Neque ergo hinc neque ex majoribus valoribus ipsi m tribuendis numeros amicabiles elicere licet.

Regula. IV.

§. LI. Possunt etiam aliae expressiones pro factore communi a inveniri, ex quibus fractionis $\frac{b}{c}$ denominator c vel unitati, vel potestati binarii fiat aequalis. Fingamus namque $a = 2^n (g-1)(h-1)$, ut sint $g-1$ & $h-1$ numeri primi; erit $\int a = (2^{n+1} - 1) gh = 2^{n+1} gh - gh$; at est $2a = 2^{n+1} gh - 2^{n+1} g - 2^{n+1} h + 2^{n+1}$ unde fit

$$2a - \int a = gh - 2^{n+1} g - 2^{n+1} h + 2^{n+1}$$

Ponatur $2a - \int a = d$, erit $gh - 2^{n+1}(g+h) + 2^{n+1} = d$ & $(g - 2^{n+1})(h - 2^{n+1}) = d - 2^{n+1} + 2^{2n+2}$: unde per resolutionem in factores ejusmodi valores pro g & h elici debent, ut $g-1$ & $h-1$ fiant numeri primi, eritque tum $a = 2^n (g-1)(h-1)$ & $\frac{b}{c} = \frac{a}{d}$.

I. Ponamus $n = 1$, erit $(g-4)(h-4) = d + 12$, ubi ut $d + 12$ duos obtineat factores pares, sequentes prodibunt valores:

Sit $d = 4$: erit $(g-4)(h-4) = 16 = 2.8$, unde $g = 6$, $h = 12$; $a = 2.5.11$ atque $\frac{b}{c} = \frac{2.5.11}{4}$ ergo $b = 5.11$ & $c = 2$.

Sit

Sit $d = 8$; erit $(g-4)(h-4) = 20 = 2.10$, unde $g = 6$, $h = 14$;

$a = 2.5.13$, atque $\frac{b}{c} = \frac{2.5.13}{8}$; ergo $b = 5.13$ & $c = 4$.

Sit $d = 16$; erit $(g-4)(h-4) = 28 = 2.14$; unde $g = 6$, $h = 18$

$a = 2.5.17$, atque $\frac{b}{c} = \frac{2.5.17}{16}$; ergo $b = 5.17$ & $c = 8$.

II. Ponamus $n = 2$, erit $(g-8)(h-8) = d + 56$; atque $a = 4(g-1)(h-1)$, unde sequentes casus resultant:

Sit $d = 4$, erit $(g-8)(h-8) = 60 = 6.10$, unde $g = 14$ & $h = 18$

$a = 4.13.17$, atque $\frac{b}{c} = \frac{4.13.17}{4}$; ergo $b = 13.17$ & $c = 1$

Sit $d = 8$, erit $(g-8)(h-8) = 64 = 4.16$; unde $g = 12$, & $h = 24$

$a = 4.11.23$, atque $\frac{b}{c} = \frac{4.11.23}{8}$; ergo $b = 11.23$ & $c = 2$

Sit $d = 16$, erit $(g-8)(h-8) = 72 = 6.12$; unde $g = 14$, & $h = 20$

$a = 4.13.19$, atque $\frac{b}{c} = \frac{4.13.19}{16}$; ergo $b = 13.19$ & $c = 4$

III. Ponamus $n = 3$, ut sit $a = 8(g-1)(h-1)$, oportebitque esse $(g-16)(h-16) = d + 240$

Sit $d = 4$, erit $(g-16)(h-16) = 244 = 2.122$; unde $g = 18$, $h = 138$; $a = 8.17.137$ & $\frac{b}{c} = \frac{8.17.137}{4}$; ergo $b = 2.17.137$ & $c = 1$

Sit $d = 8$, erit $(g-16)(h-16) = 248 = 2.124$; unde $g = 18$, $h = 140$

$a = 8.17.139$ & $\frac{b}{c} = \frac{8.17.139}{8}$; ergo $b = 17.139$ & $c = 1$

Sit $d = 16$, erit $(g-16)(h-16) = 256 = 4.64$; unde $g = 20$, $h = 80$

$a = 8.19.79$; $\frac{b}{c} = \frac{8.19.79}{16}$; ergo $b = 19.79$ & $c = 2$

Sit iterum $d = 16$; $(g-16)(h-16) = 8.32$; unde $g = 24$ & $h = 48$

$a = 8.23.47$; $\frac{b}{c} = \frac{8.23.47}{16}$ ergo $b = 23.47$ & $c = 2$.

Sumtis autem hinc valoribus pro a, si numeri amicabiles statuantur $a(x-1)(y-1)$ & $a(xy-1)$, ut sint $x-1$, $y-1$ & $xy-1$ numeri primi, efficiendum est ut sit $(cx-b)(cy-b)=bb$.

Exempl. 1.

§. LII. Sit $a=2.5.11$, erit $b=5.11=55$ & $c=2$, unde fiet $(2x-55)(2y-55)=5^2.11^2$.

