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NOVA METHODUS
Inveniendi Trajectorias reciprocas Algebraicas.

I.

Tabula. III.

Viginti abhinc annis & quod excurrit, hoc problema de trajectoriis reciprocis primum a Nicolao Bernoullio Johannis Filio non solum est propositum; sed etiam tam variæ & elegantes solutiones jam eo tempore sunt exhibitæ, ut hoc problema jam penitus exhaustum videri possit. Cum enim præcipua difficultas in inveniendis curvis algebraicis huic quæstioni satisfacientibus versaretur, tradideram equidem in secundo Comment. Petropol. Tomo methodum ex quolibet linearum curvarum ordine unam ad minimum inveniendi, quæ præscripta proprietate sit affecta. Tanto præterea studio hoc problema illo tempore a pluribus Geometris fuit pertractatum, ut etiamsi ad solam Geometriam pertineret, tamen inde universa Analysis tam eximia acceperit augmenta, ut pluribus aliis quæstionibus majoris momenti enodandis apta sit reddita, quæ sine his subsidiis intactæ essent relictæ.

2. Hanc igitur quæstionem maxime famosam denuo aggredior, non quo aliorum solutiones minus idoneas vel insufficientes censeam: sed quoniam tum temporis curvæ algebraicæ, quibus præcipua problematis vis continetur, non levi labore ac per operosas integrationes sunt erutæ, neque omnes in formulis generalibus comprehendi potuerunt; explicabo hic methodum singularem, cujus ope non solum

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folum curvæ algebraicæ, quibus problema folvitur, facili negotio, & quidem quod maxime paradoxon videatur, fine ulla integratione inveniri, fed etiam omnes fimul finitis formulis contentæ repræfentari queant. Quanquam autem hac methodo jam fæpius in aliis quæftionibus folvendis fum ufus, tamen eam nusquam adhuc expofui; ejusque applicatio ad præfens negotium peculiare requirit artificium, quod in aliis cafibus haud parum utilitatis afferre poterit.

3. Problema autem hoc fequenti modo proponi eft folitum: Fig. 1.

„*Circa datum axem* ACB *defcribere ejusmodi lineam* „*curvam* ECF, *quæ circa axem in fitu inverfo* eCf *conftituta, ac* „*fecundum directionem axis motu fibi parallelo promota, in quovis* „*fitu* c′e′f′ *priorem curvam* ECF *fub dato angulo in* M *interfecet.*

Solutio vero fequenti modo a Celeb. Joh. Bernoullio ad Analyfin eft perducta. Cum angulus EMe' ubique debeat effe datæ magnitudinis, erit is æqualis angulo ECe, ideoque duplus anguli ECA. Per M ducatur recta MP axi AB parallela; eritque EMP + e'MP = ECe; at ob motum parallelum eft angulus e'MP = enP; & ob fitum inverfum fi ducatur QN axi AB parallela, ab eoque æquidiftans ac recta PM, erit ang: ENQ = enP. Quare requiritur, ut ductis binis quibusque rectis MP & NQ axi parallelis ab eoque æquidiftantibus, fumma angulorum EMP + ENQ fit ubique eadem atque æqualis duplo anguli ECA.

4. Cum igitur natura quæftionis ad unam lineam curvam fit revocata, ducatur ad axem AB recta GH, quæ cum eo faciat angulum CAH æqualem duplo angulo ECA Fig. 2.

feu

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seu ipsi angulo intersectionis proposito æqualem. In hacque recta capiantur utrinque abscissæ AP, AQ, quæ ob æquales applicatarum PM & QN ab axe AB distantias erunt æquales; eritque EMP + ENQ = CAH. Ductis autem utrinque applicatis infinite propinquis pm & qn, rectæque GH parallelis $M\mu$ & $n\nu$; ob angulum $M\mu m = N\nu n =$ GAC erit: $Mm\mu + mM\mu = nN\nu + Nn\nu =$ CAH = EMP + ENQ. At $Mm\mu =$ EMP & $nN\nu =$ ENQ, unde sequitur fore $mM\mu = nN\nu$ & $Nn\nu = Mm\mu$. Erunt ergo triangula $Mm\mu$ & $Nn\nu$ æquiangula ac propterea similia; ex quo habebitur hæc proportio $m\mu : M\mu = n\nu : N\nu$, ideoque hæc æqualitas $m\mu . N\nu = M\mu . n\nu$, qua natura problematis continetur.

5. Vocemus abscissam $AP = x$, eique respondentem applicatam $PM = y$; erit abscissa ex altera parte sumta $AQ = -x$, cui respondens applicata ponatur $QN = z$; quæ talis erit functio ipsius $-x$, qualis y est ipsius $+x$; seu ex valore ipsius y prodibit valor ipsius z, si loco x ubique scribatur $-x$. His positis erit $Pp = M\mu = dx$; $\mu m = dy$; $Qq = n\nu = -dx$, & $N\nu = -dz$; atque æquatio modo inventa $m\mu . N\nu = M\mu . n\nu$ dabit hanc formulam

$$-dydz = -dx^2 \text{ seu } \frac{dy}{dx} \cdot \frac{dz}{dx} = 1.$$

Ponatur $\frac{dy}{dx} = M$, & $\frac{dz}{dx} = N$, eritque N talis functio ipsius $-x$, qualis M est ipsius $+x$, seu ex functione M proveniet functio N, si loco x ponatur $-x$. Quocirca ad problema resolvendum ejusmodi functiones pro M investigari oportet, ut fiat $MN = 1$: hocque facto erit $dy = Mdx$, quæ æquatio naturam curvæ exprimet.

6. Quæ-

6. Quæstio itaque huc est perducta, ut pro M ejusmodi investigetur functio ipsius x, quæ, si loco x ponatur $-x$, abeat in N, ita ut sit $MN = 1$. Manifestum autem est huic conditioni satisfacere hujusmodi valores $M = a^{x}$; $M = a^{x^3}$, similesque alios; sed cum curvas algebraicas requiramus, hujusmodi valores exponentiales excludi oportet. Sit igitur P functio par ipsius x, quæ scilicet eundem valorem retineat, posito $-x$ loco $+x$; deinde sit Q functio impar ipsius x, quæ abeat in $-$ Q, si loco x ponatur $-x$: hincque evidens est conditionem problematis impleri, si ponatur $M = \frac{P+Q}{P-Q}$, fiet enim $N = \frac{P-Q}{P+Q}$ ideoque $MN = 1$. Ponatur $\frac{Q}{P} = u$, ut sit u functio quæcunque ipsius x impar; ex quo erit $M = \frac{1+u}{1-u}$ & $dy = \frac{1+u}{1-u}\,dx$, quæ æquatio solutionem problematis in latissimo sensu complectitur, dummodo sub littera u omnes functiones ipsius x comprehendantur.

7. Quamquam hæc æquatio jam est generalis omnesque solutiones includit, tamen ex ea aliæ formari possunt, quæ latius patere videntur; & quæ in inventione curvarum algebraicarum usum commodiorem præstant. Hujusmodi est ista formula $M = \left(\frac{1+u}{1-u}\right)^{n}$; quicunque enim numerus pro exponente n assumatur, erit semper $N = \left(\frac{1-u}{1+u}\right)^{n}$,

 ideo-

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ideoque $MN = 1$. Quare si u sumatur pro functione quacunque impari ipsius x, natura curvae trajectoriae reciprocae cujuscunque hac exprimetur aequatione: $dy = \left(\frac{1+u}{1-u}\right)^n dx$. Manifestum autem est, si pro n sumantur numeri fracti, facile ejusmodi curvas obtineri, quae ex priori forma difficulter erui queant, etiamsi revera in ea contineantur.

