum huius Theoriae cum vero Apogei motu demonstrare sunt conati: verum plerumque non difficile erat paralogismum in ipsorum ratiociniis deprehendere. Maximam autem hoc loco attentionem meretur iudicium profundissimi Geometrae Clairaut, qui cum primum validissimis argumentis statuisset, Newtoni Theoriam non vltra dimidium veri apogei Lunaris motus suppeditare, subito in contrariam abiit sententiam statuens hanc Theoriam elegantissime cum veritate conspirare: neque certe tantae perspicaciae Vir a pristina sententia, quam omni studio propugnauerat, recessisse est putandus, nisi firmissimis argumentis eo esset adactus.

Cum autem omnes rationes, quae Ipsum ad hanc retractationem impulerint, nondum publice exposuerit, liceat mihi quidem, qui semper contrariae sententiae fui addictus, tantisper arduam hanc quaestionem tanquam nondum decisam spectare, donec per propriam inuestigationem inuenero, quid de ea sit statuendum. Postquam enim iam a longo temporis interuallo plurimum studii in indagatione motus Lunae consumsissem, ac variis methodis insistens semper

ſemper concluſionem Theoriae Newtonianae minus fauentem eſſem adeptus; quam tamen pro rite demonſtrata venditare non eram auſus, propterea quod approximatione eſſem vſus, ac ſemper ſuſpicio quaedam ratione terminorum praetermiſſorum remaneret: nuper in aliam incidi viam hanc inueſtigationem ſuſcipiendi, quae mihi multo certior videtur, ita vt per eam nulla dubitatione interiecta ad veritatem penetrare confidam. Ne autem ſi forte Theoriam Newtonianam minus ſufficientem inuenero, calculum ſecundum aliam Theoriam de nouo inſtituere cogar, ſtatim meam inueſtigationem in latiori ſenſu exordiar, viresque quibus Luna ad terram ſollicitatur, non exacte ſed proxime tantum quadratis diſtantiarum reciproce proportionales aſſumam: deinceps ſcilicet innoteſcet, vtrum haec aberratio a regula Newtoniana locum habeat nec ne? Calculum autem ita adornabo, vt quicquid euenerit, non ſolum verum apogei motum aſſequar, ſed etiam omnes Lunae inaequalitates inde elicere valeam, quas deinceps Aſtronomorum more tabulis complecti licebit.

Primum ergo problema in latiſſimo ſignificatu concipiam, vt corporis a viribus quibuscunque ſollicitati motum ſim inueſtigaturus: deinde vires, quibus Luna actu vrgeri cenſenda eſt, in calculum introducam, et aequationes Lunae motum determinantes exhibebo. Has porro aequationes variis modis in alias formas transmutabo, donec eas eo perduxero, vbi ad finem propoſitum maxime accommodatae videbuntur: quo cum peruenero, tandem tam motum Apogei, quam cunctas Lunae inaequalitates motus ex calculo deriuare ſtudebo.

CAPUT

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CAPUT I.

DE MOTU CORPORIS A VIRIBUS QUIBUSCUNQUE SOLLICITATI.

§. 1.

Quoniam corpus a viribus quibuscunque ſollicitari ponimus, fieri poteſt, vt eius motus non in eodem plano abſoluatur. Hinc ad eius motum per calculum ita repraeſentandum, vt ad quoduis tempus verus locus, in quo corpus verſabitur, aſſignari queat, conueniet corporis motum ad planum quoddam fixum pro lubitu aſſumtum referri. Exhibeat igitur Tabula hoc planum, atque corpus iam verſetur extra hoc planum in puncto L, vnde ad planum demittatur perpendiculum L M; eritque punctum M locus corporis ad planum relatus. Quod ſi ergo ad quoduis tempus propoſitum hunc corporis locum relatum M, ſimulque eius a plano diſtantiam L M indicare valeamus, verus corporis locus L ad hoc tempus innoteſcet. Fig. 2.

§. 2. Ad locum autem puncti M commodius determinandum, aſſumamus in plano rectam quandam fixam CQ pro axe, ita vt ducta ex M ad hanc rectam perpendiculari M P, locus puncti M more apud Geometras recepto per coordinatas orthogonales definiatur. Aſſumto ergo porro in axe puncto quodam fixo C, vnde abſciſſae C P computentur, erit P M applicata puncto M reſpondens, & ipſum punctum L determinabitur per tres coordinatas inter ſe normales C P, P M &

 M L.

M L. Cum igitur praesenti temporis momento corpus in L versari ponatur, vocentur istae tres coordinatae eo spectantes:

$$CP = p;\ PM = q \quad \text{et} \quad ML = r$$

elapso autem temporis elemento, quod per dt indicemus, coordinatae ternae tum locum corporis indicantes erunt:

$$p + dp;\quad q + dq;\quad \text{et}\quad r + dr.$$

§. 3. Quaecunque nunc fuerint vires, quibus corpus sollicitatur, eae semper per cognitam virium resolutionem reduci poterunt ad ternas vires secundum directiones ternarum coordinatarum vrgentes. Ponamus ergo vim $= P$, qua corpus in L secundum directionem ipsi P C parallelam trahitur: eam porro vim $= Q$, qua corpus secundum directionem ipsi M P parallelam trahitur: eamque denique vim $= R$, qua corpus secundum directionem L M sollicitatur. Has scilicet vires ita directas concipio, vt, si corpus earum actioni libere obediret, valores coordinatarum p, q, r inde diminuerentur. His positis, ex principiis Mechanicae constat, si elementum temporis dt pro constanti assumatur, motum corporis his tribus formulis differentio-differentialibus determinari

$$\text{I. } ddp = -\tfrac{1}{2} P dt^2;\ \text{II. } ddq = -\tfrac{1}{2} Q dt^2;\ \text{III. } ddr = -\tfrac{1}{2} R dt^2.$$

§. 4. Verum hae coordinatae ad vsum astronomicum, ad quem hic potissimum respicimus, non satis sunt accommodatae. Nam si spectatorem in C constitutum assumimus, locus L, vbi corpus cernetur, commodissime

me per quantitatem rectae C M, et angulum Q C M vna cum angulo M C L repraesentatur: atque si tabula planum eclipticae referat, rectaque C Q ad principium arietis sit directa, angulus Q C M in Astronomia vocari solet sideris longitudo, angulus M C L vero eius latitudo, et recta C M eius distantia curtata. Vocemus ergo porro:

I. Distantiam curtatam seu rectam $CM = x$

II. Longitudinem seu angulum $QCM = \phi$

III. Latitudinem seu angulum $MCL = \psi$

ac posito constanter sinu toto $= 1$, erunt valores coordinatarum ante adhibitarum:

$CP = p = x \cos\phi$; $PM = q = x \sin\phi$ & $ML = r = x \tang\psi$ atque distantia corporis vera a puncto C erit $CL = x \sec.\psi = \frac{x}{\cos\psi}$.

§. 5. Sumtis nunc differentialibus more consueto obtinebimus:

$$dp = dx \cos\phi - x d\phi \sin\phi;\ dq = dx \sin\phi + x d\phi \cos\phi$$

$$\text{et } dr = dx \tang\psi + \frac{x d\psi}{\cos\psi^2}$$

atque hinc denuo differentialibus sumendis reperietur,

$$ddp = ddx \cos\phi - 2dx\,d\phi \sin\phi - x dd\phi \sin\phi - x d\phi^2 \cos\phi$$

$$ddq = ddx \sin\phi + 2dx\,d\phi \cos\phi + x dd\phi \cos\phi - x d\phi^2 \sin\phi$$

$$ddr = ddx \tang\psi + \frac{2dx\,d\psi}{\cos\psi^2} + \frac{x dd\psi}{\cos\psi^2} + \frac{2x d\psi^2 \sin\psi}{\cos\psi^3}$$

Binae

Binae priores formulae rite combinatae ſuppeditabunt ſequentes multo concinniores

$$ddp \cos\phi + ddq \sin\phi = ddx - x\,d\phi^2$$
$$ddq \cos\phi - ddp \sin\phi = 2\,dx\,d\phi + x\,dd\phi$$

ſicque habebitur:

$$ddx - x\,d\phi^2 = -\tfrac{1}{2}\,dt^2\ (P \cos\phi + Q \sin\phi)$$
$$2dx\,d\phi + x\,dd\phi = -\tfrac{1}{2}\,dt^2\ (Q \cos\phi - P \sin\phi)$$

Tertiam vero aequationem deinceps magis tractabilem efficiemus.

§. 6 Manifeſtum autem eſt formulam $P\cos\phi + Q \sin\phi$ praebere vim ex viribus P et Q compoſitam, qua corpus in L ſecundum directionem rectae MC vrgetur, formulam vero alteram $Q \cos\phi - P \sin\phi$ exprimere vim ex eadem reſolutione ſecundum directionem MQ ad MC normalem directam. Cum igitur hae duae vires aſſumtis binis P et Q aequiualeant, ponamus eſſe

I. Vim corpus L ſecundum MC trahentem $= V$

II. Vim corpus L ſecundum MQ trahentem $= T$

manente tertia, vi corpus ad planum normaliter ſecundum LM vrgente $= R$. Atque ſequentes habebimus aequationes:

I. $2dx\,d\phi + x\,dd\phi = -\tfrac{1}{2}\,T\,dt^2$

II. $ddx - x\,d\phi^2 = -\tfrac{1}{2}\,V\,dt^2$

III. $ddx \tang\psi + \dfrac{2dx\,d\psi}{\cos\psi^2} + \dfrac{x\,dd\psi}{\cos\psi^2} + \dfrac{2x\,d\psi^2 \sin\psi}{\cos\psi^3} = -\tfrac{1}{2}\,R\,dt^2$

§. 7. Quo autem effectum tertiae vis R commodius ad calculum reuocemus, more apud Aſtronomos recepto contemplemur planum, in quo corpus durante elemento

nt ſe-)) abilem + Q , qua rgetur, re vim ad MC aſſum- = V = T ſecun- habebi- R dt² nodius os re- lurante mento

elemento temporis dt mouetur, et quod ſimul per punctum C tranſeat. Hoc igitur planum cum plano aſſumto interſectionem alicubi formabit, quae ſit recta C ☊, ac linea nodorum appellari ſolet; ſicque erit ☊ C L planum orbitae, in qua corpus L praeſenti inſtanti mouetur, et angulus, quo hoc planum ☊ C L ad planum fixum Q C M inclinatur, vocatur inclinatio orbitae ad eclipticam pro tempore praeſenti. Cum igitur ex his duabus rebus latitudo ſideris definiri ſoleat, ponamus.

Longitudinem nodi aſcendentis ſeu angulum Q C ☊ $= \pi$ ac inclinationem orbitae ☊ C L ad eclipticam $= \varrho$ atque loco latitudinis ψ has duas quantitates π et ϱ definire oportebit.

§. 8. Tertia ergo aequatio in duas diſpertietur, ad quas inueniendas ex M et L ad lineam nodorum C ☊ ducantur normales M N et L N, eritque angulus L N M menſura inclinationis orbitae ad eclipticam; ideoque L N M $= \varrho$. Tum vero ob angulum ☊ C M $= \phi - \pi$ et C M $= x$ erit:

$$CN = x \cos(\phi - \pi) \text{ et } MN = x \sin(\phi - \pi)$$

hinc elicietur $ML = x \tang\varrho \sin(\phi - \pi)$, vnde prodit $\tang MCL = \tang\psi = \tang\varrho \sin(\phi - \pi)$, quae formula inſeruit latitudini ψ ex cognita inclinatione ϱ et loco nodi eiusue longitudine π inueniendae, ſi quidem iam cognita fuerit longitudo ſideris ϕ. Quoniam autem ſidus elemento temporis dt in eodem plano manet, in differentiatione formulae $\tang\psi = \tang\varrho \sin(\phi - \pi)$, quanti-

quantitates π et ϱ tanquam constantes spectari poterunt, eritque idcirco.

$$\frac{d\,\psi}{\cos\psi^2} = d\,\Phi \text{ tang } \varrho \cos(\Phi - \pi)$$

§. 9. Interim tamen nihil impedit, quominus in eadem differentiatione quantitates π et ϱ tanquam variabiles tractemus, quales reuera esse possunt successu temporis; vnde orietur haec aequatio:

$$\frac{d\,\psi}{\cos\psi^2} = \frac{d\,\varrho}{\cos\varrho^2}\sin(\Phi-\pi) + (d\Phi - d\pi)\text{tang}\,\varrho\cos(\Phi-\pi)$$

hicque valor ipsius $\frac{d\,\psi}{\cos\psi^2}$ collatus cum praecedente praebebit hanc aequalitatem:

$$\frac{d\,\varrho}{\cos\varrho^2}\sin(\Phi-\pi) = d\pi \text{ tang } \varrho \cos(\Phi-\pi)$$

vnde obtinemus $\frac{d\,\varrho}{\sin\varrho\cos\varrho} = \frac{d\pi\cos(\Phi-\pi)}{\sin(\Phi-\pi)} = \frac{d\,\pi}{\text{tang}(\Phi-\pi)}$.

Cum iam sit $\frac{d\,\varrho}{\sin\varrho\cos\varrho} = \frac{d.\ \text{tang}\,\varrho}{\text{tang}\,\varrho} = d.\ l\,\text{tang}\,\varrho$, erit

$$d.\ l\,\text{tang}\,\varrho = \frac{d\,\pi}{\text{tang}(\Phi-\pi)}$$

ex quo, si longitudo nodi iam fuerit reperta, sine labore inclinatio ad eclipticam ϱ investigari poterit.

§. 10. Differentiemus formulam primo inuentam

$$\frac{d\,\psi}{\cos\psi^2} = d\,\Phi \text{ tang } \varrho \cos(\Phi-\pi)$$

iterum, et cum sit $d.\ \text{tang}\,\varrho = \frac{d\pi\,\text{tang}\,\varrho\cos(\Phi-\pi)}{\sin(\Phi-\pi)}$ erit:

erit:

$$\frac{d\,d\psi}{\cos\psi^2} + \frac{2d\psi^2\sin\psi}{\cos\psi^3} = d\,d\,\Phi \text{ tang } \varrho \cos(\Phi-\pi)$$
$$+ \frac{d\Phi d\pi\,\text{tang}\,\varrho\cos(\Phi-\pi)^2}{\sin(\Phi-\pi)} - d\Phi(d\Phi - d\pi)\text{tang}\,\varrho\sin(\Phi-\pi)$$

seu

$$\frac{dd\psi}{\cos\psi^2} + \frac{2d\psi^2\sin\psi}{\cos\psi^3} = d\,d\Phi \text{ tang } \varrho\cos(\Phi-\pi)$$
$$+ \frac{d\Phi d\pi\,\text{tang}\,\varrho}{\sin(\Phi-\pi)} - d\Phi^2\,\text{tang}\,\varrho\sin(\Phi-\pi)$$

qui valores pro ψ in tertia aequatione superiori substituti suppeditabunt:

$$ddx\,\text{tang}\,\varrho\sin(\Phi-\pi) + 2dxd\Phi\,\text{tang}\,\varrho\cos(\Phi-\pi) + xdd\Phi\,\text{tang}\,\varrho\cos(\Phi-\pi)$$
$$+ \frac{xd\Phi d\pi\,\text{tang}\,\varrho}{\sin(\Phi-\pi)} - x\,d\Phi^2\,\text{tang}\,\varrho\sin(\Phi-\pi) = -\tfrac{1}{2}\,R\,dt^2$$

quae transmutatur in hanc:

$$(ddx - xd\Phi^2)\text{tang}\,\varrho\sin(\Phi-\pi) + (2dxd\Phi + xdd\Phi)\,\text{tang}\,\varrho\cos(\Phi-\pi)$$
$$+ \frac{xd\Phi d\pi\,\text{tang}\,\varrho}{\sin(\Phi-\pi)} = -\tfrac{1}{2}R\,dt^2$$

§. 11. Commode, hic euenit vt in ista formula illae ipsae expressiones differentio-differentiales $ddx - xd\Phi^2$ et $2\,dxd\Phi + x\,dd\Phi$ occurrant, quae ex actione duarum reliquarum virium sunt enatae: vnde si formularum harum valores aequiualentes $-\frac{1}{2}\,V\,dt^2$ et $-\frac{1}{2}\,T\,dt^2$ substituamus, impetrabimus

$$-\tfrac{1}{2}V dt^2\,\text{tang}\,\varrho\sin(\Phi-\pi) - \tfrac{1}{2}T dt^2\,\text{tang}\,\varrho\cos(\Phi-\pi) + \frac{xd\Phi d\pi\,\text{tang}\,\varrho}{\sin(\Phi-\pi)} = -\tfrac{1}{2}R\,dt^2$$

qua differentiale $d\,\pi$, quo promotio elementaris lineae nodorum indicatur, ita determinabitur, vt sit

$$d\,\pi = \tfrac{1}{2}\,d\,t^2.\ \frac{\sin(\Phi-\pi)}{x\,d\,\Phi}\left(V\sin(\Phi-\pi) + T\cos(\Phi-\pi) - \frac{R}{\text{tang}\,\varrho}\right)$$

Deinde

Deinde cum sit $d.\,l\,\mathrm{tang}\,\varrho = \frac{d\pi}{\mathrm{tang}\,(\varphi-\pi)}$; erit

$d.\,l\,\mathrm{tang}\,\varrho = \frac{1}{2}dt^2.\,\frac{\mathrm{cos}(\varphi-\pi)}{x\,d\varphi}(V\,\mathrm{sin}(\varphi-\pi)+T\,\mathrm{cos}(\varphi-\pi)-\frac{R}{\mathrm{tang}\,\varrho})$

Duas ergo has aequationes loco superioris tertiae, ex qua latitudo ψ inueniri debebat, in calculum introduci conueniet; inuentis enim π et ϱ erit $\mathrm{tang}\,\psi = \mathrm{tang}\,\varrho$ $\mathrm{sin}\,(\varphi-\pi)$.

§. 12. Hinc patet lineam nodorum nunquam esse mobilem, quin simul inclinatio ϱ variationi sit obnoxia. Eadem enim vis $V\,\mathrm{sin}\,(\varphi-\pi) + T\,\mathrm{cos}(\varphi-\pi) - \frac{R}{\mathrm{tang}\,\varrho}$, quae lineae nodorum motum imprimit eius longitudinem π immutando, simul in inclinatione ϱ variationem generat. Nulli autem plane immutationi tam linea nodorum, quam inclinatio erunt subiectae, si vis illa euanescat, quod euenit si media directio omnium virium corpus L sollicitantium in ipsum planum ☊CL, in quo corpus semel moueri coepit, perpetuo incidat; hicque est casus, quo corpus continuo in eodem plano moueri pergit. Generatim ergo corporis a tribus viribus V, T, R sollicitati motus quatuor sequentibus aequationibus determinatur.

I. $2\,dx\,d\varphi + x\,dd\varphi = -\frac{1}{2}\,T\,dt^2$

II. $ddx - x\,d\varphi^2 = -\frac{1}{2}\,V\,dt^2$

III. $d\pi = \frac{1}{2}dt^2.\,\frac{\mathrm{sin}\,(\varphi-\pi)}{x\,d\varphi}(V\,\mathrm{sin}(\varphi-\pi)+T\,\mathrm{cos}(\varphi-\pi)-\frac{R}{\mathrm{tang}\,\varrho})$

IV. $d.\,l\,\mathrm{tang}\,\varrho = \frac{d\pi}{\mathrm{tang}\,(\varphi-\pi)}$

quas ergo quouis casu oblato resolui oportet.

CAPUT

CAPUT II.
INUESTIGATIO VIRIUM LUNAM SOLLICITANTIUM.

§. 13.

Cum Lunae motus, qualis ex centro terrae spectaretur, definiri debeat, sit C terrae centrum, ad quod etiam praecipua vis, qua Luna vrgetur, directa concipitur; atque tabula exhibeat planum eclipticae, in quo nunc quidem Sol existat in S, Luna vero supra hoc planum versetur in L latitudinem habens borealem, vnde ad planum eclipticae perpendiculum demittatur LM. Hinc ductis rectis CL, CM, CS, itemque CQ initium arietis versus, vnde longitudines computari solent, fiant sequentes denominationes. Fig. 2.

1. Longitudo Solis seu angulus ACS $= \theta$
2. Longitudo Lunae seu angulus ACM $= \varphi$
3. Latitudo Lunae seu angulus MCL $= \psi$
4. Distantia Solis a Terra CS $= y$
5. Distantia Lunae a Terra curtata CM $= x$

§. 14. Sit iam AM proiectio orbitae lunaris in planum eclipticae; ac planum, in quo Luna nunc mouetur, per centrum terrae ductum, planum eclipticae intersecet secundum rectam C☊, quae lineam nodorum pro tempore praesenti exhibebit: ac terminus ☊ quidem nodum ascendentem referet, siquidem lunam secundum regionem AM promoueri ponamus. Quod si ergo porro vocemus

6. Lon-

6. Longitudinem nodi asc: $AC☊ = \pi$

7. Incl. orbitae Lunae ad eclipticam $= \rho$

hinc latitudo Lunae geocentrica ita definietur, vt sit $\tang \psi = \tang . \sin (\phi - \pi)$. Vnde incrementum latitudinis $d\psi$ commode assignabitur, cum sit, vt supra vidimus $d. \tang \psi = \frac{d\psi}{\cos \psi^2} = d\phi \tang \rho \cos(\phi - \pi)$: ac praeterea ex motu nodi cognito variatio inclinationis ita definietur, vt sit $d. \tang \rho = \frac{d\pi \tang \rho}{\tang(\phi - \pi)}$, seu $d\rho = \frac{d\pi \sin\rho\cos\rho}{\tang(\phi - \pi)}$.

§. 15. Cum nunc primum Luna ad centrum terrae C secundum directionem LC attrahatur, sit haec vis $= M$. Deinde sit vis, qua Luna ad solem S vrgetur secundum $LS = N$; atque his duabus viribus Luna proprie vrgeri censenda est. Praeterea vero cum Terra ipsa, ad quam motum Lunae referimus, in motu versetur, ut eam tanquam quiescentem considerare queamus, non solum motum Terrae, sed etiam vires, quibus Terra vrgetur, toti mundo secundum plagas oppositas imprimi concipiamus. Sit igitur vis qua Terra ad Solem vrgetur $= S$, & vis qua ad Lunam trahitur $= L$, his viribus contrario modo in lunam translatis, Luna sequentibus viribus sollicitata habebitur

1. Secundum directionem LC vi $= M + L$
2. Secundum directionem LS vi $= N$
3. Secundum directionem MR ipsi SC parallelam vi $= S$.

§. 16. Hunc

, vt sit
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ıpra vi-
·π) : ac
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g(φ-π)
n terrae
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etur se-
ına pro-
rra ipsa,
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:quenti-
L
. Hunc

§. 16. Nunc primo has vires ad ternas directiones supra assumtas MC, MQ et LM reducamus; ac primo quidem vis $M + L$ dabit

pro directione MC vim $= (M + L) \cos \psi$

pro directione LM vim $= (M + L) \sin \psi$

Secunda vis N vero dabit

pro directione LM vim $= \frac{LM}{LS} . N$

pro directione MS vim $= \frac{MS}{LS} . N$

at haec vlterius resoluta ducta Mr ipsi CS parallela dabit:

pro directione MC vim $= \frac{MC}{LS} . N$

pro directione Mr vim $= \frac{CS}{LS} . N$

Haec postrema a vi tertia S subtracta relinquet vim $S - \frac{CS}{LS} . N$, qua Luna secundum directionem MR sollicitatur, quae ob angulum $CMR = SCM = \phi - \theta$ dabit

pro directione MC vim $= (S - \frac{CS}{LS} . N) \cos(\phi - \theta)$

pro directione Mq vim $= (S - \frac{CS}{LS} . N) \sin(\phi - \theta)$

vbi directio Mq contraria est directione MQ.