$2x-55$	1	5	25
$2y-55$	3025	605	125
x	28	30	40
y	1540	330	90
$x-1$	27*	29	39*
$y-1$	- - -	329*	- -
$xy-1$	- - -	- - -	- -

Hinc ergo nulli obtinentur numeri amicabiles.

Exemplum. 2.

§. LIII. Sit $a=2.5.13$, erit $b=5.13=65$ & $c=4$; unde fit $(4x-65)(4y-65)=5^2.13^2$.

At hic numerus $5^2.13^2$ non resolvi potest in duos factores, qui 65 aucti fiant per 4 divisibiles: quod idem in valore $a=2.5.17$ usu venit.

Exempl. 3.

LIV. Sit $a=4.13.17$, erit $b=13.17=221$ & $c=1$ esseque oportet $(x-221)(y-221)=13^2.17^2$. unde

$x-221$	13	17	169
$y-221$	3757	- -	289
$x-1$	233	237*	389
$y-1$	3977*	- - -	509
$xy-1$	- - -	- - -	198899

in

In resolutione ultima fit $x-1$ & $y-1$ numerus primus, quaestio ergo huc redit utrum $xy-1 = 198899$ sit numerus primus nec ne? Etiamsi autem hic numerus terminum 100000 excedat, tamen demonstrare possum eum esse primum, unde numeri amicabiles erunt $\begin{cases} 4.\,13.\,17.\,389.\,509 \\ 4.\,13.\,17.\,198899 \end{cases}$

Scholion.

§. LV. Numerum autem hunc 198899 esse primum inde colligo, quod observavi esse $198899 = 2.\,47^2 + 441^2$, ita ut 198899 sit numerus in hac forma $2aa + bb$ contentus. Certum autem est, si quis numerus unico modo in forma $2aa + bb$ contineatur, tum eum esse primum, sin autem duplici vel pluribus modis ad formam $2a + bb$ redigi queat, tum esse compositum. Quaesivi ergo utrum a numero hoc 198899 aliud quadratum duplum praeter 47^2 subtrahi queat, ut residuum evadat quadratum, nullum que subducto calculo inveni: ex quo tuto conclusi hunc numerum esse primum, ideoque numeros inventos esse amicabiles. Ex reliquis autem valoribus ipsius a, quos exhibui, nulli reperiuntur numeri amicabiles.

Regula. V.

§. LVI. Possunt etiam alii numeri idonei pro a assumi, ex quibus numeros amicabiles eruere liceat. Cum autem pro iis regula generalis tradi nequeat, aliquos tantum hic evolvam, ad quorum imitationem non erit difficile alios excogitare.

I. Sit ergo $a = 3^2.\,5.\,13$, erit $\int a = 13.\,6.\,14$ & ob $2a = 90.\,13$ & $\int a = 84.\,13$, erit $2a - \int a = 6.\,13$ atque $\frac{b}{c} = \frac{a}{2a - \int a} = \frac{3^2.\,5.\,13}{6.\,13} = \frac{15}{2}$ ideoque $b = 15$ & $c = 2$.

II. Sit

II. Sit $a = 3^2 \cdot 7 \cdot 13$ erit $\int a = 13 \cdot 8 \cdot 14 = 16 \cdot 7 \cdot 13$ unde ob $2a = 18 \cdot 7 \cdot 13$, erit $2a - \int a = 2 \cdot 7 \cdot 13$, ideoque $\frac{b}{c} = \frac{3^2 \cdot 7 \cdot 13}{2 \cdot 7 \cdot 13} = \frac{9}{2}$; unde $b = 9$ & $c = 2$.

III. Sit $a = 3^2 \cdot 7^2 \cdot 13$, erit $\int a = 13 \cdot 3 \cdot 19 \cdot 14 = 2 \cdot 3 \cdot 7 \cdot 13 \cdot 19$ & $2a = 42 \cdot 3 \cdot 7 \cdot 13$, unde $2a - \int a = 4 \cdot 3 \cdot 7 \cdot 13$, ideoque $\frac{b}{c} = \frac{3^2 \cdot 7^2 \cdot 13}{4 \cdot 3 \cdot 7 \cdot 13} = \frac{21}{4}$, ergo $b = 21$ & $c = 4$.

IV. Sit $a = 3^3 \cdot 5$ erit $\int a = 5 \cdot 8 \cdot 6 = 16 \cdot 3 \cdot 5$. Ergo ob $2a = 18 \cdot 3 \cdot 5$ erit $2a - \int a = 2 \cdot 3 \cdot 5$; hincque $\frac{b}{c} = \frac{3^3 \cdot 5}{2 \cdot 3 \cdot 5} = \frac{9}{2}$ & $b = 9$ & $c = 2$.