8. Tametsi functiones irrationales ob ambiguitatem neque functionibus paribus neque imparibus proprie annumerari queant: tamen in hoc negotio hujusmodi expressiones $\sqrt{(1+uu)}$ pro functionibus paribus haberi possunt, dummodo uu sit functio par neque ex $(1+uu)$ radix quadrata actu extrahi queat. At si u sit functio ipsius x impar, erit uu ac propterea $\sqrt{(1+uu)}$ ejusdem x functio par. Quo notato facile patet, hunc valorem $M = \sqrt{(1+uu)} + u$ quaesito satisfacere debere; fiet enim inde $N = \sqrt{(1+uu)} - u$, ideoque $MN = 1$. Idem evenit, si statuatur $M = (\sqrt{(1+uu)}+u)^n$, quia fit $N = (\sqrt{(1+uu)}-u)^n$ & $MN = 1$. Hinc ergo duae novae aequationes generales pro trajectoriis reciprocis oriuntur:

$$dy = (\sqrt{(1+uu)}+u)\,dx \quad \&$$
$$dy = (\sqrt{(1+uu)}+u)^n\,dx$$

9. Potest etiam nova quaedam variabilis t introduci, a qua x ita pendeat, ut posito $-t$ loco t, abscissa x abeat in $-x$; seu sit x functio impar ipsius t. Ponatur $dx = v\,dt$, eritque v functio par ipsius t; statuatur autem ut ante u

functio

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functio impar ipsius t. His positis, si fiat $M = \frac{dy}{dx} = \frac{1+u}{1-u}$, evadet denuo $N = \frac{1-u}{1+u}$, ideoque $MN = 1$. Hancobrem problemati satisfiet, si sumatur:

$$dx = vdt \quad \& \quad dy = \frac{1+u}{1-u}\, vdt$$

Simili modo problema solvetur his formulis generalibus

$$dx = vdt \quad \& \quad dy = \left(\frac{1+u}{1-u}\right)^n vdt$$

itemque his ex irrationalibus ortis:

$$dx = vdt \quad \& \quad dy = (\sqrt{(1+uu)}+u)\, vdt \quad \text{atque}$$

$$dx = vdt \quad \& \quad dy = (\sqrt{(1+uu)}+u)^n\, vdt.$$

Quæcunque autem formulæ ex his assumantur, necesse est ut inde solutio problematis generalis obtineatur.

10. Si jam curvæ algebraicæ desiderentur, totum negotium huc redit, ut qualitas functionis u, & in his posterioribus binarum functionum u & v determinetur, qua hæ formulæ integrabiles reddentur. Plures autem imo infinitæ hujusmodi functiones, cum a Celeb. Bernoullio, tum a me sunt notatæ, quæ curvas algebraicas præbeant; sed hic modus maxime est particularis, neque omnes curvas algebraicas satisfacientes in se complectitur. Deinde assumtis illiusmodi functionibus idoneis, integratio harum formularum demum actu institui debet; sicque pro quovis curvarum genere peculiari operatione est opus, quæ sæpe non sine molesto calculo absolvitur. Cui incommodo ita

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occurram, ut non solum formulas generales pro omnibus curvis algebraicis sim exhibiturus; sed etiam quæ sine prævia integratione solutionem suppeditent. Quin etiam has ipsas formulas algebraicas ex superioribus differentialibus sine actuali integratione sum derivaturus, id quod plerisque maxime paradoxon videbitur. Methodum autem meam ad singulas formulas differentiales ante inventas seorsim accommodabo.

I. Modus inveniendi trajectorias reciprocas algebraicas ex formula

$$dy = \frac{1+u}{1-u}\,dx$$

11. Quæritur ergo hic, non solum qualis functio ipsius x debeat esse u, ut formula $\frac{1+u}{1-u}\,dx$ integrationem admittat, sed etiam quænam ipsa sit futura integralis forma. Cum autem u sit functio impar ipsius x, erit vicissim x functio impar ipsius u: hincque investigabo, qualis functio quantitas x esse debeat ipsius u, ut quoque y per functionem algebraicam ipsius u exprimi queat. Quam investigationem ita instituo: quia est $y = \int \frac{1+u}{1-u}\,dx$, erit per notam integralium reductionem:

$$y = \frac{1+u}{1-u}\,x - 2\int \frac{x\,du}{(1-u)^2}.$$

Superest ergo, ut formula $\int \frac{x\,du}{(1-u)^2}$ reddatur integrabi-

lis,

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lis, in quo nulla foret difficultas, nisi x deberet esse functio impar ipsius u.

12. Denotet p functionem quamcunque parem, & q functionem imparem ipsius u, statuaturque:

$$2\int\frac{x\,du}{(1-u)^2}=\frac{(p+q)(1+u)}{1-u}$$

duas scilicet novas quantitates p & q introduco, ut non solum integrabilitas procuretur, sed etiam functioni x praescripta proprietas inducatur. Sumtis ergo differentialibus, erit:

$$\frac{2x\,du}{(1-u)^2}=\frac{(dp+dq)(1+u)}{1-u}+\frac{2(p+q)\,du}{(1-u)^2}$$

multiplicationeque per $(1-u)^2$ instituta orietur:

$$2x\,du=(1-uu)\,dp+(1-uu)\,dq+2p\,du+2q\,du$$

cujus aequationis alii termini pares ipsius u continebunt dimensiones, alii impares: quamobrem necesse est ut termini tam parium quam imparium dimensionum seorsim inter se aequentur.

13. Quia vero p est functio par ipsius u; reliquae vero quantitates q & x functiones impares, earumque differentialia eandem naturam sequuntur, erit $2x\,du$ functio par; $(1-uu)\,dp$ par; $(1-uu)\,dq$ impar; $2p\,du$ impar; & $2q\,du$ par. Aequatis ergo paribus & imparibus seorsim sequentes duae orientur aequationes:

$$2x\,du=(1-uu)\,dp+2q\,du:$$

$$\&\quad 2p\,du+(1-uu)\,dq=0.$$

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Poſterior æquatio ſtatim definit $p = -\frac{(1-uu)dq}{2du}$; hinc enim ob q functionem imparem ipſius u fiet p functio par. Cognito jam valore functionis p, ex priore æquatione deducitur:

$$x = \frac{(1-uu)dp}{2du} + q: \text{ cui abſciſſæ reſpondebit}$$

applicata $y = \frac{1+u}{1-u}x - \frac{(p+q)(1+u)}{1-u} = \frac{1+u}{1-u}(x-q-p)$

14. Si hic pro x valor ante inventus ſubſtituatur, invenietur:

$$y = \frac{(1+u)^2 dp}{2du} - \frac{p(1+u)}{1-u} \text{ ſeu ob } p = -\frac{(1-uu)dq}{2du}$$

$$y = \frac{(1+u)^2 dp + (1+u)^2 dq}{2du} = \frac{(dp+dq)(1+u)^2}{2du}$$

Quocirca hinc naſciſcimur

Primam regulam generalem pro inveniendis trajectoriis reciprocis algebraicis:

Sumatur q *functio quæcunque imparium dimenſionum ipſius* u*; indeque quæratur quantitas* $\mathrm{p} = -\frac{(1-uu)dq}{2du}$. *qua inventa erit curvæ quæſitæ:*

abſciſſa $AP = x = q + \frac{(1-uu)dp}{2du}$

appli-

applicata PM :

qui valores ſemp
braica ipſius u

15. Si

que capiatur p

$x = q' +$

Atque cum hin

ſi ex quapiam ip
$x = X$
ex alia autem h
problemati quo
$x = X$
atque generalius
$x = \alpha$
ſicque ex duabu
biles novæ inve
fuerit inventa
$\alpha X + \beta X' +$
que porro.

