DE SERIEBVS QVIBVSDAM CONSIDERATIONES.

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§. 1.

Postquam inueniſſem ſerierum reciprocarum hac forma contentarum

$$1 \pm \frac{1}{3^n} + \frac{1}{5^n} \pm \frac{1}{7^n} + \frac{1}{9^n} \pm \frac{1}{11^n} + \text{etc.}$$

vbi ambiguorum ſignorum ſuperiora valent, ſi n eſt numerus par, inferiora vero ſi n eſt numerus impar, ſummas a quadratura circuli pendere, ac per tantam peripheriae circuli π poteſtatem determinari, cuius exponens ſit $= n$; nonnullae ſe mihi obtulerunt obſeruationes, cum ad has ipſas ſeries, tum ad earum vſum in ſummandis aliis ſeriebus ſpectantes. Quae cum non admodum ſint obuiae, ac fortaſſe ad alia negotia vtilitatem non ſpernendam afferre queant, eas hic exponere non abs re fore ſum arbitratus.

§. 2. Poſita conſtanter ratione diametri ad circuli peripheriam vt 1 ad π, conſidero circulum, cuius radius ſeu ſemidiameter ſit $= 1$, et denotabit π eius ſemicircumferentiam ſeu arcum 180 graduum. Quod ſi nunc accipiatur in hoc circulo arcus $= s$, cuius ſinus ſit $= y$; coſinus $= x$, et tangens $= t$; erit

$$y = s - \frac{s^3}{1.2.3} + \frac{s^5}{1.2.3.4.5} - \frac{s^7}{1.2....7} + \text{etc.}$$

$$x = 1 - \frac{s^2}{1.2} + \frac{s^4}{1.2.3.4} - \frac{s^6}{1.2....6} + \text{etc.}$$

atque hinc

$$0 = t - s - \frac{s^2 t}{1.2} + \frac{s^3}{1.2.3} + \frac{s^4 t}{1.2.3.4} - \frac{s^5}{1.2...5} - \frac{s^6 t}{1.2...6} + \text{etc.}$$

seu

$$0 = 1 - \frac{s}{t} + \frac{s^2}{1.2} + \frac{s^3}{1.2.3 t} + \frac{s^4}{1.2.3.4} - \frac{s^5}{1.2...5t} - \frac{s^6}{1.2...6} + \text{etc.}$$

§. 3. Consideremus primo aequationem, qua relatio inter sinum y et arcum s continetur, ac manifestum est, valorem s pro dato y non esse constantem, sed omnes eos arcus denotare, quorum idem est communis sinus y. Sit arcuum horum minimus $= \frac{m}{n}\pi$, habebunt omnes sequentes arcus

$$\frac{m}{n}\pi,\quad \frac{n-m}{n}\pi,\quad \frac{2n+m}{n}\pi,\quad \frac{3n-m}{n}\pi,\quad \frac{4n+m}{n}\pi,\ \text{etc.}$$

$$-\frac{n-m}{n}\pi;\ -\frac{2n+m}{n}\pi;\ -\frac{3n-m}{n}\pi,\ -\frac{4n+m}{n}\pi,\ -\frac{5n-m}{n}\pi\ \text{etc.}$$

eundem communem sinum y, Quocirca huius aequationis:

$$0 = 1 - \frac{s}{1 \cdot y} + \frac{s^3}{1.2.3y} - \frac{s^5}{1.2...5y} + \frac{s^7}{1.2...7y} - \text{etc.}$$

habebuntur sequentes innumerabiles factores:

$$\left(1 - \frac{ns}{m\pi}\right)\left(1 + \frac{ns}{(n+m)\pi}\right)\left(1 - \frac{ns}{(n-m)\pi}\right)\left(1 + \frac{ns}{(2n-m)\pi}\right)\left(1 - \frac{ns}{(2n+m)\pi}\right)\text{etc.}$$

§. 4. Hinc itaque valores ipsius $\frac{1}{s}$ constituent sequentem seriem:

$$\frac{n}{m\pi} + \frac{n}{(n-m)\pi} - \frac{n}{(n+m)\pi} - \frac{n}{(2n-m)\pi} + \frac{n}{(2n+m)\pi} + \frac{n}{(3n-m)\pi} - \text{etc.}$$

Horum itaque summa aequalis erit coefficienti ipsius $-s$ in aequatione, qui est $= \frac{1}{1y}$: Summa factorum ex binis erit $= 0$, summa ex ternis $= -\frac{1}{1.2.3y}$, etc. vti sequitur:

summa terminorum	$=$	$\frac{1}{1y}$
summ. fact. ex binis	$=$	0
summ. fact. ex ternis	$=$	$\frac{-1}{1.2.3y}$
summ. fact. ex quaternis	$=$	0

sum.

ſumm. fact. ex quinis $= \frac{1}{1\cdot2\cdot\ldots\cdot5y}$
ſumm. fact. ex ſenis $= 0$
ſumm. fact. ex ſeptenis $= \frac{-1}{1\cdot2\cdot\ldots\cdot5\cdot7y}$
ſumm. fact. ex octonis $= 0$
etc.

§. 5. Quod ſi autem generatim ſeriei cuiuscunque $a + b + c + d + e +$ etc. fuerit

ſumma ipſorum terminorum $= \alpha$
ſumma factorum ex binis $= \beta$
ſumma factorum ex ternis $= \gamma$
ſumma factorum ex quaternis $= \delta$
ſumma factorum ex quinis $= \varepsilon$
ſumma factorum ex ſenis $= \zeta$
etc.

poterunt ex his ſummae quadratorum, cuborum, biquadratorum, et poteſtatum quarumuis terminorum huius ſeriei aſſignari. Quodſi enim ſit

$a + b + c + d$ etc. $=$ A
$a^2 + b^2 + c^4 + d^2 +$ etc. $=$ B
$a^3 + b^3 + c^3 + d^3 +$ etc. $=$ C
$a^4 + b^4 + c^4 + d^4 +$ etc. $=$ D
$a^5 + b^5 + c^5 + d^5 +$ etc. $=$ E
$a^6 + b^6 + c^6 + d^6 +$ etc. $=$ F
etc.

ſequenti modo iſtarum ſummarum valores determinabuntur.

$\mathrm{A} = \alpha$
$\mathrm{B} = \alpha\mathrm{A} - 2\beta$
$\mathrm{C} = \alpha\mathrm{B} - \beta\mathrm{A} + 3\gamma$
$\mathrm{D} = \alpha\mathrm{C} - \beta\mathrm{B} + \gamma\mathrm{A} - 4\delta$

E =

$$E = \alpha D - \beta C + \gamma B - \delta A + 5\varepsilon$$
$$F = \alpha E - \beta D + \gamma C - \delta B + \varepsilon A - 6\zeta$$
etc.

Quae progreſſio cum facilem legem teneat, et ex terminis praecedentibus quiuis terminus expedite definiri poſſit, poterimus ſeriei ſuperioris valores ipſius $\frac{x}{z}$ exhibentis ſummam poteſtatum quarumcunque terminorum definire.

§. 6. Antequam autem hanc generalem progreſſionem reliquamus, notari conueniet ſingularem proprietatem, quam valores litterarum A, B, C, D etc. inter ſe tenent. Oriuntur ii ſcilicet ex evolutione huius expreſſionis

$$\frac{\alpha - 2\beta z + 3\gamma z^2 - 4\delta z^3 + 5\varepsilon z^4 - 6\zeta z^5 + 7\eta z^6 - \text{etc.}}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 - \varepsilon z^5 + \zeta z^6 - \text{etc.}}$$

ſi quidem per diuiſionem actualem quotus ſecundum poteſtates ipſius z eruatur. Prodibit namque diuiſione conſueto more inſtituta ſequens quotus $A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 +$ etc. ita vt iſta ſeries aequalis ſit illi fractioni. Praeterea notandum eſt, ſi ſeriei $1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4 -$ etc. ſumma ponatur $= Z$, ita vt ſit Z denominator illius fractionis, fore numeratorem $= \frac{-dZ}{dz}$. Ex quo ſeriei $A + Bz + Cz^2 + Dz^3 + Ez^4 +$ etc. ſumma erit $= \frac{-dZ}{Z\,dz}$. Non ſolum itaque ex datis factis binorum, ternorum, quaternorum etc. ſummae poteſtatum ſeriei propoſitae $a + b + c + d +$ etc. ſcilicet valores litterarum A, B, C, D, etc. poterunt inueniri, ſed etiam ſumma ſeriei, quam hae ipſae poteſtates in nouam progreſſionem geometricam reſpecti-

ſpectiue ducti, nimirum huius ſeriei
$A + Bz + Cz^2 + Dz^3 + Ez^4 +$ etc. ſumma poterit aſſignari. Hancque proprietatem probe notaſſe in ſequentibus plurimum iuuabit, vbi in nouas ſeries ſumus inquiſituri.

§. 7. Cum igitur huius ſeriei:

$$\frac{n}{\pi}\left(\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} - \text{etc.}\right)$$

dentur primo ipſorum terminorum ſumma, tum etiam ſummae factorum ex binis, ternis, quaternis et ita porro,

$$A = \frac{1}{1y}$$

$$B = \frac{A}{1y}$$

$$C = \frac{B}{1y} - \frac{1}{1\cdot 2y}$$

$$D = \frac{C}{1y} - \frac{A}{1\cdot 2\cdot 3y}$$

$$E = \frac{D}{1y} - \frac{B}{1\cdot 2\cdot 3y} + \frac{1}{1\cdot 2\cdot 3\cdot 4y}$$

$$F = \frac{E}{1y} - \frac{C}{1\cdot 2\cdot 3y} + \frac{A}{1\cdot 2\cdot 3\cdot 4\cdot 5y}$$

$$G = \frac{F}{1y} - \frac{D}{1\cdot 2\cdot 3y} + \frac{B}{1\cdot 2\cdot 3\cdot 4\cdot 5y} - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6y} \quad \text{etc.}$$

erit vt ſequitur

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \text{etc.} = \frac{A\pi}{n}$$

$$\frac{1}{m^2} + \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \text{etc.} = \frac{B\pi^2}{n^2}$$

$$\frac{1}{m^3} + \frac{1}{(n-m)^3} - \frac{1}{(n+m)^3} - \frac{1}{(2n-m)^3} + \frac{1}{(2n+m)^3} + \text{etc.} = \frac{C\pi^3}{n^2}$$

$$\frac{1}{m^4} + \frac{1}{(n-m)^4} + \frac{1}{(n+m)^4} + \frac{1}{(2n-m)^4} + \frac{1}{(2n+m)^4} + \text{etc.} = \frac{D\pi^4}{n^4}$$

$$\frac{1}{m^5} + \frac{1}{(n-m)^5} - \frac{1}{(n+m)^5} - \frac{1}{(2n-m)^5} + \frac{1}{(2n+m)^5} + \text{etc.} = \frac{E\pi^5}{n^5}$$

$$\frac{1}{m^6} + \frac{1}{(n-m)^6} + \frac{1}{(n+m)^6} + \frac{1}{(2n-m)^6} + \frac{1}{(2n+m)^6} + \text{etc.} = \frac{F\pi^6}{n^6}$$

etc.

 vbi

vbi pro poteſtatibus paribus omnes termini habent ſignum $+$, pro imparibus vero ſigna conueniunt cum ſignis ipſius ſeriei primae.

§. 8. Retineant litterae A, B, C, D, E, etc. valores, quos ipſis modo tribuimus, ſitque nobis haec ſeries propoſita

$$A + Bz + Cz^2 + Dz^3 + Ez^4 \text{ etc.}$$

cuius ſummam ex regula §. 6 data inueſtigemus. Huius autem ſeriei ſumma inde eſt $= \frac{-dZ}{Z\,dz}$ exiſtente $Z = 1 - \frac{z}{1y} + \frac{z^3}{1\cdot2\cdot3y} - \frac{z^5}{1\cdot2\ldots5y} + \frac{z^7}{1\cdot2\ldots7y} - \text{etc.} = 1 - \frac{1}{y}\,\text{ſin. A}\,.\,z$. Ex quo ob y hoc loco conſtans ponendum erit $dZ = \frac{-dz\,\text{coſ. A}\,z}{y}$ ac propterea ſumma ſeriei propoſitae

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \text{etc}$$

erit $= \frac{\text{coſ. A}\,.\,z}{y - \text{ſin. A}\,z}$. Hinc erit iſtius ſeriei ſumma $Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc} = \frac{z\,\text{coſ. A}\,.\,z}{y - \text{ſin. A}\,z}$.

§. 9. Sit $z = \frac{p\pi}{n}$, exprimet haec ſeries ſummam omnium harum ſerierum:

$$+ \frac{p}{m} + \frac{p}{n-n} - \frac{p}{n+m} - \frac{p}{2n-m} + \frac{p}{2n+m} + \text{etc.}$$

$$+ \frac{p^2}{m^2} + \frac{p^2}{(n-m)^2} + \frac{p^2}{(n+n)^2} + \frac{p^2}{(2n-m)^2} + \frac{p^2}{(2n+m)^2} + \text{etc.}$$

$$+ \frac{p^3}{m^3} + \frac{p^3}{(n-m)^3} - \frac{p^3}{(n+m)^3} - \frac{p^3}{(2n-m)^3} + \frac{p^3}{(2n+m)^3} + \text{etc.}$$

etc.