V. Sit $a = 3^2 \cdot 5 \cdot 13 \cdot 19$, erit $\int a = 13 \cdot 6 \cdot 14 \cdot 20 = 16 \cdot 3 \cdot 5 \cdot 7 \cdot 13$ & ob $2a = 114 \cdot 3 \cdot 5 \cdot 13$ & $\int a = 112 \cdot 3 \cdot 5 \cdot 13$ erit $\frac{b}{c} = \frac{3^2 \cdot 5 \cdot 13 \cdot 19}{2 \cdot 3 \cdot 5 \cdot 13} = \frac{3 \cdot 19}{2}$ & $b = 3 \cdot 19 = 57$ & $c = 2$.

VI. Sit $a = 3^2 \cdot 7^2 \cdot 13 \cdot 19$, erit $\int a = 13 \cdot 3 \cdot 19 \cdot 14 \cdot 20 = 8 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19$ & ob $2a = 42 \cdot 3 \cdot 7 \cdot 13 \cdot 19$ erit $\frac{b}{c} = \frac{3^2 \cdot 7^2 \cdot 13 \cdot 19}{2 \cdot 3 \cdot 7 \cdot 13 \cdot 19} = \frac{21}{2}$, unde fit $b = 21$ & $c = 2$.

Positis autem numeris amicabilibus $a(x-1)(y-1)$ & $a(xy-1)$ fieri debet $(cx-b)(cy-b) = bb$.

Exem-

Exemplum. 1.

§. LVII. Sit $b = 15$, $c = 2$; erit $a = 3^2 . 5 . 13$ & satisfieri oportet huic æquationi $(2x - 15)(2y - 15) = 225$:

$2x - 15$	1	5	9
$2y - 15$	225	45	25
x	8	10	12
y	120	30	20
$x - 1$	7	9*	11
$y - 1$	119*	- -	19
$xy - 1$	- - -	- -	239

Numeri ergo amicabiles erunt $\left\{ \begin{matrix} 3^2 . 5 . 13 . 11 . 19 \\ 3^2 . 5 . 13 . 239 \end{matrix} \right)$

Exempl. 2.

§. LVIII. Sit $b = 9$, $c = 2$; erit vel $a = 3^2 . 7 . 13$ vel $a = 3^2 . 5$; & æquatio resolvenda $(2x - 9)(2y - 9) = 81$.

$2x - 9$	3
$2y - 9$	27
x	6
y	18
$x - 1$	5
$y - 1$	17
$xy - 1$	107

Unde cum sit $x - 1 = 5$, hic valor cum $a = 3^2 . 5$ combinari nequit. Erunt ergo numeri amicabiles:

$$\left\{ \begin{matrix} 3^2 . 7 . 13 . 5 . 17 \\ 3^2 . 7 . 13 . 107 \end{matrix} \right)$$

Exempl. 3.

§. LIX. Sit $b = 21$ & $c = 4$, erit $a = 3^2 . 7^2 . 13$ & æquatio resolvenda $(4x - 21)(4y - 21) = 441$.

4x

$4x - 21$	3
$4y - 21$	147
x	6
y	42
$x - 1$	5
$y - 1$	41
$xy - 1$	251

Quia x & y debent esse numeri pares alia resolutio locum non habet.

Ex hac ergo prodeunt numeri amicabiles hi: $\left\{ \begin{array}{l} 3^2 \cdot 7^2 \cdot 13 \cdot 5 \cdot 41 \\ 3^2 \cdot 7^2 \cdot 13 \cdot 251 \end{array} \right)$

Exempl. 4.

§. LX. Sit $b = 21$ & $c = 2$, erit $a = 3^2 \cdot 7^2 \cdot 13 \cdot 19$ & æquatio resolvenda $(2x - 21)(2y - 21) = 441$.

$2x - 21$	3	7
$2y - 21$	147	63
x	12	12
y	84	42
$x - 1$	11	13
$y - 1$	83	41
$xy - 1$	1007*	587

Quia autem valor $x - 1 = 13$ jam in valore a continetur, hinc nulli obtinentur numeri amicabiles.

Exemplum. 5.

§. LXI. Sit $b = 57$ & $c = 2$, erit $a = 3 \cdot 5 \cdot 13 \cdot 19$, & æquatio resolvenda $(2x - 57)(2y - 57) = 3249$

$2x - 57$	3	19
$2y - 57$	1083	171
x	30	38
y	570	114
$x - 1$	29	34
$y - 1$	569	113
$xy - 1$	17099	4331*

Hinc ergo oriuntur numeri amicabiles hi:

$$\left\{ \begin{array}{l} 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 29 \cdot 569 \\ 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 17099 \end{array} \right)$$

Exem-

Exemplum. 6.

§. LXII. Sit $b = 45$ & $c = 2$, erit $a = 3^4 . 5 . 11$, & aequatio resolvenda. $(2x - 45)(2y - 45) = 2025$

$2x - 45$	3	15
$2y - 45$	675	135
x	24	30
y	360	90
$x - 1$	23	29
$y - 1$	359	89
$xy - 1$	8639*	2699

Hinc ergo oriuntur numeri amicabiles:

$$\left\{ \begin{array}{l} 3^4 . 5 . 11 . 29 . 89 \\ 3^4 . 5 . 11 . 2699 \end{array} \right)$$

Exempl. 7.