16. Pon
exiſtente λ nu
fractio, cujus ta
eritque
$\frac{dq}{du} = \lambda u^{\lambda-1}$
porro

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applicata $PM = y = \frac{(dp + dq)(1+u)^2}{2\,du} = \frac{1+u}{1-u}(x-p-q)$

qui valores semper sunt algebraici, si quidem q *fuerit functio algebraica ipsius* u

15. Si ipsius u alia sumatur functio impar q', ex eaque capiatur $p' = -\frac{(1-uu)dq'}{2\,du}$, habebitur simili modo:

$$x = q' + \frac{(1-uu)dp'}{2\,du}, \ \& \ y = \frac{(dp' + dq')(1+u)^2}{2\,du}$$

Atque cum hinc æque fiat $\frac{dy}{dx} = \frac{1+u}{1-u}$: manifestum est si ex quapiam ipsius q hypothesi inventum fuerit:

$$x = X \ \& \ y = Y$$

ex alia autem hypothesi prodierit $x = X'$ & $y = Y'$, problemati quoque satisfieri his valoribus:

$$x = X + X' \ \& \ y = Y + Y'$$

atque generalius etiam, si ponatur:

$$x = \alpha X + \beta X' \ \& \ y = \alpha Y + \beta X'$$

sicque ex duabus curvis algebraicis jam inventis innumerabiles novæ inveniri poterunt. Sin autem præterea tertia fuerit inventa $x = X''$ & $y = Y''$, erit quoque $x = \alpha X + \beta X' + \gamma X''$ & $y = \alpha Y + \beta Y' + \gamma X''$ sicque porro.

16. Ponatur ut ad exempla descendamus $q = u^{\lambda}$ existente λ numero quocunque impari, sive integro sive fracto, cujus tam numerator quam denominator sit impar: eritque

$\frac{dq}{du} = \lambda u^{\lambda-1}$ & $p = -\frac{1}{2}\lambda u^{\lambda-1} + \frac{1}{2}\lambda u^{\lambda+1}$ hincque porro

dp

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$\frac{dp}{du} = -\frac{1}{2}\lambda(\lambda-1)u^{\lambda-2} + \frac{1}{2}\lambda(\lambda+1)u^{\lambda}$. Unde habebitur:

$$x = u^{\lambda} - \frac{1}{2}\lambda(\lambda-1)u^{\lambda-2} + \frac{1}{4}\lambda(\lambda+1)u^{\lambda} + \frac{1}{4}\lambda(\lambda-1)u^{\lambda} - \frac{1}{4}\lambda(\lambda+1)u^{\lambda+2}$$

seu $x = -\frac{1}{4}\lambda(\lambda-1)u^{\lambda-2} + \frac{1}{2}(\lambda\lambda+2)u^{\lambda} - \frac{1}{4}\lambda(\lambda+1)u^{\lambda+2}$

& $y = \frac{1}{2}\lambda(u^{\lambda-1} - \frac{1}{2}(\lambda-1)u^{\lambda-2} + \frac{1}{2}(\lambda+1)u^{\lambda})(1+u)^2$ seu

$$y = \frac{1}{4}\lambda u^{\lambda-2}(1+u)^4 - \frac{1}{4}\lambda\lambda u^{\lambda-2}(1-u)(1+u)^3 \text{ vel etiam}$$

$$y = \frac{1}{4}\lambda u^{\lambda-2}(1-\lambda+u+\lambda u)(1+u)^3$$

Hinc sequitur fore:

si $\lambda = 1$: $x = \frac{3}{2}u - \frac{1}{2}u^3$; $y = \frac{1}{2}(1+u)^3$ pro parabola cubicali secunda

si $\lambda = 3$; $x = -\frac{1}{2}u(1-3uu)(3-2uu)$: &

$$y = -\frac{3}{2}u(1-2u)(1+u)^3$$

si $\lambda = 5$; $x = -\frac{1}{2}u^3(10-27u^2+15u^4)$: &

$$y = -\frac{5}{2}u^3(2-3u)(1+u)^3$$

si $\lambda = 7$; $x = -\frac{1}{2}u^5(21-51u^2+28u^4)$: &

$$y = -\frac{7}{2}u^5(3-4u)(1+u)^3$$

&c.

II. Se-

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II. Secundus Modus Inveniendi Trajectorias reciprocas Algebraicas.

Ex formula $dy = \left(\frac{1+u}{1-u}\right)^n dx$

17. Quia ut ante ostensum est, abscissa x debet esse functio impar ipsius u, quaeratur qualis esse debeat, ut applicata y prodeat algebraice expressa. In hunc finem ponatur:

$$y = \int \left(\frac{1+u}{1-u}\right)^n dx = \left(\frac{1+u}{1-u}\right)^n x - 2n \int \frac{x\,du}{1-uu}\left(\frac{1+u}{1-u}\right)^n$$

Sumtisque p functione pari & q functione impari ipsius u statuatur:

$$\int \left(\frac{1+u}{1-u}\right)^n \frac{x\,du}{1-uu} = (p+q)\left(\frac{1+u}{1-u}\right)^n$$

ex qua differentiatione instituta habebitur:

$$\frac{x\,du}{1-uu} = dp + dq + \frac{2n(p+q)\,du}{1-uu} \quad \text{seu}$$

$$x\,du = (1-uu)\,dp + (1-uu)\,dq + 2npdu + 2nqdu$$

18. Discerpatur haec in duas aequationes, quarum altera contineat functiones parium, altera vero imparium dimensionum, sicque fiet:

$$x\,du = (1-uu)\,dp + 2nqdu$$

$$\& \; 0 = (1-uu)\,dq + 2npdu$$

qua-

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quarum posterior dat $p = -\frac{(1-uu)\,dq}{2n\,du}$, quo valore ipsius p invento erit ex priori:

$$x = 2nq + \frac{(1-uu)\,dp}{du}$$

Praeterea autem ex supra facta hypothesi orietur

$$y = \left(\frac{1+u}{1-u}\right)^{n}(r-2np-2nq) = \left(\frac{1+u}{1-u}\right)^{n}\frac{(1-uu)(dp+dq)}{du}$$

seu $y = \frac{(dp+dq)(1+u)^{n+1}}{du(1-u)^{n-1}}$. Sumta ergo pro q functione quacunque imparium dimensionum ipsius u, hae formulae praebebunt curvas algebraicas, quae erunt trajectoriae reciprocae.

19. Hinc ergo adepti sumus:

Secundam regulam generalem pro inveniendis trajectoriis reciprocis algebraicis.