Hae autem ſeries verticaliter additae dant

$$\frac{p}{m-p} + \frac{p}{n-m-p} - \frac{p}{n+m+p} - \frac{p}{2n-m+p} + \frac{p}{2n+m-p} + \text{etc.}$$

cuius ſeriei igitur ſumma eſt $= \frac{p\pi\,\text{coſ. A}\,.\,\frac{p\pi}{n}}{ny - n\,\text{ſin. A}\,.\,\frac{p\pi}{n}}$ ſeu cum

y ſit

y ſit ſinus arcus $\frac{m\pi}{n}$, habebitur iſtius ſeriei ſumma $= \frac{p\,\pi\, coſ.\, A.\, \frac{p\pi}{n}}{n\, ſin.\, A \frac{m\pi}{n} - n\, ſin.\, A.\, \frac{p\pi}{n}}$. Quod ſi ponatur $m - p = a$ et $m + p = b$ ita vt ſit $m = \frac{a+b}{2}$ et $p = \frac{b-a}{2}$ prodibit huius ſeriei

$$\frac{1}{a} + \frac{1}{n-b} - \frac{1}{n+b} - \frac{1}{2n-a} + \frac{1}{2n+a} + \frac{1}{3n-b} - \frac{1}{3n+b} - \text{etc.}$$

ſiue huius

$$\frac{a}{a} + \frac{2b}{n^2-b^2} - \frac{2a}{4n^2-a^2} + \frac{2b}{9n^2-b^2} - \frac{2a}{16n^2-a^2} + \frac{2b}{25n^2-b^2} - \text{etc.}$$

$$\text{ſumma} = \frac{\pi\, coſ.\, A \frac{(b-a)\pi}{2n}}{n\, ſin.\, A.\, \frac{(b+a)\pi}{2n} - n\, ſin.\, A \frac{(b-a)\pi}{2n}}$$

§. 10. Verum haec nimis ſunt generalia, vt difficulter omnia, quae in iis comprehenduntur, perſpici queant. Quamobrem ad ſpecialiora deſcendamus, ac ponamus ſinum $y =$ ſinui toti $= 1$: erit $m = 1$ et $n = 2$. Hinc igitur ſequentes nanciſcimur ſeries

$$\frac{1}{1} + \frac{1}{1} - \frac{1}{3} - \frac{1}{3} + \frac{1}{5} + \frac{1}{5} - \frac{1}{7} - \text{etc.} = \frac{A\pi}{2}$$

$$\frac{1}{1^2} + \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} = \frac{B\pi^2}{2^2}$$

$$\frac{1}{1^3} + \frac{1}{1^3} - \frac{1}{3^3} - \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{5^3} - \frac{1}{7^3} - \text{etc.} = \frac{C\pi^3}{2^3}$$

$$\frac{1}{1^4} + \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} = \frac{D\pi^4}{2^4}$$

etc.

ſeu hae

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.} = \frac{A\pi}{2^2}$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.} = \frac{B\pi^2}{2^3}$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.} = \frac{C\pi^3}{2^4}$$

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.} = \frac{D\pi^4}{2^5}$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \text{etc.} = \frac{E\pi^5}{2^6}$$
$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.} = \frac{F\pi^6}{2^7}$$
$$1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \text{etc.} = \frac{G\pi^7}{2^8}$$
$$1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.} = \frac{H\pi^8}{2^9}$$
etc.

Valores autem litterarum A. B, C, D etc. ex sequenti lege inuenientur.

$$A = 1$$
$$B = \frac{A}{1}$$
$$C = \frac{B}{1} - \frac{1}{1\cdot 2}$$
$$D = \frac{C}{1} - \frac{A}{1.2.3}$$
$$E = \frac{D}{1} - \frac{B}{1.2.3} + \frac{1}{1.2.3.4}$$
$$F = \frac{E}{1} - \frac{C}{1.2.3} + \frac{A}{1.2.3.4.5}$$
$$G = \frac{F}{1} - \frac{D}{1.2.3} + \frac{B}{1.2.3.4.5} - \frac{1}{1.2\ldots 6}$$
$$H = \frac{G}{1} - \frac{E}{1.2.3} + \frac{C}{1.2.3.4.5} - \frac{A}{1.2\ldots 7}.$$
etc.

vnde reperiuntur sequentes valores litterarum

$$A = 1 \cdot \frac{\pi}{2^2} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.}$$
$$B = \frac{1}{1} \cdot \frac{\pi^2}{2^3} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.}$$
$$C = \frac{1}{1.2} \cdot \frac{\pi^3}{2^4} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \text{etc.}$$
$$D = \frac{2}{1.2.3} \cdot \frac{\pi^4}{2^5} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.}$$
$$E = \frac{5}{1.2.3.4} \cdot \frac{\pi^5}{2^6} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \text{etc.}$$
$$F = \frac{16}{1.2.3.4.5} \cdot \frac{\pi^6}{2^7} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.}$$
$$G = \frac{61}{1.2.3.4.5.6} \cdot \frac{\pi^7}{2^8} = 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \text{etc.}$$
$$H = \frac{272}{1.2.3\ldots 7} \cdot \frac{\pi^8}{2^9} = 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.}$$

I =

$$I = \frac{1385}{1.2.3\ldots.8} \cdot \frac{\pi^9}{2^{10}} = 1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \text{etc.}$$
$$K = \frac{7936}{1.2.3\ldots.9} \cdot \frac{\pi^{10}}{2^{11}} = 1 + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \text{etc.}$$
$$L = \frac{50521}{1.2.3\ldots.10} \cdot \frac{\pi^{11}}{2^{12}} = 1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} + \frac{1}{7^{11}} + \text{etc.}$$
$$M = \frac{353792}{1.2.3\ldots.11} \cdot \frac{\pi^{12}}{2^{13}} = 1 + \frac{1}{3^{12}} + \frac{1}{5^{12}} + \frac{1}{7^{12}} + \text{etc.}$$
$$N = \frac{2702765}{1.2.3\ldots.12} \cdot \frac{\pi^{13}}{2^{14}} = 1 - \frac{1}{3^{13}} + \frac{1}{5^{13}} + \frac{1}{7^{13}} + \text{etc.}$$
$$O = \frac{22368256}{1.2.3\ldots.13} \cdot \frac{\pi^{14}}{2^{15}} = 1 + \frac{1}{3^{14}} + \frac{1}{5^{14}} + \frac{1}{7^{14}} \text{ etc.}$$

§. 11. Denotant hic litterae A, B, C, etc. numerales tantum coefficientes potestatum π per potestates binarii diuisarum: quarum valores etsi satis commode ex lege data definiri possunt, tamen alia lex potest exhiberi, quae magis ad calculum videtur expedita. Considero scilicet seriem $A + Bz + Cz^2 + Dz^3 + Ez^4 +$ etc. cuius summa, quae tantisper designetur littera s, est per §. 8. $= \frac{cos. A.z}{1 - sin. AZ}$, ob $y = 1$. Quod si igitur ex hac aequatione $s = \frac{cos. A.z}{1 - sin. A.z}$ valor ipsius s in serie exprimatur, quae secundum potestates ipsius z progrediatur, prodire debebit ipsa series $A + Bz + Cz^2 + Dz^3 +$ etc. Nulla enim alia series similis formae puta $P + Qz + Rz^2 + Sz^3$ + etc. assignari potest aequalis illi $A + Bz + Cz^2 + Dz^3 +$ etc quin simul coefficientes potestatum z congruant, sitque $P = A$; $Q = B$; $R = C$; $S = D$, etc. At vero exprimit $\frac{cos. A z}{1 - sin. A z}$ tangentem arcus $\frac{\pi}{4} + \frac{z}{2}$ seu erit $s =$ tang. $A(\frac{\pi}{4} + \frac{z}{2})$ et hancobrem conuertendo $\frac{\pi}{4} + \frac{z}{2} =$ A tang. $s = \int \frac{ds}{1 + ss}$ sumtisque differentialibus ob $\frac{\pi}{4}$ constans seu arcum 45 graduum, habebitur $\frac{dz}{2} = \frac{ds}{1 + ss}$ siue $dz + ssdz = 2ds$. Nunc ponatur $s = A + Bz + Cz^2 + Dz^3 + Ez^4 +$ etc. erit

$$\frac{2ds}{dz} = 2B + 4Cz + 6Dz^2 + 8Ez^3 + 10Fz^4 + \text{etc.}$$
$$ss = A^2 + 2ABz + 2ACz^2 + 2ADz^3 + 2AEz^4 + \text{etc.}$$
$$1 = +1 \quad + B^2z^2 + 2BCz^3 + 2BDz^4 + \text{etc.}$$
$$+ C^2z^4 + \text{etc.}$$

Comparatis nunc terminis homogeneis inter ſe valores litterarum ita definiantur, vt coefficientes ſingularum poteſtatum ipſius z euaneſcant; atque ſequentes litterarum A, B, C, D, E, etc. obtinebuntur determinationes, exiſtente vt iam inuenimus $A = 1$.

$$A = 1$$
$$B = \frac{A^2+1}{2}$$
$$C = \frac{2AB}{4}$$
$$D = \frac{2AC+B^2}{6}$$
$$E = \frac{2AD+2BC}{8}$$
$$F = \frac{2AE+2BD+C^2}{10}$$
$$G = \frac{2AF+2BE+2CD}{12}$$
$$H = \frac{2AG+2BF+2CE+D^2}{14}$$
etc.

Atque hinc eaedem prorſus determinationes litterarum A, B, C, D etc. prodibunt, quas altera lex ſupra data §. 10 ſuppeditat.

§. 12. Cum denominatores fractionum, quibus litterae A, B, C, D, etc. aequales ſint inuentae, ſatis regulariter progrediantur, poteſt hinc peculiaris regula ad inueniendos numeratores reperiri: Ponamus enim

A =

$A = \alpha$	$F = \frac{\zeta}{1.2.3.4.5}$
$B = \frac{\beta}{1}$	$G = \frac{\eta}{1.2.3.4.5.6}$
$C = \frac{\gamma}{1.2}$	$H = \frac{\theta}{1.2.3 \ldots\ldots 7}$
$D = \frac{\delta}{1.2.3}$	$I = \frac{\iota}{1.2.3 \ldots 8}$
$E = \frac{\varepsilon}{1.2.3.4}$	$K = \frac{\varkappa}{1.2.3 \ldots\ldots 9}$ etc.

eritque factis substitutionibus haec lex:

$\alpha = 1$	$\varepsilon = \alpha\delta + 3\beta\gamma$
$\beta = \frac{\alpha^2+1}{2}$	$\zeta = \alpha\varepsilon + 4\beta\delta + 3\gamma^2$
$\gamma = \alpha\beta$	$\eta = \alpha\zeta + 5\beta\varepsilon + \frac{5.4}{1.2}\gamma\delta$
$\delta = \alpha\gamma + \beta^2$	$\theta = \alpha\eta + 6\beta\zeta + \frac{6.5}{1.2}\gamma\varepsilon + \frac{6.5.4}{1.2.3}\cdot\frac{\delta^2}{2}$

$$\iota = \alpha\theta + 7\beta\eta + \frac{7.6}{1.2}\gamma\zeta + \frac{7.6.5}{1.2.3}\delta\varepsilon$$

$$\varkappa = \alpha\iota + 8\beta\theta + \frac{8.7}{1.2}\gamma\eta + \frac{8.7.6}{1.2.3}\delta\zeta + \frac{8.7.6.5}{1.2.3.4}\cdot\frac{\varepsilon^2}{2}.$$

Lex haec perspicua est, si hoc modo notetur, quoties postremus terminus sit quadratum, eum insuper per binarium diuidi debere.

§. 13. Consideremus nunc hanc seriem:
$Az + Bz^2 + Gz^3 + Dz^4 + Ez^5 +$ etc.
cuius summam constat esse $= \frac{z\,cos.\,A.z}{1-sin.\,A.z}$, ac ponatur

$z = \frac{p\pi}{2}$, erit $\frac{p\pi\,cos.\,A : \frac{p\pi}{2}}{2-2\,sin.\,A\,.\,\frac{p\pi}{2}} =$

$\frac{A\pi}{2^2}\,.\,2p + \frac{B\pi^2}{2^3}\,.\,2p^2 + \frac{C\pi^3}{2^4}\,.\,2p^3 + \frac{D\pi^4}{2^5}\,.\,2p^4 +$ etc.

seu $\frac{\pi\,cos.\,A\,\frac{p\pi}{2}}{4-4\,sin.\,A\,.\,\frac{p\pi}{2}} = \frac{A\pi}{2^2} + \frac{pB\pi^2}{2^3} + \frac{p^2C\pi^3}{2^4} + \frac{p^3D\pi^4}{2^5} +$ etc.

Quodsi ergo loco singulorum terminorum substituantur series

ries ex §. 10 prodibit $\frac{\pi \cos. A. \frac{p\pi}{2}}{4-4\sin. A. \frac{p\pi}{2}} =$

$$+1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.}$$
$$+p+\frac{p}{3^2}+\frac{p}{5^2}+\frac{p}{7^2}+\frac{p}{9^2}+\text{etc.}$$
$$+p^2-\frac{p^2}{3^3}+\frac{p^2}{5^3}+\frac{p^2}{7^3}+\frac{p^2}{9^3}-\text{etc.}$$
$$+p^3+\frac{p^3}{3^4}+\frac{p^3}{5^4}+\frac{p^3}{7^4}+\frac{p^3}{9^4}+\text{etc.}$$
etc.

Omnes autem hae feries deorfum additae abeunt in hanc:

$$\frac{1}{1-p}-\frac{1}{3+p}+\frac{1}{5-p}-\frac{1}{7+p}+\frac{1}{9-p}-\text{etc.}$$

cuius adeo fumma eft $= \frac{\pi \cos. A. \frac{p\pi}{2}}{4-4\sin. A. \frac{p\pi}{2}}$.