§. LXIII. Sit $b = 77$ & $c = 2$, erit $a = 3^2 . 7^2 . 11 . 13$, & aequatio resolvenda $(2x - 77)(2y - 77) = 49 . 121$.

$2x - 77$	7	11
$2y - 77$	847	539
x	42	44
y	462	308
$x - 1$	41	43
$y - 1$	461	307
$xy - 1$	19403	13551*

Hinc ergo oriuntur numeri amicabiles:

$$\left\{ \begin{array}{l} 3^2 . 7^2 . 11 . 13 . 41 . 461 \\ 3^2 . 7^2 . 11 . 13 . 19403 \end{array} \right)$$

Exempl. 8.

§. LXIV. Sit $b = 105$, $c = 2$, erit $a = 3^2 . 5 . 7$. & aequatio resolvenda $(2x - 105)(2y - 105) = 105^2$.

$2x-105$	3	7	15	35
$2y-105$	3675	--	735	--
x	54	56	60	70
y	1890	--	420	--
$x-1$	53	55*	59	69*
$y-1$	1889	--	419	
$xy-1$	102059	--	25199*	

Cum 102059 sit numerus primus, quia continetur in forma $8a+3$ & unico modo ad formam $2aa+bb$ reducitur, numeri amicabiles hinc orti erunt

$$\begin{cases} 3^2.5.7.53.1889 \\ 3^2.5.7.102059 \end{cases}$$

Scholion.

§. LXV. Numeri ergo amicabiles, quos hactenus ex forma *epq; ar* invenimus, sunt:

I. $\begin{cases} 2^2.5.11 \\ 2^2.71 \end{cases}$

II. $\begin{cases} 2^4.23.47 \\ 2^4.1151 \end{cases}$

III. $\begin{cases} 2^7.191.393 \\ 2^7.73727 \end{cases}$

IV. $\begin{cases} 4.23.5.137 \\ 4.23.827 \end{cases}$

V. $\begin{cases} 4.13.17.389.509 \\ 4.13.17.198899 \end{cases}$

VI. $\begin{cases} 3^2.5.13.11.19 \\ 3^2.5.13.239 \end{cases}$

VII. $\begin{cases} 3^2.7.13.5.17 \\ 3^2.7.13.107 \end{cases}$

VIII. $\begin{cases} 3^2.7^2.13.5.41 \\ 3^2.7^2.13.251 \end{cases}$

IX.

IX. $\begin{cases} 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 29 \cdot 569 \\ 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 17099 \end{cases}$

X. $\begin{cases} 3^4 \cdot 5 \cdot 11 \cdot 29 \cdot 89 \\ 3^4 \cdot 5 \cdot 11 \cdot 2699 \end{cases}$

XI. $\begin{cases} 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 41 \cdot 461 \\ 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19403 \end{cases}$

XII. $\begin{cases} 3^2 \cdot 5 \cdot 7 \cdot 53 \cdot 1889 \\ 3^2 \cdot 5 \cdot 7 \cdot 102059 \end{cases}$

Problema 2.

§. LXVI. *Invenire numeros amicabiles secundae formae* apq ars; *positis* p, q, r, s *numeris primis & factore communi* a *dato.*

Solutio.

Cum factor communis a detur, quaeratur ex eo valor fractionis $\frac{b}{c} = \frac{a}{2a - \int a}$ in minimis terminis, hincque erit $a : \int a = b : 2b - c$. Deinde cum esse debeat $\int p . \int q = \int r . \int s$ seu $(p+1)(q+1) = (r+1)(s+1)$, ponatur uterque valor $= \alpha\beta xy$, & sumatur:

$$p = \alpha x - 1;\quad q = \beta y - 1;\quad r = \beta x - 1;\quad s = \alpha y - 1.$$

Ubi manifestum est hos numeros α, β, x, y, ejusmodi esse debere, ut p, q, r, s fiant numeri primi, & numeri amicabiles erunt

$$a(\alpha x - 1)(\beta y - 1) \;\&\; a(\beta x - 1)(\alpha y - 1)$$

Praeterea vero ex natura numerorum amicabilium esse debet:

$$\alpha\beta xy \int a = a(\alpha x - 1)(\beta y - 1) + a(\beta x - 1)(\alpha y - 1)$$

seu ob $\int a : a = 2b - c : b$ erit

$$\left.\begin{array}{l} 2b\alpha\gamma xy \\ -\alpha\gamma xy \end{array}\right\} = 2b\alpha\gamma xy - b\alpha x - b\gamma y + 2b \atop \qquad - b\gamma x - b\alpha y$$

vel $c\alpha\gamma xy = b(\alpha+\gamma)(x+y) - 2b$. Unde fit

$$\begin{array}{l} cc\alpha^2\gamma^2 xy - bc\alpha\gamma(\alpha+\gamma)x \\ \qquad - bc\alpha\gamma(\alpha+\gamma)y + bb(\alpha+\gamma)^2 = -2bc\alpha\gamma + bb(\alpha+\gamma)^2 \end{array}$$