Sumatur q functio quaecunque imparium dimensionum ipsius u, denotanteque n numerum quemcunque, quaeratur inde quantitas $p = -\frac{(1-uu)\,dq}{2n\,du}$; *hinc erit curvae quaesitae;*

Abscissa $AP = x = 2nq + \frac{(1-uu)\,dp}{du}$

Appli-

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Applicata $PM = y = \frac{(dp+dq)(1+u)^{n+1}}{du(1-u)^{n-1}}$

Ubi iterum notandum est, si pro x & y jam aliquot inventi fuerint valores X & Y, X' & Y', X'' & Y'', &c. problemati quoque satisfieri, si capiatur

$$x = \alpha X + \beta X' + \gamma X'' \text{ \&c. } \quad \& \; y = \alpha Y + \beta Y' + \gamma Y'' \text{ \&c.}$$

20. Sit λ numerus quicunque impar ac statuatur: $q = 2nu^{\lambda}$, erit $\frac{dq}{du} = 2n\lambda u^{\lambda-1}$ ideoque $p = -\lambda u^{\lambda-1} + \lambda u^{\lambda+1}$; unde fit porro $\frac{dp}{du} = -\lambda(\lambda-1)u^{\lambda-2} + \lambda(\lambda+1)u^{\lambda}$. Ex his valoribus colligitur:

$$x = 4nnu^{\lambda} - \lambda(\lambda-1)u^{\lambda-2} + 2\lambda\lambda u^{\lambda} - \lambda(\lambda+1)u^{\lambda+2} \; \&$$

$$y = -(\lambda(\lambda-1)u^{\lambda-2} - 2n\lambda u^{\lambda-1} - \lambda(\lambda+1)u^{\lambda}) \frac{(1+u)^{n+1}}{(1-u)^{n-1}}$$

seu mutatis signis, sumtisque semissibus erit

$$x = u^{\lambda-2}\left(\frac{\lambda(\lambda-1)}{2} - (\lambda\lambda+2nn)u^2 + \frac{\lambda(\lambda+1)}{2}u^4\right);$$

$$y = \lambda u^{\lambda-2}\left(\frac{\lambda-1}{2} - nu - \frac{(\lambda+1)uu}{2}\right)\frac{(1+u)^{n+1}}{(1-u)^{n-1}}$$

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Hinc ergo erit:

$$\text{si } \lambda = 1:\; x = u^3 - (1+2nn)u;\; y = -(n+u)\frac{(1+u)^{n+1}}{(1-u)^{n-1}}$$

$$\text{si } \lambda = 3:\; x = u(3 - (9+2nn)u^2 + 6u^4);\; \&$$

$$y = 3u(1 - nu - 2uu)\frac{(1+u)^{n+1}}{(1-u)^{n-1}}$$

$$\text{si } \lambda = 5;\; x = u^3(10 - (15+2nn)u^2 + 15u^4);\; \&$$

$$y = 5u^3(2 - nu - 3uu)\frac{(1+u)^{n+1}}{(1-u)^{n-1}}$$

&c.

21. Sit porro m numerus quicunque, ac ponatur $q = 2nu^{\lambda}(1-uu)^{m}$, erit $\frac{dq}{du} = 2n\lambda u^{\lambda-1}(1-uu)^{m}$ $- 4mnu^{\lambda+1}(1-uu)^{m-1}$ ac propterea

$p = -\lambda u^{\lambda-1}(1-uu)^{m+1} + 2mu^{\lambda+1}(1-uu)^{m}$. Fiet itaque

$$\frac{dp}{du} = -\lambda(\lambda-1)u^{\lambda-2}(1-uu)^{m+1} + 2\lambda(m+1)u^{\lambda}(1-uu)^{m} + 2m(\lambda+1)u^{\lambda}(1-uu)^{m} - 4mmu^{\lambda+2}(1-uu)^{m-1}$$

ex quibus colligitur:

$x =$

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$$x = 4mmu^{\lambda}(1-uu)^{m} - \lambda(\lambda-1)u^{\lambda-2}(1-uu)^{m+2}$$
$$+ 2(2m\lambda+m+\lambda)u^{\lambda}(1-uu)^{m+1}$$
$$- 4mmu^{\lambda+2}(1-uu)^{m}$$

seu

$$x = u^{\lambda-2}(1-uu)^{m}\left(-\lambda(\lambda-1) + 2(2mm+\lambda\lambda+2m\lambda+m)u^{2}\right.$$
$$\left.-(4mm+4m\lambda+\lambda\lambda+2m+\lambda)u^{4}\right)$$

sive hoc modo brevius:

$$x = -u^{\lambda-2}(1-uu)^{m}\left(\lambda(\lambda-1) - 2(2mm+\lambda\lambda+2m\lambda+m)u^{2}\right.$$
$$\left.+(2m+\lambda)(2m+\lambda+1)u^{4}\right)$$

Deinde erit:

$$y = \frac{(1+u)^{n+1}}{(1-u)^{n-1}}\left(2n\lambda u(1-uu) - 4mnu^{3} - \lambda(\lambda-1)(1-uu)^{2}\right.$$
$$\left.-4mmu^{4} + 2(2m\lambda+m+\lambda)u^{2}(1-uu)\right)u^{\lambda-2}(1-uu)^{m-1}$$

seu $y = -\left(\frac{1+u}{1-u}\right)^{n} u^{\lambda-2}(1-uu)^{m}\left(\lambda(\lambda-1) - 2(\lambda\lambda+\right.$

$$2m\lambda+m)u^{2} + (2m+\lambda)(2m+\lambda+1)u^{4}$$
$$\left.-2n\lambda u + 2n(2m+\lambda)u^{3}\right)$$

mutatis ergo signis erit:

I 3 $x=$

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$$x = u^{\lambda-2}(1-uu)^m\big(\lambda(\lambda-1) - 2(2mm+\lambda\lambda+2m\lambda+m)u^2 + (2m+\lambda)(2m+\lambda+1)u^4\big)$$

$$y = u^{\lambda-2}(1-uu)^m\left(\begin{array}{l}\lambda(\lambda-1) - 2(\lambda\lambda+2m\lambda+m)u^2 \\ +(2m+\lambda)(2m+\lambda+1)u^4 \\ -2n\lambda u + 2n(2m+\lambda)u^3\end{array}\right)\left(\frac{1+u}{1-u}\right)^n$$

22. Hinc duae prodeunt solutiones prae ceteris simpliciores, quarum altera prodit, si $\lambda = 1$ & $m = -1$: tum enim fiet:

$$x = -\frac{4(nn-1)u}{1-uu} \;\&\; y = -\frac{2(n-2u+nuu)}{1-uu}\left(\frac{1+u}{1-u}\right)^n$$

seu si utrinque per constantem $\frac{-a}{2(nn-1)}$ multiplicetur erit

$$x = \frac{2au}{1-uu} \;\&\; y = \frac{a(n-2u+nuu)}{(nn-1)(2-uu)}\left(\frac{1+u}{1-u}\right)^n$$

unde non difficulter eliminatur variabilis u; est enim ex priori $u = \frac{-a+\sqrt{(aa+xx)}}{x}$: quo valore in altera substituto, irrationalibusque ex denominatore sublatis pervenietur tandem ad hanc aequationem:

$$(nn-1)a^n y = (n\sqrt{(aa+xx)}-x)(\sqrt{(aa+xx)}+x)^n$$

Ad

Ad quam reducendam notetur esse: $(\sqrt{(aa+xx)}+x)^n =$

$$a^n + \frac{nn}{1.2}a^{n-2}xx + \frac{nn(nn-4)}{1.2.3.4}a^{n-4}x^4 + \frac{nn(nn-4)(nn-16)}{1.2.3.4.5.6}a^{n-6}x^6 + \&c.$$

$$+(na^{n-2}x + \frac{n(nn-4)}{1.2.3.}a^{n-4}x^3 + \frac{n(nn-4)(nn-16)}{1.2.3.4.5}a^{n-6}x^5 + \&c.)\sqrt{(aa+xx)}$$

unde reperietur fore:

$$(nn-1)a^n y = \begin{cases} (na^n + \frac{n(nn-2)}{1.2}a^{n-2}x^2 + \frac{n(nn-4)(nn-4)}{1.2.3.4} \\ \quad a^{n-4}x^4 + \frac{n(nn-6)(nn-4)(nn-16)}{1.2.3.4.5.6} \\ \quad a^{n-6}x^6 + \&c.)\sqrt{(aa+xx)} \\ +(nn-1)a^n x + \frac{nn(nn-1)}{1.2.3}a^{n-2}x^3 + \\ \quad \frac{nn(nn-1)(nn-4)}{1.2.3.4.5}a^{n-4}x^5 + \\ \quad \frac{nn(nn-1)(nn-4)(nn-16)}{1.2.3.4.5.6.7}a^{n-6}x^7 \ \&c. \end{cases}$$

quae abrumpitur, quoties est n numerus par. Ponatur $a = 1$ fitque exempli gr.