§. 14. Plures huius generis feries fummabiles derivare licebit ex §. 9. ferie fub finem expofita. Ponamus $a = b = m$: et habebimus hanc

$$\frac{2m}{n^2-m^2}-\frac{2m}{4n^2-m^2}+\frac{2m}{9n^2-m^2}-\frac{2m}{16n^2-m^2}+\frac{2m}{25n^2-m^2}-\text{etc.}$$

cuius fumma erit $= \frac{\pi}{n \sin. A. \frac{m\pi}{n}} - \frac{1}{m}$ ob $\cos. A. 0\pi = 1$ et $\sin. A. 0\pi = 1$. Quamobrem habebitur, fi per $2m$ diuidatur

$$\frac{1}{n^2-m^2}-\frac{1}{4n^2-m^2}+\frac{1}{9n^2-m^2}-\frac{1}{16n^2-m^2}+\frac{1}{25n^2-m^2}-\text{etc.}$$

$= \frac{\pi}{2mn \sin. A. \frac{m\pi}{n}} - \frac{1}{2mm}$. Ponamus porro $a = -m$ et $b = +m$, ac proueniet $-\frac{\pi \cos. A. \frac{m\pi}{n}}{n \sin. A. \frac{m\pi}{n}} + \frac{1}{m} =$

$$\frac{2m}{n^2-m^2}+\frac{2m}{4n^2-m^2}+\frac{2m}{9n^2-m^2}+\frac{2m}{16n^2-m^2}+\frac{2m}{25n^2-m^2}+\text{etc.}$$

feu

seu facta diuisione per $2\,m$, erit $\frac{1}{2m^2} - \frac{\pi\, cos\, A\, .\, \frac{m\pi}{n}}{2\,m\,n\, sin.\, A\, .\, \frac{m\pi}{n}}$

$= \frac{1}{n^2-m^2} + \frac{1}{4n^2-m^2} + \frac{1}{9n^2-m^2} + \frac{1}{16n^2-m^2} + \frac{1}{25n^2-m^2} + \text{etc.}$

Quoties itaque euenit vt $cos.\, A\, .\, \frac{m\pi}{n}$ euanescat, toties seriei summa algebraice erit assignabilis quippe $= \frac{1}{2m^2}$. Fit autem hoc, si fuerit $\frac{m}{n} = \frac{2i+1}{2}$ seu $m = 2\,i + 1$, et $n = 2$ vnde erit:

$\frac{1}{2(2i+1)^2} = \frac{1}{4-(2i+1)^2} + \frac{1}{16-(2i+1)^2} + \frac{1}{36-(2i+1)^2} + \frac{1}{64-(2i+1)^2}$ etc.

Ex quo sequens oritur propositio paradoxa: esse scilicet

$\frac{1}{4-p} + \frac{1}{16-p} + \frac{1}{36-p} + \frac{1}{64-p} + \frac{1}{100-p} + \text{etc.} = \frac{1}{2p}$

quoties fuerit p numerus quadratus integer et impar.

§. 15. Ponamus $n = 1$, atque $m^2 = p$, erit

$\frac{1}{1-p} - \frac{1}{4-p} + \frac{1}{9-p} + \frac{1}{16-p} + \frac{1}{25-p} + \text{etc.} = \frac{\pi\sqrt{p}}{2p\, sin.\, A\cdot\pi\sqrt{p}} - \frac{1}{2p}$

$\frac{1}{1-p} + \frac{1}{4-p} + \frac{1}{9-p} + \frac{1}{16-p} + \frac{1}{25-p} + \text{etc.} = \frac{1}{2p} - \frac{\pi\sqrt{p}\, cos.\, A\cdot\pi\sqrt{p}}{2p\, sin.\, A\cdot\pi\sqrt{p}}$

quae series si addantur sequitur fore:

$\frac{1}{1-p} + \frac{1}{9-p} + \frac{1}{25-p} + \text{etc.} = \frac{\pi\sqrt{p}\, sin.\, vA.\pi\sqrt{p}}{4p\, sin.\, A\cdot\pi\sqrt{p}}$

at si eaedem a se inuicem subtrahantur; erit

$\frac{1}{4-p} + \frac{1}{16-p} + \frac{1}{36-p} + \text{etc.} = \frac{1}{2p} - \frac{\pi\sqrt{p}(1+cos.\, A\, .\pi\sqrt{p})}{4p\, sin.\, A\cdot\pi\sqrt{p}}$

At est $\frac{sin.\, vA.\pi\sqrt{p}}{sin.\, A\cdot\pi\sqrt{p}} = \text{tang. } A.\, \frac{\pi\sqrt{p}}{2}$ et $\frac{1+cos.\, A\pi\sqrt{p}}{sin.\, A\pi\sqrt{p}} = cot.\, A.\frac{\pi\sqrt{p}}{2}$

ex quo summae posteriores simpliciores reddentur.

§. 16. Possumus itaque hinc summare sequentes series $\frac{1}{1-p} \pm \frac{1}{4-p} + \frac{1}{9-p} \pm \frac{1}{16-p} + \text{etc.}$ si quidem p significet numerum affirmatiuum quemcunque. At si loco p substituatur numerus negatiuus puta $-q$, tum fiunt tam sinus et cosinus, quam ipsi arcus $\pi\sqrt{p}$ seu $\pi\sqrt{-q}$ quantitates imaginariae. Cum autem summae serierum nihilo minus

minus maneant reales et finitae, imaginaria sese destruent. Quamobrem inuestigari conueniet, cuiusmodi quantitates reales in his formis $\frac{\pi\sqrt{-q}}{\text{sin. A}\pi\sqrt{-q}}$ et $\frac{\pi\sqrt{-q}}{\text{tang. A.}\pi\sqrt{-q}}$ continentur. Ad hoc ponamus $u = \frac{\pi\sqrt{-q}}{\text{sin. A.}\pi\sqrt{-q}}$ eritque $\text{sin. A.}\pi\sqrt{-q} = \frac{\pi\sqrt{-q}}{u}$ et $\pi\sqrt{-q} = \text{A sin.}\frac{\pi\sqrt{-q}}{u}$; sumantur differentialia positis π et u variabilibus, habebitur $d\pi = \frac{ud\pi - \pi - \pi du}{u\sqrt{(uu + q\pi^2)}}$. Ponatur $u = \pi v$, prodibit $d\pi = \frac{-dv}{v\sqrt{(q + vv)}}$ et $\pi = \frac{1}{\sqrt{q}} l \frac{\sqrt{q} + \sqrt{(q + vv)}}{cv}$. Hinc erit $e^{\pi\sqrt{q}} c v = \sqrt{q} + \sqrt{(q + vv)}$ et $v = \frac{2 e^{\pi\sqrt{q}} c \sqrt{q}}{e^{2\pi\sqrt{q}} c^2 - 1}$ atque $u = \frac{2\pi e^{\pi\sqrt{q}} c\sqrt{q}}{e^{2\pi\sqrt{q}} c^2 - 1}$. Constans autem c ita debet esse comparata vt facto $\pi = 0$ fiat $u = 1$ ex quo fit $c = 1$ Quamobrem erit $\frac{\pi\sqrt{-q}}{\text{sin. A.}\pi\sqrt{-q}} = \frac{2 e^{\pi\sqrt{q}}\pi\sqrt{q}}{e^{2\pi\sqrt{q}} - 1}$. Simili modo ponatur $\frac{\pi\sqrt{-q}}{\text{tang. A.}\pi\sqrt{-q}} = \frac{\pi}{v}$ erit $v\sqrt{-q} = \text{tang. A.}\sqrt{-q}$ et $\pi\sqrt{-q} = \text{A tang.}\, v\sqrt{-q}$ ac differentiando $d\pi = \frac{dv}{1 - qvv}$. Integretur denuo, erit $\pi = \frac{1}{2\sqrt{q}} l \frac{1 + v\sqrt{q}}{1 - v\sqrt{q}}$ et $e^{2\pi\sqrt{q}} - e^{2\pi\sqrt{q}} v\sqrt{q} = 1 + v\sqrt{q}$, vnde fit $v = \frac{e^{2\pi\sqrt{q}} - 1}{(e^{2\pi\sqrt{q}} + 1)\sqrt{q}}$ atque

$$\frac{\pi\sqrt{-q}}{\text{tang. A.}\pi\sqrt{-q}} = \frac{(e^{2\pi\sqrt{q}} + 1)\pi\sqrt{q}}{e^{2\pi\sqrt{q}} - 1}.$$

§. 17. Nacti igitur sumus octo sequentes series, quarum summae assignari possunt, quas cum summis conspectui exponemus

$$\frac{1}{1-p} - \frac{1}{4-p} + \frac{1}{9-p} - \frac{1}{16-p} + \frac{1}{25-p} - \text{etc.} = \frac{\pi\sqrt{p}}{2p\,\text{sin. A.}\pi\sqrt{p}} - \frac{1}{2p}$$

$$\frac{1}{1-p} + \frac{1}{4-p} + \frac{1}{9-p} + \frac{1}{16-p} + \frac{1}{25-p} + \text{etc.} = \frac{1}{2p} - \frac{\pi\sqrt{p}}{2p\,\text{tang. A.}\pi\sqrt{p}}$$

$$\frac{1}{1-p} + \frac{1}{9-p} + \frac{1}{25-p} + \frac{1}{49-p} + \text{etc.} = \frac{\pi\sqrt{p}}{4p\,\text{cos. A.}\frac{\pi\sqrt{p}}{2}}$$

$\frac{1}{4-p}$

$$\frac{1}{4-p}+\frac{1}{16-p}+\frac{1}{36-p}+\frac{1}{64-p}+\text{etc.}=\frac{1}{2p}-\frac{\pi\sqrt{p}}{4p\,tang.\,A.\frac{\pi\sqrt{p}}{2}}$$

$$\frac{1}{1+q}-\frac{1}{4+q}+\frac{1}{9+q}-\frac{1}{16+q}+\text{etc.}=\frac{1}{2q}-\frac{e^{\pi\sqrt{q}}\pi\sqrt{q}}{(e^{2\pi\sqrt{q}}-1)q}$$

$$\frac{1}{1+q}+\frac{1}{4+q}+\frac{1}{9+q}+\frac{1}{16+q}+\text{etc.}=\frac{(e^{2\pi\sqrt{q}}+1)\pi\sqrt{q}}{2(e^{2\pi\sqrt{q}}-1)q}-\frac{1}{2q}$$

$$\frac{1}{1+q}+\frac{1}{9+q}+\frac{1}{25+q}+\frac{1}{49+q}+\text{etc.}=\frac{(e^{\pi\sqrt{q}}-1)\pi\sqrt{q}}{4(e^{\pi\sqrt{q}}+1)q}$$

$$\frac{1}{4+q}+\frac{1}{16+q}+\frac{1}{36+q}+\frac{1}{64+q}+\text{etc.}=\frac{(c^{\pi\sqrt{q}}+1)\pi\sqrt{q}}{4(e^{\pi\sqrt{q}}-1)q}-\frac{1}{2q}$$

§. 18. Cum supra legem exhibuerim, qua summae potestatum omnium terminorum huius seriei

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.}$$

progrediuntur, inuestigabo nunc legem, quam potestates impares tantum inter se tenent, quo hae summae sine cognitione parium, quousque libuerit, continuari possint; sit itaque

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.}=A\,\pi$$
$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\text{etc.}=B\,\pi^3$$
$$1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\text{etc.}=C\,\pi^5$$
$$1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}-\text{etc.}=D\,\pi^7$$
$$1-\frac{1}{3^9}+\frac{1}{5^9}-\frac{1}{7^9}+\frac{1}{9^9}-\text{etc.}=E\,\pi^9$$
etc.

atque inuestiganda erit lex, qua coefficientes A, B, C, D etc. progrediuntur. Hunc in finem considero hanc seriem $A\pi z+B\pi^3z^3+C\pi^5z^5+D\pi^7z^7+\text{etc.}$ cuius summa sit $=s$; erit ergo his seriebus per respondentes potestates ipsius z multiplicatis respectiue:

$s = \frac{z}{1-zz} - \frac{3z}{9-zz} + \frac{5z}{25-zz} - \frac{7z}{49-zz} + \text{etc.}$ et

$\frac{2s}{z} = \frac{1}{1-z} + \frac{1}{1+z} - \frac{1}{3-z} - \frac{1}{3+z} + \frac{1}{5-z} + \frac{1}{5+z}$ etc.

Cum autem ex §. 9. sit $\frac{\pi \cos A \frac{(b-a)\pi}{2n}}{n \sin A . \frac{(b+a)\pi}{2n} - n \sin A . \frac{(b-a)\pi}{2n}} =$

$\frac{1}{a} + \frac{1}{n-b} - \frac{1}{n+b} - \frac{1}{2n-a} + \frac{1}{2n+a} + \frac{1}{3n-b} - \frac{1}{3n+b} - \text{etc.}$ fiat $a = 1-z$; $n = 2$; et $b = 1-z$; atque haec series transibit in illam; ex quo prodibit

$\frac{2s}{z} = \frac{\pi}{2 \sin A . \frac{(1-z)\pi}{2}}$ et $s = \frac{\pi z}{4 \sin A . \frac{(1-z)\pi}{2}}$ siue $s =$

$$\frac{\pi z}{4 \cos A . \frac{\pi z}{2}} = \frac{\frac{\pi z}{4}}{1 - \frac{\pi^2 z^2}{1 \cdot 2 \cdot 4} + \frac{\pi^4 z^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 2^4} - \frac{\pi^6 z^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 2^6} + \text{etc.}}$$

quae fractio cum, si actu diuidatur, ipsam assumtam seriem $A\pi z + B\pi^3 z^3 + C\pi^5 z^5 + \text{etc.}$ reproducere debeat, erit:

$A = \frac{1}{4}$

$B = \frac{A}{2.4}$

$C = \frac{B}{2.4} - \frac{A}{2.4.6.8}$

$D = \frac{C}{2.4} - \frac{B}{2.4.6.8.} + \frac{A}{2.4.6.8.10.12}$

$E = \frac{D}{2.4} - \frac{C}{2.4.6.8.} + \frac{B}{2.4.6.8.10.12} - \frac{A}{2.4.6 \ldots 16}$

etc.