Quare satisfieri debet huic æquationi:

$$(c\alpha\gamma x - b(\alpha+\gamma))(c\alpha\gamma y - b(\alpha+\gamma)) = bb(\alpha+\gamma)^2 - 2bc\alpha\gamma$$

Numerus ergo $bb(\alpha+\gamma)^2 - 2bc\alpha\gamma$ quovis casu in duo ejusmodi factores, qui sint PQ, resolvi debet, ut positis

$$x = \frac{P + b(\alpha+\gamma)}{c\alpha\gamma} \;\&\; y = \frac{Q + b(\alpha+\gamma)}{c\alpha\gamma}$$

hi numeri x & y non solum fiant integri, sed etiam $\alpha x - 1$; $\gamma y - 1$; $\gamma x - 1$; & $\alpha y - 1$ numeri primi. Erit igitur

$$p = \frac{P + b\alpha + (b-c)\gamma}{c\gamma}; \; q = \frac{Q + b\gamma + (b-c)\alpha}{c\alpha}$$

$$r = \frac{P + b\gamma + (b-c)\alpha}{c\alpha}; \; s = \frac{Q + b\alpha + (b-c)\gamma}{c\gamma}$$

Quovis ergo valore ipsius a proposito, unde reperitur $\frac{b}{c} = \frac{a}{2a - \int : a}$, dispiciendum est, utrum cum numeri α & γ ita assumi, tum resolutio hæc:

$$bb(\alpha+\gamma)^2 - 2bc\alpha\gamma = PQ$$

ita institui queat, ut valores modo traditi pro p, q, r & s fiant numeri primi, & tales quidem, ut factor communis a nullum eorum involvat. Quoties autem his conditionibus satisfieri poterit, erunt numeri amicabiles: apq & ars.

Co-

Coroll.

§. LVII. Quoniam esse nequit $a = c$, pro his numeris a & c ponantur numeri simpliciores, hincque orientur casus sequentes:

I. Sit $a = 1$; $c = 2$; erit $PQ = 9bb - 4bc$; &

$$p = \frac{P + 3b - 2c}{2c}; \quad q = \frac{Q + 3b - c}{c}$$

$$r = \frac{P + 3b - c}{c}; \quad s = \frac{Q + 3b - 2c}{2c}$$

II. Sit $a = 1$; $c = 3$; erit $PQ = 16bb - 6bc$ &

$$p = \frac{P + 4b - 3c}{3c}; \quad q = \frac{Q + 4b - c}{c}$$

$$r = \frac{P + 4b - c}{c}; \quad s = \frac{Q + 4b - 3c}{3c}$$

III. Sit $a = 2$; $c = 3$; erit $PQ = 25bb - 12bc$

$$p = \frac{P + 5b}{3c} - 1; \quad q = \frac{Q + 5b}{2c} - 1$$

$$r = \frac{P + 5b}{2c} - 1; \quad s = \frac{Q + 5b}{3c} - 1$$

IV. Sit $a = 1$; $c = 4$; erit $PQ = 25bb - 8bc$ &

$$p = \frac{P + 5b}{4c} - 1; \quad q = \frac{Q + 5b}{c} - 1$$

$$r = \frac{P + 5b}{c} - 1; \quad s = \frac{Q + 5b}{4c} - 1$$

V. Sit

V. Sit $a = 3$; $c = 4$; erit $PQ = 49bb - 24bc$

$$p = \frac{P+7b}{4c} - 1; \quad q = \frac{Q+7b}{3c} - 1$$

$$r = \frac{P+7b}{3c} - 1; \quad s = \frac{Q+7b}{4c} - 1$$

VI. Sit $a = 1$; $c = 5$; erit $PQ = 36bb - 10bc$

$$p = \frac{P+6b}{5c} - 1; \quad q = \frac{Q=6b}{c} - 1; \quad r = \frac{P+6b}{c} - 1; \quad s = \frac{Q+6b}{5c} - 1$$

VII. Sit $a = 2$; $c = 5$; erit $PQ = 49bb - 20bc$

$$p = \frac{P+7b}{5c} - 1; \quad q = \frac{Q+7b}{2c} - 1; \quad r = \frac{P+7b}{2c} - 1; \quad s = \frac{Q+7b}{5c} - 1$$

VIII. Sit $a = 3$; $c = 5$; erit $PQ = 64bb - 30bc$

$$p = \frac{P+8b}{5c} - 1; \quad q = \frac{Q+8b}{3c} - 1; \quad r = \frac{P+8b}{3c} - 1;$$

$$s = \frac{Q+8b}{5c} - 1$$

IX. Sit $a = 4$; $c = 5$; erit $PQ.\ 81bb - 40bc$

$$p = \frac{P+9b}{5c} - 1; \quad q = \frac{Q+9b}{4c} - 1; \quad r = \frac{P+9b}{4c} - 1;$$

$$s = \frac{Q+9b}{5c} - 1$$

X. Sit

X.