§. 17. His iam viribus cum ternis initio assumtis V, T et R comparandis, inueniemus pro his viribus sequentes valores:

1. pro-

1. pro directione MC vim V =

$$(M+L)\cos\psi + \frac{MC}{LS}.N + (S - \frac{CS}{LS}.N)\cos(\varphi-\theta)$$

2. pro directione MQ vim $T = -(S - \frac{CS}{LS}.N)\sin(\varphi-\theta)$

3. pro directione LM vim $R = (M+L)\sin\psi + \frac{LM}{LS}.N.$

Cum nunc ſit $CM = x$, $CS = y$, et angulus $SCM = \varphi - \theta$; erit $MS = \sqrt{(xx - 2xy\cos(\varphi-\theta) + yy)}$, et ob $LM = x\tang\psi$ erit $LS = \sqrt{(yy - 2xy\cos(\varphi-\theta) + xx\sec.\psi^2)}$, quae diſtantia Solis a Luna LS breuitatis gratia ponatur $= u$, vt ſit $u = \sqrt{(yy - 2xy\cos(\varphi-\theta) + xx\sec.\psi^2)}$. His ergo valoribus introductis erunt vires noſtrae:

1. $V = (M+L)\cos\psi + \frac{Nx}{u} + S\cos(\varphi-\theta) - \frac{Ny}{u}\cos(\varphi-\theta)$

2. $T = -S\sin(\varphi-\theta) + \frac{Ny}{u}\sin(\varphi-\theta).$

3. $R = -(M+L)\sin\psi + \frac{Nx\tang\psi}{u}$

§. 18. Quia nunc eſt $\tang\psi = \tang\varrho\sin(\varphi-\pi)$ et $\sin\psi = \tang\psi\cos\psi$ erit:

$$\frac{R}{\tang\varrho} = (M+L)\cos\psi\sin(\varphi-\pi) + \frac{Nx\sin(\varphi-\pi)}{u}$$

tum vero habebitur

$$V\sin(\varphi-\pi) + T\cos(\varphi-\pi) = (M+L)\cos\psi\sin(\varphi-\pi) + \frac{Nx\sin(\varphi-\pi)}{u}$$
$$+ S\cos(\varphi-\theta)\sin(\varphi-\pi) - \frac{Ny\cos(\varphi-\theta)\sin(\varphi-\pi)}{u}$$
$$- S\sin(\varphi-\theta)\cos(\varphi-\pi) + \frac{Ny\sin(\varphi-\theta)\cos(\varphi-\pi)}{u}$$

quae ob $\cos(\varphi-\theta)\sin(\varphi-\pi) - \sin(\varphi-\theta)\cos(\varphi-\pi) = \sin(\varphi-\pi)$, dat

V ſin

coſ(φ-θ) ı (φ - θ) M S N. =φ-θ; r tang ψ liſtantia vt ſit rgo va- ɔſ(φ-θ) -π) et π) (φ-π) (φ-π) (φ-π) r), dat V ſin

$$V\sin(\varphi-\pi) + T\cos(\varphi-\pi) - \frac{R}{\tang\varrho} = S\sin(\theta-\pi) - \frac{Ny}{u}\sin(\theta-\pi)$$

ex quo aequationes motum Lunae continentes erunt:

I. $2dxd\varphi + xdd\varphi = -\frac{1}{2}dt^2(\frac{Ny}{u} - S)\sin(\varphi-\theta)$

II. $ddx - xd\varphi^2 = -\frac{1}{2}dt^2((M+L)\cos\psi + \frac{Nx}{u} - (\frac{Ny}{u} - S)\cos(\varphi-\theta))$

III. $d\pi = -\frac{1}{2}dt^2.\frac{\sin(\varphi-\pi)\sin(\theta-\pi)}{xd\varphi}(\frac{Ny}{u} - S)$

IV. $d.l\tang\varrho = \frac{dx}{\tang(\varphi-\pi)}$

vbi $\theta - \pi$ exprimit angulum ☊ CS ſeu diſtantiam Solis a nodo aſcendente.

§. 19. Jam ſecundum Theoriam Newtoni, ſi maſſam Terrae ponamus = ♁ ac Lunae = ☾, ob diſtantiam $CL = \frac{x}{\cos\psi}$, foret vis $M = \frac{♁\cos\psi^3}{xx}$, et vis $L = \frac{☾\cos\psi^2}{xx}$, ſicque vis tota $M + L = (♁ + ☾).\frac{\cos\psi^2}{xx}$. Quo autem, ſi forte haec Theoria inſufficiens deprehendatur, rem generalius complectamur, ponamus hanc vim:

$$M + L = (♁ + ☾)\cos\psi^2\left(\frac{1}{xx} - \frac{1}{bb}\right)$$

vbi terminus $\frac{1}{bb}$ defectum huius vis a Theoria Newtoniana exhibeat; qui cum ſit minimus, pro conſtanti haberi poterit ſaltem pro exigua variabilitate, quam diſtantia x ſubit. Vim autem Solis exacte Theoriae Newtonianae conformem aſſumere poterimus; quoniam etiamſi inde recederet, differentia non ſolum foret

 quam

quam minima, ſed quia pro Luna aeque diſcreparet ac pro Terra, in noſtris formulis nullius plane eſſet momenti.

§. 20. Poſita ergo Solis maſſa $= \odot$, erit vis, qua Terram ad ſe attrahit $S = \frac{\odot}{yy}$, vis autem qua Lunam ad ſe trahit $N = \frac{\odot}{uu}$. His ergo valoribus virium in calculum inductis, motus lunae ex quatuor ſequentibus aequationibus determinari debet:

$$\text{I. } 2dxd\varphi + xdd\varphi = -\tfrac{1}{2}dt^2 \left(\frac{\odot y}{u^3} - \frac{\odot}{yy}\right) \sin(\varphi - \theta)$$

$$\text{II. } ddx - xd\varphi^2 = -\tfrac{1}{2}dt^2 (\text{♁} + \text{☾}) \cos\psi^3 \left(\frac{1}{xx} - \frac{1}{bb}\right)$$

$$-\tfrac{1}{2}dt^2\left(\frac{\odot x}{u^3} - \frac{\odot y}{u^3}\cos(\varphi-\theta) + \frac{\odot}{yy}\cos(\varphi-\theta)\right)$$

$$\text{III. } d\pi = -\tfrac{1}{2}dt^2 . \frac{\sin(\varphi-\pi)\sin(\theta-\pi)}{xd\varphi}\left(\frac{\odot y}{u^3} - \frac{\odot}{yy}\right)$$

$$\text{IV. } d.l\ \tang\rho = \frac{d\pi}{\tang(\varphi-\pi)}$$

Hic iam primum curandum eſt, vt elementum temporis, quod eſt quantitas heterogenea, ex calculo eliminemus; id quod commodiſſime fiet, ſi motum medium ſolis vtpote tempori proportionalem, loco temporis in calculum introducemus.

§. 21. Cum igitur etiam motus Solis in his aequationibus ſit ratio habenda, eum prius inueſtigemus: et quoniam pro terra quieſcente Sol a ſola vi $\frac{\odot}{yy}$ ad terram

ram ſollicitari concipiendus eſt, ſi formulas pro luna inuentas ad ſolem accommodemus, obtinebimus:

$$2dyd\theta + ydd\theta = 0$$

$$ddy - yd\theta^2 = -\tfrac{1}{2}dt^2 . \frac{\odot}{yy}$$

ſi iam diſtantiam Solis a terra mediam ponamus $= b$ ejusque anomaliam mediam $= q$; caſu quo excentricitas orbitae ſolaris eſſet nulla, foret ſemper $y = b$ & $d\theta = dq$: vnde altera aequatio dabit $-bdq^2 = -\tfrac{1}{2}dt^2 . \frac{\odot}{bb}$. Quare loco elementi temporis dt elementum anomaliae mediae ſolis ita in calculum introduci debet, vt vbique loco $\tfrac{1}{2}dt^2$ ſcribatur $\frac{b^3 dq^2}{\odot}$, id quod tam in his formulis pro Sole, quam in ſuperioribus pro Luna fieri poterit.

§. 22. Cum iam b denotet diſtantiam ſolis a terra mediam, ſit eius vera diſtantia $y = b\,\omega$, et anomalia eius vera $= s$, erit $d\theta = ds$, quandoquidem a motu apogei ſolis animum abſtrahimus. Hinc itaque erit

$$2d\omega ds + \omega dds = 0$$

$$dd\omega - \omega ds^2 = -\frac{dq^2}{\omega\omega},$$

quarum prior integrata dat $\omega\omega ds = C dq$ ob dq conſtans, ideoque $\omega ds^2 = \frac{CCdq^2}{\omega^3}$; qui valor in altera aequatione ſubſtitutus praebet,

$$dd\omega = \frac{CCdq^2}{\omega^3} - \frac{dq^2}{\omega\omega}.$$

quae

quae per $2\,d\dot{\omega}$ multiplicata et integrata dat:

$$\frac{d\omega^2}{dq^2} = D - \frac{CC}{\omega\omega} + \frac{2}{\omega}$$

vnde fit $dq = \frac{\omega\, d\omega}{\sqrt{(D\omega\omega + 2\omega - CC)}}$

ac proinde $ds = \frac{C\, d\omega}{\omega\sqrt{(D\omega\omega + 2\omega - CC)}}$

§. 23. Quanquam autem hinc valores finiti haud difficulter deduci possent, tamen alia vtar methodo, quae in motu Lunae maiorem praestabit vtilitatem. Inuento autem $\omega\omega\, ds = C\, dq$, alteram aequationem ita transformo, vt elementi constantis dq ratio non amplius habeatur:

$$dq.\, d.\, \frac{d\omega}{dq} - \omega\, ds^2 = -\frac{dq^2}{\omega\omega}$$

Sit nunc $\omega = \frac{1}{u}$, vt habeat $ds = C\, uu\, dq$, et $d\omega = -\frac{du}{uu}$, et ob $dq = \frac{ds}{C\, uu}$ erit $\frac{d\omega}{dq} = -\frac{C\, du}{ds}$; hinc sumto iam elemento ds constante, erit

$$\frac{-d's}{C\, uu} \cdot \frac{C\, ddu}{ds} - \frac{ds^2}{u} = -\frac{ds^2}{CC\, uu} \text{ seu}$$

$$ddu + u\, ds^2 = \frac{ds^2}{CC}$$

vnde statim elicitur $u = \frac{1 - e\cos s}{CC}$, vbi e excentricitatem orbitae solaris indicabit.

§. 24. Hinc porro habebitur $\omega = \frac{CC}{1 - e\cos s}$, et $y = \frac{CCb}{1 - e\cos s}$, anomalia vera s ab apogeo computata; vnde distantia apogei

apogei a terra posito $s = 0$ erit $= \frac{CCb}{1 - e}$, et distantia perigei posito $s = 180°$ prodit $= \frac{CCb}{1 + e}$; sicque distantia media fiet $= \frac{CCb}{1 - ee}$, quae cum per hypothesin aequalis esse debeat ipsi b, statui oportet $CC = 1 - ee$: hincque erit

$$y = \frac{b(1 - ee)}{1 - e\cos s} \quad \text{et} \quad \omega = \frac{1 - ee}{1 - e\cos s}$$

Porro autem aequatio $\omega\omega\, ds = C\, dq = dq\sqrt{(1 - ee)}$ abibit in hanc:

$$dq = \frac{(1 - ee)^{\frac{3}{2}}\, ds}{(1 - e\cos s)^2} \text{ et } q = \int \frac{(1 - ee)^{\frac{3}{2}}\, ds}{(1 - e\cos s)^2}$$

ex qua, vti satis constat, vel data anomalia vera s inueniri potest anomalia media q; vel viciſſim. His itaque formulis motum Solis continentibus in determinatione motus Lunae vtamur.

§. 25. Primo ergo loco $\frac{1}{2} ds^2$ vbique scribamus $\frac{b^3\, dq^2}{\odot}$ et $b\omega$ loco y, quo facto nostrae aequationes fient

I. $2\, dx\, d\phi + x\, dd\phi = -b^3 dq^2 \left(\frac{b\omega}{u^3} - \frac{1}{bb\omega\omega}\right) \sin(\phi - \theta)$

II. $ddx - x\, d\phi^2 = -\frac{(♁ + ☾)\, b^3 dq^2}{\odot} \cos\psi^3 \left(\frac{1}{xx} - \frac{1}{bb}\right)$

$$- b^3 dq^2 \left(\frac{x}{u^3} - \frac{b\omega\cos(\phi - \theta)}{u^3} + \frac{\cos(\phi - \theta)}{bb\omega\omega}\right)$$

III. $d\pi = -b^3 dq^2 . \frac{\sin(\phi - \pi)\sin(\theta - \pi)}{x\, d\phi} \left(\frac{b\omega}{u^3} - \frac{1}{bb\omega\omega}\right)$

Ponatur porro $u = bv$, adque in calculum quoque introducaducatur distantia media lunae a terra, quae sit $= a$, positoque $x = az$, prodibit:

I. $2dzd\phi + zdd\phi = -\frac{bdq^2}{a}\left(\frac{\omega}{v^3} - \frac{1}{\omega\omega}\right)\sin(\phi-\theta)$

II. $ddz - zd\phi^2 = -\frac{(\male + \leftmoon)b^3}{\odot\, a^3}dq^2\cos\psi^3\left(\frac{1}{zz} - \frac{aa}{bb}\right)$

$$-\frac{zdq^2}{v^3} + \frac{b\omega dq^2\cos(\phi-\theta)}{av^3} - \frac{bdq^2\cos(\phi-\theta)}{a\omega\omega}$$

III. $d\pi = -\frac{bdq^2}{azd\phi}\sin(\phi-\pi)\sin(\theta-\pi)\left(\frac{\omega}{v^3} - \frac{1}{\omega\omega}\right)$

§. 26. Ponamus nunc ad abbreuiandum:

$\frac{(\male + \leftmoon)b^3}{\odot\, a^3} = m$; $\frac{(\male + \leftmoon)b^3}{\odot\, abb} = \mu$, seu $\mu = \frac{maa}{bb}$

quarum litterarum valores m et μ per obseruationes definiri debent; tum vero sit $\frac{a}{b} = \nu$, quae est quantitas valde parua a parallaxi solis pendens. Hisque valoribus introductis, aequationes nostrae sequentes induent formas:

I. $2dzd\phi + zdd\phi = -\frac{1}{\nu}dq^2\left(\frac{\omega}{v^3} - \frac{1}{\omega\omega}\right)(\sin\phi - \theta)$

II. $ddz - zd\phi^2 = -\frac{mdq^2\cos\psi^3}{zz} + \mu dq^2\cos\psi^3$

$$-\frac{zdq^2}{v^3} + \frac{1}{\nu}dq^2\left(\frac{\omega}{v^3} - \frac{1}{\omega\omega}\right)\cos(\phi-\theta)$$

III. $d\pi = -\frac{dq^2}{\nu zd\phi}\sin(\phi-\pi)\sin(\theta-\pi)\left(\frac{\omega}{v^3} - \frac{1}{\omega\omega}\right)$

IV. $d.l\tan\rho = \frac{d\pi}{\tan(\phi-\pi)}$

a, posito- ·θ) aa/bb) cos(φ−θ)/aωω) ones defi- titas val- ribus in- t formas; −θ) §. 27.

§. 27. Cum iam posuerimus:

$x = az$; $y = b\omega$; $v = \frac{u}{b}$ et $\nu = \frac{a}{b}$

erit $u = \sqrt{(bb\omega\omega - 2ab\omega z\cos(\phi-\theta) + aazz\sec.\psi^2)}$ atque $v = \sqrt{(\omega\omega - 2\nu\omega z\cos(\phi-\theta) + \nu\nu zz\sec.\psi^2)}$ vbi notandum est quantitates ω, q et s ex motu solis ita inter se pendere, vt sit

$$\omega = \frac{1-ee}{1-e\cos s} \text{ et } dq = \frac{(1-ee)^{\frac{3}{2}}ds}{(1-e\cos s)^2} = \frac{\omega\omega ds}{\sqrt{(1-ee)}}$$

ita vt huic ad datum quoduis tempus tam valor ipsius ω quam anomaliae verae s definiri possit. Modum autem has formulas ad calculum reuocandi hic non trado, quia eum alias fusius iam exposui: hoc solum hic notari conueniet, excentricitatis orbitae solaris valorem ex obseruationibus colligi $e = 0,01680$.

§. 28. Nunc antequam vlterius progredi queamus, valorem irrationalem ipsius v tolli conueniet, quod facile per seriem praestabitur maxime conuergentem, ob ν fractionem valde paruam; sumta enim parallaxi solis $= 12''$, quia parallaxis lunae media est $= 3380''$, erit $\frac{a}{b} = \nu = \frac{12}{3380} = \frac{1}{282}$. Hinc sufficit seriei illius conuergentis, quam reperiemus, aliquot tantum terminos ab initio assumsisse; quia reliqui ob paruitatem continuo magis crescentem tuto omitti poterunt. Cum autem angulus $\phi - \theta$, qui distantiam solis a luna secundum longitudinem denotat, in hac resolutione frequentissime occurret, breuitatis gratia ponamus $\phi - \theta = \eta$

ita vt pro v sequentem habeamus valorem irrationalem

$$v = \sqrt{(\omega\omega - 2\nu\omega z\cos\eta + \nu\nu zz\sec.\psi^2)}.$$

§. 29. Quoniam ergo in nostris formulis occurrit $\frac{1}{v^3}$ ob $\frac{1}{v^3} = (\omega\omega - 2v\omega z \cos\eta + vvzz \sec\psi^2)^{-\frac{3}{2}}$, nanciscemur

$$\frac{1}{v^3} = \frac{1}{\omega^3} + \frac{3vz\cos\eta}{\omega^4} - \frac{3vvzz\sec\psi^2}{2\omega^5} + \frac{15vvzz}{2\omega^5}\cos\eta^2$$

vbi terminos altiores ipsius v potestates inuoluentes sine haesitatione reiicere possumus; in ipsis aequationibus autem tantum in prima ipsius v potestate subsistemus. Habebimus ergo:

$$\frac{1}{v}\left(\frac{\omega}{v^3} - \frac{1}{\omega\omega}\right) = \frac{3z\cos\eta}{\omega^3} + \frac{3vzz}{2\omega^4}(5\cos\eta^2 - \sec\psi^2) \text{ seu}$$

$$\frac{1}{v}\left(\frac{\omega}{v^3} - \frac{1}{\omega\omega}\right) = \frac{3z\cos\eta}{\omega^3} + \frac{3vzz}{4\omega^4}(5 + 5\cos 2\eta - 2\sec\psi^2)$$

hincque porro:

$$\frac{1}{v}\left(\frac{\omega}{v^3} - \frac{1}{\omega\omega}\right)\sin(\phi - \theta) = \frac{3z\sin 2\eta}{2\omega^3} + \frac{3vzz}{8\omega^4}(5\sin\eta + 5\sin 3\eta - 4\sin\eta\sec\psi^2)$$

$$\frac{1}{v}\left(\frac{\omega}{v^3} - \frac{1}{\omega\omega}\right)\cos(\phi - \theta) = \frac{3z}{2\omega^3}(1 + \cos 2\eta) + \frac{3vzz}{8\omega^4}(15\cos\eta + 5\cos 3\eta - 4\cos\eta\sec\psi^2)$$

$$\text{atque } \frac{z}{v^3} = \frac{z}{\omega^3} + \frac{3vzz}{\omega^4}\cos\eta$$

§. 30. Substituamus hos valores in nostris aequationibus atque obtinebimus:

$$\text{I. } 2dzd\phi + zdd\phi = -dq^2\left(\frac{3z\sin 2\eta}{2\omega^3} + \frac{3vzz}{8\omega^4}(5\sin\eta + 5\sin 3\eta - 4\sin\eta\sec\psi^2)\right)$$

$$\text{II. } ddz - zd\phi^2 = -\frac{mdq^2\cos\psi^3}{zz} + \mu dq^2\cos\psi^3 + \frac{zdq^2}{2\omega^3} + \frac{3zdq^2}{2\omega^3}\cos 2\eta$$
$$+ \frac{3vzzdq^2}{8\omega^4}(7\cos\eta + 5\cos 3\eta - 4\cos\eta\sec\psi^2)$$

III. $d\pi$

s occurrit
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$\frac{vzz}{\omega^5}\cos\eta^2$
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nus. Ha-
c. ψ^2) seu
$-2\sec\psi^2$)
$\eta\sec\psi^2$)
$\cos\eta\sec\psi^2$)
is aequa-
$\eta\sec\psi^2$))
$\frac{dq^2}{\omega^3}\cos 2\eta$
ec. ψ^2)
III. $d\pi$

$$\text{III. } d\pi = -\frac{dq^2}{zd\phi}\sin(\phi - \pi)\sin(\theta - \pi)\left(\frac{3z\cos\eta}{\omega^3} + \frac{3vzz}{4\omega^4}(5 + 5\cos 2\eta - 2\sec\psi^2)\right)$$

$$\text{IV. } d.\,l\,\tang\varrho = \frac{d\pi}{\tang(\phi - \pi)}.$$

Hic iam obseruare licet, cum angulus ψ nunquam fere 5° superet, eiusque secans nonnisi in terminis iam per v multiplicatis, ac propterea respectu reliquorum valde paruis occurrat, sine vllius erroris sensibilis metu in his terminis poni posse sec. $\psi = 1$.

§. 31. Deinde vt etiam ex maioribus terminis $\cos\psi$ eliminemus; consideremus formulam $\tang\psi = \tang\varrho \sin(\phi - \pi)$, eritque $\sec\psi = \frac{1}{\cos\psi} = \sqrt{(1 + \tang\varrho^2 \sin(\phi - \pi)^2)}$

Hinc ergo habebimus:

$$\cos\psi^3 = (1 + \tang\varrho^2.\sin(\phi - \pi)^2)^{-\frac{3}{2}}$$

et cum $\tang\varrho^2$ nunquam fere fractionem $\frac{1}{144}$ superet erit satis exacte:

$$\cos\psi^3 = 1 - \tfrac{3}{2}\tang\varrho^2.\sin(\phi - \pi)^2 \text{ — vel etiam}$$
$$\cos\psi^3 = 1 - \tfrac{3}{4}\tang\varrho^2 + \tfrac{3}{4}\tang\varrho^2\cos 2(\phi - \pi)$$

qui valor pro $\cos\psi^3$ in termino maiore $\frac{mdq^2\cos\psi^3}{zz}$ substitui potest: in altero autem termino $\mu dq^2\cos\psi^3$ quia per se est valde paruus, atque adeo secundum Theoriam Newtoni euanesceret, nihil impedit, quo minus loco $\cos\psi^3$ scribamus vnitatem.

 §. 31. Hoc

§. 32. Hoc ergo modo ſi aequationes noſtras a conſideratione latitudinis Lunae ψ liberemus ad ſequentes perueniemus aequationes:

$$\text{I. } 2dzd\Phi + zdd\Phi = -dq^2\left(\frac{3z\sin 2\eta}{2\omega^3} + \frac{3vzz}{8\omega^4}(\sin\eta + 5\sin 3\eta)\right)$$

$$\text{II. } ddz - zd\Phi^2 = -\frac{mdq^2}{zz}(1 - \tfrac{3}{4}\tan g\varrho^2 + \tfrac{3}{4}\tan g\varrho^2 \cos 2(\Phi - \pi)) + \mu dq^2$$

$$+ \frac{zdq^2}{2\omega^3} + \frac{3zdq^2}{2\omega^3}\cos 2\eta + \frac{3vzzdq^2}{8\omega^4}(3\cos\eta + 5\cos 3\eta)$$

$$\text{III. } d\pi = -\frac{dq^2}{zd\Phi}\sin(\Phi - \pi)\sin(\theta - \pi)\left(\frac{3z\cos\eta}{\omega^3} + \frac{3vzz}{4\omega^4}(3 + 5\cos 2\eta)\right)$$

$$\text{IV. } d.\,l\tan g\,\varrho = \frac{d\pi}{\tan g(\Phi - \pi)}.$$

Nunc igitur in hoc erit incumbendum, vt ex his quatuor aequationibus omnia motus phaenomena, quae in Luna ſecundum Theoriam adeſſe debent, ſollicite eruantur, atque tum cum obſeruationibus conferantur.

CAPUT

oſtras a ſequen- ſin3η))))+μdq² -5coſ3η) coſ2η)) is qua- quae in ite eru- ur. APUT

CAPUT III.

INTRODUCTIO ANOMALIAE VERAE LUNAE IN PRAECEDENTES AEQUATIONES.

§. 33.