§. 19. Vel si ponatur:

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.} = \frac{A\pi}{2^2}$

$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \text{etc.} = \frac{B\pi^3}{2^4}$

$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \text{etc.} = \frac{C\pi^5}{2^6}$

$1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \text{etc.} = \frac{D\pi^7}{2^8}$

etc.

coeffi-

coefficientes A, B, C, etc. hanc tenebunt legem:

$$A = 1$$

$$B = \frac{A}{1.2.}$$

$$C = \frac{B}{1.2} - \frac{A}{1.2.3.4}$$

$$D = \frac{C}{1.2} - \frac{B}{1.2.3.4} + \frac{A}{1.2.\ldots.6}$$

$$E = \frac{D}{1.2} - \frac{C}{1.2.3.4} + \frac{B}{1.2.\ldots.6} - \frac{A}{1.2.\ldots.8}$$

Quodsi autem illae series retro continuentur, vt ad potestates affirmatiuas deueniatur, erunt omnium illarum serierum summae $= 0$; ita vt etiamsi in his formis vlterius progrederemur, tamen alii valores non prodituri essent. Est scilicet

$$1 - 3 + 5 - 7 + 9 - \text{etc.} = 0$$

$$1 - 3^3 + 5^3 - 7^3 + 9^3 - \text{etc.} = 0$$

$$1 - 3^5 + 5^5 - 7^5 + 9^5 - \text{etc.} = 0$$

$$1 - 3^7 + 5^7 - 7^7 + 9^7 - \text{etc.} = 0$$

§. 20. Quemadmodum autem summae potestatum imparium peculiarem inter se tenent progressionis legem, ita etiam potestates pares simili proprietate gaudent, vt omnes ex se ipsis sine subsidio potestatum imparium definiri queant. Quam legem vt eruamus, simili vtamur operatione. Sit igitur

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.} = A\pi^2$$

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.} = B\pi^4$$

$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.} = C\pi^6$$

$$1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.} = D\pi^8$$

ac inuestigetur summa huius seriei:

$$A\pi^2 z^2 + B\pi^4 z^4 + C\pi^6 z^6 + D\pi^8 z^8 + \text{etc.} = s,$$

erit $s = \frac{z^2}{1-z^2} + \frac{z^2}{9-z^2} + \frac{z^2}{25-z^2} + \frac{z^2}{49-z^2} + \text{etc.}$ vnde ex

§. 17. fiet $s = \frac{\pi z}{4 \cos. A \frac{\pi z}{2}}$; siue per seriem

$$s = \frac{\frac{\pi^2 z^2}{1.2^3} - \frac{\pi^4 z^4}{1.2.3.2^5} + \frac{\pi^6 z^6}{1.2.3.4.5.2^7} - \text{etc.}}{1 - \frac{\pi^2 z^2}{1.2.2^2} + \frac{\pi^4 z^4}{1.2.3.4.2^4} - \frac{\pi^6 z^6}{1.2.3.4.5.6.2^6} + \text{etc.}}$$

ex qua diuisione cum ipsa series assumta oriri debeat, erit

$$A = \frac{1}{8}$$
$$B = \frac{A}{2.4} - \frac{1}{2.4.6.4}$$
$$C = \frac{B}{2.4} - \frac{A}{2.4.6.8} + \frac{2}{2.4.6.8.10.4}$$
$$D = \frac{C}{2.4} - \frac{B}{2.4.6.8} + \frac{A}{2.4.6\ldots12} - \frac{1}{2.4\ldots14.4}$$
etc.

§. 21. Lex haec facilius inspicietur, si ponatur

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} = A \frac{\pi^2}{2^3}$$
$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} = B \frac{\pi^4}{2^5}$$
$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.} = C \frac{\pi^6}{2^7}$$
$$1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.}\ D \frac{\pi^8}{2^9}$$
etc.

Hic enim coefficientes A, B, C etc. sequentem tenebunt progressionem:

$$A = 1$$
$$B = \frac{A}{1.2} - \frac{1}{1.2.3}$$
$$C = \frac{B}{1.2} - \frac{A}{1.2.3.4} + \frac{1}{1.2.3.4.5}$$
$$D = \frac{C}{1.2} - \frac{B}{1.2.3.4} + \frac{A}{1.2.3.4.5.6} - \frac{1}{1.2\ldots7}$$
etc.

Quodsi ergo fingatur series haec: $s = Az + Bz^3 + Cz^5 + Dz^7 + Ez^9 +$ etc. erit

$$s = \frac{z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \frac{z^7}{1.2\ldots7} + \text{etc.}}{1 - \frac{z^2}{1.2} + \frac{z^4}{1.2.3.4} - \frac{z^6}{1.2.3\ldots6} + \text{etc.}}$$

atque

atque hinc $s =$ tang A. z, seu $z =$ tang. s. Habebimus ergo $dz = \frac{ds}{1+ss}$ et $dz + ss\,dz = ds$ cui aequationi cum satisfacere debeat valor hic

$$s = Az + Bz^3 + Cz^5 + Dz^7 + Ez^9 + \text{etc.}$$

substituantur valores loco ds et ss, eritque

$$\begin{aligned} \tfrac{ds}{dz} &= A + 3Bz^2 + 5Cz^4 + 7Dz^6 + 9Ez^8 + \text{etc.} \\ ss &= \quad + A^2z^2 + 2ABz^4 + 2ACz^6 + 2ADz^8 + \text{etc.} \\ & \qquad\qquad\qquad + B^2z^6 + 2BCz^8 + \text{etc.} \\ 1 &= 1 \end{aligned}$$

Hinc itaque formatis aequationibus aliae sequentes prodibunt determinationes litterarum A, B, C, D, etc.

$$\begin{aligned} A &= 1 \\ B &= \tfrac{A^2}{3} \\ C &= \tfrac{2AB}{5} \\ D &= \tfrac{2AC+B^2}{7} \\ E &= \tfrac{2AD+2BC}{9} \\ F &= \tfrac{2AE+2BD+C^2}{11} \\ &\text{etc.} \end{aligned}$$

§. 22. Ab his seriebus potestatum parium pendent summae serierum sub hac forma generali contentarum

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc.}$$

denotante n numerum parem. Quodsi enim fuerit

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \text{etc.} = N\pi^n$$

erit $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \text{etc.} = \frac{2^n N \pi^n}{2^n - 1}$

vnde omnium harum serierum dummodo sit n numerus

par,

par, summae per quadraturam circuli poterunt inueniri; atque ex iam inuentis summis similium potestatum parium pro numeris imparibus solis. Verum vt hae summae directe inueniri queant, in legem peculiarem qua istae summae progrediuntur, inquiramus. Sit itaque

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = A\pi^2$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = B\pi^4$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \text{etc.} = C\pi^6$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \text{etc.} = D\pi^8$$

etc.

ac contemplabor hanc seriem: $s =$
$A\pi^2 z^2 + B\pi^4 z^4 + C\pi^6 z^6 + D\pi^8 z^8 + E\pi^{10} z^{10} + \text{etc.}$
quae substitutis loco $A\pi^2$, $B\pi^4$, $C\pi^6$, etc. seriebus quas denotant, additisque terminis homologis, prodibit

$$s = \frac{zz}{1-zz} + \frac{zz}{4-zz} + \frac{zz}{9-zz} + \frac{zz}{16-zz} + \frac{zz}{25-zz} + \text{etc.}$$

quae series per §. 17. summata dat $s = \frac{1}{2} - \frac{\pi z}{2\,tang\,A.\,\pi z}$ vel si tangens arcus πz per seriem exprimatur:

$$s = \frac{1}{2} - \frac{1}{2}.\frac{1 - \frac{\pi^2 z^2}{1.2} + \frac{\pi^4 z^4}{1.2.3.4} - \frac{\pi^6 z^6}{1.2.3.4.5.6} + \text{etc.}}{1 - \frac{\pi^2 z^2}{1.2.3} + \frac{\pi^4 z^4}{1.2.3.4.5} - \frac{\pi^6 z^6}{1.2\ldots7} + \text{etc.}}$$

$$s = \frac{\frac{\pi^2 z^2}{1.2.3} - \frac{2\pi^4 z^4}{1.2.3.4.5} + \frac{3\pi^6 z^6}{1.2\ldots7} - \frac{4\pi^8 z^8}{1.2\ldots9} + \text{etc.}}{1 - \frac{\pi^2 z^2}{1.2.3} + \frac{\pi^4 z^4}{1.2.3.4.5} - \frac{\pi^6 z^6}{1.2\ldots7} + \frac{\pi^8 z^8}{1.2\ldots9} - \text{etc.}}$$

qua expressione euoluta, cum ipsam seriem assumtam $A\pi^2 z^2 + B\pi^4 z^4 + C\pi^6 z^6 + D\pi^8 z^8 + \text{etc.}$ praebere debeat, sequentur hae coefficientium determinationes

$$A = \frac{1}{6}$$

$$B = \frac{A}{1.2.3} - \frac{2}{1.2.3.4.5}$$

$$C = \frac{B}{1.2.3} - \frac{A}{1.2.3.4.5} + \frac{3}{1.2\ldots7}$$

$$D = \frac{C}{1.2.3} - \frac{B}{1.2.3.4.5} + \frac{A}{1.2\ldots7} - \frac{4}{1.2\ldots9}$$

etc.

§. 23.

§. 23. At pro iisdem his coefficientibus alia lex progressionis potest exhiberi, cuius ope eos multo expeditius reperire licebit. Cum enim sit $s = \frac{1}{2} - \frac{\pi z}{2\,\text{tang.}\,A\,\pi z}$ erit tang. A. $\pi z = \frac{\pi z}{1-2s}$ et $\pi z = A$ tang. $\frac{\pi z}{1-2s}$; ponatur $\pi z = u$, erit $u = A$ tang. $\frac{u}{1-2s}$ et differentiando $du = \frac{du - 2s\,du + 2u\,ds}{1-4s+4ss+uu}$ vel $uu\,du + 4ss\,du = 2s\,du + 2u\,ds$: cui aequationi satisfacit valor hic $s = Au^2 + Bu^4 + Cu^6 + Du^8 + Eu^{10} +$ etc. quo substituto fiet

$$uu = uu$$

$$\begin{aligned} 4ss = \; & 4A^2u^4 + 8ABu^6 + 8ACu^8 + 8ADu^{10} + 8AEu^{12} \\ & \qquad\qquad + 4B^2u^8 + 8BCu^{10} + 8BDu^{12} \\ & \qquad\qquad\qquad\qquad + 4C^2u^{12} \end{aligned}$$

$$2s = 2Au^2 + 2Bu^4 + 2Cu^6 + 2Du^8 + 2Eu^{10} + 2Fu^{12}$$

$$\frac{2u\,ds}{du} = 4Au^2 + 8Bu^4 + 12Cu^6 + 16Du^8 + 20Eu^{10} + 24Fu^{12}$$

vnde sequentes consequuntur determinationes:

$A = \frac{1}{6}$ | $E = \frac{4AD+4BC}{11}$

$B = \frac{2A^2}{5}$ | $F = \frac{4AE+4BD+2C^2}{13}$

$C = \frac{4AB}{7}$ | $G = \frac{4AF+4BE+4CD}{15}$

$D = \frac{4AC+2B^2}{9}$ | $H = \frac{4AG+4BF+4CE+2D^2}{17}$

etc.

§. 24. Ipsae autem huiusmodi serierum summae, quousque quidem eas supputaui, sequentes sunt:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = \frac{2}{1\cdot 2\cdot 3} \cdot \frac{1}{2}\pi^2$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = \frac{2^3}{1\cdot 2\cdot 3\cdot 4\cdot 5} \cdot \frac{1}{6}\pi^4$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \text{etc.} = \frac{2^5}{1\cdot 2 \cdots 7} \cdot \frac{1}{6}\pi^6$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \text{etc.} = \frac{2^7}{1\cdot 2 \cdots 9} \cdot \frac{3}{10}\pi^8$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\text{etc.}=\frac{2^{9}}{1\cdot2\cdot\ldots\cdot11}\cdot\frac{5}{6}\pi^{10}$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\text{etc.}=\frac{2^{11}}{1\cdot2\cdot\ldots\cdot13}\cdot\frac{691}{210}\pi^{12}$$

$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\frac{1}{5^{14}}+\text{etc.}=\frac{2^{13}}{1\cdot2\cdot\ldots\cdot15}\cdot\frac{35}{2}\pi^{14}$$

$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\frac{1}{5^{16}}+\text{etc.}=\frac{2^{15}}{1\cdot2\cdot\ldots\cdot17}\cdot\frac{3617}{30}\pi^{16}$$

$$1+\frac{1}{2^{18}}+\frac{1}{3^{18}}+\frac{1}{4^{18}}+\frac{1}{5^{18}}+\text{etc.}=\frac{2^{17}}{1\cdot2\cdot\ldots\cdot19}\cdot\frac{43867}{42}\pi^{18}$$

$$1+\frac{1}{2^{20}}+\frac{1}{3^{20}}+\frac{1}{4^{20}}+\frac{1}{5^{20}}+\text{etc.}=\frac{2^{19}}{1\cdot2\cdot\ldots\cdot21}\cdot\frac{1222277}{110}\pi^{20}$$

$$1+\frac{1}{2^{22}}+\frac{1}{3^{22}}+\frac{1}{4^{22}}+\frac{1}{5^{22}}+\text{etc.}=\frac{2^{21}}{1\cdot2\cdot\ldots\cdot23}\cdot\frac{854513}{6}\pi^{22}$$

$$1+\frac{1}{2^{24}}+\frac{1}{3^{24}}+\frac{1}{4^{24}}+\frac{1}{5^{24}}+\text{etc.}=\frac{2^{23}}{1\cdot2\cdot\ldots\cdot25}\cdot\frac{1181820455}{546}\pi^{24}.$$

In his expreſſionibus fractionum mediarum tantum lex non eſt manifeſta, reliquae partes vero perſpicue progrediuntur. Cum autem iſtas fractiones medias $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{3}{10}$, $\frac{5}{6}$ etc. attentius eſſem contemplatus, easdem deprehendi occurrere in expreſſione generali, quam olim tradidi pro ſumma cuiuscunque ſeriei ex dato termino generali inuenienda, ita vt alterius expreſſionis ope altera poſſit confici.