$p =$

$s =$

XI.

$p =$

s

S

tos, qui

apti, eve

numeros

§.

cafus feca

2'rr, fie

PQ

$p =$

Q

factores e

fiunt per 3

$P =$

$Q =$

Euleri C

X. Sit $a = 1$; $c = 6$; erit $PQ = 49bb - 12bc$

$$p = \frac{P+7b}{6c} - 1;\ q = \frac{Q+7b}{c} - 1;\ r = \frac{P+7b}{c} - 1;$$

$$s = \frac{Q+7b}{6c} - 1;$$

XI. Sit $a = 5$; $c = 6$, erit $PQ = 121bb - 60bc$

$$p = \frac{P+11b}{6c} - 1;\ q = \frac{Q+11b}{5c} - 1;\ r = \frac{P+11b}{5c} - 1;$$

$$s = \frac{Q+11b}{6c} - 1;$$

Secundum hos igitur casus valores ipsius a jam ante adhibitos, quia prae ceteris ad numeros amicabiles inveniendos videntur apti, evolvam, ex iis autem potissimum eos eligam, qui actu ad numeros amicabiles deducunt.

Exemplum. 1.

§. LXVIII. Sit $a = 2^2$; erit $b = 4$, & $c = 1$. Sumatur casus secundus quo $a = 1$, $c = 3$, ut numeri amicabiles sint 2^2pq & 2^2rs, fierique debet

$$PQ = 16.16 - 6.4 = 232, \text{ atque}$$

$$p = \frac{P+16}{3} - 1;\ q = Q + 16 - 1;\ r = P + 16 - 1 \ \&\ s = \frac{Q+16}{3} - 1$$

factores ergo numeri 232 ita debent esse comparati, ut 16 aucti fiant per 3 divisibiles:

$P = 2$ Alia resolutio nulla succedit, si enim poneretur
$Q = 116$ $P = 8$, fieret Q numerus impar, neque ergo q &

$P + 16 =$	18
$Q + 16 =$	132
$p =$	5
$q =$	131
$r =$	17
$s =$	43

& s numeri primi esse possent. Hine ergo obtinentur hi numeri amicabiles:

$$\begin{Bmatrix} 2^2 . 5 . 131 \\ 2^2 . 17 . 43 \end{Bmatrix}$$

Exemplum. 2.

§. LXIX. Si $u = 1$ & $t = 3$, & a potestas binarii altior: inventio numerorum amicabilium non succedit, donec perveniatur ad $a = 2^8$. Tum autem erit $b = 2^1$ & $t = 1$: atque

$$PQ = 16.\, 2^{16} - 6.\, 2^8 = 2^9 (2^{11} - 3) = 512.\, 2045 = 512.5.409;$$

$$p = \frac{P + 1024}{3} - 1;\; q = Q + 1024 - 1;\; r = P + 1024 - 1;\; s = \frac{Q + 1024}{3} - 1$$

unde factores P & Q ita debent esse comparati, ut quaternario aucti per 3, (vel ut quoti fiant pares) per 6 sint divisibiles.

$P =$	2	8	20	32	80	128	320	1280
$Q =$	- -				13088	8180		
$P + 1024 =$	1026	1032	1044	1056	1104	1152	1344	2304
$Q + 1024 =$	- -	- -			14112	9204		
$p =$	341	343*	347		367	383	447*	767*
$q =$	- - -	- -			14111*	9203		
$r =$	1025*	- -	1043*	1055*	1103	1151	1343	2303
$s =$	- - -	- -	- -	- -	4703	3067		

Erunt ergo numeri amicabiles $\begin{Bmatrix} 2^8 . 383 . 9203 \\ 2^8 . 1151 . 3067 \end{Bmatrix}$

Exem-

Exempl. 3.

§. LXX. Sit $a = 2$ & $c = 3$ & sumatur $a = 3^2.5.13$ ut sit $b = 15$ & $c = 2$; erit $PQ = 25.225 - 12.30 = 3^4.5.13$

$$p = \frac{P+75}{6} - 1;\ q = \frac{Q+75}{4} - 1;\ r = \frac{P+75}{4} - 1;$$

$$s = \frac{Q+75}{6} - 1$$

unde factores P Q ejusmodi esse debent, ut ternario aucti fiant per 24 divisibiles.

P	$=$	45
Q	$=$	117
$P+75$	$=$	120
$Q+75$	$=$	192
p	$=$	19
q	$=$	47
r	$=$	29
s	$=$	31

Aliæ resolutiones non inveniunt locum; unde hinc numeri amicabiles prodeunt.

$$\begin{Bmatrix} 3^2.5.13.19.47 \\ 3^2.5.13.29.31 \end{Bmatrix}$$

Exempl. 3.