$n = 1$ erit $y = x$

$n = 2;\ 3y = (1+2xx)\sqrt{(1+xx)} + 3x + 2x^3$

$n =$

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$$n = 4;\ 15y = (4 + 28xx + 24x^4)\sqrt{(1+xx)} + 15x + 40x^3 + 24x^5$$

$$n = 6;\ 35y = (6 + 102xx + 256x^4 + 160x^6)\sqrt{(1+xx)} + 35x + 210x^3 + 336x^5 + 160x^7$$

&c.

23. Sin autem n sit numerus impar, erit

$$(\sqrt{(aa+xx)} + x)^n = na^{n-1}x + \frac{n(nn-1)}{1.2.3}a^{n-3}x^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}a^{n-5}x^5 + \&c. + \left(a^{n-1} + \frac{(nn-1)}{1.2.}a^{n-3}x^2 + \frac{(nn-1)(nn-9)}{1.2.3.4}a^{n-5}x^4 + \frac{(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6}a^{n-7}x^6 + \&c.\right)\sqrt{(aa+xx)}$$

unde sequitur fore:

$$(nn-1)a^n y = na^{n+1} + \frac{n(nn-1)}{1.2}a^{n-1}x^2 + \frac{n(nn-1)(nn-1)}{1.2.3.4}a^{n-3}x^4 + \frac{n(nn-1)(nn-1)(nn-9)}{1.2.3.4.5.6}a^{n-5}x^6 + \&c.$$

$$+ \left((nn-1)a^{n-1}x + \frac{(nn-3)(nn-1)}{1.2.3}a^{n-3}x^3 + \frac{(nn-5)(nn-1)(nn-9)}{1.2.3.4.5}a^{n-5}x^5 + \&c.\right)\sqrt{(aa+xx)}$$

fit

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Sit $a = 1$, ac ponatur successive $n = 1, 3, 5$, &c.

$n = 1$; $\quad 0y = 1$

$n = 3$; $\quad 8y = 3 + 12xx + 8x^4 + (8x + 8x^3)\sqrt{(1+xx)}$

$n = 5$; $\quad 24y = 5 + 60x^2 + 120x^4 + 64x^6 + (24x + 88x^3 + 64x^5)\sqrt{(1+xx)}$

Generaliter autem posito $a = 1$ erit: $(nn - 1)y =$

$$(n-1)2^{n-1}x^{n+1} + \frac{(n+1)n}{1}2^{n-3}x^{n-1} + \frac{(n+1)n(n-1)}{1.\,2}2^{n-5}x^{n-3} + \frac{(n+1)n(n-1)(n-4)}{1.\,2.\,3}2^{n-7}x^{n-5} +$$

$$\frac{n(nn-1)(n-5)(n-6)}{1.\,2.\,3.\,4}2^{n-9}x^{n-7} + \frac{n(nn-1)(n-6)(n-7)(n-8)}{1.\,2.\,3.\,4.\,5}2^{n-11}x^{n-9} + \&c. +$$

$$\Big((n-1)2^{n-1}x^{n} + \frac{nn-n+2}{1}2^{n-3}x^{n-2} + \frac{(nn-n+4)(n-3)}{1.\,2}2^{n-5}x^{n-4} + \frac{(nn-n+6)(n-4)(n-5)}{1.\,2.\,3}2^{n-7}x^{n-5}$$

$$+ \frac{(nn-n+8)(n-5)(n-6)(n-7)}{1.\,2.\,3.\,4}2^{n-9}x^{n-8} + \&c.\Big)\sqrt{(1+xx)}$$

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24. Si n fuerit numerus *fractus* puta $= \frac{n}{m}$, habebitur pro trajectoria reciproca hæc æquatio:

$$(nn-mm)y = m\left(n\sqrt{(1+xx)}-mx\right)\left(\sqrt{(1+xx)}+x\right)^{n:m}$$

seu

$$\left(\frac{nn-mm}{m}\right)^{m} y^{m} = \left(n\sqrt{(1+xx)}-mx\right)^{m}\left(\sqrt{(1+xx)}+x\right)^{n}$$

quæ maxime fœcunda est in curvis simplicioribus suppeditandis: exceptis enim ordinibus 2 & 3, ex quovis ordine adminimum unam largitur trajectoriam reciprocam. Convenit autem hæc æquatio cum ea, quam in Tomo Comment. II. exhibueram. Scilicet si $m = 1$, curva erit ordinis $n + 2$: æquatio enim ab irrationalitate liberata non plures dimensiones obtinet quam $n + 2$: sin autem denominator m fuerit numerus quicunque, æquatio ad rationalitatem reducta assurget ad $n + 2m$ dimensiones. Desideratur autem adhuc methodus has æquationes rationales sine operosa elevatione ad potestates ex traditis formulis statim eliciendi; cujusmodi methodus sine dubio in Analysi datur, cum in sublatione surditatis plurimi termini se mutuo destruant. Æquatio vero rationalis ita erit comparata, ut sit $y^{2m} + Py^{m} + Q = 0$, ubi sint P & Q functiones rationales ipsius x.

25. Per inductionem autem ex pluribus casibus collegi, si sit $m = 1$, fore

(*nn*-

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$$(nn-1)y^2 - 2(nn-1)y\left\{\begin{array}{l}(n-1)2^{n-1}x^{n+1} + \frac{(n+1)n}{1}2^{n-3}x^{n-1} \\ + \frac{n(nn-1)}{1.2}2^{n-5}x^{n-3} + \\ \frac{n(nn-1)(n-4)}{1.2.3}2^{n-7}x^{n-5} + \\ \frac{n(nn-1)(n-5)(n-6)}{1.2.3.4}2^{n-9}x^{n-7} + \\ \&c.\end{array}\right\}$$

$$= \pm nn \pm (nn-1)xx$$

ubi ſignorum ambiguorum valet ſuperius, ſi n ſit numerus par, inferius ſi ſit impar. Hinc ergo ſequentes oriuntur æquationes ſimpliciores pro trajectoriis reciprocis;

$$9y^2 = 6y(2x^3 + 3x) + 3xx + 4$$
$$64y^2 = 16y(8x^4 + 12xx + 3) - 8xx - 9$$
$$225y^2 = 30y(24x^5 + 40x^3 + 15x) + 15xx + 16$$
$$576y^2 = 48y(64x^6 + 120x^4 + 60x^2 + 5) - 24xx - 25$$
$$1225y^2 = 70y(16x^7 + 336x^5 + 210x^3 + 35x) + 35xx + 36$$
$$2304y^2 = 96y(384x^8 + 896x^6 + 672x^4 + 168x^2 + 7) - 48xx - 49$$
$$3969y^2 = 126y(896x^9 + 2304x^7 + 2016x^5 + 672x^3 + 63x) + 63xx + 64$$

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quae ergo aequationum series, quousque libuerit facile continuabitur.