§. 25. Operae pretium igitur erit in conſenſum harum duarum expreſſionum inter ſe tantopere diuerſarum diligentius inquirere. Altera quidem expreſſio quam pro ſummatione ſerierum dedi, ita ſe habet: ſi ſeriei cuiuscunque terminus generalis, ſeu is qui reſpondet exponenti indefinito numerico x fuerit $=X$; et ſumma ſeriei a termino primo vsque ad hunc X incluſiue ponatur $=S$, erit $S=$

$$\int X\,dx+\frac{X}{1\cdot2}+\frac{dX}{1\cdot2\cdot3\cdot2dx}-\frac{d^3X}{1\cdot2\cdot3\cdot4\cdot5\cdot6dx^3}$$
$$+\frac{d^5X}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot6dx^5}-\frac{3d^7X}{1\cdot2\cdot3\cdot\ldots\cdot9\cdot10\,dx^7}+\frac{5d^9X}{1\cdot2\cdot\ldots\cdot11\cdot6\,dx^9}$$
$$-\frac{691d^{11}X}{1\cdot2\cdot\ldots\cdot13\cdot210\,dx^{11}}+\frac{35d^{13}X}{1\cdot2\cdot\ldots\cdot15\cdot2dx^{13}}-\frac{3617d^{15}X}{1\cdot2\cdot\ldots\cdot17\cdot30\,dx^{15}}$$

+

$$+ \frac{43867 d^{17} X}{1 \cdot 2 \cdots 19 \cdot 42 dx^{17}} - \frac{1222277 d^{19} X}{1 \cdot 2 \cdots 21 \cdot 110 dx^{19}}$$
$$+ \frac{854513 d^{21} X}{1 \cdot 2 \cdots 23 \cdot 6 dx^{21}} - \frac{1181820455 d^{23} X}{1 \cdot 2 \cdots 25 \cdot 546 dx^{23}} \text{ etc.}$$

in qua expreſſione apparet easdem omnino fractiones irregulares $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{3}{10}$, $\frac{5}{6}$, etc. ineſſe, quae ante in expreſſionibus ſummarum occurrerunt, hoc tantum diſcrimine, quod hic ſigna habeant alternantia, cum ibi omnia ſigno $+$ eſſent affectae. Atque hic ipſe conſenſus mihi hanc praeſtitit vtilitatem, vt iſtam generalem ſummae S expreſſionem eousque continuare potuerim, cum hoc per eam legem, quae tum temporis mihi pro iſtorum terminorum progreſſione inuenta erat, ſine multo maiore labore praeſtare non potuiſſem.

§. 26. Tametſi autem haec tanti conſenſus mera obſeruatio ſufficere poſſet, ad conſenſum in ſequentibus terminis, qui nondum conſtant, euincendum, tamen praeſtabit ex ipſa rei natura eandem conuenientiam eruere, vt ea non caſu, ſed neceſſario accidiſſe intelligatur. Hanc vero poſteriorem expreſſionem ſequenti modo ſum aſſecutus. Cum S denotet ſummam tot terminorum in ſerie quacunque, quot vnitates continentur in exponente x, vltimusque horum terminorum ſit $= X$: manifeſtum eſt, ſi in S ponatur $x - 1$ loco x, tum prodire debere ſummam eandem S vltimo termino minutam, ſeu $S - X$. At poſito $x - 1$ loco x quantitas S abibit in hanc:

$$S - \frac{dS}{1 \cdot dx} + \frac{ddS}{1 \cdot 2 dx^2} - \frac{d^3 S}{1 \cdot 2 \cdot 3 dx^3} + \frac{d^4 S}{1 \cdot 2 \cdot 3 \cdot 4 \cdot dx^4} - \text{etc.}$$

quae propterea aequalis eſt ipſi $S - X$; vnde habetur haec aequatio:

$$X = \frac{dS}{1 dx} - \frac{ddS}{1 \cdot 2 dx^2} + \frac{d^3 S}{1 \cdot 2 \cdot 3 dx^3} - \frac{d^4 S}{1 \cdot 2 \cdot 3 \cdot 4 dx^4} + \text{etc.}$$

Vt nunc ex hac aequatione S in X exprimatur, accipio hanc aequationem:

$$S = \int X dx + \alpha X + \frac{\beta dX}{dx} + \frac{\gamma ddX}{dx^2} + \frac{\delta d^3X}{dx^3} + \text{etc.}$$

cuius in illa facta substitutione habebitur

$$\begin{aligned} X = X &+ \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} + \frac{\gamma d^3X}{dx^3} + \frac{\delta d^4X}{dx^4} \\ &- \frac{dX}{1\cdot 2dx} - \frac{\alpha ddX}{1\cdot 2dx^2} - \frac{\beta d^3X}{1\cdot 2dx^3} - \frac{\gamma d^4X}{1\cdot 2dx^4} \\ &+ \frac{ddX}{1\cdot 2\cdot 3dx^2} + \frac{\alpha d^3X}{1\cdot 2\cdot 3dx^3} + \frac{\beta d^4X}{1\cdot 2\cdot 3dx^4} \\ &- \frac{d^3X}{1\cdot 2\cdot 3\cdot 4dx^3} - \frac{\alpha d^4X}{1\cdot 2\cdot 3\cdot 4dx^4} \\ &+ \frac{d^5X}{1\cdot 2\cdot 3\cdot 4\cdot 5dx^5} \end{aligned}$$

§. 27. Ex hac aequalitate nascuntur sequentes coefficientium α, β, γ, δ, etc. determinationes.

$$\alpha = \frac{1}{1\cdot 2}$$

$$\beta = \frac{\alpha}{1\cdot 2} - \frac{1}{1\cdot 2\cdot 3}$$

$$\gamma = \frac{\beta}{1\cdot 2} - \frac{\alpha}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4}$$

$$\delta = \frac{\gamma}{1\cdot 2} - \frac{\beta}{1\cdot 2\cdot 3} + \frac{\alpha}{1\cdot 2\cdot 3\cdot 4} - \frac{1}{1\cdot 2 \cdots 5}$$

$$\varepsilon = \frac{\delta}{1\cdot 2} - \frac{\gamma}{1\cdot 2\cdot 3} + \frac{\beta}{1\cdot 2\cdot 3\cdot 4} - \frac{\alpha}{1\cdot 2 \cdots 5} + \frac{1}{1\cdot 2 \cdots 6}$$

etc.

atque ex formulis tum temporis valores harum litterarum definiui, idque multo labore. Neque, quod hic euenit, aliter nisi sola obseruatione cognoui valores alternos γ, ε, η, etc. euanescere. At ex principiis nunc stabilitis idem luculenter ostendi poterit, si alia huius progressionis lex inuestigetur. Considero ad hoc istam seriem:

$$s = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.}$$

eritque ex praecedente coefficientium lege:

$s =$

$$s = \cfrac{1}{1 - \frac{z}{1\cdot 2} + \frac{zz}{1\cdot 2\cdot 3} - \frac{z^3}{1\cdot 2\cdot 3\cdot 4} + \frac{z^5}{1\cdot\ \cdot\ \cdot 4\cdot 5} - \text{etc.}}$$

quae aequatio abit in hanc $s = \frac{z}{1 - e^{-z}}$ seu $s = \frac{e^z z}{e^z - 1}$. Hinc oritur $e^z s - s = e^z z$ et $e^z = \frac{s}{s-z}$ atque $z = ls - l(s-z)$. Differentiando autem habebitur $dz = \frac{ds}{s} - \frac{ds + dz}{s-z}$ siue

$ssdz - szdz = sdz - zds$

cui aequationi satisfacere debet valor assumtus

$s = 1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc.}$

substituatur itaque hic valor in hac aequatione

$\frac{zds}{dz} - s - sz + ss = 0$

atque obtinebitur:

$$\begin{array}{rcllllll}
\frac{zds}{dz} & = & & + \alpha z & + 2\beta z^2 & + 3\gamma z^3 & + 4\delta z^4 & + 5\varepsilon z^5 \\
-s & = & -1 & - \alpha z & - \beta z^2 & - \gamma z^3 & - \delta z^4 & - \varepsilon z^5 \\
-sz & = & & - z & - \alpha z^2 & - \beta z^3 & - \gamma z^4 & - \delta z^5 \\
+s^2 & = & 1 & + 2\alpha z & + 2\beta z^2 & + 2\gamma z^3 & + 2\delta z^4 & + 2\varepsilon z^5 \\
& & & & + \alpha^2 & + 2\alpha\beta & + 2\alpha\gamma & + 2\alpha\delta \\
& & & & & & + \beta^2 & + 2\beta\gamma
\end{array}$$

Hinc igitur colligitur fore:

$\alpha = \frac{1}{2}$

$\beta = \frac{\alpha - \alpha^2}{3}$

$\gamma = \frac{\beta - 2\alpha\beta}{4}$

$\delta = \frac{\gamma - 2\alpha\gamma - \beta\beta}{5}$

$\varepsilon = \frac{\delta - 2\alpha\delta - 2\beta\gamma}{6}$

$\zeta = \frac{\varepsilon - 2\alpha\varepsilon - 2\beta\delta - \gamma\gamma}{7}$

$\eta = \frac{\zeta - 2\alpha\zeta - 2\beta\varepsilon - 2\gamma\delta}{8}$

§. 28. Cum igitur sit $\alpha = \frac{1}{2}$ erit $1 - 2\alpha = 0$, qui valor cum in omnibus terminis sequentibus occurrat, erit:

$$\begin{aligned}
\alpha &= \tfrac{1}{2} \\
\beta &= \tfrac{1}{12} \\
\gamma &= 0 \\
\delta &= -\tfrac{\beta\beta}{5} \\
\varepsilon &= -\tfrac{2\beta\gamma}{6} \\
\zeta &= -\tfrac{2\beta\delta - \gamma\gamma}{7} \\
\eta &= -\tfrac{2\beta\varepsilon - 2\gamma\delta}{8} \\
\theta &= -\tfrac{2\beta\zeta - 2\gamma\varepsilon - \delta\delta}{9} \\
\iota &= -\tfrac{2\beta\eta - 2\gamma\zeta - 2\delta\varepsilon}{10}
\end{aligned}$$

etc.

cum nunc sit $\gamma = 0$, perspicuum est fore etiam $\varepsilon = 0$, hincque porro $\eta = 0$, $\iota = 0$, etc. ita vt omnes termini alterni incipiendo ab γ sint $= 0$, id quod ex praecedente lege tantum per obseruationes patebat, nunc vero id necessario euenire debere intelligitur. Hinc ergo manente $\alpha = \frac{1}{2}$ erit vt sequitur

$$\begin{aligned}
\beta &= \tfrac{1}{12} \\
\delta &= -\tfrac{\beta\beta}{5} \\
\zeta &= -\tfrac{2\beta\delta}{7} \\
\theta &= -\tfrac{2\beta\zeta - \delta\delta}{9} \\
\varkappa &= -\tfrac{2\beta\theta - 2\delta\zeta}{11}
\end{aligned}$$

Quod si ergo ponatur $\beta = \frac{A}{2}$; $\delta = -\frac{B}{2^3}$; $\zeta = \frac{C}{2^5}$; $\theta = -\frac{D}{2^7}$, $\varkappa = \frac{E}{2^9}$, etc. ita vt sit $S = \int X dx + \frac{X}{2} + \frac{A dX}{2 dx} - \frac{B d^3X}{2^3 dx^3} + \frac{C d^5X}{2^5 dx^5} - \frac{D d^7X}{2^7 dx^7} + \frac{E d^9X}{2^9 dx^9} - \frac{F d^{11}X}{2^{11} dx^{11}} +$ etc. tenebunt coefficientes A, B, C, D, etc. hanc legem

A =

$$A = \frac{1}{6} \qquad E = \frac{4AD + 4BC}{11}$$
$$B = \frac{2A^2}{5} \qquad F = \frac{4AE + 4BD + 2C^2}{13}$$
$$C = \frac{4AB}{7} \qquad G = \frac{4AF + 4BE + 4CD}{15}$$
$$D = \frac{4AC + 2B^2}{7} \qquad \text{etc.}$$

Obtinent ergo litterae A, B, C, D, etc. eos ipſos valores, quos ipſis ſupra in §. §. 22 et 23 tribuimus. Atque hinc de conſenſu coefficientium in his expreſſionibus maxime diuerſis plene ſummus certi, neque eum caſui amplius adſcribere conueniet.

§. 29. Quanquam autem hoc modo ſatis expedite ſummam huius ſeriei $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} +$ etc. aſſignare valemus, ſi n eſt numerus par, tamen ex his iisdem principiis nihil omnino concludere poſſumus ad ſummas inueniendas, ſi n ſit numerus impar. Videri quidem poſſet, etiam has ſeries a quadratura circuli ita pendere, vt ſumma earum ſit $= N\pi^n$ caſibus ſcilicet quoque, quibus eſt n numerus impar: verum ſi has ſummas actu per approximationes ſumamus, videbimus coefficientem N non fieri numerum rationalem, niſi ſit n numerus par, id quod ex hac tabella clarius elucebit:

$$1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \text{etc.} = 1,644934066 = \tfrac{\pi^2}{6}$$
$$1 + \tfrac{1}{2^3} + \tfrac{1}{3^3} + \text{etc.} = 1,202056903 = \tfrac{\pi^3}{25,79436}$$
$$1 + \tfrac{1}{2^4} + \tfrac{1}{3^4} + \text{etc.} = 1,082323233 = \tfrac{\pi^4}{90}$$
$$1 + \tfrac{1}{2^5} + \tfrac{1}{3^5} + \text{etc.} = 1,036927755 = \tfrac{\pi^5}{295,1215}$$
$$1 + \tfrac{1}{2^6} + \tfrac{1}{3^6} + \text{etc.} = 1,017343062 = \tfrac{\pi^6}{945}$$
$$1 + \tfrac{1}{2^7} + \tfrac{1}{3^7} + \text{etc.} = 1,008349329 = \tfrac{\pi^7}{2995,286}$$
$$1 + \tfrac{1}{2^8} + \tfrac{1}{3^8} + \text{etc.} = 1,004077356 = \tfrac{\pi^8}{9450}$$

1 +

$$1 + \frac{1}{2^9} + \frac{1}{3^9} + \text{etc.} = 1,002008392 = \frac{\pi^9}{29749,35}$$
$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \text{etc.} = 1,000994575 = \frac{\pi^{10}}{93555}$$
$$1 + \frac{1}{2^{11}} + \frac{1}{3^{11}} + \text{etc.} = 1,000494188 = \frac{\pi^{11}}{294058,5}$$
$$1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \text{etc.} = 1,000246086 = \frac{\pi^{12}}{924041\frac{544}{691}}$$

Neque vero vlla relatio inter ſummas poteſtatum imparium cernitur ſimilis ei, quam ſummae poteſtatum parium inter ſe tenent.