§. LXXI. Sit $a = 1$ & $c = 4$, sumatur $a = 3^2.5$, ut sit $p = 9$, $c = 2$, erit $PQ = 25.81 - 8.18 = 9.11.19$ &

$$p = \frac{P+45}{8} - 1;\ q = \frac{Q+45}{2} - 1;\ r = \frac{P+45}{2} - 1;$$

$$s = \frac{Q+45}{8} - 1$$

unde P & Q ejusmodi debent esse numeri, ut quinario aucti per 8 fiant divisibiles:

P	=	3	19
Q	=	627	99
P+45	=	28	64
Q+45	=	672	144
p	=	5	7
q	=	335*	71
r	=	23	31
s	=	83	17

Hinc ergo oriuntur numeri amicabiles:

$$\left\{ \begin{array}{l} 3^2 \cdot 5 \cdot 7 \cdot 71 \\ 3^2 \cdot 5 \cdot 31 \cdot 17 \end{array} \right.$$

Scholion.

§. LXXII. Hæ autem operationes nimis sunt incertæ, ac plerumque plures frustra instituuntur, antequam numeri amicabiles se offerunt. Labor quoque foret vehementer prolixus, si singulis valoribus ipsius a, quos quidem supra exhibui, per singulos casus litterarum α & ϵ percurrere velimus; atque nimis raro evenit, ut quatuor numeri pro p, q, r & s resultantes simul fiant primi. Tum vero etiam inventio numerorum amicabilium per determinationem rationis α & ϵ nimis restringitur, atque dantur casus hujusmodi numerorum, in quibus ratio α & ϵ tam est complicata, ut nulla probabili ratione eligi potuisset, cujusmodi sunt numeri amicabiles $2^4 \cdot 19 \cdot 8563$ & $2^4 \cdot 83 \cdot 2039$, ad quos hac via inveniendos ratio $\alpha : \epsilon$ assumi debuisset $5 : 21$ vel $1 : 102$. Hanc ob rem huic methodo nimis sterili & operosæ diutius non immoror, sed aliam viam aperiam, qua facilius & expeditius numeros amicabiles tam hujus secundæ formæ, quam aliarum magis compositarum investigare liceat; & quæ similis sit præcedenti, quæ tribus tantum numeris primis reperiendis absolvitur.

Problema. 2.

§. LXXIII. *Invenire numeros amicabiles hujus formæ* a p q *&* a r s, *fr, ubi p, q, & r sint numeri primi, f sive primus sive compositus, qui perinde ac factor communis a sit datus.*

Solutio.

Quaerantur iterum ex cognito factore communi a valores b & c ut sit $\frac{b}{c} = \frac{a}{2a - \int a}$; & sit numeri f summa divisorum $\int f = gh$. Cum igitur primo requiratur ut sit $\int p . \int q = \int f . \int r$, erit $(p+1)(q+1) = gh(r+1)$. Ponatur $r+1 = xy$, $p+1 = hx$ & $q+1 = gy$, & necesse erit, ut sint hi tres numeri primi, scilicet $p = hx - 1$; $q = gy - 1$ & $r = xy - 1$. Deinde opus est, ut sit $\int apq = ghxy \int a = a(hx-1)(gy-1) + af(xy-1) = a((gh+f)xy - hx - gy + 1 - f)$; seu $2bghxy - cghxy = b(gh+f)xy - bhx - bgy + b(1-f)$ vel $(bf - bgh + cgh)xy - bhx - bgy = b(f-1)$

Ponamus brevitatis gratia $bf - bgh + cgh = e$ erit $eexy - ebhx - ebgy = eb(f-1)$ sive:

$$(ex - bg)(ey - bh) = bbgh + be(f-1)$$

Numerus ergo $bbgh + be(f-1)$ in duos ejusmodi factores, qui sint P & Q resolvi debet, ut fiant

$$x = \frac{P + bg}{e} \ \& \ y = \frac{Q + bh}{e}$$

numeri integri, tum vero $hx - 1$, $gy - 1$ & $xy - 1$ numeri primi. Quae conditio, quoties impleri poterit, erunt numeri amicabiles $\begin{cases} a(hx-1)(gy-1) \\ af(xy-1) \end{cases}$

Notandum est, neque ullum horum numerorum primorum $hx - 1$, $gy - 1$, $xy - 1$, neque ullum factorem ipsius f divisorem esse debere ipsius a: nec non f & $xy - 1$ esse debere numeros primos inter se.

Coroll. 1.

§. LXXIV. Si f fit numerus primus, uti fecunda forma numerorum amicabilium postulat; erit $f+1 = gh$, & propterea $f = gh - 1$. Hoc ergo casu erit $e = cgh - b$ & $PQ = bbgh + bc(gh-1)$ seu $PQ = bcgghh - 2bcgh + 2bb$. Unde quæri debent numeri x & y supra memoratis proprietatibus prædit, ut fit $x = \frac{P+bg}{e}$ & $y = \frac{Q+bh}{e}$,

Coroll. 2.