Quoniam noſtra quaeſtio circa Lunam verſatur, loco anomaliae mediae ſolis, quam pro tempore in calculum introduximus, magis conueniet motu Lunae medio vti, qui itidem tempori eſt proportionalis. Verum ex ſequentibus patebit calculum commodiorem reddi, ſi loco motus medii adhibeamus anomaliam Lunae mediam, cuius incrementa itidem tempori ſunt proportionalia. Sit itaque ad datum tempus anomalia media Lunae $= p$; et cum eius incrementum dp ad incrementum anomaliae mediae ſolis eodem tempuſculo acceptum datum ac per obſeruationes cognitam teneat rationem, ponamus $dp = ndq$. Tabulae autem Aſtronomicae pro interuallo 365 dierum praebent:

Motum anomaliae mediae Solis $11^s, 29^\circ, 44', 39'' = 1295079''$

Motum anomaliae mediae Lunae

$13^{\text{Rev.}}\ 2^s, 28^\circ, 43'\ 13'' = 17167393''$

vnde fit $n = \frac{17167393}{1295079} = 13,\ 25586$

§. 34. Poſito ergo $\frac{dp}{n}$ loco dq, aequationes noſtrae erunt

$$\text{I. } 2dzd\Phi + zdd\Phi = -\frac{dp^2}{nn}\left(\frac{3z\sin 2\eta}{2\omega^3} + \frac{3vzz}{8\omega^4}(\sin\eta + 5\sin 3\eta)\right)$$

II. ddz

II. $ddz - zd\Phi^2 = -\frac{mdp^2}{nnzz}(1 - \frac{3}{4}\mathrm{tang}\,\varrho^2 + \frac{3}{4}\mathrm{tang}\,\varrho^2 \cos 2(\Phi - \pi)) + \frac{\mu dp^2}{nn}$

$$+ \frac{z\,dp^2}{2n^2\omega^3} + \frac{3z\,dp^2}{2n^2\omega^3}\cos 2\eta + \frac{3\nu zz dp^2}{8nn\omega^4}(3\cos\eta + 5\cos 3\eta)$$

III. $d\pi = \frac{-dp^2}{nn:d\Phi}\sin(\Phi-\pi)\sin(\theta-\pi)\left(\frac{3z\cos\eta}{\omega^3} + \frac{3\nu zz}{4\omega^4}(3+5\cos 2\eta)\right)$

IV. $d.\,l\,\mathrm{tang}\,\varrho = \frac{d\pi}{\mathrm{tang}(\Phi-\pi)}$

atque hic elementum dp assumtum est constans: simul autem patet terminos, qui per nn sunt diuisi, prae ceteris satis esse paruos, cum sit $nn = 175,71795$. Quae circumstantia sequentes approximationes non mediocriter adiuuabit.

§. 35. Nunc antequam vlterius progrediamur, aequationem primam per z multiplicemus, atque integratione in priori parte instituta obtinebimus

$$zzd\Phi = C dp - \frac{dp}{nn}\int dp\left(\frac{3z^2\sin 2\eta}{2\omega^3} + \frac{3\nu z^3}{8\omega^4}(\sin\eta + 5\sin 3\eta)\right)$$

ponamus breuitatis gratia hoc membrum integrale

$$\int dp\left(\frac{3zz}{2\omega^4}\sin 2\eta + \frac{3\nu z^3}{8\omega^4}(\sin\eta + 5\sin 3\eta)\right) = S$$

quod integrale, ne introductio constantis incertitudinem pariat, ita capi assumo, vt nullum terminum mere constantem contineat, quippe qui iam in C esset comprehensus. Hoc ergo circa determinationem integrationis probe obseruato, erit $zzd\Phi = dp\left(C - \frac{S}{nn}\right)$: vbi terminus S aequabilem arearum descriptionem, quam Regula Kepleri in planetis primariis infert, perturbat; est enim $\frac{1}{2}zzd\Phi$ elementum areae descriptae, quod si ipsi C dp esset aequale, tempori exacte esset proportionale.

§. 36. Cum igitur sit $d\Phi = \frac{dp}{zz}\left(C - \frac{S}{nn}\right)$, erit $zd\Phi^2 = \frac{dp^2}{z^3}\left(CC - \frac{2}{nn}CS + \frac{1}{n^4}SS\right)$, quo valore substituto reliquae nostrae aequationes sequentes induent formas:

II. $ddz = \frac{dp^2}{z^3}(CC - \frac{2}{nn}CS + \frac{1}{n^4}SS)$

$$-\frac{m\,dp^2}{nnzz}(1 - \frac{3}{4}\mathrm{tang}\,\varrho^2 + \frac{3}{4}\mathrm{tang}\,\varrho^2\cos 2(\Phi-\pi)) + \frac{\mu dp^2}{nn}$$

$$+ \frac{z\,dp^2}{2nn\omega^3} + \frac{3z\,dp^2}{2nn\omega^3}\cos 2\eta + \frac{3\nu zz dp^2}{8nn\omega^4}(3\cos\eta + 5\cos 3\eta)$$

III. $d\pi = -\frac{z\,dp}{Cnn-S}\sin(\Phi-\pi)\sin(\theta-\pi)\left(\frac{3z\cos\eta}{\omega^3} + \frac{3\nu zz}{4\omega^4}(3+5\cos 2\eta)\right)$

et quarta manet $d.\,l\,\mathrm{tang}\,\varrho = \frac{d\pi}{\mathrm{tang}(\Phi-\pi)}$ vt ante.

Eo igitur pertigimus, vt inuestigari oporteat quantitates z, π et ϱ, quibus inuentis obtinebitur Φ ex formula primum eruta. Cum autem sit $d\eta = d\Phi - d\theta$, ob $d\theta = ds = \frac{dq\sqrt{(1-ee)}}{\omega\omega} = \frac{dp\sqrt{(1-ee)}}{n\omega\omega}$, erit $d\eta = \frac{dp}{zz}\left(C - \frac{S}{nn}\right) - \frac{dp\sqrt{(1-ee)}}{n\omega\omega}$. Tum vero est vti vidimus $\omega = \frac{1-ee}{1-e\cos s}$, vnde et huius differentiale ad dp reduci poterit.

§. 37. Si hunc calculum prosequi vellemus, tota inuestigatio tandem eo rediret, vt definiretur quantum longi-

longitudo Lunae vera ab eius longitudine media, quae ex anomalia media p haberetur, discreparet: hoc autem discrimen nonnunquam vltra 8 gradus exsurgere posset, ideoque correctiones admodum notabiles requireret. Vt igitur nobis quam minimae correctiones inuestigandae relinquantur, expediet differentiam inter locum Lunae verum, et locum corporis quod secundum regulas Kepleri in ellipsi circa Terram reuolueretur, ita tamen mobili, vt eius motus absidum cum motu apogei Lunae per obseruationes cognito conueniret. Seu quod eodem redit, quaeramus primo ex anomalia Lunae media p secundum regulas Kepleri anomaliam eius veram quae sit $= r$, vnde si longitudo apogei fuerit $= \varpi$, quantita, $\varpi + r$ nunquam multum vltra gradum a longitudine Lunae vera differet: vnde discrimen multo facilius inueniri poterit, si quidem debita orbitae lunaris excentricitas in calculum inducatur. Hinc loco anomaliae Lunae mediae p eius anomaliam veram, quae scilicet mediae pro excentricitate rite assumta conueniat, in aequationes nostras inferamus.

§. 38. Tabulae quidem astronomicae excentricitatem orbitae lunaris plerumque variabilem statuunt; sed cum hic non de vera huius orbitae excentricitate quaestio fit, quam de excentricitate illius orbitae ellipticae mobilis, in qua corpus motum proxime motum Lunae referat; huius excentricitas media erit statuenda inter maximam ac minimam, quae vulgo orbitae lunari tribuuntur: vnde ista excentricitas media colligitur $= 0,05445$. Ne autem huic conclusioni nimium fidamus generatim hanc excentricitatem ponamus $= k$; atque anomalia vera per mediam ita determinabitur, vt sit

$$dp = \frac{(1-kk)^{\frac{3}{2}}\,dr}{(1-k\cos r)^2},$$ vel sit breuitatis gratia $\frac{1-kk}{1-k\cos r} = t$,

vt sit $dp = \frac{tt\,dr}{\sqrt{(1-kk)}}$. Porro autem reliqua differentialia ita ad elementum dr reuocabuntur, vt sit:

$$ds = \frac{tt\,dr\sqrt{(1-ee)}}{n\omega\omega\sqrt{(1-kk)}} = d\theta, \text{ et } d\eta = \frac{tt\,dr}{zz\sqrt{(1-kk)}}\left(C - \frac{S}{nn}\right) - \frac{tt\,dr\sqrt{(1-ee)}}{n\omega\omega\sqrt{(1-kk)}}.$$

§. 39. Si motus Lunae cum motu huius corporis, quod imaginamur, perfecte conueniret, tum vbique foret $z = \frac{1-kk}{1-k\cos r}$ seu $z = t$: quoniam autem hi duo motus inter se non conueniunt, non erit $z = t$. Ponamus ergo esse:

$$z = tu = \frac{(1-kk)\,u}{1-k\cos r} \text{ seu } x = \frac{(1-kk)\,au}{1-k\cos r}$$

vbi primum obseruo, quantitatem u valde parum ab vnitate recedere. Erit autem quantitas variabilis, quae alium terminum constantem praeter vnitatem non inuoluet: nam si alium terminum constantem contineret, is in a posset comprehendi, idque indicio esset distantiam mediam a non recte esse assumtam. Habebit ergo u huiusmodi formam $1 + Z$, vbi Z ex terminis nonnisi variabilibus constabit. Praeterea autem animaduerto, hanc quantitatem Z nullum terminum huius formae $\alpha \cos r$ complecti debere; quoniam hoc indicio esset excentricitatem k non recte esse assumtam, sed eam vel maiorem vel minorem accipi oportuisse.

 §. 40. His

§. 40. His igitur notatis, quod quantitas u primo terminum constantem $= 1$ contineat, cum vero nullum terminum formae $\alpha \cos r$ inuoluat, statuamus $z = tu$ seu $z = \frac{(1-kk)u}{1-k\cos r}$ posito breuitatis gratia $t = \frac{1-kk}{1-k\cos r}$. Atque cum supra elementum dp constans posuissemus, hac conditione exuenda erit $ddz = dp\, d.\frac{dz}{dp}$, et $\frac{ddz}{dp^2} = \frac{1}{dp}\, d.\frac{dz}{dp}$. Diuisa ergo secunda aequatione per dp^2, erit:

$$\text{II. } \frac{1}{dp}\, d.\frac{dz}{dp} = \frac{CC}{t^3u^3} - \frac{2CS}{mt^3u^3} + \frac{SS}{n^4t^3u^3}$$
$$-\frac{m}{nnttuu}(1-\tfrac{3}{2}\tang\varrho^2+\tfrac{3}{2}\tang\varrho^2\cos 2(\phi-\pi)) + \frac{\mu}{nn} + \frac{tu}{2nn\omega^3}$$
$$+\frac{3tu\cos 2\eta}{2nn\omega^3} + \frac{3\nu ttuu}{8nn\omega^4}(3\cos\eta + 5\cos 3\eta)$$

$$\text{III. } d\pi = \frac{-tu\,dp}{Cnn-S}\sin(\phi-\pi)\sin(\theta-\pi)\left(\frac{3tu\cos\eta}{\omega^3} + \frac{3\nu ttuu}{4\omega^4}(3\mp 5\cos 2\eta)\right)$$

vbi nunc nullum differentiale assumtum est constans, sed iam pro lubitu quoduis differentiale constans assumi poterit.

§. 41. Posito autem $z = tu$ et $dp = \frac{ttdr}{\sqrt{(1-kk)}}$ existente $t = \frac{1-kk}{1-k\cos r}$ erit primo:

$$S = \int \frac{ttdr}{\sqrt{(1-kk)}}\left(\frac{3ttuu}{2\omega^3}\sin 2\eta + \frac{3\nu t^3u^3}{8\omega^4}(\sin\eta + 5\sin 3\eta)\right) \text{ seu}$$
$$S = \int \frac{dr}{\sqrt{(1-kk)}}\left(\frac{3t^4uu}{2\omega^3}\sin 2\eta + \frac{3\nu t^5u^3}{8\omega^4}(\sin\eta + 5\sin 3\eta)\right)$$

Hinc

pri- vero imus ratia con- z = ndá $\frac{u}{\nu\omega^3}$ 2η)) sed umi $\frac{r}{kk)}$ 1 η)) inc

Hinc fiet $d\phi = \frac{dr}{uu\sqrt{(1-kk)}}\left(C - \frac{S}{nn}\right)$ atque

$$d\eta = \frac{dr}{uu\sqrt{(1-kk)}}\left(C - \frac{S}{nn}\right) - \frac{ttdr\sqrt{(1-ee)}}{n\omega\omega\sqrt{(1-kk)}}$$

Porro autem ob $dz = tdu + udt$, erit $\frac{dz}{dp} = \frac{tdu+udt}{ttdr}\sqrt{(1-kk)}$; at est $dt = -\frac{(1-kk)kdr\sin r}{(1-k\cos r)^2} = -\frac{kttdr\sin r}{1-kk}$; sicque fiet $\frac{dz}{dp} = \frac{du\sqrt{(1-kk)}}{tdr} - \frac{ku\sin r}{\sqrt{(1-kk)}}$; ac posito elemento dr constante erit $d.\frac{dz}{dp} = \frac{ddu\sqrt{(1-kk)}}{tdr} - \frac{dudt\sqrt{(1-kk)}}{ttdr} - \frac{kdu\sin r}{\sqrt{(1-kk)}} - \frac{kudr\cos r}{\sqrt{(1-kk)}}$ hincque ob $\frac{dt}{tt} = -\frac{kdr\sin r}{1-kk}$ habebitur:

$$d.\frac{dz}{dp} = \frac{ddu\sqrt{(1-kk)}}{tdr} - \frac{kudr\cos r}{\sqrt{(1-kk)}}.$$

§. 42. Hinc iam porro obtinemus pro secunda aequatione $\frac{1}{dp}\, d.\frac{dz}{dp} = \frac{(1-kk)ddu}{t^3dr^2} - \frac{ku\cos r}{tt}$ qui valor substitutus in aequatione per $\frac{t^3}{1-kk}$ multiplicata orietur haec aequatio:

$$\text{II. } \frac{ddu}{dr^2} - \frac{ktu\cos r}{1-kk} = \frac{CC}{(1-kk)u^3} - \frac{2CS}{(1-kk)nnu^3} + \frac{SS}{n^4(1-kk)u^3}$$
$$-\frac{mt}{nn(1-kk)uu}(1-\tfrac{3}{2}\tang\varrho^2+\tfrac{3}{2}\tang\varrho^2\cos 2(\phi-\pi)) + \frac{\mu t^3}{nn(1-kk)} + \frac{t^4u}{2nn(1-kk)\omega^3}$$
$$+\frac{3t^4u\cos 2\eta}{2nn(1-kk)\omega^3} + \frac{3\nu t^5uu}{8nn\omega^4(1-kk)}(3\cos\eta + 5\cos 3\eta)$$

$$\text{III. } d\pi = -\frac{udr\sin(\phi-\pi)\sin(\theta-\pi)}{(Cnn-S)\sqrt{(1-kk)}}\left(\frac{3t^4\cos\eta}{\omega^3} + \frac{3\nu t^5u}{4\omega^4}(3+5\cos 2\eta)\right)$$

Quartam

Quartam aequationem $d.\, l \tang \rho = \frac{d\pi}{\tang(\Phi-\pi)}$, cum nullam mutationem subeat, superfluum foret continuo repetere.

§. 43. Conueniet autem quantitates constantes C et m, quarum valores nondum nouimus, saltem vero proxime indagare, quo facilius deinceps ipsam aequationum resolutionem dirigere queamus. Perspicuum autem est, si omnes quantitates a situ solis pendentes ex calculo deleantur, tum vtique fieri debere $u=1$. Cum igitur primum S ab angulo η pendeat, terminos tam S quam η inuoluentes omittamus, ac pro ω quidem scribamus 1; quia tantum determinationem ad verum accedentem requirimus, quem in finem quoque inclinationem orbitae negligamus. Hinc aequatio secunda dabit:

$$s = \frac{kt\cos r}{1-kk} = \frac{CC}{1-kk} - \frac{mt}{nn(1-kk)} + \frac{\mu t^3}{n^2(1-kk)} + \frac{t^4}{2nn(1-kk)} \text{ siue}$$

$$CC = \frac{mt}{nn} - \frac{\mu t^3}{nn} - kt\cos r - \frac{t^4}{2nn}$$

Cum autem sit $t = \frac{1-kk}{1-k\cos r} = 1 + k\cos r$ proxime, ob k valde paruum habebitur.

$$CC = \frac{m}{nn} - \frac{\mu}{nn} - \frac{1}{2nn} + \frac{m}{nn}k\cos r - \frac{3\mu k}{nn}\cos r - k\cos r - \frac{2k}{nn}\cos r$$

vnde perspicuum esse oportere.

$$\frac{m}{nn} = 1 + \frac{2+3\mu}{nn} \text{ et } CC = 1 + \frac{3+4\mu}{2nn}$$

§. 44. His

cum ntinuo tes C vero juatio- m au- tes ex Cum tam S n scri- acce- inatio- dabit: siue e, ob k 4. His

§. 44. His igitur constantium $\frac{m}{nn}$ et CC valoribus proximis inuentis ponamus esse reuera:

$$\frac{m}{nn} = 1 + \frac{2+3\mu+\gamma}{nn} \text{ et } CC = 1 + \frac{3+4\mu+\delta}{2nn} = \lambda\lambda$$

scribamus enim λ pro C, quia litteris maiusculis A, B, C, D etc. deinceps in operationibus sequentibus vtemur: sicque fiet

$$S = \frac{1}{\sqrt{(1-kk)}} \int dr \left(\frac{3t^4uu}{2\omega^3}\sin 2\eta + \frac{3\nu t^5u^3}{8\omega^4}(\sin\eta + 5/3\eta)\right)$$

$$d\Phi = \frac{dr}{uu\sqrt{(1-kk)}}\left(\lambda - \frac{S}{nn}\right)$$

$$d\eta = \frac{dr}{uu\sqrt{(1-kk)}}\left(\lambda - \frac{S}{nn}\right) - \frac{ttdr\sqrt{(1-ee)}}{n\omega\omega\sqrt{(1-kk)}}$$

$$\text{II. } \frac{(1-kk)\,ddu}{dr^2} = ktu\cos r + \frac{\lambda\lambda}{u^3} - \frac{2\lambda S}{nnu^3} + \frac{SS}{n^4u^3} + \frac{\mu t^3}{nn} + \frac{t^4u}{2nn\omega^3}$$

$$- \frac{mt}{nnuu}\left(1 - \tfrac{3}{4}\tang\rho^2 + \tfrac{3}{4}\tang\rho^2\cos 2(\Phi-\pi)\right)$$

$$+ \frac{3t^4u\cos 2\eta}{2nn\omega^3} + \frac{3\nu t^5uu}{8nn\omega^4}(3\cos\eta + 5\cos 3\eta)$$

$$\text{III. } d\pi = -\frac{uudr\sin(\Phi-\pi)\sin(\theta-\pi)}{(\lambda nn-S)\sqrt{(1-kk)}}\left(\frac{3t^4}{\omega^3}\cos\eta + \frac{3\nu t^5u}{4\omega^4}(3+5\cos 2\eta)\right)$$

§. 45. Ponatur $\lambda = u\sqrt{(1-kk)}$, vt sit $uu = 1 + \frac{3+4\mu+\delta}{2nn}$ defectum enim in termino indefinito δ complecti licet, existente $m = nn + 2 + 3\mu + \gamma$; tum vero ponatur $S = (1-kk)^{\frac{1}{2}}\int R\,dr$, vt sit $R = \frac{dS}{dr\sqrt{(1-kk)}}$; ac si pro t et ω valores restituamus, qui erant,

$t = \frac{1-kk}{1-k\cos r}$ et $\omega = \frac{1-ee}{1-e\cos s}$ habebimus:

R =

$$R = \frac{3\,(1-kk)^3\,(1-e\cos s)^3}{2\,(1-ee)^3\,(1-k\cos r)^4}\,uu\,\sin 2\eta + \frac{3\nu(1-kk)^4\,(1-e\cos s)^4}{8\,(1-ee)^4\,(1-k\cos r)^5}\,u^3\,(\sin\eta + 5\sin 3\eta)$$

$$d\Phi = \frac{dr}{uu}\left(u - \frac{1}{nn}\int R\,dr\right);\; d\theta = ds = \frac{(1-kk)^{\frac{3}{2}}(1-e\cos s)^2}{n(1-ee)^{\frac{3}{2}}(1-k\cos r)^2}\,dr$$

$$\frac{d\eta}{ds} = \frac{u}{uu} - \frac{\int R\,dr}{nnuu} - \frac{(1-kk)^{\frac{3}{2}}(1-e\cos s)^2}{n(1-ee)^{\frac{3}{2}}(1-k\cos r)^2}$$

§. 46. Aequatio autem secunda facta hac substitutione, si per $1-kk$ diuidatur, abibit in sequentem:

$$\text{II. } \frac{ddu}{dr^2} = \frac{ku\cos r}{1-k\cos r} + \frac{nn}{u^3} - \frac{2n\int R\,dr}{nnu^3} + \frac{(\int R\,dr)^2}{n^4u^3} + \frac{\mu(1-kk)^2}{nn(1-k\cos r)^3} - \frac{m}{nn(1-k\cos r)\,uu}\left(1-\tfrac{3}{4}\tan\rho^2 + \tfrac{3}{4}\tan\rho^2\cos 2(\Phi-\pi)\right) + \frac{(1-kk)^3(1-e\cos s)^3}{2nn(1-ee)^3(1-k\cos r)^4}\,u\,(1+3\cos 2\eta) + \frac{3\nu(1-kk)^4(1-e\cos s)^4}{8nn(1-ee)^4(1-k\cos r)^5}\,u^2\,(3\cos\eta + 5\cos 3\eta)$$

$$\text{III. } d\pi = -\frac{dr\sin(\Phi-\pi)\sin(\theta-\pi)}{(unn - R\int dr)}\left(\frac{3(1-kk)^3(1-e\cos s)^3}{(1-ee)^3(1-k\cos r)^4}\,uu\cos\eta + \frac{3\nu(1-kk)^4(1-e\cos s)^4}{4(1-ee)^4(1-k\cos r)^5}\,u^3\,(3+5\cos 2\eta)\right)$$

Ac si ϵ denotet inclinationem mediam orbitae lunaris, quantitas $1-\frac{3}{4}\tan\rho^2 + \frac{3}{4}\tan\rho^2\cos 2(\Phi-\pi)$ in has duas partes discespi poterit:

$$\left(1-\tfrac{3}{4}\tan\epsilon^2\right) + \tfrac{3}{4}\left(\tan\epsilon^2 - \tan\rho^2 + \tan\rho^2\cos 2(\Phi-\pi)\right)$$

quarum illa est constans, haec vero proprie a nodo et inclinatione pendet.

§. 47.

§. 47. Euoluamus autem producta illa ex s et ω orta, et quoniam excentricitates k et e sunt valde paruae, sufficit ad eos vsque terminos tantum progredi, qui coefficientes habeant kk, ek et ee, eosque qui per altiores potestates sint multiplicati omittere. Hinc erit:

$$\frac{1}{1-k\cos r} = 1 + \tfrac{1}{2}kk + k\cos r + \tfrac{1}{2}k^2\cos 2r$$

$$\frac{k\cos r}{1-k\cos r} = \tfrac{1}{2}kk + k\cos r + \tfrac{1}{2}k^2\cos 2r$$

$$\frac{(1-kk)^2}{(1-k\cos r)^3} = 1 + 3k\cos r,$$ quia hic terminus per μ multiplicatur.

$$\frac{(1-kk)^{\frac{3}{2}}}{(1-k\cos r)^2} = 1 + 2k\cos r + \tfrac{3}{2}kk\cos 2r$$

$$\frac{(1-kk)^3}{(1-k\cos r)^4} = 1 + 2kk + 4k\cos r + 5kk\cos 2r$$

$$\frac{(1-kk)^4}{(1-k\cos r)^5} = 1 + 5k\cos r,$$ quia hic terminus iam per ν est multiplicatus.