§. 30. Videtur autem aliquid circa ſummas poteſtatum imparium concludi poſſe, ſi ſigna ponantur alternantia. Cum enim imparium poteſtatum prima $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \text{etc.}$ cognitam habeat ſummam ſcilicet $l\,2$: valde veriſimile videtur, etiam ſequentium poteſtatum imparium ſummas a logarithmo binarii pendere, ac fortaſſe inſuper a quadratura circuli. Sed antequam hic aliquid concludere queamus, inueſtigemus ſummas poteſtatum parium: ſitque

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \text{etc.} = A\pi^2$$
$$1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \text{etc.} = B\pi^4$$
$$1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \text{etc.} = C\pi^6$$
$$1 - \frac{1}{2^8} + \frac{1}{3^8} - \frac{1}{4^8} + \text{etc.} = D\pi^8$$
$$1 - \frac{1}{2^{10}} + \frac{1}{3^{10}} - \frac{1}{4^{10}} + \text{etc.} = E\pi^{10}$$
etc.

vbi valores litterarum A, B, C, D etc. facile ex cognitis valoribus pro ſeriebus iisdem, dum omnes termini accipiuntur affirmatiui, concludi poſſunt, ſed praeſtabit peculiarem legem pro his elicere. Conſidero itaque ſequentem ſeriem:

$s =$

$$s = A\pi^2 z^2 + B\pi^4 z^4 + C\pi^6 z^6 + D\pi^8 z^8 + \text{etc.}$$

quae substitutis ipsis seriebus abibit in hanc:

$$s = \frac{zz}{1-zz} - \frac{zz}{4-zz} + \frac{zz}{9-zz} - \frac{zz}{16-zz} + \text{etc.}$$

quae series per §. 17. summata dabit

$$s = \frac{\pi z}{2 \sin. A. \pi z} - \frac{1}{2} \text{ siue sinu expresso}$$

$$s = -\frac{1}{2} + \frac{\frac{1}{2}}{1 - \frac{\pi^2 z^2}{1.2.3} + \frac{\pi^4 z^4}{1.2.3.4.5} - \frac{\pi^6 z^6}{1.2.\ldots 7} - \text{etc.}}$$

Quodsi nunc ponatur terminus precedens in serie litterarum A, B, C, D, E, etc. seu ante primum A existens $= \Delta$ erit

$$\Delta = \frac{1}{2}$$
$$A = \frac{\Delta}{1.2.3} = \frac{1}{12}$$
$$B = \frac{A}{1.2.3} - \frac{\Delta}{1.2.3.4.5}$$
$$C = \frac{B}{1.2.3} - \frac{A}{1.2.3.4.5} + \frac{\Delta}{1.2.\ldots 7}$$
$$D = \frac{C}{1.2.3} - \frac{B}{1.2.3.4.5} + \frac{A}{1.2.\ldots 7} - \frac{\Delta}{1.2.\ldots 9}$$

At valor ipsius Δ non est mere assumtitius, sed reipsa summam seriei praecedentis exprimit, quae est

$$1 - 1 + 1 - 1 + 1 - 1 + \text{etc.} = \Delta\,\pi^0 = \frac{1}{2}$$

serierum vero reliquarum, quae hanc praecedunt, omnium summae sunt $= 0$: scilicet

$$1 - 2^2 + 3^2 - 4^2 + \text{etc.} = 0$$
$$1 - 2^4 + 3^4 - 4^4 + \text{etc.} = 0$$
$$1 - 2^6 + 3^6 - 4^6 + \text{etc.} = 0$$
etc.

§. 31. Ex his igitur sequitur summam cuiusuis seriei ex praecedentibus omnibus recte concludi posse hoc modo, si fuerit

$$1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \text{etc.} = \alpha\pi^n$$

$$1 - \frac{1}{2^{n-2}} + \frac{1}{3^{n-2}} - \frac{1}{4^{n-2}} + \text{etc.} = \beta\pi^{n-2}$$

$$1 - \frac{1}{2^{n-4}} + \frac{1}{3^{n-4}} - \frac{1}{4^{n-4}} + \text{etc.} = \gamma\pi^{n-4}$$

$$1 - \frac{1}{2^{n-6}} + \frac{1}{3^{n-6}} - \frac{1}{4^{n-6}} + \text{etc} = \delta\pi^{n-6}$$

erit $\alpha = \frac{\beta}{1.2.3} - \frac{\gamma}{1.2.3.4.5} + \frac{\delta}{1.2\ldots7} - \frac{\varepsilon}{1.2\ldots9} + \frac{\zeta}{1.2\ldots11} - \text{etc.}$

Vt igitur summam huius seriei inueniamus

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\frac{1}{5^2} - \frac{1}{6^2} + \text{etc.}$$

omnes, quae secundum hanc legem ipsam antecedunt considerari oportebit, quae sunt:

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \text{etc.} = A\pi^2$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \text{etc.} = A\pi$$

$$1 - 2 + 3 - 4 + \text{etc.} = \frac{\alpha}{\pi}$$

$$1 - 2^3 + 3^3 - 4^3 + \text{etc.} = \frac{\beta}{\pi^3}$$

$$1 - 2^5 + 3^5 - 4^5 + \text{etc.} = \frac{\gamma}{\pi^5}$$

eritque $B = \frac{A}{1.2.3} - \frac{\alpha}{1.2.3.4.5} + \frac{\beta}{1.2\ldots7} - \frac{\gamma}{1.2\ldots9} + \text{etc.}$

At harum serierum omnium summae exhiberi possunt, est enim

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \text{etc.} = l\,2$$

$$1 - 2 + 3 - 4 + \text{etc.} = \frac{1}{4} = \frac{2.1}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \text{etc.}\right)$$

$$1 - 2^3 + 3^3 - 4^3 + \text{etc.} = \frac{-1}{8} = \frac{-2.1.2.3}{\pi^4}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \text{etc.}\right)$$

$$1 - 2^5 + 3^5 - 4^5 + \text{etc.} = \frac{1}{4} = \frac{2.1.2.3.4.5}{\pi^6}\left(1 + \frac{1}{3^6} + \frac{1}{5^6} + \text{etc.}\right)$$

$$1 - 2^7 + 3^7 - 4^7 + \text{etc.} = \frac{-17}{16} = \frac{-2.1.2\ldots7}{\pi^8}\left(1 + \frac{1}{3^8} + \frac{1}{5^8} + \text{etc.}\right)$$

etc.

Atque

Atque hinc erit

$A = \frac{l2}{\pi}$

$\alpha = \frac{2.1}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.}\right)$

$\beta = \frac{2.1.2.3}{\pi}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.}\right)$

$\gamma = \frac{2.1.2.3.4.5}{\pi}\left(1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.}\right)$

$\delta = \frac{-2.1.2.\ldots 7}{\pi}\left(1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.}\right)$

$\varepsilon = \frac{2.1.2.\ldots 9}{\pi}\left(1 + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \text{etc.}\right)$

etc.

§. 32. Summas autem poteſtatum parium fractionum, quarum denominatores ſunt numeri impares ſupra exhibuimus. Sit

$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} = P\pi^2$

$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} = Q\pi^4$

$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.} = R\pi^6$

$1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.} = S\pi^8$

Erit per §. 21: $P\pi^2 + Q\pi^4 + R\pi^6 + S\pi^8 + \text{etc.} = \frac{\pi}{4}\,\text{tang}\,A.\frac{\pi}{2}$ At litterae α, β, γ, δ, etc. ſequentes obtinebunt valores.

$\alpha = \frac{2.1}{\pi} P\pi^2$

$\beta = \frac{-2.1.2.3}{\pi} Q\pi^4$

$\gamma = \frac{2.1.2.3.4.5}{\pi} R\pi^6$

$\delta = \frac{-2.1.2.3.4.5.6.7}{\pi} S\pi^8$

etc.

Ex lege ergo progreſſionis, ſi ponatur

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.} = A\pi = l2$

$1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{6^3} + \text{etc.} = B\pi^3$

$$1 - \frac{1}{2^5} + \frac{1}{3^5} - \frac{1}{4^5} + \frac{1}{5^5} - \frac{1}{6^5} + \text{etc.} = C\,\pi^5$$

$$1 - \frac{1}{2^7} + \frac{1}{3^7} - \frac{1}{4^7} + \frac{1}{5^7} - \frac{1}{6^7} + \text{etc.} = D\,\pi^7$$

etc.

Poterimus hos coefficientes A, B, C, D, etc. ita determinare, vt ſit:

$$A = \frac{l2}{\pi}\left(\frac{P\pi^2}{2.3} + \frac{Q\pi^4}{4.5} + \frac{R\pi^6}{6.7} + \frac{S\pi^8}{8.9} + \text{etc.}\right)$$

$$B = \frac{A}{1.2.3} - \frac{2}{\pi}\left(\frac{P\pi^2}{2.3.4.5} + \frac{Q\pi^4}{4.5.6.7} + \frac{R\pi^6}{6.7.8.9} + \text{etc.}\right)$$

$$C = \frac{B}{1.2.3} - \frac{A}{1.2.3.4.5} + \frac{2}{\pi}\left(\frac{P\pi^2}{2.3\ldots7} + \frac{Q\pi^4}{4.5\ldots9} + \frac{R\pi^6}{6.7\ldots11} + \text{etc.}\right.$$

$$D = \frac{C}{1.2.3}\ \frac{-B}{1.2.3.4.5} + \frac{A}{1.2\ldots7} - \frac{2}{\pi}\left(\frac{P\pi^2}{2.3\ldots9} + \frac{Q\pi^4}{4.5\ldots11} + \text{etc.}\right.$$

§. 33. Antequam autem quicquam hinc concludere ſuſcipiamus, exemplo vno doceamus regulam hanc inuentam recte ſe habere; ac valores veros litterarum inde prodire. Sumamus igitur primam formulam, et cum ſit $A = \frac{l2}{\pi}$ habebitur iſta aequatio

$$\frac{l2}{2} = \frac{P\pi^2}{2.3} + \frac{Q\pi^4}{4.5} + \frac{R\pi^6}{6.7} + \frac{S\pi^8}{8.9} + \text{etc.}$$

Eſt vero ad veros valores appropinquando

$l2 = 0{,}69314718 0$
$P\pi^2 = 1{,}233700550$
$Q\pi^4 = 1{,}014678031$
$R\pi^6 = 1{,}001447077$
$S\pi^8 = 1{,}000155179$
$T\pi^{10} = 1{,}000017041$
$V\pi^{12} = 1{,}000001885$
$W\pi^{14} = 1{,}000000209$
$X\pi^{16} = 1{,}000000023$
$Y\pi^{18} = 1{,}000000002$

Sumamus primum vnitates integras, pro $P\pi^2$, $Q\pi^4$ etc. habe-

habebimus $\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \frac{1}{8.9} +$ etc. cuius seriei summa constat, quippe quae est $= 1 - l\,2$ seu

0,306852810.

nunc sumamus eorundem terminorum fractiones annexas, quae per respectiuos denominatores diuisae dabunt:

0,038950091
0,000733901
34454
2155
155
12
1

0,039720771 addatur $1 - l\,2$
0,306852819

0,346573590 At vero est $\frac{l2}{2} =$
0,346573590

vnde aequalitas luculenter perspicitur.

§. 34. Cum igitur certo nunc constet de veritate propositionis §. 32. assertae, legem habemus, secundum quam summae serierum $1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} +$ etc. denotante n numerum imparem quemcunque progrediuntur. Verum quia ex obseruatione tantum nobis constat esse

$\frac{l2}{2} = \frac{P\pi^2}{2 \cdot 3} + \frac{Q\pi^4}{4 \cdot 5} + \frac{R\pi^6}{6 \cdot 7} + \frac{S\pi^8}{8 \cdot 9} +$ etc. siue

$$l2 = \begin{cases} \frac{1}{3}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{etc.}\right) \\ \frac{1}{10}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \text{etc.}\right) \\ \frac{1}{21}\left(1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \text{etc.}\right) \\ \frac{1}{36}\left(1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \text{etc.}\right) \\ \frac{1}{55}\left(+ \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \frac{1}{9^{10}} + \text{etc.}\right) \\ \text{etc.} \end{cases}$$

 operae

operae pretium erit in demonstrationem huius veritatis inquirere. Ponamus igitur

$$s = \frac{P\pi^2}{2.3} + \frac{Q\pi^4}{4.5} + \frac{R\pi^6}{6.7} + \frac{S\pi^8}{8.9} + \text{etc.}$$

atque instituantur sequentes transformationes.

$$\frac{d.\pi s}{d\pi} = \frac{P\pi^2}{2} + \frac{Q\pi^4}{4} + \frac{R\pi^6}{6} + \text{etc.}$$

$$\frac{dd.\pi s}{d\pi^2} = P\pi + Q\pi^3 + R\pi^5 + \text{etc.}$$

Quia haec vltima series si per π multiplicetur, summam habet $\frac{\pi}{4}$ tang. A. $\frac{\pi}{2}$, quae expressio locum habet, etiamsi π quantitas sit variabilis, quemadmodum hic posuimus. Erit itaque

$$dd.\pi s = \frac{d\pi^2}{4} \text{ tang. A. } \frac{\pi}{2} \text{ et hinc}$$

$$d.\pi s = \frac{d\pi}{4} \int d\pi \text{ tang. A. } \frac{\pi}{2} \text{ ac denique}$$

$$s = \frac{1}{4\pi} \int d\pi \int d\pi \text{ tang. A. } \frac{\pi}{2}$$

cuius aequationis radix iam patet, quippe est $s = \frac{l2}{2}$.

§. 35. Consideremus primum formulam hanc $\int d\pi$ tang. A. $\frac{\pi}{2}$ quae abit in $\int \frac{d\pi \sin A. \frac{\pi}{2}}{\cos A \frac{\pi}{2}} = -2 l \cos. A \frac{\pi}{2}$ hoc vero integrali substituto habebimus $s = \frac{-1}{\pi} \int \frac{d\pi}{2} l \cos. A. \frac{\pi}{2}$. Ad hanc formulam integrandam pono tang. A. $\frac{\pi}{2} = t$, erit cos. A $\frac{\pi}{2} = \frac{1}{\sqrt{(1+tt)}}$ et $-l \cos. A. \frac{\pi}{2} = l\sqrt{(1+tt)} = \frac{1}{2} l(1+tt)$ et $\frac{d\pi}{2} = \frac{dt}{1+tt}$; ex quo erit $s = \frac{1}{2\pi} \int \frac{dt}{1+tt} l(1+tt)$ atque ideo quaestio huc est reducta, vt definiatur integrale $\int \frac{dt\, l(1+tt)}{1+tt}$ tali adhibita constante vt integrale euanescat posito, $t = 0$; quo facto restitui oportet $t =$ tang. A. $\frac{\pi}{2}$ seu ob $\frac{\pi}{2} =$ arcui 90°, erit $t = \infty$. Formula autem haec, quia est $l(1+tt) = \frac{tt}{1+tt} + \frac{t^4}{2(1+tt)^2} + \frac{t^6}{3(1+tt)^3} + \frac{t^8}{4(1+tt)^4}$ + etc.