26. Supra duarum solutionum simpliciorum, quae ex formulis (21) sequuntur, mentionem fecimus, alteramque hic fusius sumus prosecuti. Altera igitur solutio expendenda est, in qua $\lambda = 1$ & $m = -\frac{1}{2}$. Hinc ergo oritur:

$$x = \frac{1}{u\sqrt{(1-uu)}}\left(-(4nn-1)uu\right) = -\frac{(4nn-1)u}{\sqrt{(1-uu)}}$$

$$y = \frac{1}{u\sqrt{(1-uu)}}(uu-2nu)\left(\frac{1+u}{1-u}\right)^n = \frac{u-2n}{\sqrt{(1-uu)}}\left(\frac{1+u}{1-u}\right)^n$$

Multiplicentur hae formulae per $\frac{-a}{4nn-1}$, habebiturque

$$x = \frac{au}{\sqrt{(1-uu)}}; \ \& \ y = \frac{a(2n-u)}{(4nn-1)\sqrt{(1-uu)}}\left(\frac{1+u}{1-u}\right)^n$$

unde fit $\frac{(4nn-1)y}{x} = \frac{2n-u}{u}\left(\frac{1+u}{1-u}\right)^n$. Ex priore autem valore oritur $u = \frac{x}{\sqrt{(aa+xx)}}$, qui in hac substitutus dat:

$$(4nn-1)y = \left(2n\sqrt{(aa+xx)}-x\right)\left(\frac{\sqrt{(aa+xx)}+x}{\sqrt{(aa+xx)}-x}\right)^n$$

sive:

$$(4nn-1)a^{2n}y = \left(2n\sqrt{(aa+xx)}-x\right)\left(\sqrt{(aa+xx)}+x\right)^{2n},$$

quae aequatio non differt ab ea, quam § 22. nacti sumus, nisi quod hic sit $2n$, quod ibi erat n.

Inveni

27. O

$y = x(\sqrt{(1-}$

ubi notandum
rem ipsius u.
ctio impar, sta

$\int\frac{x\,du}{\sqrt{(1+uu)}}$ (1

erit summis diffe
peractis.

$\frac{x\,du}{\sqrt{(1+uu)}} = d$

Jam hinc forme
nes pares, alter

$\frac{x\,du}{\sqrt{(1+uu)}} = dp$

quarum posteric

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III. Tertius Modus Inveniendi Trajectorias reciprocas Algebraicas ex formula

$$dy = (\sqrt{(1+uu)}+u)^n dx$$

27. Ob $d.(\sqrt{(1+uu)}+u)^n = \frac{n\,du}{\sqrt{(1+uu)}}(\sqrt{(1+uu)}+u)^n$ erit:

$$y = x(\sqrt{(1+uu)}+u)^n - n\int\frac{x\,du}{\sqrt{(1+uu)}}(\sqrt{(1+uu)}+u)^n$$

ubi notandum eſt, ut ante, eſſe oportere x functionem imparem ipſius u. Sit p functio quæcunque par ipſius u & q functio impar, ſtatuaturque

$$\int\frac{x\,du}{\sqrt{(1+uu)}}(\sqrt{(1+uu)}+u)^n = (p+q)(\sqrt{(1+uu)}+u)^n$$

erit ſumtis differentialibus, diviſioneque per $(\sqrt{(1+uu)}+u)^n$ peracta.

$$\frac{x\,du}{\sqrt{(1+uu)}} = dp + dq + \frac{npdu + nqdu}{\sqrt{(1+uu)}}$$

Jam hinc formentur duæ æquationes, quarum altera functiones pares, altera impares complectatur:

$$\frac{x\,du}{\sqrt{(1+uu)}} = dp + \frac{nq\,du}{\sqrt{(1+uu)}} \quad \& \quad dq + \frac{np\,du}{\sqrt{(1+uu)}} = 0$$

quarum poſterior ſponte dat: $p = -\frac{dq\sqrt{(1+uu)}}{n\,du}$

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Inventa autem functione p aequatio prior praebet:

$$x = nq + \frac{dp\sqrt{(1+uu)}}{du}$$ & ex substitutione habebimus

$$y = (x - np - nq)(\sqrt{(1+uu)}+u)^n = \frac{(dp+dq)\sqrt{(1+uu)}}{du}(\sqrt{(1+uu)}+u)^n$$

28. Hinc itaque adipiscimur

Tertiam regulam pro inveniendis trajectoriis reciprocis algebraicis.

Sumatur q *functio quaecunque imparium dimensionum ipsius* u, *indeque formetur functio* $p = -\frac{dq\sqrt{(1+uu)}}{ndu}$; *qua inventa erit curvae quaesitae*

Abscissa $AP = x = nq + \frac{dp\sqrt{(1+uu)}}{du}$

Applicata $PM = y = \frac{(dp+dq)(1+uu)}{du}(\sqrt{(1+uu)}+u)^n$

Vel cum sumto elemento du constante fit $dp = -\frac{ddq\sqrt{(1+uu)}}{ndu} - \frac{udq}{n\sqrt{(1+uu)}}$, erit curvae quaesitae:

Abscissa $AP = x = nq - \frac{udq}{ndu} - \frac{ddq(1+uu)}{ndu^2}$

Applicata $PM = y = \left(\frac{dq\sqrt{(1+uu)}}{du} - \frac{udq}{ndu} - \frac{ddq(1+uu)}{ndu^2}\right)(\sqrt{(1+uu)}+u)^n$

Ubi

Ubi iterum
factis, pro x
Y'; X'' &
x =
y =
quo pacto
rum augetu
29.
u, cujus ex
erit $\frac{dq}{du} =$
$x = nu^{\lambda}$ —
$y = (\lambda u^{\lambda-1}$
seu $y = -$
Unde expon
postquam hi

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Ubi iterum notandum est, si ex aliquot hypothesibus pro q factis, pro x & y jam inventi fuerint valores X & Y; X′ & Y′; X″ & Y″, &c. quaestioni quoque satisfieri his valoribus

$$x = \alpha X + \beta X' + \gamma X'' + \delta X''' \ \&c.$$
$$y = \alpha Y + \beta Y' + \gamma Y'' + \delta Y''' \ \&c.$$

quo pacto numerus curvarum inventarum facile in infinitum augetur.

29. Ponamus pro q potestatem quamcunque ipsius u, cujus exponens λ sit numerus impar: sit scilicet $q = u^{\lambda}$ erit $\frac{dq}{du} = \lambda u^{\lambda-1}$ & $\frac{ddq}{du^2} = \lambda(\lambda-1)u^{\lambda-2}$; hinc ergo fiet

$$x = nu^{\lambda} - \frac{\lambda u^{\lambda}}{n} - \frac{\lambda(\lambda-1)}{n}(u^{\lambda-2} + u^{\lambda}) = \frac{(nn-\lambda\lambda)}{n}u^{\lambda} - \frac{\lambda(\lambda-1)}{n}u^{\lambda-2}$$

$$y = \left(\lambda u^{\lambda-1}\sqrt{(1+uu)} - \frac{\lambda\lambda}{n}u^{\lambda} - \frac{\lambda(\lambda-1)}{n}u^{\lambda-2}\right)(\sqrt{(1+uu)}+u)^{n}$$

$$\text{seu } y = -\frac{\lambda u^{\lambda-2}}{n}\left(\lambda - 1 - nu\sqrt{(1+uu)} + \lambda u^{2}\right)(\sqrt{(1+uu)}+u)^{n}$$

Unde exponenti λ variis valoribus tribuendis reperietur, postquam hi valores per n fuerint multiplicati,

$\lambda =$

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$$\lambda = 1;\ \begin{cases} x = (nn-1)u \\ y = (n\sqrt{(1+uu)}-u)(\sqrt{(1+uu)}+u)^n \end{cases}$$

$$\lambda = 3;\ \begin{cases} x = (nn-9)u^3 - 6u \\ y = 3u(nn\sqrt{(1+uu)}-2-3uu)(\sqrt{(1+uu)}+u)^n \end{cases}$$

$$\lambda = 5;\ \begin{cases} x = (nn-25)u^5 - 20u^3 \\ y = 5u^3(nn\sqrt{(1+uu)}-4-5uu)(\sqrt{(1+uu)}+u)^n \end{cases}$$

$$\lambda = 7;\ \begin{cases} x = (nn-49)u^7 - 42u^5 \\ y = 7u^5(nn\sqrt{(1+uu)}-6-7uu)(\sqrt{(1+uu)}+u)^n \end{cases}$$

&c.