§. 48. Porro vero pro terminis ex ω enatis est:

$$\frac{(1-e\cos s)^2}{(1-ee)^{\frac{3}{2}}} = 1 + 2ee - 2e\cos s + \tfrac{1}{2}ee\cos 2s$$

$$\frac{(1-e\cos s)^3}{(1-ee)^3} = 1 + \tfrac{9}{2}ee - 3e\cos s + \tfrac{3}{2}ee\cos 2s$$

$$\frac{(1-e\cos s)^4}{(1-ee)^4} = 1 - 4e\cos s,$$ quia hic factor tantum in minimis terminis occurrit.

Hinc ergo colligimus:

$$\frac{(1-kk)^{\frac{3}{2}}(1-e\cos s)^2}{(1-ee)^{\frac{3}{2}}(1-k\cos r)^2} = 1 + 2ee + 2k\cos r + \tfrac{3}{2}kk\cos 2r - 2e\cos s - 2ek\cos(r+s) - 2ek\cos(r-s) + \tfrac{1}{2}ee\cos 2s$$

$$\frac{(1-kk)^3(1-e\cos s)^3}{(1-ee)^3(1-k\cos r)^4} = 1 + 2kk + \tfrac{9}{2}ee + 4k\cos r + 5kk\cos 2r - 3e\cos s - 6ek\cos(r-s) - 6ek\cos(r+s) + \tfrac{3}{2}ee\cos 2s$$

$$\frac{(1-kk)^4(1-e\cos s)^4}{(1-ee)^4(1-k\cos r)^5} = 1 + 5k\cos r - 4e\cos s,$$

atque hinc fiet: $d\Phi = \frac{dr}{uu}\left(n - \frac{1}{nn}\int R\,dr\right)$ atque

$$\frac{ds}{dr} = \frac{1+2ee}{n} + \frac{2k}{n}\cos r - \frac{2e}{n}\cos s - \frac{2ek}{n}(\cos r - s)$$
$$+ \frac{3kk}{2n}\cos 2r + \frac{ee}{2n}\cos 2s - \frac{2ek}{n}\cos(r+s)$$

$$\frac{d\eta}{dr} = \frac{n}{uu} - \frac{\int R\,dr}{nnuu} - \frac{1-2ee}{n} - \frac{2k}{n}\cos + r\frac{2e}{n}\cos s + \frac{2ek}{n}\cos(r-s)$$
$$- \frac{3kk}{2n}\cos 2r - \frac{ee}{2n}\cos 2s + \frac{2ek}{n}\cos(r+s)$$

§. 49. Introductis nunc his valoribus euolutis in formulas nostras, iisque, qui per sinum cosinumue alterius anguli sunt multiplicati, pariter secundum simplices angulos explicatis, obtinebimus primum valorem ipsius R, qui erit:

$$R = \tfrac{3}{2}u^2\begin{cases}(1+2kk+\tfrac{9}{2}ee)\sin 2\eta + 2k\sin(2\eta-r) + 2k\sin(2\eta+r)\\ + \tfrac{5}{2}kk\sin(2\eta-2r) + \tfrac{5}{2}kk\sin(2\eta+2r)\\ - \tfrac{3}{2}e\sin(2\eta-s) - \tfrac{3}{2}e\sin(2\eta+s)\\ + \tfrac{3}{4}ee\sin(2\eta-2s) + \tfrac{3}{4}ee\sin(2\eta+2s)\\ - 3ek\sin(2\eta-r+s) - 3ek\sin(2\eta+r-s)\\ - 3ek\sin(2\eta-r-s) - 3ek\sin(2\eta+r+s)\end{cases}$$

$$+ \tfrac{3}{8}v u^3\begin{cases}\sin\eta + \tfrac{3}{2}k\sin(\eta-r) - 2e\sin(\eta-s)\\ 5\sin 3\eta + \tfrac{3}{2}k\sin(\eta+r) - 2e\sin(\eta+s)\\ + \tfrac{25}{2}k\sin(3\eta-r) - 10e\cos(3\eta-s)\\ + \tfrac{25}{2}k\sin(3\eta+r) - 10e\cos(3\eta+s)\end{cases}$$

§. 50.

cos 2r
ecos2s
s(r−s)
(r+s)
itis in
alte-
impli-
lorem
§. 50.

§. 50. Aequatio autem secunda principalis sequentem induet formam:

$$\text{II.}\quad \frac{ddu}{dr^2} = \frac{nn}{u^3} - \frac{2n\int R\,dr}{nnu^3} + \frac{(\int R\,dr)^2}{n^4u^3}$$
$$+ \frac{3m\,\text{tang}\,\rho^2}{4nnuu}\left(1-\cos 2(\Phi-\pi)\right)(1+k\cos r)$$
$$- \frac{m}{nnuu}\left(1+\tfrac{1}{2}kk+k\cos r+\tfrac{1}{2}k^2\cos 2r\right) + \frac{\mu}{nn}(1+3k\cos r)$$
$$+ u\left(\tfrac{1}{2}kk + k\cos r + \tfrac{1}{2}kk\cos 2r\right)$$
$$+ \frac{u}{2nn}\begin{cases}1+2kk+\tfrac{9}{2}ee+4k\cos r-3e\cos s-6ek\cos(r-s)\\ +5kk\cos 2r+\tfrac{3}{2}ee\cos 2s-6ek\cos(r+s)\end{cases}$$
$$+ \frac{3u}{2nn}\begin{cases}(1+2kk+\tfrac{9}{2}ee)\cos 2\eta + 2k\cos(2\eta-r) + 2k\cos(2\eta+r)\\ + \tfrac{5}{2}kk\cos(2\eta-2r) + \tfrac{5}{2}kk\cos(2\eta+2r)\\ - \tfrac{3}{2}e\cos 2(2\eta-s) - \tfrac{3}{2}e\cos(2\eta+s).\\ + \tfrac{3}{4}ee\cos(2\eta-2s) + \tfrac{3}{4}ee\cos(2\eta+2s)\\ - 3ek\cos(2\eta-r+s) - 3ek\cos(2\eta+r-s)\\ - 3ek\cos(2\eta-r-s) - 3ek\cos(2\eta+r+s)\end{cases}$$
$$+ \frac{3vuu}{8nn}\begin{cases}3\cos\eta + 5\cos 3\eta + \tfrac{15}{2}k\cos(\eta-r) + \tfrac{15}{2}k\cos(\eta+r)\\ -6e\cos(\eta-s) - 6e\cos(\eta+s) + \tfrac{25}{2}k\cos(\eta-r) + \tfrac{25}{2}k\cos(\eta+r)\\ - 10e\cos(3\eta-s) - 10e\cos(3\eta+s)\end{cases}$$

vbi terminos, qui adhuc vlteriori euolutione indigent, primo loco posui, et cum terminus tang ρ^2 implicans iam sit valde paruus, in eius multiplicatore secundam ipsius k potestatem omisi: sin autem alicuius momenti videantur, loco $1 + k\cos r$ scribi poterit $1 + \tfrac{1}{2}kk + k\cos r + \tfrac{1}{2}kk\cos 2r$.

§. 51. Pro longitudine vero nodi inuenienda aequatio sequens prodibit resoluenda:

 $d\pi =$

$$d\pi = -\frac{3uudr\sin(\phi-\pi)\sin(\theta-\pi)}{\varkappa nn - \int R dr}\cos\eta\,(1 + 2kk + \tfrac{9}{2}ee + 4k\cos r$$
$$+ 5kk\cos 2r - 3e\cos s + \tfrac{3}{2}ee\cos 2s)$$
$$- \frac{3vu^3 dr\sin(\phi-\pi)\sin(\theta-\pi)}{4(\varkappa nn - \int R dr)}(3 + 5\cos 2\eta)(1 + 5k\cos r)$$

At est $\sin(\phi-\pi)\sin(\theta-\pi) = \frac{1}{2}\cos\eta - \frac{1}{2}(\phi+\theta-2\pi)$; vnde
$\sin(\phi-\pi)\sin(\theta-\pi)\cos\eta = \frac{1}{4} + \frac{1}{4}\cos 2\eta - \frac{1}{4}\cos 2(\phi-\pi) - \frac{1}{4}\cos 2(\theta-\pi)$
et $\sin(\phi-\pi)\sin(\theta-\pi)\cos 2\eta = \frac{1}{4}\cos\eta + \frac{1}{4}\cos 3\eta$
$- \frac{1}{4}\cos(3\phi-\theta-2\pi) - \frac{1}{4}\cos(3\theta-\phi-2\pi)$

Tum vero ob $\int R dr$ valde paruum prae $\varkappa nn$, erit satis exacte

$$\frac{1}{\varkappa nn - \int R dr} = \frac{1}{\varkappa nn} + \frac{\int R dr}{\varkappa\varkappa n^4} + \frac{(\int R dr)^2}{\varkappa^3 n^6}$$

vbi quidem postremus terminus tuto omitti potest.

§. 52. Praeterea vero ponatur $u = 1 + \frac{v}{nn}$, vt fit $ddu = \frac{ddv}{nn}$, et reiectis terminis per n^4 diuisis, qui iam per exiguam quantitatem sunt multiplicati, erit

$$\frac{d\phi}{dr} = \varkappa - \frac{2\varkappa v}{nn} + \frac{3\varkappa v^2}{n^4} - \frac{\int R dr}{nn} + \frac{2v\int R dr}{n^4}$$

$$\frac{d\eta}{dr} = \varkappa - \frac{1-2ee}{n} - \frac{2k}{n}\cos r + \frac{2e}{n}\cos s - \frac{3kk}{2n}\cos 2r - \frac{ee}{2n}\cos 2s$$
$$+ \frac{2ek}{n}\cos(r-s) + \frac{2ek}{n}\cos(r+s)$$
$$- \frac{2\varkappa v - \int R dr}{nn} + \frac{3\varkappa v^2 + 2v\int R dr}{n^4} \quad \text{atque}$$

$$R = \tfrac{3}{2}(1 + 2kk \mp \tfrac{9}{2}ee)\sin 2\eta + 3k\sin(2\eta-r) - \tfrac{9}{4}e\sin(2\eta-s)$$
$$+ 3k\sin(2\eta+r) - \tfrac{9}{4}e\sin(2\eta+s)$$
$$+ \frac{3v}{nn}\sin 2\eta \quad + \tfrac{15}{4}kk\sin(2\eta-2r) + \tfrac{9}{8}ee\sin(2\eta-2s)$$
$$+ \tfrac{15}{4}kk\sin(2\eta+2r) + \tfrac{9}{8}ee\sin(2\eta+2s)$$
$$+ \frac{3vv}{2n^4}\sin 2\eta \quad - \tfrac{9}{2}ek\sin(2\eta-r+s) - \tfrac{9}{2}ek\sin(2\eta+r-s)$$
$$- \tfrac{9}{2}ek\sin(2\eta-r-s) - \tfrac{9}{2}ek\sin(2\eta+r+s)$$
+

$$+ \frac{6kv}{nn}\sin(2\eta-r) + \frac{6kv}{nn}\sin(2\eta+r)$$
$$- \frac{9ev}{2nn}\sin(2\eta-s) - \frac{9ev}{2nn}\sin(2\eta+s)$$
$$+ \tfrac{3}{8}v\sin\eta + \tfrac{15}{16}vk\sin(\eta-r) - \tfrac{3}{4}ve\sin(\eta-s)$$
$$+ \tfrac{75}{16}vk\sin(3\eta-r) - \tfrac{15}{4}ve\sin(3\eta-s)$$
$$+ \tfrac{15}{8}v\sin 3\eta + \tfrac{15}{16}vk\sin(\eta+r) - \tfrac{3}{4}ve\sin(\eta+s)$$
$$+ \tfrac{75}{16}vk\sin(3\eta+s) - \tfrac{15}{4}ve\sin(3\eta+s)$$

§. 53. Ipsa vero aequatio secunda per hanc substitutionem, postquam per nn fuerit multiplicata, in formam sequentem abibit.

II. $$\frac{ddv}{dr^2} = \varkappa\varkappa nn - 3\varkappa\varkappa v + \frac{6\varkappa\varkappa vv}{nn} - 2\varkappa\int R dr$$
$$+ \frac{6\varkappa v}{nn}\int R dr + \frac{1}{nn}(\int R dr)^2$$
$$+ \frac{3m\tang\rho^2}{4}(1 - \cos 2(\phi-\pi))(1 + \tfrac{1}{2}kk + k\cos r + \tfrac{1}{2}kk\cos 2r)$$
$$- \frac{3mv\tang\rho^2}{2nn}(1 - \cos 2(\phi-\pi))\quad(1 + k\cos r)$$
$$- \frac{3mvv}{n^4}(1 + k\cos r) - m(1 + \tfrac{1}{2}kk + k\cos r + \tfrac{1}{2}kk\cos 2r)$$
$$+ \frac{2mv}{nn}(1 + \tfrac{1}{2}kk + k\cos r + \tfrac{1}{2}kk\cos 2r) + nn(\tfrac{1}{2}kk + k\cos r + \tfrac{1}{2}kk\cos 2r)$$
$$+ v(\tfrac{1}{2}kk + k\cos r + \tfrac{1}{2}kk\cos 2r) + \mu(1 + 3k\cos r)$$
$$+ \tfrac{1}{2} + kk + \tfrac{9}{4}ee + 2k\cos r - \tfrac{3}{2}e\cos s - 3ek\cos(r-s)$$
$$+ \tfrac{5}{2}kk\cos r + \tfrac{3}{4}ee\cos 2s - 3ek\cos(r+s)$$
$$+ \frac{v}{2nn}(1 + 4k\cos r - 3e\cos s)$$
$$+ \tfrac{3}{2}(1 + 2kk + \tfrac{9}{2}ee)\cos 2\eta + 3k\cos(2\eta-r) - \tfrac{9}{4}e\cos(2\eta-s)$$
$$+ 3\cos(2\eta+r) - \tfrac{9}{4}e\cos(2\eta+s)$$

+

$$+\tfrac{15}{4}kk\cos(2\eta-2r)+\tfrac{9}{8}ee\cos(2\eta-2s)$$
$$+\tfrac{15}{4}kk\cos(2\eta+2r)+\tfrac{9}{8}ee\cos(2\eta+2s)$$
$$-\tfrac{9}{2}ek\cos(2\eta-r+s)-\tfrac{9}{2}ek\cos(2\eta+r-s)$$
$$-\tfrac{9}{2}ek\cos(2\eta-r-s)-\tfrac{9}{2}ek\cos(2\eta+r+s)$$

$$+\frac{3v}{2nn}\begin{cases}\cos 2\eta+2k\cos(2\eta-r)+2k\cos(2\eta+r)\\ -\tfrac{3}{2}e\cos(2\eta-s)-\tfrac{3}{2}e\cos(2\eta+s)\end{cases}$$

$$+\tfrac{3}{8}v\begin{cases}+3\cos\eta+\tfrac{15}{2}k\cos(\eta-r)-6e\cos(\eta-s)\\ +5\cos 3\eta+\tfrac{15}{2}k\cos(\eta+r)-6e\cos(\eta+s)\\ +\tfrac{25}{2}k\cos(3\eta-r)-10e\cos(3\eta-s)\\ +\tfrac{25}{2}k\cos(3\eta+r)-10e\cos(3\eta+s)\end{cases}$$

$$+\frac{3\nu v}{4nn}(3\cos\eta+5\cos 3\eta)$$

§. 54. Cum autem fit $m = nn + 2 + 3\mu + \gamma$ et $\varkappa\varkappa = 1 + \frac{3+4\mu+\delta}{2nn}$, fi hi valores fubftituantur, plures termini fe mutuo deftruent, aequatioque prodibit fequenti forma concinniori contenta: vbi quidem in terminis per fe minimis loco m fcribi licebit nn, et 1 loco $\varkappa\varkappa$ vel $\varkappa$.

II. Aequatio.

$$\frac{ddv}{dr^2}=\tfrac{1}{2}\delta-\gamma+\tfrac{9}{4}ee-\gamma k\cos r+\tfrac{3}{2}kk\cos 2r$$
$$-2\left(1+\frac{3+4\mu+\delta}{4nn}\right)\int R\,dr+\frac{1}{nn}\left(\int R\,dr\right)^2$$
$$-v\left(1-\tfrac{3}{2}kk-3k\cos r-\tfrac{3}{2}kk\cos 2r\right)$$
$$+\frac{vv}{nn}(3-3k\cos r)-\tfrac{3}{2}e\cos s+\tfrac{3}{4}ee\cos 2s-3ek\cos(r-s)$$
$$-3ek\cos(r+s)+\tfrac{3}{2}(1+2kk+\tfrac{9}{2}ee)\cos 2\eta$$

+

$$+3k\cos(2\eta-r)+\tfrac{15}{4}kk\cos(2\eta-2r)-\tfrac{9}{4}e\cos(2\eta-s)$$
$$+3k\cos(2\eta+r)+\tfrac{15}{4}kk\cos(2\eta+2r)-\tfrac{9}{4}e\cos(2\eta-s)$$
$$+\tfrac{9}{8}ee\cos(2\eta-2s)-\tfrac{9}{2}ek\cos(2\eta-r+s)-\tfrac{9}{2}ek\cos(2\eta-r-s)$$
$$+\tfrac{9}{8}ee\cos(2\eta+2s)-\tfrac{9}{2}ek\cos(2\eta+r-s)-\tfrac{9}{2}ek\cos(2\eta+r+s)$$

$$+\frac{v}{nn}\begin{cases}2\gamma-\tfrac{3}{2}\delta+6k\cos r+2(3\mu+\gamma)k\cos r+6\int R\,dr\\ -\tfrac{3}{2}e\cos s+\tfrac{3}{2}\cos 2\eta+3k\cos(2\eta-r)+3k\cos(2\eta+r)\\ -\tfrac{9}{4}e\cos(2\eta-s)-\tfrac{9}{4}e\cos(2\eta+s)\end{cases}$$

$$+\tfrac{3}{8}v\begin{cases}3\cos\eta+\tfrac{15}{2}k\cos(\eta-r)-6e\cos(\eta-s)\\ +5\cos 3\eta+\tfrac{15}{2}k\cos(\eta+r)-6e\cos(\eta+s)\\ +\tfrac{25}{2}k\cos(3\eta-r)-10e\cos(3\eta-s)\\ +\tfrac{25}{2}k\cos(3\eta+r)-10e\cos(3\eta+s)\end{cases}$$

$$+\frac{3\nu v}{4nn}(3\cos\eta+5\cos 3\eta)$$
$$+\tfrac{3}{4}(nn+2+3\mu+\gamma)\frac{(1-2v)}{nn}\tan g\,\rho^2(1-\cos 2(\Phi-\pi))$$
$$(1+\tfrac{1}{2}kk+k\cos r+\tfrac{1}{2}kk\cos 2r)$$

§. 55. Pro loco nodi autem inveniendo prodibit fequens aequatio.

$$\frac{d\pi}{dr}=\frac{-3}{\varkappa nn}\left(1+\frac{2\varkappa v+\int R\,dr}{\varkappa nn}\right)\left(1+2kk+\tfrac{9}{2}ee\right)$$
$$\begin{cases}\tfrac{1}{4}+\tfrac{1}{4}\cos 2\eta-\tfrac{1}{4}\cos 2(\Phi-\pi)-\tfrac{1}{4}\cos 2(\theta-\pi)\\ +k\cos r+\tfrac{1}{2}k\cos(2\eta-r)-\tfrac{1}{2}k\cos(2\Phi-2\pi-r)\\ -\tfrac{3}{4}e\cos s+\tfrac{1}{2}k\cos(2\eta+r)-\tfrac{1}{2}k\cos(2\Phi-2\pi+r)\\ \qquad-\tfrac{3}{8}e\cos(2\eta-s)-\tfrac{1}{2}k\cos(2\theta-2\pi-r)\\ \qquad-\tfrac{3}{8}e\cos(2\eta+r)-\tfrac{1}{2}k\cos(2\theta-2\pi+r)\end{cases}$$
$$-\frac{3\nu}{4\varkappa nn}\begin{cases}\tfrac{11}{4}\cos\eta+\tfrac{5}{4}\cos 3\eta-\tfrac{3}{2}\cos(3\Phi+\theta-2\pi)\\ -\tfrac{5}{4}\cos(3\Phi-\theta-2\pi)-\tfrac{5}{4}\cos(3\theta-\Phi-2\pi)\end{cases}$$

At

At pro inclinatione orbitae habebitur:

$$\frac{d.l\,\mathrm{tang}\,\varrho}{dr} = \frac{-3}{\varkappa nn}\left(1+\frac{2\varkappa v+\int R dr}{\varkappa nn}\right)\;(1+2kk+\tfrac{9}{8}ee)$$

$$\begin{cases} \tfrac{1}{4}\sin 2(\varphi-\pi) + \tfrac{1}{4}\sin 2(\theta-\pi) - \tfrac{1}{4}\sin 2\eta) \\ -\tfrac{1}{2}k\sin(2\eta-r) + \tfrac{1}{2}k\sin(2\varphi-2\pi-r) \\ -\tfrac{1}{2}k\sin(2\eta+r) + \tfrac{1}{2}k\sin(2\varphi-2\pi+r) \\ +\tfrac{3}{8}e\sin(2\eta-s) + \tfrac{1}{2}k\sin(2\theta-2\pi-r) \\ +\tfrac{3}{8}e\sin(2\eta+s) + \tfrac{1}{2}k\sin(2\theta-2\pi+r) \end{cases}$$

$$-\frac{3v}{4\varkappa nn}\left(-\tfrac{1}{4}\sin\eta-\tfrac{5}{4}\sin 3\eta+\tfrac{3}{2}\sin(\varphi+\theta-2\pi)\right.$$

$$\left.+\tfrac{5}{4}\sin(3\varphi-\theta-2\pi)+\tfrac{5}{4}\sin(3\theta-\varphi-2\pi)\right)$$

Quomodo igitur his aequationibus ad motum Lunae cognoscendum vti conueniat, in sequentibus capitibus videamus.

CAPUT

CAPUT IV.

INUESTIGATIO INAEQUALITATIS LUNAE ABSOLUTAE, QUAE VARIATIO DICITUR.

§. 33.

Ex his aequationibus perspicitur in determinationem motus Lunae plurimorum angulorum vel sinus vel cosinus ingredi, qui anguli formantur per variam combinationem sequentium 4 angulorum:

1. ex distantia Solis a Luna, quem angulum posuimus $= \eta$
2. ex anomalia Lunae vera $= r$
3. ex anomalia Solis vera $= s$
4. ex distantia Lunae a nodo ascendente $= \varphi - \pi$.

Ne igitur a tanta angulorum multitudine obruamur, a casibus simplicioribus ordiamur: ac primo quidem in eas tantum motus inaequalitates inquiramus, quae a solo angulo η pendeant, neque idcirco excentricitatem vel Solis vel Lunae implicent, neque ab orbitae lunaris inclinatione ad eclipticam afficiantur.

§. 57. Has igitur inaequalitates, quae a solo situ Solis respectu Lunae nascuntur, atque ab Astronomis sub nomine variationis comprehendi solent, ex praecedentibus aequationibus eliciemus, si tam excentricitatem Lunae k quam solis e pro nihilo habeamus, atque inclinationem orbitae lunaris ad eclipticam euanescentem statuamus, ita vt sit $k = 0$, $e = 0$ et tang $\varrho = 0$. Sic enim obtinebimus eas inaequalitates Lunae, quae ab his elementis non pendent, ideoque tantum per angu-

 lum

lum η determinantur; quae cum vnica tabula comprehendi queant, haec tabula variationem Lunae indicare dicitur. Interim tamen hic animaduerti oportet, partem quandam exiguam variationis quoque ab excentricitate orbitae Lunae k pendere, quam partem deinceps fupplebimus, cum huius excentricitatis rationem fumus habituri.