$+$ etc. transit in sequentem ita vt sit $\int \frac{dt}{1+tt}\, l(1+tt) =$ $\int \frac{ttdt}{(1+tt)^2} + \frac{1}{2}\int \frac{t^4 dt}{(1+tt)^3} + \frac{1}{3}\int \frac{t^6 dt}{(1+tt)^4} + \frac{1}{4}\int \frac{t^8 dt}{(1+tt)^5} +$ etc. Per reductionem autem formularum integralium est generaliter.

$$\int \frac{t^{2m}\, dt}{(1+tt)^{m+1}} = \frac{-t^{2m-1}}{2m(1+tt)^m} + \frac{2m-1}{2m}\int \frac{t^{2m-2}\, dt}{(1+tt)^m}$$

Quare cum sit $\int \frac{dt}{1+tt} = \frac{\pi}{2}$ erit

$$\int \frac{ttdt}{(1+tt)^2} = \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot\frac{t}{1+tt}$$

$$\int \frac{t^4 dt}{(1+tt)^3} = \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2} - \frac{1\cdot 3}{2\cdot 4}\cdot\frac{t}{1+tt} - \frac{1}{4}\cdot\frac{t^3}{(1+tt)^2}$$

$$\int \frac{t^6 dt}{(1+tt)^4} = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{t}{1+tt} - \frac{1\cdot 5}{4\cdot 6}\cdot\frac{t^3}{(1+tt)^2} - \frac{1t^5}{6(1+tt)^3}$$

$$\int \frac{t^8 dt}{(1+tt)^5} = \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{\pi}{2} - \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{t}{1+tt} - \frac{1\cdot 5\cdot 7}{4\cdot 6\cdot 8}\cdot\frac{t^3}{(1+tt)^2} - \frac{1\cdot 7}{6\cdot 8}\cdot\frac{t^5}{(1+tt)^3} - \frac{1}{8}\cdot\frac{t^7}{(1+tt)^4}$$

$$\int \frac{t^{10} dt}{(1+tt)^6} = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\cdot\frac{\pi}{2} - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\cdot\frac{t}{1+tt} - \frac{1\cdot 5\cdot 7\cdot 9}{4\cdot 6\cdot 8\cdot 10}\cdot\frac{t^3}{(1+tt)^2} - \frac{1\cdot 7\cdot 9}{6\cdot 8\cdot 10}\cdot\frac{t^5}{(1+tt)^3} - \frac{1\cdot 9}{8\cdot 10}\cdot\frac{t^7}{(1+tt)^4} - \frac{1}{10}\frac{t^9}{(1+tt)^5} \text{ etc.}$$

Ex his substitutis orietur $\int \frac{dt}{1+tt}\, l(1+tt) =$

$$+\frac{\pi}{2}\left(\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4} + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 5} + \text{etc.}\right)$$

$$-\frac{t}{1+tt}\left(\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4} + \text{etc.}\right)$$

$$-\frac{t^3}{4(1+tt)^2}\left(\frac{1}{2} + \frac{5}{6\cdot 3} + \frac{5\cdot 7}{6\cdot 8\cdot 4} + \frac{5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 5} + \text{etc.}\right)$$

$$-\frac{t^5}{6(1+tt)^3}\left(\frac{1}{3} + \frac{7}{8\cdot 4} + \frac{7\cdot 9}{8\cdot 10\cdot 5} + \frac{7\cdot 9\cdot 11}{8\cdot 10\cdot 12\cdot 6} + \text{etc.}\right)$$

$$-\frac{t^7}{8(1+tt)^4}\left(\frac{1}{4} + \frac{9}{10\cdot 5} + \frac{9\cdot 11}{10\cdot 12\cdot 6} + \frac{9\cdot 11\cdot 13}{10\cdot 12\cdot 14\cdot 7} + \text{etc.}\right)$$

etc.

§. 36. Quaeramus primum summam seriei huius,

$$\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4} + \text{etc.}$$

ponamusque:

$S =$

$$s = \frac{x}{2\cdot 1} + \frac{1\cdot 3\cdot x^2}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5\, x^3}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 3\cdot 5\cdot 7\cdot x^4}{2\cdot 4\cdot 6\cdot 8\cdot 4} + \text{etc.}$$

erit $s = \int \frac{dx}{x\sqrt{(1-x)}}\ lx$, vt euoluenti facile patebit. At eſt $\int \frac{dx}{x\sqrt{(1-x)}} = c - l(1+\sqrt{(1-x)}) + l(1-\sqrt{(1-x)})$ hincque $s = c - l(1+\sqrt{(1-x)}) + l(1-\sqrt{(1-x)}) - lx$. vbi conſtans c ita debet eſſe comparata, vt poſito $x=0$ fiat $s=0$. Fiat igitur x infinite paruum, erit $\sqrt{(1-x)} = 1 - \frac{x}{2}$ et $l(1-\sqrt{(1-x)}) = l\frac{x}{2} = lx - l2$ et $l(1+\sqrt{(1-x)}) = l2$; vnde $c = 2\,l2$. Ponatur nunc $x=1$, erit $s = 2\,l2$. atque

$$\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4} + \text{etc.} = 2\,l\,2$$

Ex hac autem ſerie reliquarum ſerierum ſummae ita determinabuntur vt ſit:

$$\frac{1}{2} + \frac{5}{6\cdot 3} + \frac{5\cdot 7}{6\cdot 8\cdot 4} + \frac{5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 5} + \text{etc.} = \frac{2\cdot 4}{1\cdot 3}\,.\,2\,l\,2 - \frac{2\cdot 4}{1\cdot 3\cdot 2}$$

$$\frac{1}{3} + \frac{7}{8\cdot 4} + \frac{7\cdot 9}{8\cdot 10\cdot 5} + \frac{7\cdot 9\cdot 11}{8\cdot 10\cdot 12\cdot 6} + \text{etc.} = \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\,2\,l2 - \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 2} - \frac{6}{5\cdot 2}$$

$$\frac{1}{4} + \frac{9}{10\cdot 5} + \frac{9\cdot 11}{10\cdot 12\cdot 6} + \text{etc.} = \frac{2\cdot 4\cdot 6\cdot 8}{1\cdot 3\cdot 5\cdot 7}\,.\,2\,l\,2 - \frac{2\cdot 4\cdot 6\cdot 8}{1\cdot 3\cdot 5\cdot 7\cdot 2} - \frac{6\cdot 8}{5\cdot 7\cdot 2} - \frac{8}{7\cdot 3}$$

etc.

Quibus ſummis ſubſtitutis proueniet $\int \frac{dt}{1+tt}\ l(1+tt)$

$$= + \frac{\pi}{2}\,.\,2\,l\,2$$
$$- \frac{t}{1+tt}\,.\,2\,l\,2$$
$$- \frac{t^3}{(1+tt)^2}\left(\frac{2}{3}\,.\,2\,l\,2 - \frac{1}{3\cdot 1}\right)$$
$$- \frac{t^5}{(1+tt)^3}\left(\frac{2\cdot 4}{3\cdot 5}\,2\,l\,2 - \frac{4}{3\cdot 5\cdot 1} - \frac{1}{5\cdot 2}\right)$$
$$- \frac{t^7}{(1+tt)^4}\left(\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\,2\,l\,2 - \frac{4\cdot 6}{3\cdot 5\cdot 7\cdot 1} - \frac{6}{5\cdot 7\cdot 2} - \frac{8}{7\cdot 3}\right)$$
$$- \frac{t^9}{(1+tt)^5}\left(\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9}\,2\,l\,2 - \frac{4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9\cdot 1} - \frac{6\cdot 8}{5\cdot 7\cdot 9\cdot 2} - \frac{8}{7\cdot 9\cdot 3} - \frac{8}{9\cdot 4}\right)$$

etc.

§. 37.

§. 37. Quoniam vero ad institutum nostrum post integrationem absolutam poni debet $t = \infty$, fiet $\int \frac{dt}{1+tt} l(1+tt) = \pi l 2$ atque $s = \frac{1}{2}\pi \int \frac{dt}{1+tt} l(1+tt) = \frac{l2}{2}$, qui est ille ipse valor quem praeuidimus prodire debere (§. 34.). Reliqui enim termini in expressione, quam pro $\int \frac{dt}{1+tt} l(1+tt)$ inuenimus, si ponatur $t = \infty$ omnes euanescunt, quia in denominatoribus singulorum terminorum t plures habet dimensiones quam in numeratoribus, atque insuper coefficientes numerici decrescunt. Nisi enim hoc eueniret, tuto concludere non possemus summam omnium terminorum, quorum quisque euanescit esse $= 0$. Nam si verbi gratia priores tantum coefficientium numericorum partes accipiantur, vt prodiret haec series:

$$\frac{t}{1+tt} + \frac{2t^3}{3(1+tt)^2} + \frac{2.4t^5}{3.5(1+tt)^3} + \frac{2.4.6.t^7}{3.5.7(1+tt)^4} \text{ etc.}$$

summa ipsius casu quo $t = \infty$, fit finita et $= \frac{\pi}{2}$ etiamsi singuli termini euanescant, quodsi autem integri coefficientes capiantur ob seriem eorum valde conuergentium, tota quoque series euadit $= 0$.

§. 38. Inquiramus nunc in summam huius seriei $1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \text{etc.} = B\pi^3$ quae summa per §. 32. erit $B\pi^3 = \frac{\pi^2 l2}{1.2.3} - 2\pi^2 \left(\frac{P\pi^2}{2.3.4.5} + \frac{Q\pi^4}{4.5.6.7} + \frac{R\pi^6}{6.7.8.9} + \text{etc.}\right)$ Ad valorem huius quantitatis inueniendum sit $s = \frac{P\pi^2}{2.3.4.5} + \frac{Q\pi^4}{4.5.6.7} + \frac{R\pi^6}{6.7.8.9} + \text{etc.}$

erit $\frac{d.\pi^3 s}{d\pi} = \frac{P\pi^4}{2.3.4} + \frac{Q\pi^6}{4.5.6} + \frac{R\pi^8}{6.7.8} + \text{etc.}$

$$\frac{dd.\pi^3 s}{d\pi^2} = \frac{P\pi^3}{2.3} + \frac{Q\pi^4}{4.5} + \frac{R\pi^7}{6.7} + \text{etc.}$$

$$\frac{d^3.\pi^3 s}{d\pi^3} = \frac{P\pi^2}{2} + \frac{Q\pi^4}{4} + \frac{R\pi^6}{6} + \text{etc.}$$

$$\frac{d^4.\pi^3 s}{d\pi^4} = P\pi + Q\pi^3 + R\pi^5 + \text{etc.} = \frac{1}{4}\text{tang. A}\frac{\pi}{2}.$$

Regrediendo erit ergo.

$\frac{d^3 . \pi^3 s}{d\pi^3} = \frac{1}{4} \int d\pi \text{ tang. A } \frac{\pi}{2}.$

$\frac{dd . \pi^3 s}{d\pi^2} = \frac{1}{4} \int d\pi \int d\pi \text{ tang. A } \frac{\pi}{2}.$

$\frac{d . \pi^3 s}{d\pi} = \frac{1}{4} \int d\pi \int d\pi \int d\pi \text{ tang. A } \frac{\pi}{2}.$

$\pi^3 s = \frac{1}{4} \int d\pi \int d\pi \int d\pi \int d\pi \text{ tang. A } \frac{\pi}{2}.$

Atque hinc habebitur ſumma ſeriei propoſitae $B\pi^3 = \frac{\pi^2 l 2}{6} - \frac{1}{2\pi} \int d\pi \int d\pi \int d\pi \int d\pi \text{ tang. A } \frac{\pi}{2}$, quae omnia integralia ita debent accipi, vt euaneſcant poſito $\pi = 0$.

§. 39. Ponatur $\frac{\pi}{2} = q$, ita vt integrationibus abſolutis q denotet quartam peripheriae partem circuli, cuius radius $= 1$ ſeu arcum 90. graduum. Sitque porro ſin. A $q = y$ et coſ. A $q = x = \sqrt{(1 - yy)}$, erit tang. A. $\frac{\pi}{2} = \frac{y}{x}$. Quare ob $\pi = 2q$, erit ſumma noſtrae ſeriei $B\pi^3 = \frac{2qq l 2}{3} - \frac{4}{q} \int dq \int dq \int dq \int \frac{y dq}{x}$. Ponamus tantiſper $\int dq \int dq \int dq \int \frac{y dq}{x} = u$ erit $B\pi^3 = \frac{2qq l 2}{3} - \frac{4u}{q}$, vbi in quantitate u inuenienda omnes integrationes ita debent accipi, vt integralia ſingula euaneſcant poſito $q = 0$ et $y = 0$, integralibus vero abſolutis erit $y = 1$ et $x = 0$. Eſt vero $\int \frac{y dq}{x} = \int \frac{y dy}{1 - yy} = -l\sqrt{(1 - yy)} = l \frac{1}{x}$.
et $l \frac{1}{x} = \frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \frac{y^{10}}{10} + \text{etc.}$
Cum nunc ſit $u = \int dq \int dq \int dq \; l \frac{1}{x}$, erit per reductionem integralium

$$u = q \int dq \int dq \; l \frac{1}{x} - \int q dq \int dq \; l \frac{1}{x}$$

atque porro

$$\int dq \int dq \; l \frac{1}{x} = q \int dq \; l \frac{1}{x} - \int q dq \; l \frac{1}{x}$$

$$\int q dq \int dq \; l \frac{1}{x} = \frac{qq}{2} \int dq \; l \frac{1}{x} - \frac{1}{2} \int qq dq \; l \frac{1}{x}$$

ergo

$$u = \frac{1}{2} qq \int dq \; l \frac{1}{x} - q \int q dq \; l \frac{1}{x} + \frac{1}{2} \int qq dq \; l \frac{1}{x}$$

ita

ita vt nunc tres formulas habeamus simpliciter differentiales, quas integrare debemus.