30. Dividantur formulæ primi casus per $nn-1$, atque prodibunt isti valores:

$$x = u \ \& \ (nn-1)y = (n\sqrt{(1+uu)}-u)(\sqrt{(1+uu)}+u)^n$$

unde ob $u = x$ variabilis u facillime eliminatur; orieturque sequens æquatio inter x & y

$$(nn-1)y = (n\sqrt{(1+xx)}-x)(\sqrt{(1+xx)}+x)^n$$

quæ est eadem æquatio, quam jam supra bis elicuimus, & quam maxime fœcundam esse linearum algebraicarum simpliciorum, fusius ostendi. Hæc enim æquatio ad rationalitatem reducta, si fuerit n numerus integer, ascendet ad $n+2$ dimensiones, sin autem n sit numerus fractus, puta $= \frac{\mu}{m}$ numerus dimensionum erit $= \mu + m$. Cum autem posi-

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positio $n = 1$ fit inanis, curva simplicissima hinc orienda erit linea quarti ordinis.

31. Jam conjunctim sumantur formulae casuum: $\lambda = 1$ & $\lambda = 3$, ex iisque reperietur:

$$x = (nn-1)\alpha u + (nn-9)\beta u^3 - 6\beta u$$

$$y = \left\{ \begin{array}{l} \alpha(n\sqrt{(1+uu)} - u \\ 2\beta u(nu\sqrt{(1+uu)} - 2 - 3uu) \end{array} \right\} (\sqrt{(1+uu)}+u)^n$$

Quo hinc facilius variabilis u eliminari possit, ponatur $\beta = \frac{1}{nn-9}$ & $(nn-1)\alpha = 6\beta = \frac{6}{nn-9}$; ut sit

$\alpha = \frac{6}{(nn-1)(nn-9)}$, & $x = u^3$, ideoque $u = \sqrt[3]{x}$.

Quare ob $(\sqrt{(1+uu)}+u)^n = (\sqrt{(1+uu)}-u)^{-n}$ erit:

$$y(\sqrt{(1+uu)}-u)^n = \frac{6n\sqrt{(1+uu)} - 6u}{(nn-1)(nn-9)} +$$

$$\frac{3nuu\sqrt{(1+uu)} - 6u - 9u^3}{nn-9} = \frac{3n(2+(nn-1)uu)\sqrt{(1+uu)}}{(nn-1)(nn-9)}$$

$- \frac{3u(2nn+3(nn-1)uu)}{(nn-1)(nn-9)}$. Hancobrem habebimus:

$$\frac{(nn-1)(nn-9)}{3} y(\sqrt{(1+uu)}-u)^n = \begin{array}{l} +n(2+(nn-1)uu)\sqrt{(1+uu)} \\ -u(2nn+3(nn-1)uu) \end{array}$$

existente $u = \sqrt[3]{x}$.

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32. Conjungamus cum his formulis casus $\lambda = 5$ eritque

$$x = \alpha u^3 + (nn-25)\beta u^5 - 20\beta u^3$$

$$\frac{(nn-1)(nn-9)}{3}\, y\, (\sqrt{(1+uu)} - u)^n =$$

$$+ n\alpha(2+(nn-1)uu)\sqrt{(1+uu)}$$

$$- \alpha u(2nn + 3(nn-1)uu)$$

$$+ \tfrac{5}{3}\beta(nn-1)(nn-9)u^3(nu\sqrt{(1+uu)} - 4 - 5uu)$$

Sit $\beta = \frac{1}{nn-25}$ & $\alpha = 20\beta = \frac{20}{nn-25}$, erit $x = u^5$ &

$$\frac{(nn-1)(nn-9)(nn-25)}{5}\, y(\sqrt{(1+uu)} - u)^n =$$

$$+n(24+12(nn-1)uu+(nn-1)(nn-9)u^4)\sqrt{(1+uu)}$$

$$-u(24nn+4nn(nn-1)uu+5(nn-1)(nn-9)u^4)$$

Simili autem modo ulterius pergere licet, his formulis cum casu $\lambda = 7$ conjungendis: prodibit autem

$x = u^7$ atque

$$\frac{(nn-1)(nn-9)(nn-25)(nn-49)}{7}\, y(\sqrt{(1+uu)} - u)^n =$$

$$n(720+360(nn-1)u^2+30(nn-1)(nn-9)u^4+(nn-1)(nn-9)$$
$$(nn-25)u^6)\sqrt{(1+uu)}$$

$$-u$$

—u(720nn…

7(…

unde non …
potest.

33. …
esse eliminat…
prærogativa …

x = −(nn…

y = λu…
quoties expo…
fractus, statu…
par, ac fiat λ…

x = n(n—…

y = nu…

seu per n(n—…

x = u…

(n—1), y = u…

34. …
ergo, ut exem…

y = 3x(2—3…

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$$-u(720mm+120mm(mm-1)u^2+6mm(mm-1)(mm-9)u^4+7(mm-1)(mm-9)(mm-25)u^6)$$

unde non difficulter lex sequentium formularum colligi potest.

33. Præter hos vero casus, quibus variabilis u facile eliminatur, ex formulis ante §. 29 inventis, alii eadem prærogativa gaudentes derivari possunt. Cum enim sit:

$$x=-(mm-\lambda\lambda)u^{\lambda}+\lambda(\lambda-1)u^{\lambda-2}$$

$$y=\lambda u^{\lambda-2}(\lambda-1-nu\surd(1+uu)+\lambda uu)(\surd(1+uu)+u)^{n}$$

quoties exponens n est numerus impar sive integer sive fractus, statuere licebit $\lambda=n$. Sit igitur n numerus impar, ac fiat $\lambda=n$ erit:

$$x=n(n-1)u^{n-2} \quad \&$$

$$y=nu^{n-2}(n-1-nu\surd(1+uu)+nuu)(\surd(1+uu)+u)^{n}$$

seu per $n(n-1)$, dividendo habebitur:

$$x=u^{n-2} \quad \&$$

$$(n-1),y=u^{n-2}(n-1-nu\surd(1+uu)+nuu)(\surd(1+uu)+u)^{n}$$

34. Si, $n=1$ ex his formulis nihil oritur; fiat ergo, ut exemplum afferamus, $n=3$, eritque $x=u$ &

$$y=3x(2-3x\surd(1+xx)+3xx)(\surd(1+xx)+x)^{3}.$$

Cum igitur sit:

$$(\sqrt{(1+xx)}+x)^3 = 3x+4x^3+(1+4xx)\sqrt{(1+xx)}, \text{ erit}$$

$$y = 3x(3x+2x^3+2(1+xx)\sqrt{(1+xx)}) \quad \text{ideoque}$$

$$y-9xx-6x^4 = 6x(1+xx)\sqrt{(1+xx)}$$

Quæ æquatio ad rationalitatem perducta fit:

$$yy-18xxy-12x^4y = 36xx+27x^4$$

quæ est nova linea quinti ordinis in numerum trajectoriarum reciprocarum referenda. Innumerabiles autem aliæ æquationes rationales inter x & y hinc erui possunt; ponendis pro n aliis numeris imparibus, quæ autem ad multo altiores dimensiones assurgent.