§. 58. Reiectis ergo terminis k, e, et tang ρ continentibus, habebimus:

$$\frac{d\Phi}{dr} = \varkappa - \frac{2\varkappa v - \int R\,dr}{nn} + \frac{3\varkappa v^2 + 2v\int R\,dr}{n^4}$$

$$\frac{d\eta}{dr} = \varkappa - \frac{1}{n} - \frac{2\varkappa v - \int R\,dr}{nn} + \frac{3\varkappa v^2 + 2v\int R\,dr}{n^4}$$

$R = \frac{3}{2}\sin 2\eta + \frac{3v}{nn}\sin 2\eta + \frac{3}{8}v\sin\eta + \frac{15}{8}v\sin 3\eta$, ac denique

$$\frac{ddv}{dr^2} = \tfrac{1}{2}\delta - \gamma - v(1 + \frac{3+4\mu+\delta}{4nn})\int R\,dr + \frac{1}{nn}(\int R\,dr)^2 - v + \frac{3vv}{nn}$$
$$+ \tfrac{3}{2}\cos 2\eta + \frac{v}{nn}(2\gamma - \tfrac{1}{2}\delta) + \frac{3v\cos 2\eta}{2nn} + \frac{6v}{nn}\int R\,dr$$
$$+ \tfrac{3}{8}\cos\eta + \tfrac{15}{8}v\cos 3\eta$$

Hic autem notandum eft effe $\varkappa = \sqrt{(1 + \frac{3+4\mu+\delta}{2nn})}$; quoniam vero valores litterarum μ et δ demum cum per confenfum obferuationum, tum per indolem calculi definire inftituimus, hic ex obferuationibus petamus valores ipfius $\varkappa$; cum enim fit $\varkappa : 1 = d\Phi : dr$, hoc eft vt motus Lunae medius ad motum anomaliae, erit $\varkappa = 1,0085272$. Fieri quidem poteft, vt hic valor aliquantulum a vero differat, fed errorem fi quis lateat infra detegemus, facillimeque emendabimus.

§. 59.

compre-
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, partem
ntricitate
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i fumus
ontinen-
denique
$-v + \frac{3vv}{nn}$
$\int R\,dr$
$\frac{\mu+\delta}{nn}$);
n cum
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ius va-
: eft vt
: $\varkappa$ =
liquan-
: infra
§. 59.

§. 59. Cum igitur iam fupra inuenerimus effe $n = 13,25586$ ac proinde $nn = 175,71795$ erit $\frac{1}{n} = 0,075438$, ideoque $\varkappa - \frac{1}{n} = 0,933089$. Hic autem numerus, qui iam quafi medium valorem rationis $\frac{d\eta}{dr}$ exprimit, in omnibus operationibus, quae fequuntur, frequentiffime occurret, hincque breuitatis gratia ponamus

$$\varkappa - \frac{1}{n} = \alpha, \text{ vt fit } \alpha + \frac{1}{n} = \sqrt{(1 + \frac{3+4\mu+\delta}{2nn})}$$

eritque ergo $\alpha = 0,933089$, qui valor quam minime a vero difcrepat, vti mox patebit. Quod autem verus ipfius α valor aliquantulum diuerfus effe poffit, inde primo patet, quod minutias, quae ex terminis $\frac{3\varkappa v^2 + 2v\int R\,dr}{n^4}$ quantitati conftanti accrefcere potuiffent, hic neglex imus; tum vero fieri poteft, vt ratio media differentialium $d\eta$ ad dr alia fit atque quantitatum finitarum η et r.

§. 60. Si has formulas attente contemplemur, mox deprehendemus valorem integralis $\int R\,dr$ conftare ex cofinibus angulorum 2η, η, 3η, et 4η. Quanquam enim altiora quoque multipla huius anguli ingredientur, tamen facile patet, coefficientes eorum continuo fieri minores, ita vt in quadruplo tuto fubfiftere poffimus: fimilis autem erit ratio valoris ipfius v. Hinc ponamus:

$$\int R\,dr = \mathfrak{A}\cos 2\eta + \mathfrak{B}\cos 4\eta + \mathfrak{a}\,v\cos\eta + \mathfrak{b}\,v\cos 3\eta$$
$$v = A\cos 2\eta + B\cos 4\eta + a\,v\cos\eta + b\,v\cos 3\eta$$

atque hos valores fictitios in formulis noftris fubftituamus:

mus, vt inde valores istorum coefficientium assumtorum determinare possimus: quippe qui modus aptissimus videtur ad cognitionem integralium perueniendi. Quia autem est circiter $v = \frac{1}{282}$, patet terminos per v multiplicatos prae reliquis tam esse exiguos, vt eos qui multo fuerint minores, sine haesitatione praetermittere possimus.

§. 61. Per hos ergo valores assumtos consequemur:

$$\frac{d\Phi}{dr} = \varkappa - \frac{(2\varkappa A + \mathfrak{A})}{nn}\cos 2\eta - \frac{(2\varkappa B + \mathfrak{B})}{nn}\cos 4\eta$$
$$+ \frac{A(3\varkappa A + 2\mathfrak{A})}{2n^4}\cos 2\eta + \frac{A(3\varkappa A + 2\mathfrak{A})}{2n^4}\cos 4\eta$$
$$- \frac{(2\varkappa a + \mathfrak{a})}{nn}v\cos\eta - \frac{(2\varkappa b + \mathfrak{b})}{nn}v\cos 3\eta$$

atque ob $\varkappa - \frac{1}{n} = \alpha$ erit minimis terminis omissis, quia hi in operatione multo magis diminuerentur:

$$\frac{d\eta}{dr} = \alpha - \frac{(2\varkappa A + \mathfrak{A})}{nn}\cos 2\eta - \frac{(2\varkappa B + \mathfrak{B})}{nn}\cos 4\eta$$
$$- \frac{(2\varkappa a + \mathfrak{a})}{nn}v\cos\eta - \frac{(2\varkappa b + \mathfrak{b})}{nn}v\cos 3\eta$$

His positis erit:

$$\frac{d.\cos 2\eta}{dr} = -\sin 2\eta.\frac{2d\eta}{dr} = -2\alpha\sin 2\eta - \frac{(2\varkappa B + \mathfrak{B})}{nn}\sin 2\eta + \frac{(2\varkappa A + \mathfrak{A})}{nn}\sin 4\eta$$
$$+ \frac{(2\varkappa a + \mathfrak{a})}{nn}v\sin\eta + \frac{(2\varkappa a + \mathfrak{a})}{nn}v\sin 3\eta$$
$$- \frac{(2\varkappa b + \mathfrak{b})}{nn}v\sin\eta$$

d.

mto- ptissi- iendi. per v s qui ittere mur: η η iissis, sin4η d.

$$\frac{d.\cos 4\eta}{dr} = -\sin 4\eta.\frac{4d\eta}{dr} = -4\alpha\sin 4\eta + \frac{2(2\varkappa A + \mathfrak{A})}{nn}\sin 2\eta$$
$$\frac{d.\cos\eta}{dr} = -\sin\eta.\frac{d\eta}{dr} = -\alpha\sin\eta$$
$$\frac{d.\cos 3\eta}{dr} = -\sin 3\eta.\frac{3d\eta}{dr} = -3\alpha\sin 3\eta$$

§. 62. Quod si iam secundum has formulas quantitas integralis $\int R\,dr$ differentietur, obtinebitur:

$$R = (-2\alpha\mathfrak{A} - \frac{\mathfrak{A}(2\varkappa B + \mathfrak{B})}{nn} + \frac{2\mathfrak{B}(2\varkappa A + \mathfrak{A})}{nn})\sin 2\eta$$
$$+ (\frac{\mathfrak{A}(2\varkappa A + \mathfrak{A})}{nn} - 4\alpha\mathfrak{B})\sin 4\eta$$
$$+ (\frac{\mathfrak{A}(2\varkappa a + \mathfrak{a})}{nn} + \frac{\mathfrak{A}(2\varkappa b + \mathfrak{b})}{nn} - \alpha\mathfrak{a})\,v\sin\eta$$
$$+ (\frac{\mathfrak{A}(2\varkappa a + \mathfrak{a})}{nn} - 3\alpha\mathfrak{b})\,v\sin 3\eta$$

Cum iam sit per hypothesin

$$R = \tfrac{3}{2}\sin 2\eta + \frac{3A}{2nn}\sin 4\eta + \tfrac{3}{8}v\sin\eta + \tfrac{15}{8}v\sin 3\eta$$
$$- \frac{3B}{2nn} \quad + \frac{3a}{2nn}v \quad + \frac{3a}{2nn}v$$
$$- \frac{3b}{2nn}v$$

prodibit terminis homogeneis comparandis:

$$2\alpha\mathfrak{A} = -\tfrac{3}{2} - \frac{\mathfrak{A}(2\varkappa B + \mathfrak{B}) + 2\mathfrak{B}(2\varkappa A + \mathfrak{A})}{nn} + \frac{3B}{2nn}$$
$$4\alpha\mathfrak{B} = -\frac{3A}{2nn} + \frac{\mathfrak{A}(2\varkappa A + \mathfrak{A})}{nn}$$
$$\alpha\mathfrak{a} = -\tfrac{3}{8} - \frac{3(a-b)}{2nn} + \frac{\mathfrak{A}(2\varkappa a + \mathfrak{a}) - \mathfrak{A}(2\varkappa b + \mathfrak{b})}{nn}$$
$$3\alpha\mathfrak{b} = -\tfrac{15}{8} - \frac{3a}{2nn} + \frac{\mathfrak{A}(2\varkappa a + \mathfrak{a})}{nn}$$

§. 63.

§. 63. Aequatio autem nostra differentio-differentialis, si pro $\int R\,dr$ et v valores assumti substituantur, sequentem induet formam:

$$\frac{ddv}{dr^2}=(\tfrac{1}{2}\delta-\gamma)-2\varkappa\mathfrak{A}\cos 2\eta-2\varkappa\mathfrak{B}\cos 4\eta-2\varkappa\mathfrak{a}v\cos\eta-2\varkappa\mathfrak{b}v\cos 3\eta$$

$$\begin{array}{lllll}
+\frac{\mathfrak{A}\mathfrak{A}}{2nn} & & +\frac{\mathfrak{A}\mathfrak{A}}{2nn} & & \\
 & -A & -B & -av & -bv \\
+\frac{3AA}{2nn} & & +\frac{3AA}{2nn} & & \\
 & +\frac{1}{2} & & & \\
 & +\frac{(4\gamma-3\delta)}{2nn}A & +\frac{(4\gamma-3\delta)}{2nn}B & +\frac{3b}{4nn}v & \\
+\frac{3A}{4nn} & +\frac{3B}{4nn} & +\frac{3A}{4nn} & +\frac{3a}{4nn}v & +\frac{3a}{4nn}v \\
+\frac{3A\mathfrak{A}}{nn} & & +\frac{3A\mathfrak{A}}{nn} & +\frac{9}{8}v & +\frac{15}{8}v
\end{array}$$

vbi quidem perspicuum est, quinam termini respectu reliquorum tam sint parui, vt sine errore deleri queant.

§. 64. Quaeramus ergo primum differentiale $\frac{dv}{dr}$ ac reperietur:

$$\frac{dv}{dr}=\begin{array}{l}
\left(-2A\alpha-\frac{A(2\varkappa B+\mathfrak{B})}{nn}+\frac{2B(2\varkappa A+\mathfrak{A})}{nn}\right)\sin 2\eta \\
\left(-4\alpha B+\frac{A(2\varkappa A\mp\mathfrak{A})}{nn}\right)\sin 4\eta \\
\left(-\alpha a+\frac{A(2\varkappa a+\mathfrak{a})}{nn}-\frac{A(2\varkappa b+\mathfrak{b})}{nn}\right)v\sin\eta \\
\left(-3\alpha b+\frac{A(2\varkappa a+\mathfrak{a})}{nn}\right)v\sin 3\eta
\end{array}$$

pona-

ponatur autem breuitatis ergo:

$$\frac{dv}{dr}=-A'\sin 2\eta-B'\sin 4\eta-a'v\sin\eta-b'v\sin 3\eta$$

vt sit:

$$A'=2\alpha A+\frac{A(2\varkappa B+\mathfrak{B})}{nn}-\frac{2B(2\varkappa A+\mathfrak{A})}{nn}$$

$$B'=4\alpha B-\frac{A(2\varkappa A+\mathfrak{A})}{nn}$$

$$a'=\alpha a-\frac{A(2\varkappa a+\mathfrak{a})}{nn}+\frac{A(\varkappa b+\mathfrak{b})}{nn}$$

$$b'=3\alpha b-\frac{A(2\varkappa a+\mathfrak{a})}{nn}$$

§. 65. Hinc cum sit:

$$\frac{d.\sin 2\eta}{dr}=\cos 2\eta.\frac{2d\eta}{dr}=2\alpha\cos 2\eta-\frac{(2\varkappa B+\mathfrak{B})}{nn}\cos 2\eta-\frac{(2\varkappa A+\mathfrak{A})}{nn}\cos 4\eta$$
$$-\frac{(2\varkappa A+\mathfrak{A})}{nn}-\frac{(2\varkappa a+\mathfrak{a})}{nn}v\cos\eta-\frac{(2\varkappa b+\mathfrak{b})}{nn}v\cos\eta-\frac{(2\varkappa a+\mathfrak{a})}{nn}v\cos 3\eta$$

$$\frac{d.\sin 4\eta}{dr}=\cos 4\eta.\frac{4d\eta}{dr}=4\alpha\cos 4\eta-\frac{2(2\varkappa A+\mathfrak{A})}{nn}\cos 2\eta-\frac{2(2\varkappa B+\mathfrak{B})}{nn}$$

$$\frac{d.\sin\eta}{dr}=\cos\eta.\frac{d\eta}{dr}=\alpha\cos\eta;\ \text{et}\ \frac{d.\sin 3\eta}{dr}=\cos 3\eta.\frac{3d\eta}{dr}=3\alpha\cos 3\eta$$

prodibit

$$\frac{ddv}{dr^2}=\begin{array}{l}
+\frac{A'(2\varkappa A+\mathfrak{A})}{nn}+\frac{2B'(2\varkappa B+\mathfrak{B})}{nn} \\
\left(-2\alpha A'+\frac{A'(2\varkappa B+\mathfrak{B})}{nn}+\frac{2B'(2\varkappa A+\mathfrak{A})}{nn}\right)\cos 2\eta \\
\left(-4\alpha B'+\frac{A'(2\varkappa A+\mathfrak{A})}{nn}\right)\cos 4\eta \\
\left(-\alpha a'+\frac{A'(2\varkappa a+\mathfrak{a})}{nn}\right)v\cos\eta \\
\left(-3\alpha b'+\frac{A'(2\varkappa a+\mathfrak{a})}{nn}\right)v\cos 3\eta
\end{array}$$

seu

seu substitutis superioribus valoribus:

$$+\frac{2A\alpha(2\varkappa A+\mathfrak{A})}{nn}+\frac{8\alpha B(2\varkappa B+\mathfrak{B})}{nn}$$

$$\left(-4\alpha\alpha A+\frac{12\alpha B(2\varkappa A+\mathfrak{A})}{nn}\right)\cos 2\eta$$

$$\frac{ddv}{dr^2}=\left(-16\alpha\alpha B+\frac{8\alpha A(2\varkappa A+\mathfrak{A})}{nn}\right)\cos 4\eta$$

$$\left(-\alpha\alpha a+\frac{3\alpha A(2\varkappa a+\mathfrak{a})}{nn}-\frac{\alpha A(2\varkappa b+\mathfrak{b})}{nn}\right)v\cos\eta$$

$$\left(-9\alpha\alpha b+\frac{5\alpha A(2\varkappa a+\mathfrak{a})}{nn}\right)v\cos 3\eta$$

§ 66. Hi iam termini singulatim illis, qui §. 63. sunt exhibiti, aequales statuantur, atque sequentes prodibunt determinationes,

$$\tfrac{1}{2}\delta-\gamma+\frac{3AA+6A\mathfrak{A}+\mathfrak{A}\mathfrak{A}}{2nn}+\frac{3A}{4nn}=\frac{2\alpha A(2\varkappa A+\mathfrak{A})}{nn}+\frac{8\alpha B(2\varkappa B+\mathfrak{B})}{nn}$$

$$-A+\tfrac{3}{2}-2\varkappa\mathfrak{A}+\frac{(4\gamma-3\delta)}{2nn}A+\frac{3B}{4nn}=-4\alpha\alpha A+\frac{12\alpha B(2\varkappa A+\mathfrak{A})}{nn}$$

$$-B-2\varkappa\mathfrak{B}+\frac{3AA+6A\mathfrak{A}+\mathfrak{A}\mathfrak{A}}{2nn}+\frac{3A}{4nn}+\frac{(4\gamma-3\delta)}{2nn}B=$$

$$-16\alpha\alpha B+\frac{8\alpha A(2\varkappa A+\mathfrak{A})}{nn}$$

$$-a+\tfrac{9}{8}-2\varkappa\mathfrak{a}+\frac{3a+3b}{4nn}=-\alpha\alpha a+\frac{3\alpha A(2\varkappa a+\mathfrak{a})}{nn}-\frac{\alpha A(2\varkappa b+\mathfrak{b})}{nn}$$

$$-b+\tfrac{15}{8}-2\varkappa\mathfrak{b}+\frac{3a}{4nn}=-9\alpha\alpha b+\frac{5\alpha A(2\varkappa a+\mathfrak{a})}{nn}$$

vnde primum quaeri debent valores vero proximi, qui sunt:

$$\mathfrak{A}=-\frac{3}{4\alpha};\quad \mathfrak{a}=-\frac{3}{8\alpha};\quad \mathfrak{b}=-\frac{5}{8\alpha};$$

$$A=-\frac{\tfrac{3}{2}+2\varkappa\mathfrak{A}}{4\alpha\alpha-1};\; a=\frac{\tfrac{9}{8}-2\varkappa\mathfrak{a}}{1-\alpha\alpha};\; b=-\frac{\tfrac{15}{8}+2\varkappa\mathfrak{b}}{9\alpha\alpha-1}$$

§. 67.

§. 67. Calculus ergo sequenti modo instituatur:

$\alpha = 0,933089$;		$l\alpha = 9,969923$
$\varkappa = 1,008527$;		$l\varkappa = 0,003687$
		$l2\varkappa = 0,304717$
	Iam est	$l8\alpha = 0,873013$
	subtr. a	$l3 = 0,477121$
		$l5 = 0,698970$
$\mathfrak{a} = -0,402$; erit		$l\text{-}\mathfrak{a} = 9,604108$
$\mathfrak{b} = -0,635$;		$l\text{-}\mathfrak{b} = 9,825957$
$\mathfrak{A} = -0,804$;		$l\text{-}\mathfrak{A} = 9,905138$

atque hinc conficietur:

$$A=-\frac{3,121}{4\alpha\alpha-1};\; a=+\frac{1,936}{1-\alpha\alpha};\; b=-\frac{3,156}{9\alpha\alpha-1}$$

quarum ergo litterarum valores proximi sunt

$A = -1,2583$; $a = +14,968$; $b = -0,4613$

§. 68. Quaeramus hinc primum valores litterarum $\mathfrak{B}$ et B.

$\mathfrak{A} = -0,804$; $l-\mathfrak{A} = 9,905138$

$2\varkappa A+\mathfrak{A} = -3,341$; vnde colligitur

$4\alpha\mathfrak{B} = +\frac{4,573}{nn}$ $\quad l4,573 = 0,660201$

$l\,nn = 2,244816$

hinc erit $\quad 8,415385$

$l\,4\alpha = 0,571983$

$\mathfrak{B} = +0,00697$ $\quad l\mathfrak{B} = 7,843402$

Deinde est

$$(16\alpha\alpha-1)B=2\varkappa\mathfrak{B}-\frac{3A}{4nn}-\frac{3AA-6A\mathfrak{A}-\mathfrak{A}\mathfrak{A}}{2nn}+\frac{8\alpha A(2\varkappa A+\mathfrak{A})}{nn}$$

seu $B = +\frac{0,16819}{16\alpha\alpha-1}$; vnde reperitur

$B = +0,012792$ et $lB = 8,106947$

§. 69.

§. 69. His iam valoribus proxime veris inuentis quaerantur exacti, ac primo quidem

$$2\alpha\mathfrak{A} = -\tfrac{3}{2} + \frac{\tfrac{3}{4}B - \mathfrak{A}(2\varkappa B + \mathfrak{B}) + 2\mathfrak{B}(2\varkappa A + \mathfrak{A})}{nn}$$

vnde reperitur vt ante:

$\mathfrak{A} = -0,80378 \quad \ldots \quad l{-}\mathfrak{A} = 9,905138$

$$\alpha\mathfrak{a} = -\tfrac{3}{2} - \frac{\tfrac{3}{4}(a-b) + \mathfrak{A}(2\varkappa a + \mathfrak{a}) - \mathfrak{A}(2\varkappa b + \mathfrak{b})}{nn} = -0,65361$$

$\mathfrak{a} = -0,70048 \quad \ldots \quad l{-}\mathfrak{a} = 9,845396$

$$3\alpha\mathfrak{b} = -\tfrac{15}{8} - \frac{\tfrac{3}{4}a + \mathfrak{A}(2\varkappa a + \mathfrak{a})}{nn} = -2,13900$$

$\mathfrak{b} = -0,76413 \quad \ldots \quad l{-}\mathfrak{b} = 9,883167$

$$(4\alpha\alpha - 1)A = -\tfrac{3}{2} + 2\varkappa\mathfrak{A} - \frac{\tfrac{3}{4}B + 12\alpha B(2\varkappa A + \mathfrak{A})}{nn} = -3,12379$$

$A = -1,25826 \quad \ldots \quad l{-}A = 0,099771$

$$(1-\alpha\alpha)a = \tfrac{3}{2} - 2\varkappa\mathfrak{a} + \frac{\tfrac{3}{4}(a+b) - 3\alpha A(2\varkappa a + \mathfrak{a}) + \alpha A(2\varkappa b + \mathfrak{b})}{nn} \text{ vel}$$

$$\left(1 - \alpha\alpha - \frac{3}{4nn} + \frac{(6\alpha\varkappa A)}{nn}\right)a = \tfrac{3}{2} - 2\varkappa\mathfrak{a}$$
$$+ \frac{\tfrac{3}{4}b - 3\alpha A\mathfrak{a} + \alpha A(2\varkappa b + \mathfrak{b})}{nn} = -2,53335$$

hinc $a = +30,989$ et $l\,a = 1,491207$

Vnde patet valorem ipsius $\mathfrak{a}$ ante inuentum non satis esse exactum, exactior ergo prodibit ex hac formula

$$\left(\alpha - \frac{\mathfrak{A}}{nn}\right)\mathfrak{a} = -\tfrac{3}{2} - \frac{\tfrac{3}{4}(a-b) + \mathfrak{A}(2\varkappa a - 2\varkappa b - \mathfrak{b})}{nn} = -0,93709$$

hinc $\mathfrak{a} = -0,?99939$ et $l{-}\mathfrak{a} = 9,999735$

vnde etiam exactius valor ipsius a reperitur, ex quo denuo

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,65361
16
0
17
,12379
71
b) vel
7
n fatis
nula
93709
5
x quo
denuo

denuo valor ipsius $\mathfrak{a}$ corrigetur, sicque tandem satis exacte obtinebitur

$\mathfrak{a} = -1,2537 \quad \ldots \quad l{-}\mathfrak{a} = 0,098200$

$a = +44,48 \quad \ldots \quad l\,a = 1,648165$

$\mathfrak{b} = -0,95003 \quad \ldots \quad l{-}\mathfrak{b} = 9,977736$

§. 70. Hinc iam accuratius quaeramus valorem ipsius b

$$(9\alpha\alpha - 1)b = -\tfrac{15}{8} + 2\varkappa\mathfrak{b} - \frac{\tfrac{3}{4}a + 5\alpha A(2\varkappa a + \mathfrak{a})}{nn}$$

$b = -1,0146 \quad \ldots \quad l{-}b = 0,006314$

vnde si denuo praecedentes valores corrigantur, fiet

$\mathfrak{a} = -1,2630 \quad \ldots \quad l{-}\mathfrak{a} = 0,101403$

$\mathfrak{b} = -0,9500 \quad \ldots \quad l{-}\mathfrak{b} = 9,977736$

$a = +44,525 \quad \ldots \quad l\,a = 1,648604$

$b = -1,015 \quad \ldots \quad l{-}b = 0,006400$

$\mathfrak{A} = -0,80378 \quad \ldots \quad l{-}\mathfrak{A} = 9,905138$

$\mathfrak{B} = +0,00697 \quad \ldots \quad l\,\mathfrak{B} = 7,843402$

$A = -1,25826 \quad \ldots \quad l{-}A = 0,099771$

$B = \pm 0,01279 \quad \ldots \quad l\,B = 8,106947$

His autem valoribus inuentis colligitur fore

$\tfrac{1}{2}\delta - \gamma = +0,01742$

Hic autem valor partem insuper accipit cum ab excentricitate vtriusque orbitae, tum ab inclinatione oriundam, quam deinceps determinabimus.