§. 40. Consideremus singulas has tres formulas seorsim, ac primo quidem hanc $\int dq\, l\frac{1}{x}$, quam etsi iam supra integrauimus, tamen eandem ex consideratione sinuum et cosinuum denuo integremus, vt via facilior paretur ad reliquas integrandas. Est igitur:

$$\int dq\, l\tfrac{1}{x} = \int dq\, \left(\tfrac{yy}{2} + \tfrac{y^4}{4} + \tfrac{y^6}{6} + \tfrac{y^8}{8} + \tfrac{y^{10}}{10} + \text{etc.}\right)$$

Ad hoc integrale inueniendum consideretur membrum eius quodcunque $\int y^{n+2}\, dq$, et cum sit $dq = \frac{dy}{x} = \frac{-dx}{y}$ et $xx + yy = 1$, erit

$$\int y^{n+2}\, dq = -\int y^{n+1}\, dx = -y^{n+1} x + (n+1) \int y^n x\, dy$$

at est $\int y^n x\, dy = \int y^n x^2\, dq = \int y^n\, dq - \int y^{n+2}\, dq$

ob $xx = 1 - yy$: ex quo erit:

$$\int y^{n+2}\, dq = -y^{n+1} x + (n+1) \int y^n\, dq - (n+1) \int y^{n+2}\, dq$$

atque

$$\int y^{n+2}\, dq = \frac{-y^{n+1} x}{n+2} + \frac{n+1}{n+2} \int y^n\, dq$$

Hinc itaque integrale cuiusque membri reducitur ad integrale praecedentis, et quoniam integratione absoluta fit $x = 0$; erit pro hoc casu

$$\int y^{n+2}\, dq = \frac{n+1}{n+2} \int y^n\, dq.$$

Ex hac ergo formula reperientur singulae integralis partes vt sequitur.

$$\int y^2\, dq = \tfrac{1}{2} q$$

$$\int y^4\, dq = \tfrac{1 \cdot 3}{2 \cdot 4} q$$

$$\int y^6\, dq = \tfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} q$$

$$\int y^8\, dq = \tfrac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} q$$

Hancobrem habebitur: $\int dq\ l\frac{1}{x} = \frac{1}{2}q\left(\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4} + \text{etc.}\right)$ cuius feriei cum iam fupra inuenta fit fumma (§. 36.) $= 2\,l\,2$ erit $\int dq\ l\frac{1}{x} = q\,l\,2$.

§. 41. Progrediamur iam ad fecundam formulam integralem $\int q\,dq\ l\frac{1}{x}$, quae abit in

$$\int q\,dq\ l\frac{1}{x} = \int q\,dq\left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \text{etc.}\right)$$

cuius partem confideremus quamcunque $\int y^{n+2}q\,dq = -\int y^{n+1}q\,dx = -y^{n+1}qx + \int y^{n+1}x\,dq + (n+1)\int y^n qx\,dy$

$$= -y^{n+1}qx + \frac{y^{n+2}}{n+2} + (n+1)\int y^n q\,dq - (n+1)\int y^{n+2}q\,dq$$

Pofito itaque $y = 1$ et $x = 0$, erit

$$\int y^{n+2}q\,dq = \frac{1}{(n+2)^2} + \frac{n+1}{n+2}\int y^n q\,dq$$

Integralia igitur fingulorum membrorum ita fe habebunt:

$$\int y^2 q\,dq = \frac{1}{2^2} + \frac{1}{2}\cdot\frac{q^2}{2}$$

$$\int y^4 q\,dq = \frac{1}{4^2} + \frac{3}{4\cdot 2^2} + \frac{1\cdot 3}{2\cdot 4}\frac{q^2}{2}$$

$$\int y^6 q\,dq = \frac{1}{6^2} + \frac{5}{6\cdot 4^2} + \frac{3\cdot 5}{4\cdot 6\cdot 2^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{q^2}{2}$$

$$\int y^8 q\,dq = \frac{1}{8^2} + \frac{7}{8\cdot 6^2} + \frac{5\cdot 7}{6\cdot 8\cdot 4^2} + \frac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8\cdot 2^2} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{q^2}{2}$$

etc.

Ex quo obtinebitur integrale $\int q\,dq\ l\frac{1}{x} =$

$$+ \frac{qq}{4}\left(\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4}\ \text{etc.}\right.$$

$$+ \frac{1}{2\cdot 2^2}\left(\frac{1}{1} + \frac{3}{4\cdot 2} + \frac{3\cdot 5}{4\cdot 6\cdot 3} + \frac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8\cdot 4} + \text{etc.}\right)$$

$$+ \frac{1}{2\cdot 4^2}\left(\frac{1}{2} + \frac{5}{6\cdot 3} + \frac{5\cdot 7}{6\cdot 8\cdot 4} + \frac{5\cdot 7\cdot 9}{6\cdot 8\cdot 10\cdot 5} + \text{etc.}\right)$$

$$+ \frac{1}{2\cdot 6^2}\left(\frac{1}{3} + \frac{7}{8\cdot 4} + \frac{7\cdot 9}{8\cdot 10\cdot 5} + \frac{7\cdot 9\cdot 11}{8\cdot 10\cdot 12\cdot 6} + \text{etc.}\right)$$

etc.

Vel etiam hac forma $\int q\,dq\ l\frac{1}{x} =$

$$\frac{qq}{4}\left(\frac{1}{2\cdot 1} + \frac{1\cdot 3}{2\cdot 4\cdot 2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 3} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 4} + \text{etc.}\right.$$

+

$$+ \frac{1}{2^2 \cdot 2} + \frac{1}{4^2 \cdot 4} + \frac{1}{6^2 \cdot 6} + \frac{1}{8^2 \cdot 8} + \text{etc.}$$

$$+ \frac{3}{2^2 \cdot 4^2} + \frac{5}{4^2 \cdot 6^2} + \frac{7}{6^2 \cdot 8^2} + \frac{9}{8^2 \cdot 10^2} + \text{etc.}$$

$$+ \frac{3 \cdot 5}{2^2 \cdot 4 \cdot 6^2} + \frac{5 \cdot 7}{4^2 \cdot 6 \cdot 8^2} + \frac{7 \cdot 9}{6^2 \cdot 8 \cdot 10^2} + \frac{9 \cdot 11}{8^2 \cdot 10 \cdot 12^2} + \text{etc.}$$

$$+ \frac{3 \cdot 5 \cdot 7}{2^2 \cdot 4 \cdot 6 \cdot 8^2} + \frac{5 \cdot 7 \cdot 9}{4^2 \cdot 6 \cdot 8 \cdot 10^2} + \frac{7 \cdot 9 \cdot 11}{6^2 \cdot 8 \cdot 10 \cdot 12^2} + \text{etc.}$$

hae autem ſeries id ipſum, quod eſt in quaeſtione inuoluunt, nempe ſummationem cuborum terminorum ſeriei harmonicae.

§. 42. Quod ſi ſequamur priorem formam, omnes ſeries ſummabiles §. 36, habebiturque

$$\int q\,dq\, l\tfrac{1}{x} = \frac{qq}{2}\, l2$$
$$+ \frac{1}{2^2}\left(\frac{2}{1}\, l2\right)$$
$$+ \frac{1}{4^2}\left(\frac{2 \cdot 4}{1 \cdot 3}\, l2 - \frac{4}{3 \cdot 2}\right.$$
$$+ \frac{1}{6^2}\left(\frac{2 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5}\, l2 - \frac{4 \cdot 6}{3 \cdot 5 \cdot 2} - \frac{6}{5 \cdot 4}\right)$$
$$+ \frac{1}{8^2}\left(\frac{2 \cdot 4 \cdot 6 \cdot 8}{1 \cdot 3 \cdot 5 \cdot 7}\, l2 - \frac{4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7 \cdot 2} - \frac{6 \cdot 8}{5 \cdot 7 \cdot 4} - \frac{8}{7 \cdot 6}\right) \text{etc.}$$

quae ſi denuo ſeries deorſum capiantur, dant.

$$\int q\,dq\, l\tfrac{1}{x} = \frac{qq}{2}\, l2$$
$$+ l2\left(\frac{1}{2} + \frac{2}{3 \cdot 4} + \frac{2 \cdot 4 \cdot}{3 \cdot 5 \cdot 6} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7 \cdot 8} + \text{etc.}\right.$$
$$- \frac{1}{6}\left(\frac{1}{4} + \frac{4}{5 \cdot 6} + \frac{4 \cdot 6}{5 \cdot 7 \cdot 8} + \frac{4 \cdot 6 \cdot 8}{5 \cdot 7 \cdot 9 \cdot 10} + \text{etc.}\right.$$
$$- \frac{1}{4 \cdot 5}\left(\frac{1}{6} + \frac{6}{7 \cdot 8} + \frac{6 \cdot 8}{7 \cdot 9 \cdot 10} + \frac{6 \cdot 8 \cdot 10}{7 \cdot 9 \cdot 11 \cdot 12} + \text{etc.}\right.$$
$$- \frac{1}{6 \cdot 7}\left(\frac{1}{8} + \frac{8}{9 \cdot 10} + \frac{8 \cdot 10}{9 \cdot 11 \cdot 12} + \frac{8 \cdot 10 \cdot 12}{9 \cdot 11 \cdot 13 \cdot 14} + \text{etc.}\right.$$

etc.

At eſt $\frac{1}{2} + \frac{2}{3 \cdot 4} + \frac{2 \cdot 4}{3 \cdot 5 \cdot 6} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7 \cdot 8} + \text{etc.} = \frac{qq}{2}$

vnde erit

$$\frac{1}{4} + \frac{4}{5 \cdot 6} + \frac{4 \cdot 6}{5 \cdot 7 \cdot 8} + \text{etc.} = \frac{3}{2} \cdot \frac{qq}{2} - \frac{3}{2} \cdot \frac{1}{2}$$

$$\frac{1}{6} + \frac{8}{7 \cdot 8} + \frac{6 \cdot 8}{7 \cdot 9 \cdot 10} + \text{etc.} = \frac{3 \cdot 5}{2 \cdot 4} \cdot \frac{qq}{2} - \frac{3 \cdot 5}{2 \cdot 4} \cdot \frac{1}{2} - \frac{5}{4} \cdot \frac{1}{4}$$

$$\frac{1}{8} + \frac{8}{9 \cdot 10} + \frac{8 \cdot 10}{9 \cdot 11 \cdot 12} + \text{etc.} = \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6} \cdot \frac{qq}{2} - \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6} \cdot \frac{1}{2} - \frac{5 \cdot 7}{4 \cdot 6} \cdot \frac{1}{4} - \frac{7}{6} \cdot \frac{1}{6}$$

etc.

Hinc igitur erit:

$$u = \frac{q}{2.2}\left(\frac{qq}{6} - \frac{1}{2^2}\right)$$
$$+ \frac{1\cdot 3\, q}{2.4.4}\left(\frac{qq}{6} - \frac{1}{2^2} - \frac{1}{4^2}\right)$$
$$+ \frac{1\cdot 3\cdot 5\, q}{2.4.6.6}\left(\frac{qq}{6} - \frac{1}{2^2} - \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{6^2}\right)$$

vel serie prima verticali actu summata

$$u = \frac{q^3}{6}\, l2$$
$$- \frac{q}{2.2}\left(\frac{1}{2^2}\right)$$
$$- \frac{1\cdot 3\, q}{2.4.4}\left(\frac{1}{2^2} + \frac{1}{4^2}\right)$$
$$- \frac{1\cdot 3\cdot 5\cdot q}{2.4.6.6}\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2}\right)$$

etc.

§. 44 Cum nunc seriei nostrae propositae $1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} -$ etc. summa sit $= B\pi^3 = \frac{2qq\, l2}{3} - \frac{4u}{q}$ fiet eadem summa $=$

$$+ \frac{1}{2.2}\cdot 1$$
$$+ \frac{1\cdot 3\cdot}{2.4.4}\left(1 + \frac{1}{2^2}\right)$$
$$+ \frac{1\cdot 3\cdot 5}{2.4.6.6}\left(1 + \frac{1}{2^2} + \frac{1}{3^2}\right)$$
$$+ \frac{1\cdot 3\cdot 5\cdot 7}{2.4.6.8.8}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\right)$$

etc.

Vel cum sit $\frac{1}{2.2} + \frac{1\cdot 3}{2.4.4} + \frac{1\cdot 3\cdot 5}{2.4.6.6}$ etc. $= l2$ erit seriei propositae summa $B\pi^3 = l2 + \frac{1}{2^2}\left(l2 - \frac{1}{2.2}\right) + \frac{1}{3^2}\left(l2 - \frac{1}{2.2} - \frac{1\cdot 3}{2.4.4}\right) + \frac{1}{4^2}\left(l2 - \frac{1}{2.2} - \frac{1\cdot 5}{2.4.4} - \frac{1\cdot 3\cdot 5}{2.4.6.6}\right) +$ etc. siue haec eadem summa ita poterit exprimi vt sit

$$B\pi^3 = \frac{\pi^2}{6}\, l2 - \frac{1}{2^2}\left(\frac{1}{2.2}\right)$$
$$- \frac{1}{3^2}\left(\frac{1}{2.2} + \frac{1\cdot 3}{2.4.4}\right)$$
$$- \frac{1}{4^2}\left(\frac{1}{2.2} + \frac{1\cdot 3}{2.4.4} + \frac{1\cdot 3\cdot 5}{2.4.6.6}\right)$$
$$- \frac{1}{5^2}\left(\frac{1}{2.2} + \frac{1\cdot 3}{2.4.4} + \frac{1\cdot 3\cdot 5}{2.4.6.6} + \frac{1\cdot 3\cdot 5\cdot 7}{2.4.6.8.8}\right) \text{etc.}$$

Quoniam autem vtcunque has series transmutemus, eas ad seriem simplicem, cuius summa constet, reducere non possumus, negotium hoc abrumpamus, pluribus his expressionibus contenti, quas seriei propositae $1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} -$ etc. aequales esse inuenimus.

COM-