§. LXXV. His igitur formulis ita uti conveniet, ut pro a fucceffive alii atque alii valores ex iis, quos fupra expofui fubftituantur, atque pro fingulis litteræ f varii numeri tam primi quam compofiti fubftituantur, qui quidem ad numeros amicabiles inveniendos idonei videantur.

Cafus. 1.

§. LXXVI. Sit $a = 4$, (ex valore enim $a = 2$ nullos obtineri numeros amicabiles obfervavi) eritque $b = 4$ & $c = 1$. Tum pofitis numeris amicabilibus $4pq$ & $4fr$, fit $ff = gh$, & $e = 4f - 3gh$. Deinde per refolutionem quærantur factores P & Q ut fit:

$$PQ = 16gh + 4e(f-1)$$

Hincque eruantur numeri integri x & y, ut fit

$$x = \frac{P+4g}{e} \quad \& \; y = \frac{Q+4h}{e}$$

& ex his deriventur valores litterarum $p = hx - 1$, $q = gy - 1$ & $r = xy - 1$, qui fi fuerint numeri primi, erunt $4pq$ & $4fr$ numeri amicabiles.

Exem-

12 — 2, atq primo ($p = 3$ $q = 2$ $r = x$ P x $p = 6x$ $q = 1y$ $r = xy$

Exemplum. 1.

§. LXXVII. Sit $f = 3$, erit $ff = gh = 4$; hincque $e = 12 - 12 = 0$, unde patet ex hac hypothesi nihil obtineri.

Exempl. 2.

§. LXXVIII. Sit $f = 5$, erit $ff = gh = 6$; $e = 20 - 18 = 2$, atque $PQ = 16.6 + 8.4 = 128$. Jam ex $gh = 6$ ponatur primo $g = 2$, & $h = 3$, fietque.

$$x = \frac{P+8}{2} \ \& \ y = \frac{Q+12}{2}$$

Quare sequentes habebuntur resolutiones:

$P =$	2	4	8	16	32	64
$Q =$	64	32	16	8	4	2
$x =$	5	6	8	12	20	36
$y =$	38	22	14	10	8	7
$p = 3x - 1 =$	19*	17	23	35*	59	107
$q = 2y - 1 =$	--	43	27*	19	15*	13
$r = xy - 1 =$	--	131	111*	119*	159	251

Hinc ergo prodeunt numeri amicabiles.

$\begin{cases} 4\cdot17\cdot43 \\ 4\cdot5\cdot131 \end{cases}$ & $\begin{cases} 4\cdot13\cdot107 \\ 4\cdot5\cdot251 \end{cases}$

Ponatur secundo $g = 1$, $h = 6$, fietque:

$$x = \frac{P+4}{2} \ \& \ y = \frac{Q+24}{2}$$

$P =$	2	4	8	16	32	64
$Q =$	64	32	16	8	4	2
$x =$	3	4	6	10	18	34
$y =$	44	28	20	16	14	13
$p = 6x - 1 =$	17*	23	35*	59	107	203*
$q = 1y - 1 =$	43	27*	19	15*	13	12*
$r = xy - 1 =$	131	111*	119*	159	251	441*

Iidem ergo prodeunt bini numeri amicabiles qui ante.

Sunt

Sunt ergo hinc numeri amicabiles:

$$\begin{Bmatrix} 4\cdot 17\cdot 43 \\ 4\cdot 5\cdot 131 \end{Bmatrix} \quad \& \quad \begin{Bmatrix} 4\cdot 13\cdot 107 \\ 4\cdot 5\cdot 251 \end{Bmatrix}$$

Exempl. 3.

§. LXXIX. Sit $f = 7$, erit $ff = gh = 8$; $r = 28 - 24 = 4$. & $PQ = 16.8 + 16.6 = 224$.

Sit ergo primo $g = 2$, $h = 4$ erit

$$x = \frac{P+8}{4};\ y = \frac{Q+16}{4};\ p = 4x - 1;\ q = 2y - 1;$$
$$r = xy - 1.$$

P	4	8	28	56
Q	56	28	8	4
x	3	4	9	16
y	18	11	6	5
$4x-1$	11	15*	35*	63*
$2y-1$	35*	21*	11	9*
$xy-1$	53	43	53	79

Sit secundo $g = 1$, $h = 8$; erit $x = \frac{P+4}{4}$; $y = \frac{Q+32}{4}$ & $p = 8x - 1$; $q = y - 1$; $r = xy - 1$.

P	4	8	28	56
Q	56	28	8	4
x	2	3	8	15
y	22	15	10	9
$8x-1$	15*	23	63*	119*
$y-1$	21	14	9	8
$xy-1$	43	44*	79	134*

Hinc ergo nulli prodeunt numeri amicabiles.

Exem-