§. 71. Ex his ergo valoribus habebimus:

$$\int R\,dr = -\underset{9,905138}{0,80378}\cos 2\eta + \underset{7,843402}{0,00697}\cos 4\eta$$
$$-\underset{0,101403}{1,2630}\,v\cos\eta - \underset{9,977736}{0,9500}\,v\cos 3\eta$$

$v =$

$v = -1,25826 \cos 2\eta + 0,01279 \cos 4\eta$
  0,099771 8,106947
$+ 44,525\, \nu \cos \eta - 1,015\, \nu \cos 3\eta$
  1,648604 0,006400

hincque porro

$\frac{d\Phi}{dr} = \varkappa + 0,019015 \cos 2\eta - 0,0000762 \cos 4\eta$
  8,279096 5,881955
$+0,0001103$
$- 0,50381\, \nu \cos \eta + 0,017068\, \nu \cos 3\eta$
  9,702270 8,232184

at est $\frac{d\eta}{dr} = \frac{d\Phi}{dr} - \frac{1}{n}$, posuimusque $\varkappa - \frac{1}{n} = \alpha$, existente $\varkappa = V\left(1 + \frac{3 + 4\mu + \delta}{2nn}\right)$

§. 72. Ponatur breuitatis gratia

$\frac{d\Phi}{dr} = \mathfrak{O} + \mathfrak{P} \cos 2\eta - \mathfrak{Q} \cos 4\eta - \mathfrak{R}\nu \cos \eta + \mathfrak{S}\nu \cos 3\eta$

erit $\frac{d\eta}{dr} = \alpha + \mathfrak{P} \cos 2\eta - \mathfrak{Q} \cos 4\eta - \mathfrak{R}\nu \cos \eta + \mathfrak{S}\nu \cos 3\eta$

sitque ad integrandum:

$\Phi = \mathfrak{o} r + \mathfrak{p} \sin 2\eta - \mathfrak{q} \sin 4\eta - \mathfrak{r}\nu \sin \eta + \mathfrak{s}\nu \sin 3\eta$

vnde per differentiationem elicitur:

$\frac{d\Phi}{dr} = \mathfrak{o} + 2\alpha\mathfrak{p} \cos 2\eta - 4\alpha\mathfrak{q} \cos 4\eta - \alpha\mathfrak{r}\nu \cos \eta + 3\alpha\mathfrak{s}\nu \cos 3\eta$
$\mathfrak{P}\mathfrak{p} - \mathfrak{Q}\mathfrak{p} \cos 2\eta + \mathfrak{P}\mathfrak{p} \cos 4\eta - \mathfrak{R}\mathfrak{p}\nu \cos \eta - \mathfrak{R}\mathfrak{p}\nu \cos 3\eta$
$+ 2\mathfrak{Q}\mathfrak{q} - 2\mathfrak{P}\mathfrak{q} \cos 2\eta \qquad + 2\mathfrak{R}\mathfrak{q}\nu \cos 3\eta$

hinc ergo fit:

$\mathfrak{o} = \mathfrak{O} - \mathfrak{P}\mathfrak{p} - 2\mathfrak{Q}\mathfrak{q} = \varkappa + 0,0001103 - \mathfrak{P}\mathfrak{p} - 2\mathfrak{Q}\mathfrak{q}$
$(2\alpha - \mathfrak{Q})\mathfrak{p} = \mathfrak{P} + 2\mathfrak{P}\mathfrak{q}$; $4\alpha\mathfrak{q} = \mathfrak{Q} + \mathfrak{P}\mathfrak{p}$;
$\alpha\mathfrak{r} = \mathfrak{R} - \mathfrak{R}\mathfrak{p}$; $3\alpha\mathfrak{s} = \mathfrak{S} + \mathfrak{R}(\mathfrak{p} - 2\mathfrak{q})$

Ergo

2 cos 4η
ν cos 3η
α, exi-
5ν cos 3η
5ν cos 3η
ν sin 3η
δν cos 3η
ν cos 3η
ν cos 3η
− 2Qq
;
Ergo

Ergo $\mathfrak{p} = 0,010191 \;\ldots\; l\mathfrak{p} = 8,008208$
$\mathfrak{q} = 0,000072 \;\ldots\; l\mathfrak{q} = 5,859381$
$\mathfrak{r} = 0,53453 \;\ldots\; l\mathfrak{r} = 9,727977$
$\mathfrak{s} = 0,00790 \;\ldots\; l\mathfrak{s} = 7,897466$
$\mathfrak{o} = \varkappa - 0,000080$

§. 73. Longitudo igitur lunae Φ quatenus pendet a sola distantia lunae a sole erit

$\Phi = (\varkappa - 0,000080)r + 0,010191 \sin 2\eta - 0,53453\, \nu \sin \eta$
$- 0,000072 \sin 4\eta + 0,00790\, \nu \sin 3\eta$

Simili modo cum distantia lunae a terra posita sit $= \frac{a(1-kk)u}{1-k\cos r}$, ob $u = 1 + \frac{v}{nn}$, quatenus valor ipsius u a sola phasi lunae pendet, erit

$u = 1 - 0,00716 \cos 2\eta + 0,00287\, \nu \cos \eta$
$+ 0,00007 \cos 4\eta - 0,00009\, \nu \cos 3\eta$

Verum tamen hic valor litterae α ac praecipue ipsius α non admodum certus videtur, cum a terminis neglectis licet minimis insignem mutationem perpeti queat. Hic enim pro α non solum $\varkappa - \frac{1}{n}$ sed $\varkappa - \frac{1}{n} + 0,0001103$ accipi debuisset; quare cum valorem ipsius α propius cognoscimus, hanc determinationem repeti conueniet.

CAPUT V.

INUESTIGATIO INAEQUALITATUM LUNAE AB EIUS EXCENTRICITATE SIMPLICI SOLUM PEDENTIUM.

§. 74.

Quemadmodum in praecedenti capite inaequalitas absoluta seu variatio duabus partibus constans est inuenta, quarum posterior a littera ν seu a parallaxi solis pendebat, ac maiorem curam requirebat; ita etiam inaequalitates, quas hoc capite scrutamur, partes continent ab eadem parallaxi solis pendentes; quarum indagatio quoque accuratiorem cognitionem quorundam elementorum exigit. Hancobrem et praecedentis capitis et huius partes, quae litteram ν inuoluunt deinceps, cum reliquas inaequalitates, a parallaxi solis non pendentes determinauerimus, seorsim inuestigabimus, atque titulo inaequalitatum parallacticarum complectemur.

§. 75. In hoc ergo capite ac sequentibus, donec ad parallaxin solis perueniamus, terminos formularum nostrarum per ν multiplicatos tantisper remouebimus; et quoniam hoc loco tantum propositum est in motus lunae inaequalitates a sola excentricitate orbitae lunaris ortas inquirere, eos terminos qui vel excentricitatem solis e vel inclinationem ϱ continent, praetermittemus. Cum autem in formulis nostris duplicis generis termini relinquantur, quorum alteri per k, alteri per kk sunt affecti, inaequalitates ab excentricitate lunae k pendentes in

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in duas partes distribui conueniet; quarum altera excentricitatem tantum simplicem k implicet, cui hoc caput destinatur, altera vero excentricitatis huius quadrato kk afficiatur, de quo in sequenti capite agemus.

§. 76. Verum tam in huius generis inaequalitates, quam in sequentes, omnes inaequalitates absolutae in praecedenti capite erutae praecipue ingrediuntur; ex quo eas quoque in calculum introduci oportebit. Retinendae ergo erunt in calculo litterae $\mathfrak{A}$, $\mathfrak{B}$ et A, B, quarum valores cum iam constent, calculus vehementer contrahetur: imprimis autem quia valores litterarum $\mathfrak{B}$ et B per se sunt admodum parui, quatenus illi in valores sequentium terminorum influunt, effectum pro nihilo habendum praestabunt. Inuestigationem ergo nostram ita incipiemus, vt pro $\int R\,dr$ et v valores fictos assumamus, et quoniam $\int R\,dr$ nullum terminum constantem, v vero neque constantem neque terminum huius formae $\alpha \cos r$ continere debet, ponamus:

$$\int R\,dr = \mathfrak{A}\cos 2\eta + \mathfrak{B}\cos 4\eta + \mathfrak{C}\,k\cos r$$
$$+ \mathfrak{D}\,k\cos(2\eta - r) + \mathfrak{F}\,k\cos(4\eta - r)$$
$$+ \mathfrak{E}\,k\cos(2\eta + r) + \mathfrak{G}\,k\cos(4\eta + r)$$

$$v = A\cos 2\eta + B\cos 4\eta$$
$$+ D\,k\cos(2\eta + r) + F\,k\cos(4\eta - r)$$
$$+ E\,k\cos(2\eta + r) + G\,k\cos(4\eta + r)$$

vbi quidem facile colligere licet, coefficientes $\mathfrak{F}$, $\mathfrak{G}$, F et G fore minimos.

§. 77.

§. 77. Ex his autem valoribus aſſumtis obtinebimus ex (§. 52.) ſequentes expreſſiones.

$$\frac{d\Phi}{dr} = \begin{cases} \varkappa + \frac{A(3\varkappa A + 2\mathfrak{A})}{2n^4} - \frac{(2\varkappa A + \mathfrak{A})}{nn}\cos 2\eta \\ (-\frac{(2\varkappa B + \mathfrak{B})}{nn} + \frac{A(3\varkappa A + 2\mathfrak{A})}{2n^4})\cos 4\eta \\ (-\frac{\mathfrak{C}}{nn} + \frac{3\varkappa AD}{n^4} + \frac{A\mathfrak{D} + \mathfrak{A}D}{n^4})k\cos r \\ -\frac{(2\varkappa D + \mathfrak{D})}{nn}k\cos(2\eta - r) - \frac{(2\varkappa E + \mathfrak{E})}{nn}k\cos(2\eta + r) \\ -\frac{(2\varkappa F + \mathfrak{F})}{nn}k\cos(4\eta - r) - \frac{(2\varkappa G + \mathfrak{G})}{nn}k\cos(4\eta + r) \\ + \frac{(3\varkappa AD + A\mathfrak{D} + \mathfrak{A}D)}{n^4}k\cos(4\eta - r) \end{cases}$$

Patebit enim ex valoribus qui inuenientur, litteras D et $\mathfrak{D}$ tantum prae reliquis fore notabiles, vnde terminos ex combinatione reliquarum litterarum oriundos tuto omittere licet. Pro valore autem ipſius $\frac{d\eta}{dr}$ etiam hi termini ex combinatione orti omitti poterunt. Poſito ergo $\varkappa + \frac{A(3\varkappa A + 2\mathfrak{A})}{2n^4} - \frac{1}{n}$ ſeu $\varkappa + 0{,}000103 - \frac{1}{n} = \alpha$ erit:

$$\frac{d\eta}{dr} = \alpha - \frac{(2\varkappa A + \mathfrak{A})}{nn}\cos 2\eta - \frac{2k}{n}\cos r$$
$$- \frac{(2\varkappa D + \mathfrak{D})}{nn}k\cos(2\eta - r) - \frac{(2\varkappa E + \mathfrak{E})}{nn}k\cos(2\eta + r)$$

Cum enim haec formula differentiationibus inſtituendis inſeruiat, reliqui termini poſt primum cum aliis angulis combinantur, ſicque tanto minores terminos producunt,

cu ıebi- atq ter 2 η viſ ſ4 η tiſſ coſ r bu +r) foı +r) R: +r) η−r) et 𝔇 com- licet. mbi- +2𝔄) vt ſu R +r) endis ıngu- ·odu- :unt,

cunt, qui ex calculo ſine errore expungi poterunt: atque ob hanc cauſam in expreſſione valoris $\frac{d\eta}{dr}$, ſtatim terminos prae reliquis admodum paruos praetermittere viſum eſt.

§. 78. Valorem autem ipſius R atque $\frac{ddv}{dr^2}$ accuratiſſime exhiberi oportet, propterea quod his expreſſionibus totus calculus praecipue innititur, dum valor $\frac{d\eta}{dr}$ formulam tantum ſubſidiariam ſuppeditat. Erit ergo

$$R = \tfrac{3}{2}\sin 2\eta + \frac{3A}{2nn}\sin 4\eta + 3k\sin(2\eta - r) + 3k\sin(2\eta + r)$$
$$+ \frac{3A}{nn}k\sin(4\eta - r) + \frac{3A}{nn}k\sin(4\eta + r)$$
$$+ \frac{3D}{2nn}k\sin r - \frac{3E}{2nn}k\sin r$$
$$+ \frac{3D}{2nn}k\sin(4\eta - r) + \frac{3E}{2nn}k\sin(4\eta + r)$$
$$- \frac{3A}{nn}k\sin r + \frac{3A}{nn}k\sin r$$

vbi quidem terminos ab k non pendentes omittere poſſumus, quia illorum iam habuimus rationem, ita vt ſit

$$R = \ldots + \frac{3(D-E)}{2nn}k\sin r + 3k\sin(2\eta - r) + 3k\sin(2\eta + r)$$
$$+ \frac{3D}{2nn}k\sin(4\eta - r) + \frac{3E}{2nn}k\sin(4\eta + r)$$
$$+ \frac{3A}{nn}k\sin(4\eta - r) + \frac{3A}{nn}k\sin(4\eta + r)$$

§. 79.

§. 79. Simili modo terminis a k non pendentibus omittendis habebitur:

$$\frac{ddv}{dr^2} = -\gamma k \cos r + 3k\cos(2\eta-r) + 3k\cos(2\eta+r) - 2\kappa\mathfrak{F}k\cos(4\eta-r) - 2\kappa\mathfrak{G}k\cos(4\eta+r)$$

$$\begin{array}{lllll}
-2\kappa\mathfrak{E} & -2\kappa\mathfrak{D} & -2\kappa\mathfrak{E} & +\frac{\mathfrak{A}\mathfrak{D}}{2nn} & +\frac{\mathfrak{A}\mathfrak{E}}{2nn} \\
+\frac{\mathfrak{A}\mathfrak{D}}{2nn} & +\frac{\mathfrak{A}\mathfrak{E}}{2nn} & +\frac{\mathfrak{A}\mathfrak{E}}{2nn} & & \\
+\frac{\mathfrak{A}\mathfrak{E}}{2nn} & -D & -E & -F & -G \\
+\frac{3AD}{nn} & +\frac{1}{2}A & +\frac{1}{2}A & +\frac{1}{2}B & +\frac{1}{2}B \\
+\frac{3AE}{nn} & +\frac{(2\gamma-\frac{3}{2}\delta)}{nn}D & +\frac{(2\gamma-\frac{3}{2}\delta)}{nn}E & +\frac{3AD}{nn} & +\frac{3AE}{nn} \\
-\frac{3AA}{2nn} & +\frac{(3+3\mu+\gamma)}{nn}A & +\frac{(3+3\mu+\gamma)}{nn}A & -\frac{3AA}{4nn} & -\frac{3AA}{4nn} \\
+\frac{3A\mathfrak{D}}{nn} & & & +\frac{3A\mathfrak{D}}{nn} & +\frac{3A\mathfrak{E}}{nn} \\
+\frac{3A\mathfrak{E}}{nn} & & & +\frac{3\mathfrak{A}D}{nn} & +\frac{3\mathfrak{A}E}{nn} \\
+\frac{3\mathfrak{A}D}{nn} & & & & \\
+\frac{3\mathfrak{A}E}{nn} & & & &
\end{array}$$

Hic scilicet plures terminos, qui nullius futuri essent momenti, omisimus, ne calculus nimium implicaretur: notandum autem est esse $\kappa = \sqrt{\left(1+\frac{3+4\mu+\delta}{2nn}\right)} = 1 + \frac{3+4\mu+\delta}{4nn}$ proxime; vnde $\mu = (\kappa-1)nn - \frac{3-\delta}{4}$ et $\frac{3+3\mu+\gamma}{4nn} = 3(\kappa-1) + \frac{3-3\delta+4\gamma}{4nn}$.

§. 80.

dentibus

$k\cos(4\eta+r)$

$\frac{\mathfrak{A}\mathfrak{E}}{2nn}$

G

$\frac{1}{2}B$

$\frac{3AE}{nn}$

$\frac{3AA}{4nn}$

$\frac{3A\mathfrak{E}}{nn}$

$\frac{3\mathfrak{A}E}{nn}$

i essent

aretur:

$\frac{3+4\mu+\delta}{4nn}$

$\mu+\gamma$

§. 80.

§. 80. Quaeramus nunc quoque ex forma pro $\int R\,dr$ ficta valorem ipsius R, atque exclusis terminis ab k non pendentibus reperiemus:

$$R = \begin{array}{l}
\left(-\frac{\mathfrak{A}(2\kappa E+\mathfrak{E})}{nn} + \frac{\mathfrak{A}(2\kappa D+\mathfrak{D})}{nn} - \mathfrak{E} - \frac{\mathfrak{D}(2\kappa A+\mathfrak{A})}{nn} + \frac{\mathfrak{E}(2\kappa A+\mathfrak{A})}{nn}\right) k \sin r \\
\left(+\frac{2\mathfrak{A}}{n} - (2\alpha-1)\mathfrak{D}\right) k \sin(2\eta-r) \\
\left(+\frac{2\mathfrak{A}}{n} - (2\alpha+1)\mathfrak{E}\right) k \sin(2\eta+r) \\
\left(+\frac{\mathfrak{A}(2\kappa D+\mathfrak{D})}{nn} + \frac{4\mathfrak{B}}{n} + \frac{\mathfrak{D}(2\kappa A+\mathfrak{A})}{nn} - (4\alpha-1)\mathfrak{F}\right) k \sin(4\eta-r) \\
\left(+\frac{\mathfrak{A}(2\kappa E+\mathfrak{E})}{nn} + \frac{4\mathfrak{B}}{n} + \frac{\mathfrak{E}(2\kappa A+\mathfrak{A})}{nn} - (4\alpha+1)\mathfrak{G}\right) k \sin(4\eta+r)
\end{array}$$

atque instituta comparatione inuenietur:

$$\mathfrak{E} = \frac{\mathfrak{A}(2\kappa D+\mathfrak{D}) - (\mathfrak{D}-\mathfrak{E})(2\kappa A+\mathfrak{A}) - \frac{3}{2}(D-E) - \mathfrak{A}(2\kappa E+\mathfrak{E})}{nn}$$

$$(2\alpha-1)\,\mathfrak{D} = \frac{2\mathfrak{A}}{n} - 3; \quad (2\alpha+1)\,\mathfrak{E} = \frac{2\mathfrak{A}}{n} - 3;$$

$$(4\alpha-1)\,\mathfrak{F} = \frac{4\mathfrak{B}}{n} + \frac{\mathfrak{A}(2\kappa D+\mathfrak{D}) + \mathfrak{D}(2\kappa A+\mathfrak{A}) - \frac{3}{2}(2A+D)}{nn}$$

$$(4\alpha+1)\,\mathfrak{G} = \frac{4\mathfrak{B}}{n} + \frac{\mathfrak{A}(\kappa E+\mathfrak{E}) - \frac{3}{2}(2A+E) + \mathfrak{E}(2\kappa A+\mathfrak{A})}{nn}$$

§. 81. Pro differentiali $\frac{dv}{dr}$ inueniendo, praeter terminos supra inventos habebimus:

$$\frac{dv}{dr} = -A' \sin 2\eta - B' \sin 4\eta$$

$$\left(+\frac{A(2\kappa D+\mathfrak{D})}{nn} - \frac{A(2\kappa E+\mathfrak{E})}{nn} - \frac{D(2\kappa A+\mathfrak{A})}{nn} + \frac{E(2\kappa A+\mathfrak{A})}{nn}\right) k \sin r$$

$$\left(+\frac{2A}{n} - (2\alpha-1)D\right) k \sin(2\eta-r) + \left(\frac{2A}{n} - (2\alpha+1)E\right) k \sin(2\eta+r)$$

 +

$$\left(+\frac{A(2\kappa D+\mathfrak{D})}{n}+\frac{4B}{n}+\frac{D(2\kappa A+\mathfrak{A})}{nn}-(4\alpha-1)F\right)k\sin(4\eta-r)$$

$$\left(+\frac{A(2\kappa E+\mathfrak{E})}{nn}+\frac{4B}{n}+\frac{E(2\kappa A+\mathfrak{A})}{nn}-(4\alpha+1)G\right)k\sin(4\eta+r)$$

Ponatur autem breuitatis gratia:

$$\frac{dv}{dr}=-A'\sin 2\eta-B'\sin 4\eta-C'k\sin r-D'k\sin(2\eta-r)$$
$$-E'k\sin(2\eta+r)-F'k\sin(4\eta-r)-G'k\sin(4\eta+r)$$

vt sit:

$$A'=2A\alpha+\frac{A(2\kappa B+\mathfrak{B})-2B(2\kappa A+\mathfrak{A})}{nn};\ B'=4\alpha B-\frac{A(2\kappa A+\mathfrak{A})}{nn}$$

$$C'=\frac{-A(2\kappa D+\mathfrak{D})+A(2\kappa E+\mathfrak{E})+(D-E)(2\kappa A+\mathfrak{A})}{nn}$$

sive $C'=-\frac{A(\mathfrak{D}-\mathfrak{E})+\mathfrak{A}(D-E)}{nn}$

$$D'=(2\alpha-1)D-\frac{2A}{n};\quad E'=(2\alpha+1)E-\frac{2A}{n}$$

$$F'=(4\alpha-1)F-\frac{4B}{n}-\frac{A(2\kappa D+\mathfrak{D})-D(2\kappa A+\mathfrak{A})}{nn}$$

$$G'=(4\alpha+1)G-\frac{4B}{n}-\frac{A(2\kappa E+\mathfrak{E})-E(2\kappa A+\mathfrak{A})}{nn}$$

§. 82. Hinc denuo differentiando obtinebitur terminis tantum per k multiplicatis scribendis:

$$\frac{ddv}{dr^2}=\left(-C'+\frac{A'(2\kappa D+\mathfrak{D})}{nn}+\frac{A'(2\kappa E+\mathfrak{E})}{nn}+\frac{D'(2\kappa A+\mathfrak{A})}{nn}+\frac{E'(2\kappa A+\mathfrak{A})}{nn}\right)k\cos r$$

$$\left(+\frac{2A'}{n}-(2\alpha-1)D'\right)k\cos(2\eta-r)$$

$$\left(+\frac{2A'}{n}-(2\alpha+1)E'\right)k\cos(2\eta+r)$$

$$\left(+\frac{4B'}{n}+\frac{A'(2\kappa D+\mathfrak{D})}{nn}+\frac{D'(2\kappa A+\mathfrak{A})}{nn}-(4\alpha-1)F'\right)k\cos(4\eta-r)$$

$$\left(+\frac{4B'}{n}+\frac{A'(2\kappa E+\mathfrak{E})}{nn}+\frac{E'(2\kappa A+\mathfrak{A})}{nn}-(4\alpha+1)G'\right)k\cos(4\eta+r)$$

vnde comparatione instituta orietur:

$$\gamma=-2\kappa\mathfrak{E}-\frac{\frac{1}{2}AA+3A(D+E)+2A(\mathfrak{D}+2\mathfrak{E})+2\mathfrak{A}(2D+E)+\frac{1}{2}\mathfrak{A}(\mathfrak{D}+\mathfrak{E})}{nn}$$
$$-\frac{A'(2\kappa D+\mathfrak{D})-A'(2\kappa E+\mathfrak{E})-(D'+E')(2\kappa A+\mathfrak{A})}{nn}$$

$$\left.\begin{array}{l}(2\alpha-1)^2D-\frac{2(2\alpha-1)}{n}A-\frac{2A'}{n}+3-2\kappa\mathfrak{D}-D\\+\frac{1}{2}A+\frac{\frac{1}{2}\mathfrak{A}\mathfrak{E}+(2\gamma-\frac{1}{2}\delta)D+(3+3\mu\mp\gamma)A}{nn}\end{array}\right\}=0$$

$$\left.\begin{array}{l}(2\alpha+1)^2E-\frac{2(2\alpha+1)}{n}A-\frac{2A'}{n}+3-2\kappa\mathfrak{E}-E\\+\frac{1}{2}A+\frac{\frac{1}{2}\mathfrak{A}\mathfrak{E}+(2\gamma-\frac{1}{2}\delta)E+(3+3\mu+\gamma)A}{nn}\end{array}\right\}=0$$

$$\left.\begin{array}{l}(4\alpha-1)^2F-\frac{4(4\alpha-1)}{n}B-\frac{(4\alpha-1)A(2\kappa D+\mathfrak{D})-(4\alpha-1)D(2\kappa A+\mathfrak{A})}{nn}\\-\frac{4B'}{n}-\frac{A'(2\kappa D+\mathfrak{D})-D'(2\kappa A+\mathfrak{A})}{nn}-2\kappa\mathfrak{F}-F\\+\frac{1}{2}B+\frac{\frac{1}{2}\mathfrak{A}\mathfrak{D}+3AD-\frac{1}{2}AA+3A\mathfrak{D}+3\mathfrak{A}D}{nn}\end{array}\right\}=0$$

$$\left.\begin{array}{l}(4\alpha+1)^2G-\frac{4(4\alpha+1)}{n}B-\frac{(4\alpha+1)A(2\kappa E+\mathfrak{E})-(4\alpha+1)E(2\kappa A+\mathfrak{A})}{nn}\\-\frac{4B'}{n}-\frac{A'(2\kappa E+\mathfrak{E})-E'(2\kappa A+\mathfrak{A})}{nn}-2\kappa\mathfrak{G}-G\\+\frac{1}{2}B+\frac{\frac{1}{2}\mathfrak{A}\mathfrak{E}+3AE-\frac{1}{2}AA+3A\mathfrak{E}+3\mathfrak{A}E}{nn}\end{array}\right\}=0$$