35. Simpliciores provenient, si in prima æquatione ponatur $\lambda = n-2$, unde fit:

$$x = (n-2)(n-3)u^{n-4} - 4(n-1)u^{n-2} \quad \&$$

$$y = (n-2)u^{n-4}(n-3+(n-2)uu-(n-2)u\sqrt{(1+uu)})(\sqrt{(1+uu)}+u)^n$$

Si enim hæ formulæ cum prius inventis conjungantur, fiatque

$$x = \alpha u^{n-2} + (n-2)(n-3)\beta u^{n-4} - 4(n-1(\beta u^{n-2}$$

$$\left.\begin{aligned}(n-1)y = {} & \alpha u^{n-2}(n-1+nuu-nu\sqrt{(1+uu)}) \\ & +(n-1)(n-2)\beta u^{n-4}(n-3+(n-2)uu- \\ & (n-2)u\sqrt{(1+uu)}\,)\end{aligned}\right\}(\sqrt{(1+uu)}+u)^n$$

po-

ponamus β:

$x = u^{n-4}$

$(n-2)(n-3)$

$((n-2)^2$

Quod si jam

$6y = x(6+$

$(\sqrt{}$

Simili autem

ut posito $n =$

buerit, progre

35. Ex

tam x quam y

impar, n vero

algebraicæ con

mulis $dx = vd$

& $n = 1$, ut si

formulæ non

nerales. Cu

$x = t -$

perspicuum est

sudt fuerit int

ponamus $\beta = \frac{1}{(n-2)(n-3)}$ & $\alpha = \frac{4(n-1)}{(n-2)(n-3)}$, ut fit $x = u^{n-4}$ erit:

$$(n-2)(n-3)y = u^{n-2}((n-2)(n-3) + nnuu + 4nu^4 - u((n-2)^2 + 4nuu)\sqrt{(1+uu)})(\sqrt{(1+uu)}+u)^n$$

Quod si jam ponatur $n = 5$, fiet $x = u$ ideoque

$$6y = x(6 + 25xx + 20x^4 - (9x + 20x^3)\sqrt{(1+xx)})(\sqrt{(1+xx)} + x)^5.$$

Simili autem modo hic ita adjungi poteſt caſus $\lambda = n - 4$, ut poſito $n = 7$ fiat $x = u$, atque pari modo, quousque libuerit, progredi licet.

35. Ex reliquis formulis §. 9 exhibitis, in quibus tam x quam y per tertiam variabilem t, cujus u eſt functio impar, v vero functio par, exprimitur, in genere formulæ algebraicæ commode erui non poſſunt: ſin autem in his formulis $dx = vdt$ & $dy = \left(\frac{1+u}{1-u}\right)^n vdt$ ponatur $v = 1 - uu$, & $n = 1$, ut fiat $dx = (1 - uu)dt$ & $dy\,(1 + u)^2 dt$, quæ formulæ non obſtante hac determinatione ſunt maxime generales. Cum igitur hinc ſit:

$$x = t - \int uudt \text{ \& } y = t + 2\int udt + \int uudt$$

perſpicuum eſt curvas algebraicas prodire, ſi tam $\int udt$ quam $\int uudt$ fuerit integrabile. Ad hoc utrumque præſtandum

 in

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In formula:	ponatur
$\int u\,dt = tu - \int t\,du$	$\int t\,du = p$; ut fit $t = \frac{dp}{du}$

ubi, quia t est functio impar ipsius u, necesse est ut sit p functio par. Jam hic valor ipsius t in altera formula substituatur, eritque

$$\int uu\,dt = tuu - 2\int tu\,du = \frac{uu\,dp}{du} - 2\int u\,dp = \frac{uu\,dp}{du} - 2pu + 2\int p\,du$$

ponatur $\int p\,du = q$, ut fit $p = \frac{dq}{du}$, eritque q functio impar ipsius u; & sumto du constante fiet $dp = \frac{ddq}{du}$ atque substitutionibus retro factis erit $t = \frac{ddq}{du^2}$ &

$$\int uu\,dt = \frac{uu\,ddq}{du^2} - \frac{2u\,dq}{du} + 2q \quad \& \quad \int u\,dt = \frac{u\,ddq}{du^2} - \frac{dq}{du}$$

37. His valoribus substitutis habebimus

Quartam Regulam pro inveniendis trajectoriis reciprocis algebraicis:

Sumatur q *functio quaecunque impar ipsius* u, *positoque elemento* du *constante erit curvae quaesitae*

Abscissa AP $= x = \frac{ddq(1-uu)}{du^2} + \frac{2u\,dq}{du} - 2q$

Applicata PM $= y = \frac{ddq(1+u)^2}{du^2} - \frac{2dq(1+u)}{du} + 2q$

Ponamus $q = u^\lambda$, existente λ numero impari, erit

$x =$

$x = \lambda(\lambda -$

$y = \lambda(\lambda - 1$

$x = \lambda(\lambda -$

$y = \lambda(\lambda - 1)$

38. …minatur, si e…
$\lambda(\lambda - 1)(x -$
$(x -$

hincque $uu =$

inveniturque

$u =$

qui valor si in

$x + y = 2\lambda$
aequatio inter
dis ad rational
gulis pro inve
carum tradit
adhuc de

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$$x = \lambda(\lambda-1)u^{\lambda-2}(1-uu) + 2\lambda u^{\lambda} - 2u^{\lambda}$$

$$y = \lambda(\lambda-1)u^{\lambda-2}(1+u)^{2} - 2\lambda u^{\lambda-1}(1+u) + 2u^{\lambda}$$

five

$$x = \lambda(\lambda-1)u^{\lambda-2} - (\lambda-1)(\lambda-2)u^{\lambda}$$

$$y = \lambda(\lambda-1)u^{\lambda-2} + 2\lambda(\lambda-2)u^{\lambda-1} + (\lambda-1)(\lambda-2)u^{\lambda}$$

38. Ex his formulis variabilis u non difficulter eliminatur, fi enim altera per alteram dividatur, erit

$$\lambda(\lambda-1)(x-y) + 2\lambda(\lambda-2)xu + (\lambda-1)(\lambda-2)uu(x+y) = 0$$

hincque $uu = \dfrac{-2\lambda(\lambda-2)xu - \lambda(\lambda-1)(x-y)}{(\lambda-1)(\lambda-2)(x+y)}$;

inveniturque

$$u = \frac{-\lambda(\lambda-2)x \pm \sqrt{\lambda(\lambda-2)((\lambda-1)^{2}yy - xx)}}{(\lambda-1)(\lambda-2)(x+y)}$$

qui valor fi in altera æquatione, vel in fumma amborum $x + y = 2\lambda u^{\lambda-2}(\lambda-1+(\lambda-2)u)$ fubftituatur, orietur æquatio inter x & y, quæ femel tantum quadratis fumendis ad rationalitatem reducetur. Quatuor autem hifce regulis pro inventione trajectoriarum reciprocarum algebraicarum traditis, quicquid in folutione hujus problematis adhuc defiderari poterat, hic abunde præftitiffe mihi videor.

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