§. 83. Incipiamus a coefficientibus $\mathfrak{D}$, $\mathfrak{E}$, et D,E; et quia $\mathfrak{E}$ est quantitas admodum exigua, erit:

$$(2\alpha-1)\mathfrak{D} = -3 + \frac{2\mathfrak{A}}{n}; \quad (2\alpha+1)\mathfrak{E} = -3 + \frac{2\mathfrak{A}}{n}$$

$$\left((2\alpha-1)^2 - 1 + \frac{2\gamma-\frac{3}{2}\delta}{nn}\right) D = -3 - \tfrac{3}{2} A + 2\varkappa\mathfrak{D} + 2\frac{(2\alpha-1)}{n} A + \frac{2A'}{n} - \left(3\varkappa - 3 + \frac{3-3\delta+4\gamma}{4nn}\right) A$$

$$\left((2\alpha+1)^2 - 1 + \frac{2\gamma-\frac{3}{2}\delta}{nn}\right) E = -3 - \tfrac{3}{2} A + 2\varkappa\mathfrak{E} + 2\frac{(2\alpha+1)}{n} A + \frac{2A'}{n} - \left(3' \;\; \frac{3-3\delta+4\gamma}{4nn}\right) A$$

vnde reperitur:

$\mathfrak{D} = -3,6035 \quad . \; . \; . \quad l{-}\mathfrak{D} = 0,556724$

$\mathfrak{E} = -1,0890 \quad . \; . \; . \quad l{-}\mathfrak{E} = 0,037028$

ac porro

$$\left.\begin{aligned}\left(-0,24973 + \frac{2\gamma-\frac{3}{2}\delta}{nn}\right) D &= -1,40048 - 7,4315 \\ \left(+7,21497 + \frac{2\gamma-\frac{3}{2}\delta}{nn}\right) E &= -1,40048 - 2,7403\end{aligned}\right\} + \frac{2\gamma-\frac{3}{2}\delta}{nn}\, 0,629$$

§. 84. Quoniam autem valorem ipsius $\frac{2\gamma-\frac{3}{2}\delta}{nn}$ nondum nouimus, hunc terminum, cum certo sit valde paruus, reiiciamus. Postmodum vero cum istum terminum cognouerimus, facile erit correctionem inde oriundam, si operae pretium videbitur, inuenire.

$D = +35,3662 \quad . \; . \; . \quad l\,D = 1,548588$

$E = -0,5739 \quad . \; . \; . \quad l{-}E = 9,758848$

Porro autem litterae $\mathfrak{F}$ et $\mathfrak{G}$ ita elicientur, vt sit:

$(4\alpha-1)\mathfrak{F} = 0,000529 - 0,30976 + 0,06852 - 0,28042$

$(4\alpha+1)\mathfrak{G} = 0,000529 + 0,01028 + 0,02071 + 0,02638$

D,E; − 2𝔄/n 2κ𝔇 γ)A −2κ𝔈)A ,629 non- par- inum dam, 8042 2638 𝔉 =

$\mathfrak{F} = -0,1907 \quad . \; . \; . \quad l{-}\mathfrak{F} = 9,280416$

$\mathfrak{G} = +0,0122 \quad . \; . \; . \quad l\,\mathfrak{G} = 8,087607$

Praeterea autem colligimus fore

$\mathfrak{E} = -0,67465 \quad . \; . \; . \quad l{-}\mathfrak{E} = 9,829072$

vnde erit proxime $\frac{\mathfrak{A}\mathfrak{E}}{2nn} = 0,00154$, ex quo accuratius concluditur fore

$D = +35,3724 \quad . \quad l\,D = 1,548664$

$E = -0,5741 \quad . \quad l{-}E = 9,758988$

§. 85. Reliquae aequationes nobis praebebunt

$$6,4655\,F + 3,67820 = 0$$

$$21,3946\,G - 0,29574 = 0$$

vnde obtinebitur

$F = -0,56890 \quad ; \quad l{-}F = 9,755033$

$G = +0,01382 \quad . \quad l\,G = 8,140620$

ac denique $\gamma = 1,40673$.

Supra autem iam inuenimus $\frac{3}{2}\delta - \gamma = 0,01742$, vnde ambas istas quantitates γ et δ, quas initio ad veros valores constantium litterarum m et n determinandos assumsimus, nunc cognitas habemus, erit enim:

$$\delta = 2,84830, \quad \text{et} \quad 2\gamma - \tfrac{3}{2}\delta = -1,45899$$

ac propterea particulae illius $\frac{2\gamma-\frac{3}{2}\delta}{nn}$ hactenus neglectae valor erit $\frac{2\gamma-\frac{3}{2}\delta}{nn} = -0,00832$, cuius ope iam litterae D et E accuratius definiri poterunt.

§. 86.

§. 86. Hinc autem potiſſimum valor ipſius D mutationem patitur, fiet enim re vera

$$-0,25805\, D = -8,83698 \quad \text{ſeu}$$
$$D = 34,24520 \quad \ldots \quad l\, D = 1,534600$$
$$7,20665\, E = -4,14578 \quad \text{ſeu}$$
$$E = -0,57527 \quad \ldots \quad l{-}E = 9,759874$$

et quoniam D parte ſua triceſima diminuitur, in eadem fere ratione diminuentur valores litterarum $\mathfrak{C}$ et γ, ita vt exactius ſit:

$$\mathfrak{C} = -0,65217 \quad \ldots \quad l{-}\mathfrak{C} = 9,814361$$
$$\gamma = +1,35984 \quad \ldots \quad l\,\gamma = 0,133490$$
$$\delta = +2,75336 \quad \ldots \quad l\,\delta = 0,439863$$
$$\text{et } \frac{2\gamma - \frac{1}{2}\delta}{nn} = -0,00804$$

Deinceps autem operae erit pretium in hos valores adhuc diligentius inquirere.

§. 87. Cum igitur finxerimus ſequentes valores:

$$\int R\, dr = \mathfrak{A}\cos 2\eta + \mathfrak{B}\cos 4\eta + \mathfrak{C}\, k\cos r$$
$$+ \mathfrak{D}\, k\cos(2\eta - r) + \mathfrak{F}\, k\cos(4\eta - r)$$
$$+ \mathfrak{E}\, k\cos(2\eta + r) + \mathfrak{G}\, k\cos(4\eta + r)$$
$$v = A\cos 2\eta + B\cos 4\eta$$
$$+ D\, k\cos(2\eta - r) + F\, k\cos(4\eta - r)$$
$$+ E\, k\cos(2\eta + r) + G\, k\cos(4\eta + r)$$

horum coefficientium valores ſunt.

$\mathfrak{A} = -0,80378$	$A = -1,25826$
$\mathfrak{B} = +0,00697$	$B = -0,01279$
$\mathfrak{C} = -0,65217$	. .
$\mathfrak{D} = -3,60350$	$D = +34,24520$
$\mathfrak{E} = -1,08900$	$E = -0,57527$
$\mathfrak{F} = -0,19070$	$F = -0,56890$
$\mathfrak{G} = +0,01220$	$G = +0,01382$

vnde

D mu- v. u. eadem $\frac{d}{d}$ t γ, ita a Φ ores ad- e lores: $\frac{d}{d}$ $\eta - r)$ $\eta + r)$ $\eta - r)$ $\eta + r)$ F (vnde

vnde pro diſtantia lunae a terra $x = \frac{(1-kk)au}{1-k\cos r}$ fit

$$u = 1 - 0,007161\cos 2\eta + 0,000073\cos 4\eta$$
$$+ 0,194888\, k\cos(2\eta - r) - 0,003274\, k\cos(2\eta + r)$$
$$- 0,003238\, k\cos(4\eta - r) + 0,000078\, k\cos(4\eta + r)$$

§. 88. His valoribus in §. 77. ſubſtitutis obtinebimus:

$$\frac{d\Phi}{dr} = \begin{array}{l} x \quad + 0,019015\cos 2\eta - 0,001255\, k\cos r \\ + 0,0001103 - 0,000076\cos 4\eta \\ - 0,38410\, k\cos(2\eta - r) + 0,01278\, k\cos(2\eta + r) \\ + 0,002647\, k\cos(4\eta - r) - 0,000229\, k\cos(4\eta + r) \end{array}$$

ad cuius integrale inueniendum ponamus:

$$\Phi = O\, r + \mathfrak{A}'\sin 2\eta + \mathfrak{B}'\sin 4\eta + \mathfrak{C}' k\sin r$$
$$+ \mathfrak{D}' k\cos(2\eta - r) + \mathfrak{E}' k\cos(2\eta + r)$$
$$+ \mathfrak{F}' k\cos(4\eta - r) + \mathfrak{G}' k\cos(4\eta + r)$$

eritque differentiando et terminis iam cognitis omittendis.

$$\frac{d\Phi}{dr} = \left(\mathfrak{C}' - \frac{\mathfrak{A}'(2xD+\mathfrak{D})}{nn} - \frac{\mathfrak{A}'(2xE+\mathfrak{E})}{nn} - \frac{\mathfrak{D}'(2xA+\mathfrak{A})}{nn} - \frac{\mathfrak{E}'(2xA+\mathfrak{A})}{nn}\right) k\cos r$$
$$\left(-\frac{2\mathfrak{A}'}{n} + (2\alpha - 1)\,\mathfrak{D}'\right) k\cos(2\eta - r)$$
$$\left(-\frac{2\mathfrak{A}'}{n} + (2\alpha + 1)\,\mathfrak{E}'\right) k\cos(2\eta + r)$$
$$\left(-\frac{4\mathfrak{B}'}{n} - \frac{\mathfrak{A}'(2xD+\mathfrak{D})}{nn} - \frac{\mathfrak{D}'(2xA+\mathfrak{A})}{nn} + (4\alpha - 1)\,\mathfrak{F}'\right) k\cos(4\eta - r)$$
$$\left(-\frac{4\mathfrak{B}'}{n} - \frac{\mathfrak{A}'(2xE+\mathfrak{E})}{nn} - \frac{\mathfrak{E}'(2xA+\mathfrak{A})}{nn} + (4\alpha + 1)\,\mathfrak{G}'\right) k\cos(4\eta + r)$$

Pro terminis autem iam inuentis eſt

$$O = x - 0,000080; \quad \mathfrak{A}' = 0,010191; \quad \mathfrak{B}' = -0,000072$$
$$l\mathfrak{A}' = 8,008208; \quad l\mathfrak{B}' = -5,859381$$

§. 89. Comparatione iam instituta fiet:

$$(2\alpha-1)\mathfrak{D}'=-0{,}38410+\frac{2\mathfrak{A}'}{n};\quad (2\alpha+1)\mathfrak{E}'=+0{,}01278+\frac{2\mathfrak{A}'}{n}$$

$$(4\alpha-1)\mathfrak{F}'=+0{,}002647+\frac{4\mathfrak{B}'}{n}+\frac{\mathfrak{A}'(2\kappa D+\mathfrak{D})+\mathfrak{D}'(2\kappa A+\mathfrak{A})}{nn}$$

$$(4\alpha+1)\mathfrak{G}'=-0{,}000229+\frac{4\mathfrak{B}'}{n}+\frac{\mathfrak{A}'(2\kappa E+\mathfrak{E})+\mathfrak{E}'(2\kappa A+\mathfrak{A})}{nn}$$

$$\mathfrak{C}'=-0{,}001255+\frac{\mathfrak{A}'(2\kappa D+\mathfrak{D})+\mathfrak{D}'(2\kappa A+\mathfrak{A})}{nn}+\frac{\mathfrak{A}'(2\kappa E+\mathfrak{E})+\mathfrak{E}'(2\kappa A+\mathfrak{A})}{nn}$$

vnde colligitur fore

$\mathfrak{C}'=+0{,}01083$
$\mathfrak{D}'=-0{,}44167 \quad \ldots \quad l-\mathfrak{D}'=9{,}645092$
$\mathfrak{E}'=+0{,}00499 \quad \ldots \quad l\,\mathfrak{E}'=7{,}698640$
$\mathfrak{F}'=+0{,}00546 \quad \ldots \quad l\,\mathfrak{F}'=7{,}737733$
$\mathfrak{G}'=-0{,}00010 \quad \ldots \quad l-\mathfrak{G}'=6{,}002537$

ita vt sit

$$\Phi=(\kappa-0{,}000080r)+0{,}010191\sin 2\eta+0{,}01083\,k\sin r-0{,}000072\sin 4\eta$$
$$-0{,}44167\,k\sin(2\eta-r)+0{,}00499\,k\sin(2\eta+r)$$
$$+0{,}00546\,k\sin(4\eta-r)-0{,}00010\,k\sin(4\eta+r)$$

vnde ex comparatione motus medii ad modum anomaliae erit $\kappa=1{,}008607$, et $\alpha=0{,}933279$, qui valores iam propius ad veritatem accedunt, quam hactenus vsurpati.

CAPUT VI.

INUESTIGATIO INAEQUALITATUM LUNAE A QUADRATO EXCENTRICITATIS IPSIUS ORTARUM.

§. 90.

Peruenimus nunc ad alteram partem inaequalitatum in motu Lunae, quae ab eius excentricitate k pendent, eiusque quadratum inuoluunt, ita vt hic nonnisi eos terminos simus contemplaturi, qui per quadratum excentricitatis lunae kk sunt multiplicati. Hic autem tam in valorem ipsius $\int R dr$, quam ipsius v termini formae $kk\cos 2\eta$ et $kk\cos 4\eta$ ingredientur, qui postquam fuerint inuenti, terminis huius generis iam ante inuentis adiici debent; praeterea vero vtrinque etiam termini formae $kk\cos 2r$ accedent. Hinc ponamus:

$$\int R dr=\mathfrak{A}\cos 2\eta+\mathfrak{a}\,kk\cos 2\eta+\mathfrak{B}\cos 4\eta+\mathfrak{b}\,kk\cos 4\eta$$
$$+\mathfrak{C}\,k\cos r+\mathfrak{D}\,k\cos(2\eta-r)+\mathfrak{E}\,k\cos(2\eta+r)$$
$$+\mathfrak{F}\,k\cos(4\eta-r)+\mathfrak{G}\,k\cos(4\eta+r)$$
$$+\mathfrak{H}\,kk\cos 2r+\mathfrak{I}\,kk\cos(2\eta-2r)+\mathfrak{K}\,kk\cos(2\eta+2r)$$
$$+\mathfrak{L}\,kk\cos(4\eta-2r)+\mathfrak{M}\,kk\cos(4\eta+2r)$$

$$v=A\cos 2\eta+a\,kk\cos 2\eta+B\cos 4\eta+b\,kk\cos 4\eta$$
$$+D\,k\cos(2\eta-r)+E\,k\cos(2\eta+r)$$
$$+F\,k\cos(4\eta-r)+G\,k\cos(4\eta+r)$$
$$+H\,kk\cos 2r+J\,kk\cos(2\eta-2r)+K\,kk\cos(2\eta+2r)$$
$$+L\,kk\cos(4\eta-2r)+M\,kk\cos(4\eta+2r)$$

§. 91. Nunc ad terminos, quibus ante valorem ipsius $\frac{d\Phi}{dr}$ exprimi inuenimus, insuper sequentes per kk multiplicati accedent:

$$\frac{d\Phi}{dr} = \ldots + \frac{D(3\varkappa D+2\mathfrak{D})}{2n^4}k^2 + \left(-\frac{(2\varkappa a+\mathfrak{a})}{nn}+\frac{\mathfrak{C}D}{n^4}\right)k^2\cos 2\eta$$

$$\left(-\frac{(2\varkappa b+\mathfrak{b})}{nn}+\frac{D(3\varkappa E+2\mathfrak{E})}{2n^4}+\frac{E(3\varkappa D\pm 2\mathfrak{D})}{2n^4}\right)k^2\cos 4\eta$$

$$\left[-\frac{(2\varkappa H+\mathfrak{H})}{nn}+\frac{D(3\varkappa E+2\mathfrak{E})}{2n^4}+\frac{E(3\varkappa D+2\mathfrak{D})}{2n^4}+\frac{A(3\varkappa J+2\mathfrak{J})}{2n^4}+\frac{J(3\varkappa A+2\mathfrak{A})}{2n^4}\right]k^2\cos 2r$$

$$\left(-\frac{(2\varkappa J+\mathfrak{J})}{nn}+\frac{A(3\varkappa H+2\mathfrak{H})}{2n^4}+\frac{H(3\varkappa A+2\mathfrak{A})}{2n^4}+\frac{\mathfrak{C}D}{n^4}\right)k^2\cos(2\eta-2r)$$

$$\left(-\frac{(2\varkappa K+\mathfrak{K})}{nn}+\frac{A(3\varkappa H\pm 2\mathfrak{H})}{2n^4}+\frac{H(3\varkappa A+2\mathfrak{A})}{2n^2}\right)k\cos(4\eta\pm 2r)$$

$$\left(-\frac{(2\varkappa L\pm\mathfrak{L})}{nn}\pm\frac{A(3\varkappa J\pm 2\mathfrak{J})}{2n^4}\pm\frac{J(3\varkappa A\pm 2\mathfrak{A})}{2n^4}\pm\frac{D(3\varkappa D\pm 2\mathfrak{D})}{2n^4}\right)k^2\cos(4\eta-2r)$$

$$-\frac{(2\varkappa M+\mathfrak{M})}{nn}k^2\cos(4\eta+2r)$$

vbi quidem terminos, quos minimos fore facile est praeuidere, omisimus.

§. 92. Terminus autem constans $\frac{D(3\varkappa D+2\mathfrak{D})kk}{2n^4}$ reperitur $= 0,000175$, vnde posito $\varkappa + 0,000285 - \frac{1}{n} = \alpha$, quoniam valorem ipsius $\frac{d\eta}{dr}$ non opus est tam exacte nosse, sumamus:

$\frac{d\eta}{dr}$

valorem
per kk
$)k^2\cos 2\eta$
$k^2\cos 4\eta$
$k^2\cos 2r$
$\cos(2\eta-2r)$
$(4\eta\pm 2r)$
$(4\eta-2r)$
cile est
$2\mathfrak{D})\,kk$
000285
us est
$\frac{d\eta}{dr}$

$$\frac{d\eta}{dr} = \alpha - \frac{(2\varkappa A+\mathfrak{A})}{nn}\cos 2\eta - \left(\frac{2k}{n}+\frac{\mathfrak{C}k}{nn}\right)\cos r$$

$$-\frac{(2\varkappa D+\mathfrak{D})}{nn}k\cos(2\eta-r) - \left(\frac{3kk}{2n}\mp\frac{(2\varkappa H\mp\mathfrak{H})}{nn}\right)\cos 2r$$

$$-\frac{(2\varkappa J+\mathfrak{J})}{nn}k^2\cos(2\eta-2r)$$

Deinde vero praeter terminos iam tractatos habebitur:

$$R = \ldots 3k^2\sin 2\eta + \frac{3D}{nn}k^2\sin 4\eta + \frac{3(2D+J)}{2nn}k^2\sin 2r$$

$$+\left(\tfrac{15}{4}k^2+\frac{3(H-L)}{2nn}\right)\sin(2\eta-2r)+\left(\tfrac{15}{4}k^2+\frac{3H}{2nn}\right)\sin(2\eta+2r)$$

$$+\frac{3(2D+J)}{2nn}k^2\sin(4\eta-2r)$$

atque simili modo:

$$\frac{ddv}{dr^2} = \tfrac{1}{2}\delta-\gamma+\frac{\mathfrak{A}\mathfrak{a}+\frac{1}{2}\mathfrak{D}\mathfrak{D}+3D\mathfrak{D}+3DD+3Aa+3\mathfrak{A}a+3A\mathfrak{a}}{nn}kk$$

$$+\left(3+\tfrac{3}{2}(A+D+E)-a-2\varkappa\mathfrak{a}\right)kk\cos 2\eta$$

$$+\left(\tfrac{3}{2}(B+F+G)-b-2\varkappa\mathfrak{b}\right)kk\cos 4\eta$$

$$+\left(\tfrac{3}{2}-2\varkappa\mathfrak{H}-H+\frac{\mathfrak{A}\mathfrak{J}+3\mathfrak{A}J+3A\mathfrak{J}+3AJ}{nn}\right)kk\cos 2r$$

$$+\left(\tfrac{15}{4}-2\varkappa\mathfrak{J}-J+\tfrac{3}{2}A+\tfrac{3}{2}D\right)kk\cos(2\eta-2r)$$

$$+\left(\tfrac{15}{4}-2\varkappa\mathfrak{K}-K+\tfrac{3}{2}A+\tfrac{3}{2}E\right)k^2\cos(2\eta+2r)$$

$$+\left(-2\varkappa\mathfrak{L}-L+\tfrac{3}{2}B+\tfrac{3}{2}F\right)kk\cos(4\eta-2r)$$

$$+\left(-2\varkappa\mathfrak{M}-M+\tfrac{3}{2}B+\tfrac{3}{2}G\right)k^2\cos(4\eta+2r)$$

$$+\frac{\mathfrak{A}\mathfrak{J}+3\mathfrak{A}\mathfrak{J}+3AJ+\frac{1}{2}\mathfrak{D}\mathfrak{D}+3D\mathfrak{D}+3DD}{nn}kk\cos(4\eta-2r)$$

§. 93. Eliciamus nunc quoque valorem ipsius R per differentiationem ex formula $\int R\,dr$, ac terminis apte dispositis habebimus

$R =$

$R =$

$kk \sin 2\eta$	$kk \sin 4\eta$	$kk \sin 2r$	$k^2 \sin(2\eta - 2r)$
$-2\alpha\mathfrak{a}$	$+\frac{\mathfrak{a}(2\kappa A+\mathfrak{A})}{nn}$	$+\frac{\mathfrak{A}(2\kappa J+\mathfrak{J})}{nn}$	$+\frac{3\mathfrak{A}}{2n}$
$+\frac{2\mathfrak{b}(2\kappa A+\mathfrak{A})}{nn}$	$-4\alpha\mathfrak{b}$	$-2\mathfrak{H}$	$+\frac{\mathfrak{A}(2\kappa H+\mathfrak{H})}{nn}$
$+\frac{\mathfrak{D}(2n+\mathfrak{E})}{nn}$	$+\frac{\mathfrak{E}(2\kappa D+\mathfrak{D})}{nn}$	$+\frac{\mathfrak{E}(2\kappa D+\mathfrak{D})}{nn}$	$+\frac{\mathfrak{D}(2n+\mathfrak{E})}{nn}$
$+\frac{\mathfrak{E}(2n+\mathfrak{E})}{nn}$	$+\frac{2\mathfrak{F}(2n+\mathfrak{E})}{nn}$	$\frac{\mathfrak{J}(2\kappa A+\mathfrak{A})}{nn}$	$-2(\alpha-1)\mathfrak{J}$
$+\frac{2\mathfrak{F}(2\kappa D+\mathfrak{D})}{nn}$	$+\frac{2\mathfrak{G}(2n+\mathfrak{E})}{nn}$	$+\frac{\mathfrak{K}(2\kappa A+\mathfrak{A})}{nn}$	$+\frac{2\mathfrak{L}(2\kappa A+\mathfrak{A})}{nn}$

$kk \sin(2\eta+2r)$	$kk \sin(4\eta-2r)$	$kk \sin(4\eta+2r)$
$+\frac{3\mathfrak{A}}{2n}$	$+\frac{\mathfrak{A}(2\kappa J+\mathfrak{J})}{nn}$	
$+\frac{\mathfrak{A}(2\kappa H+\mathfrak{H})}{nn}$	$+\frac{3\mathfrak{B}}{n}$	$+\frac{3\mathfrak{B}}{n}$
$+\frac{2\mathfrak{B}(2\kappa J+\mathfrak{J})}{nn}$	$+\frac{2\mathfrak{B}(2\kappa H+\mathfrak{H})}{nn}$	$+\frac{2\mathfrak{B}(2\kappa H+\mathfrak{H})}{nn}$
$+\frac{\mathfrak{E}(2n+\mathfrak{E})}{nn}$	$+\frac{\mathfrak{D}(2\kappa D+\mathfrak{D})}{nn}$	$+\frac{\mathfrak{G}(2n+\mathfrak{E})}{nn}$
$+\frac{2\mathfrak{G}(2\kappa D+\mathfrak{D})}{nn}$	$+\frac{2\mathfrak{F}(2n+\mathfrak{E})}{nn}$	$+\frac{\mathfrak{K}(2\kappa A+\mathfrak{A})}{nn}$
$-2(\alpha+1)\mathfrak{K}$	$+\frac{\mathfrak{J}(2\kappa A+\mathfrak{A})}{nn}$	$-2(2\alpha+1)\mathfrak{M}$
$+\frac{2\mathfrak{M}(2\kappa A+\mathfrak{A})}{nn}$	$-2(2\alpha-1)\mathfrak{L}$	

vnde

(2η − 2r)
H+ℌ) nn
n+𝔈) nn
1)𝔍
κA+𝔄) nn
vnde

vnde oriuntur sequentes determinationes:

$$3 = -2\alpha\mathfrak{a} + \frac{2\mathfrak{b}(2\kappa A+\mathfrak{A})+(\mathfrak{D}+\mathfrak{E})(2n+\mathfrak{E})+2\mathfrak{F}(2\kappa D+\mathfrak{D})}{nn}$$

$$\frac{3D}{nn} = -4\alpha\mathfrak{b} + \frac{\mathfrak{a}(2\kappa A+\mathfrak{A})+\mathfrak{E}(2\kappa D+\mathfrak{D})+2(\mathfrak{F}+\mathfrak{G})(2n+\mathfrak{E})}{nn}$$

$$\frac{3(2D+J)}{2nn} = -2\mathfrak{H} + \frac{\mathfrak{A}(2\kappa J+\mathfrak{J})+\mathfrak{E}(2\kappa D+\mathfrak{D})-(\mathfrak{J}-\mathfrak{K})(2\kappa A+\mathfrak{A})}{nn}$$

$$\tfrac{1}{4}\gamma + \frac{3(H-L)}{2nn} = -2(\alpha-1)\mathfrak{J} + \frac{3\mathfrak{A}}{nn} + \frac{\mathfrak{A}(2\kappa H+\mathfrak{H})}{nn} + \frac{\mathfrak{D}(2n+\mathfrak{E}) + 2\mathfrak{L}(2\kappa A+\mathfrak{A})}{nn}$$

$$\tfrac{1}{4}\gamma + \frac{3H}{2nn} = 2(\alpha+1)\mathfrak{K} + \frac{3\mathfrak{A}}{2n} + \frac{\mathfrak{A}(2\kappa H+\mathfrak{H})}{nn} + \frac{2\mathfrak{B}(2\kappa J+\mathfrak{J})+2\mathfrak{G}(2\kappa D+\mathfrak{D})+\mathfrak{E}(2n+\mathfrak{E})+2\mathfrak{M}(2\kappa A+\mathfrak{A})}{nn}$$

$$+\frac{3(2D+J)}{2nn} = -2(2\alpha-1)\mathfrak{L} + \frac{3\mathfrak{B}}{n} + \frac{\mathfrak{A}(2\kappa J+\mathfrak{J})}{nn} + \frac{2\mathfrak{B}(2\kappa H+\mathfrak{H})+\mathfrak{D}(2\kappa D+\mathfrak{D})+2\mathfrak{F}(2n+\mathfrak{E})+\mathfrak{J}(2\kappa A+\mathfrak{A})}{nn}$$

$$0 = -2(2\alpha+1)\mathfrak{M} + \frac{3\mathfrak{B}}{n} + \frac{2\mathfrak{B}(2\kappa H+\mathfrak{H})+2\mathfrak{G}(2n+\mathfrak{E})+\mathfrak{K}(2\kappa A+\mathfrak{A})}{nn}$$

§. 94. Deinde simili modo si ponatur:

$$\frac{dv}{dr} = -A' \sin 2\eta - a'k^2 \sin 2\eta - B' \sin 4\eta - b'k^2 \sin 4\eta$$
$$- C'k \sin r - D'k \sin(2\eta-r) - F'k \sin(4\eta-r)$$
$$- E'k \sin(2\eta+r) - G'k \sin(2\eta+r)$$
$$- H'k^2 \sin 2r - J'k^2 \sin(2\eta-2r) - L'k^2 \sin(4\eta-2r)$$
$$- K'k^2 \sin(2\eta+2r) - M'k^2 \sin(4\eta+2r)$$

erit

erit praeter valores §. 81. datos:

$$a' = 2\kappa a - \frac{2b(2\kappa A+\mathfrak{A}) - (D+E)(2n+\mathfrak{E}) - 2F(2\kappa D+\mathfrak{D})}{nn}$$

$$b' = 4ab - \frac{a(2\kappa A+\mathfrak{A}) - E(2\kappa D+\mathfrak{D})}{nn} - \frac{2(F+G)(2n+\mathfrak{E}) - D(2\kappa E+\mathfrak{E})}{nn}$$

$$H' = 2H - \frac{A(2\kappa J+\mathfrak{J}) - E(2\kappa D+\mathfrak{D})}{nn} + \frac{(J-K)(2\kappa A+\mathfrak{A}) + D(2\kappa E+\mathfrak{E})}{nn}$$

$$J' = 2(\alpha-1)J - \frac{3A}{2n} - \frac{A(2\kappa H+\mathfrak{H}) - D(2n+\mathfrak{E}) - 2L(2\kappa A+\mathfrak{A})}{nn}$$

$$K' = 2(\alpha+1)K - \frac{3A}{2n} - \frac{A(2\kappa H+\mathfrak{H}) - 2B(2\kappa J+\mathfrak{J}) - 2G(2\kappa D+\mathfrak{D})}{nn} - \frac{E(2n+\mathfrak{E}) - 2M(2\kappa A+\mathfrak{A})}{nn}$$

$$L' = 2(2\alpha-1)L - \frac{3B}{n} - \frac{A(2\kappa J+\mathfrak{J}) - 2B(2\kappa H+\mathfrak{H}) - D(2\kappa D+\mathfrak{D})}{nn} - \frac{2F(2n+\mathfrak{E}) - J(2\kappa A+\mathfrak{A})}{nn}$$

$$M' = 2(2\alpha+1)M - \frac{3B}{n} - \frac{2B(2\kappa H+\mathfrak{H}) - G(2n+\mathfrak{E}) - K(2\kappa A+\mathfrak{A})}{nn}$$

vbi quidem plures terminos, quos admodum paruos fore praeuidimus, omisimus.

§. 95.

D+𝔇) +𝔇) E+𝔈) +𝔇) E+𝔈) A+𝔄) D+𝔇) A+𝔄) D+𝔇) A+𝔄) A+𝔄) paruos §. 95.

§. 95. Hinc autem denuo differentiando obtinemus

valorem ipsius $\frac{ddv}{dr^3}$ =

kk	$kk \cos 2\eta$	$kk \cos 4\eta$	$kk \cos 2r$
$+\frac{a'(2\kappa A+\mathfrak{A})}{nn}$	$-2\alpha a'$	$+\frac{a'(2\kappa A+\mathfrak{A})}{nn}$	$+\frac{A'(2\kappa J+\mathfrak{J})}{nn}$
$+\frac{D'(2\kappa D+\mathfrak{D})}{nn}$	$+\frac{2b'(2\kappa A+\mathfrak{A})}{nn}$	$-4\alpha b'$	$+\frac{E'(2\kappa D+\mathfrak{D})}{nn}$
	$+\frac{D'(2n+\mathfrak{E})}{nn}$	$+\frac{E'(2\kappa D+\mathfrak{D})}{nn}$	$-2H'$
	$+\frac{E'(2n+\mathfrak{E})}{nn}$	$+\frac{2F'(2n+\mathfrak{E})}{nn}$	$+\frac{J'(2\kappa A+\mathfrak{A})}{nn}$
	$+\frac{2F'(2\kappa D+\mathfrak{D})}{nn}$	$+\frac{2G'(2n+\mathfrak{E})}{nn}$	$+\frac{K'(2\kappa A+\mathfrak{A})}{nn}$

$kk\cos(2\eta-2r)$	$kk\cos(2\eta+2r)$	$kk\cos(4\eta-2r)$	$kk\cos(4\eta+2r)$
$+\frac{3A'}{2n}$	$+\frac{3A'}{2n}$	$+\frac{A'(2\kappa J+\mathfrak{J})}{nn}$	
$+\frac{A'(2\kappa H+\mathfrak{H})}{nn}$	$+\frac{A'(2\kappa H+\mathfrak{H})}{nn}$	$+\frac{3B'}{n}$	$+\frac{3B'}{n}$
$+\frac{D'(2n+\mathfrak{E})}{nn}$	$+\frac{2G'(2\kappa D+\mathfrak{D})}{nn}$	$+\frac{2B'(2\kappa H+\mathfrak{H})}{nn}$	$+\frac{2B'(2\kappa H+\mathfrak{H})}{nn}$
$-2(\alpha-1)J'$	$-2(\alpha+1)K'$	$+\frac{D'(2\kappa D+\mathfrak{D})}{nn}$	$+\frac{2G'(2n+\mathfrak{E})}{nn}$
$+\frac{2L'(2\kappa A+\mathfrak{A})}{nn}$	$+\frac{2M'(2\kappa A+\mathfrak{A})}{nn}$	$+\frac{2F(2n+\mathfrak{E})}{nn}$	$-2(2\alpha+1)M'$
		$-2(2\alpha-1)L'$	$+\frac{K'(2\kappa A+\mathfrak{A})}{nn}$
		$+\frac{J'(2\kappa A+\mathfrak{A})}{nn}$	

L vnde

vnde tandem nanciſcimur has determinationes:

$$\tfrac{1}{2}\delta-\gamma+\frac{\mathfrak{A}\mathfrak{a}+\frac{1}{2}\mathfrak{D}\mathfrak{D}+3D\mathfrak{D}+\frac{3}{2}DD+3A\mathfrak{a}+3\mathfrak{A}a+3Aa}{nn}kk$$
$$=\frac{a'(2nA+\mathfrak{A})+D'(2nD+\mathfrak{D})}{nn}kk$$

$$3+\tfrac{3}{2}(A+D+E)-a-2na=-2aa'+\frac{2b'(2nA+\mathfrak{A})}{nn}$$
$$+\frac{(D'+E')(2n+\mathfrak{C})+2F'(2nD+\mathfrak{D})}{nn}$$

$$\tfrac{3}{2}(B+F+G)-b-2nb=-4ab'+\frac{a'(2nA+\mathfrak{A})}{nn}$$
$$+\frac{E'(2nD+\mathfrak{D})+2(F'+G')(2n+\mathfrak{C})}{nn}$$

$$\tfrac{3}{2}-2n\mathfrak{H}-H+\frac{\mathfrak{A}\mathfrak{J}+3\mathfrak{A}J+3A\mathfrak{J}+3AJ}{nn}=-2H'+\frac{A'(2nJ+\mathfrak{J})}{nn}$$
$$+\frac{E'(2nD+\mathfrak{D})+(J'+K')(2nA+\mathfrak{A})}{nn}$$

$$\tfrac{15}{4}-2n\mathfrak{J}-J+\tfrac{3}{4}A+\tfrac{3}{4}D=-2(a-1)J'+\frac{3A'}{2n}+\frac{A'(2nH+\mathfrak{H})}{nn}$$
$$+\frac{D'(2n+\mathfrak{C})+2L'(2nA+\mathfrak{A})}{nn}$$

$$\tfrac{15}{4}-2n\mathfrak{K}-K+\tfrac{3}{4}A+\tfrac{3}{4}E=-2(a+1)K'+\frac{3A'}{2n}+\frac{A'(2nH+\mathfrak{H})}{nn}$$
$$+\frac{2G'(2nD+\mathfrak{D})+2M'(2nA+\mathfrak{A})}{nn}$$

$$-2n\mathfrak{L}-L+\tfrac{3}{4}B+\tfrac{3}{4}F+\frac{\mathfrak{A}\mathfrak{J}+3\mathfrak{A}J+3A\mathfrak{J}+3AJ+\frac{3}{2}\mathfrak{D}\mathfrak{D}+3D\mathfrak{D}+3DD}{nn}=$$
$$-2(2a-1)L'+\frac{3B'}{n}+\frac{A'(2nJ+\mathfrak{J})+2B'(2nH+\mathfrak{H})}{nn}$$
$$+\frac{D'(2nD+\mathfrak{D})+2F'(2n+\mathfrak{C})+J'(2nA+\mathfrak{A})}{nn}$$

$$-2n\mathfrak{M}-M+\tfrac{3}{4}B+\tfrac{3}{4}G=-2(2a+1)M'+\frac{3B'}{n}+\frac{2B'(2nH+\mathfrak{H})}{nn}$$
$$+\frac{2G'(2n+\mathfrak{C})+K'(2nA+\mathfrak{A})}{nn}$$

§. 96. Primum autem valoribus iam cognitis ſubſtituendis, reperitur:

$2aa=-3,837-0,038\,\mathfrak{b}$ et $4a\mathfrak{b}=-1,073-0,019\,\mathfrak{a}$

hincque $\mathfrak{a}=-2,051$ et $b=-0,277$

ex quibus porro elicimus: $a=-12,595$ et $b=-0,086$

et $a'=-23,510$. Deinde pro reliquis litteris

$\mathfrak{J}=+32,663$ - - - $l\,\mathfrak{J}=1,514059$

$\mathfrak{K}=-1,025$ $l{-}\mathfrak{K}=0,014776$

$J'=-2(1-a)\,J-5,060;$

$K'=2(1+a)\,K+0,227;$

$J=-15,555$ $l{-}J=1,191891$

$K=-0,370$ $l{-}K=9,568589$

Porro $\mathfrak{L}=-1,453$ - - - $l{-}\mathfrak{L}=0,162070$

$\mathfrak{M}=-0,000$ - - -

$L'=2(2a-1)\;L-12,786$

$M'=2(2a+1)\;M$

$L=+6,252$ $l\,L=0,796019$

$M=-0,001$ $l{-}M=7,000000$

Denique $\mathfrak{H}=-0,123$ et $H=-1,033$

atque $\frac{1}{2}\delta-\gamma=-7,459\,kk$

§. 97. His igitur valoribus inuentis innoteſcet primum diſtantia Lunae a terra curtata, quatenus a ſola

 excen-

excentricitate orbitae lunaris k pendet. Cum enim haec distantia posita sit $x = \frac{(1-kk)au}{1-k\cos r}$ ob $u = 1 + \frac{v}{nn}$, erit

$$u = 1 - 0{,}007161 \cos 2\eta - 0{,}0719\, kk \cos 2\eta$$
$$+ 0{,}000073 \cos 4\eta - 0{,}0005\, kk \cos 4\eta$$
$$+ 0{,}194888\, k \cos(2\eta - r) - 0{,}003274\, k \cos(2\eta + r)$$
$$- 0{,}003238\, k \cos(4\eta - r) + 0{,}000078\, k \cos(4\eta + r)$$
$$- 0{,}0059\, kk \cos 2r$$
$$- 0{,}0889\, kk \cos(2\eta - 2r) - 0{,}0021\, kk \cos(2\eta + 2r)$$
$$+ 0{,}0357\, kk \cos(4\eta - 2r)$$

At pro longitudine Lunae, quatenus a sola excentricitate k pendet, prodibit $\frac{d\Phi}{dr} =$

$$u + 0{,}000285 + 0{,}019015 \cos 2\eta + 0{,}000076 \cos 4\eta$$
$$+ 0{,}1562\, kk \cos 2\eta + 0{,}0008\, kk \cos 4\eta$$
$$- 0{,}001255\, k \cos r - 0{,}38410\, k \cos(2\eta - r) + 0{,}002647\, k \cos(4\eta - r)$$
$$+ 0{,}01278\, k \cos(2\eta + r) - 0{,}000229\, k \cos(4\eta + r)$$
$$+ 0{,}0118\, kk \cos 2r - 0{,}0081\, kk \cos(2\eta - 2r) - 0{,}0076\, kk \cos(4\eta - 2r)$$
$$+ 0{,}0102\, kk \cos(2\eta + 2r)$$

§. 98. Ponatur nunc:

$$\Phi = Or + \mathfrak{A}' \sin 2\eta + a' kk \sin 2\eta + \mathfrak{B}' \sin 4\eta + b' kk \sin 4\eta$$
$$+ \mathfrak{C}' k \sin r + \mathfrak{D}' k \sin(2\eta - r) + \mathfrak{F}' k \sin(4\eta - r)$$
$$+ \mathfrak{E}' k \sin(2\eta + r) + \mathfrak{G}' k \sin(4\eta + r)$$
$$+ \mathfrak{H}' kk \sin 2r - \mathfrak{J}' kk \sin(2\eta - 2r) + \mathfrak{L}' kk \sin(4\eta - 2r)$$
$$+ \mathfrak{K}' kk \sin(2\eta + 2r) + \mathfrak{M}' kk \sin(4\eta + 2r)$$

atque

atque differentiando orientur sequentes comparationes

$$u + 0{,}000285 = O - \frac{a'(2nA + \mathfrak{A}) - \mathfrak{D}'(2aD + \mathfrak{D})}{nn} kk + 0{,}000190$$

$$+ 0{,}1562 = 2\alpha a' - \frac{(\mathfrak{D}' + \mathfrak{E}')(2n + \mathfrak{E}) - 2\mathfrak{F}'(2nD + \mathfrak{D})}{nn}$$

$$+ 0{,}0008 = 4\alpha b' - \frac{a'(2nA + \mathfrak{A}) - \mathfrak{E}'(2nD + \mathfrak{D}) - 2(\mathfrak{F}' + \mathfrak{G}')(2n + \mathfrak{E})}{nn}$$

$$+ 0{,}0118 = 2\mathfrak{H}' - \frac{\mathfrak{E}'(2nD + \mathfrak{D}) - \mathfrak{A}'(2nJ + \mathfrak{J}) - (\mathfrak{J}' + \mathfrak{K}')(2nA + \mathfrak{A})}{nn}$$

$$- 0{,}0081 = 2(\alpha - 1)\mathfrak{J}' - \frac{3\mathfrak{A}'}{2n} - \frac{\mathfrak{D}'(2n + \mathfrak{E}) - \mathfrak{A}'(2nH + \mathfrak{H})}{nn} - \frac{2\mathfrak{L}'(2nA + \mathfrak{A})}{nn}$$

$$+ 0{,}0102 = 2(\alpha + 1)\mathfrak{K}' - \frac{3\mathfrak{A}'}{2n} - \frac{\mathfrak{A}'(2nH + \mathfrak{H}) - 2\mathfrak{G}'(2nD + \mathfrak{D})}{nn}$$

$$- 0{,}0076 = 2(2\alpha - 1)\mathfrak{L}' - \frac{3\mathfrak{B}'}{2n} - \frac{\mathfrak{A}'(2nJ + \mathfrak{J}) - \mathfrak{J}'(2nA + \mathfrak{A})}{nn} - \frac{\mathfrak{D}'(2nD + \mathfrak{D}) - 2\mathfrak{F}'(2n + \mathfrak{E})}{nn}$$

$$0 = 2(2\alpha + 1)\mathfrak{M}' - \frac{3\mathfrak{B}'}{n} - \frac{2\mathfrak{G}'(2n + \mathfrak{E})}{nn}$$

§. 99. Ex his comparationibus elicimus:

$a' = +0{,}0509$; $b' = 0{,}0008$
$\mathfrak{J}' = +0{,}5385$; $\mathfrak{K}' = 0{,}0028$
$\mathfrak{L}' = -0{,}1055$; $\mathfrak{M}' = 0{,}0000$
$\mathfrak{H}' = +0{,}0021$; et $O = n - 0{,}000429$

posito

posito $k = 0,05445$. Hinc autem erit $\frac{1}{2}\delta - \gamma = -0,02302$
Cum autem iam ante inuentum esset $\frac{1}{2}\delta - \gamma = -0,01742$
erit reuera $\frac{1}{2}\delta - \gamma = -0,00560$. Tum vero inueniemus:
$\gamma = 1,40673$, vnde erit $\frac{1}{2}\delta = 1,40113$, et $\delta = 2,80226$
hincque $2\gamma - \frac{3}{2}\delta = -1,38993$ et $\frac{2\gamma - \frac{3}{2}\delta}{nn} = -0,00794$.
Verum ex cognita ratione motus medii ad motum anomaliae est $O = 1,0085272$, vnde $\varkappa = 1,0089562$
Verum esse debet $\varkappa = 1 + \frac{3 + 4\mu + \delta}{4nn}$; vnde foret
$0,0089562 = 0,008289 + \frac{\mu}{nn}$; ideoque $\frac{\mu}{nn} = 0,000667$
qui valor cum sit tam exiguus, merito dubitamus, num μ non prorsus sit $= 0$.

0,02302 0,01742 C niemus: 2,80226 0,00794. m ano- 0089562 ina foret ter col 000667 pr fu itamus, $\int$R v: PUT

CAPUT VII.

CORRECTIO INAEQUALITATUM LUNAE, ANTE INUENTARUM.

§. 100.

Quoniam nunc quidem valores litterarum γ et δ ita inuenimus, vt eos pro proxime veris habere queamus, ex iis coefficientes terminorum, quibus inaequalitates lunae continentur, accuratius definire poterimus. Cum enim sit $\gamma = 1,40673$ et $\delta = 2,80226$, colligamus hic in vnum omnes formulas, quas hactenus pro inueniendis coefficientibus assumtis elicuimus. Posueramus autem:

$$\int R\,dr = \mathfrak{A}\cos 2\eta + \mathfrak{a}\,kk\cos 2\eta + \mathfrak{B}\cos 4\eta + \mathfrak{b}\,kk\cos 4\eta$$
$$+ \mathfrak{C}\,k\cos r + \mathfrak{D}\,k\cos(2\eta - r) + \mathfrak{F}\,k\cos(4\eta - r)$$
$$+ \mathfrak{E}\,k\cos(2\eta + r) + \mathfrak{G}\,k\cos(4\eta + r)$$
$$+ \mathfrak{H}\,kk\cos 2r + \mathfrak{J}\,k^2\cos(2\eta - 2r) + \mathfrak{L}\,k^2\cos(4\eta - 2r)$$
$$+ \mathfrak{K}\,k^2\cos(2\eta + 2r) + \mathfrak{M}\,k^2\cos(4\eta + 2r)$$

$$v = A\cos 2\eta + a\,kk\cos 2\eta + B\cos 4\eta + b\,kk\cos 4\eta$$
$$+ D\,k\cos(2\eta - r) + F\,k\cos(4\eta - r)$$
$$+ E\,k\cos(2\eta + r) + G\,k\cos(4\eta + r)$$
$$+ H\,kk\cos 2r + J\,kk\cos(2\eta - 2r) + L\,kk\cos(4\eta - 2r)$$
$$+ K\,kk\cos(2\eta + 2r) + M\,kk\cos(4\eta + 2r)$$