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DE
FRACTIONIBVS CONTINVIS
OBSERVATIONES.

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§. 1.

Cum anno ſuperiore incepiſſem fractiones continuas examini ſubiicere, hancque fere nouam analyſeos partem euoluere, nonnullae obſeruationes ſe interea obtulerunt, quae forte ad iſtam Theoriam excolendam non erunt incongruae. Quamobrem cum exploratio huius doctrinae non parum adiumenti analyſi allatura eſſe videatur, hoc argumentum denuo aggrediar, et quae huc ſpectantia occurrerunt, dilucide exponam. Sit igitur propoſita haec fractio continua

$$A+\cfrac{B}{C+\cfrac{D}{E+\cfrac{F}{G+\cfrac{H}{I+\text{etc.}}}}}$$

cuius valor verus reperietur continuando ſequentem ſeriem in infinitum.

$A+\frac{B}{1.P}-\frac{BD}{PQ}+\frac{BDF}{QR}-\frac{BDFH}{RS}+$ etc. in qua ſerie litterae P, Q, R, S etc. ſequentes obtinent valores:

$P=C;\ Q=EP+D;\ R=GQ+FP;\ S=IR+HQ;$ etc. Series haec autem ſemper eſt conuergens, quantumvis creſcant vel decreſcant litterae B, C, D, E, F etc. dummodo omnes ſint affirmatiuae, quilibet terminus enim minor eſt quam praecedens, maior vero quam ſequens; id quod

quod lex, qua valores P, Q, R, S etc. formantur, statim declarat,

§. 2. Si ergo vicissim haec proposita fuerit series infinita $\frac{B}{P} - \frac{BD}{PQ} + \frac{BDF}{QR} - \frac{BDFH}{RS} +$ etc. eius summa commode per fractionem continuam exprimi poterit. Cum enim sit $C = P$; $E = \frac{Q-D}{P}$; $G = \frac{R-FP}{Q}$; $I = \frac{S-HQ}{R}$, etc. habebitur fractio continua illi seriei aequalis haec:

$$\cfrac{B}{P + \cfrac{D}{\cfrac{Q-D}{P} + \cfrac{F}{\cfrac{R-FP}{Q} + \cfrac{H}{\cfrac{S-HQ}{R} + \cfrac{K}{etc.}}}}} \quad \text{seu} \quad \cfrac{B}{P + \cfrac{DP}{Q-D + \cfrac{FPQ}{R-FP + \cfrac{HQR}{S-HQ + \cfrac{KRS}{etc.}}}}}$$

Quare si data fuerit ista series $\frac{a}{p} - \frac{b}{q} + \frac{c}{r} - \frac{d}{s} + \frac{e}{t} -$ etc. ob $B = a$; $D = b:a$; $F = c:b$; $H = d:c$; $K = e:d$, etc. et $P = p$; $Q = q:p$; $R = pr:q$; $S = qs:pr$; $T = prt:qs$; etc. huius seriei $\frac{a}{p} - \frac{b}{q} + \frac{c}{r} - \frac{d}{s} + \frac{e}{t} -$ etc. summae aequalis erit sequens fractio continua:

$$\cfrac{a}{p + \cfrac{b:a}{\cfrac{aq-bp}{app} + \cfrac{c:b}{\cfrac{p^2(br-cq)}{bqq} + \cfrac{d:c}{\cfrac{q^2(cs-dr)}{cp^2r^2} + \cfrac{e:d}{\cfrac{p^2r^2(dt-es)}{dq^2s^2} + etc.}}}}} = \cfrac{a}{p + \cfrac{bp^2}{aq-bp + \cfrac{acaq}{br-cq + \cfrac{bdrr}{cs-dr + \cfrac{cess}{dt-es + etc.}}}}}$$

§. 3. Vt haec exemplis nonnullis illustremus, sumamus seriem $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} +$ etc. cuius summa est $= l2$ seu $= \int \frac{dx}{1+x}$ si post integrationem ponatur $x = 1$. erit ergo $a = b = c = d$ etc. $= 1$; $p = 1$; $q = 2$; $r = 3$; $s = 4$; etc. atque $p = 1$; $aq - bp = 1$; $br - cq = 1$; $cs - dr = 1$, etc.

Hinc

Hinc igitur fit $\int\frac{dx}{1+x} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{4}{1+\cfrac{9}{1+\cfrac{16}{1+\text{etc.}}}}}}$

seu huius fractionis continuae valor est l2.

§. 4. Contemplemur nunc hanc seriem $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-$ etc. cuius summa est area circuli, diametrum $=1$ habentis, seu $=\int\frac{dx}{1+x^2}$ posito post integrationem $x=1$. Erit ergo $a=b=c=d=$ etc. $=1$ et $p=1$; $q=3$; $r=5$; $s=7$; etc. vnde fit:

$$\int\frac{dx}{1+xx} = \cfrac{1}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2\ \text{etc.}}}}}}$$

quae est ipsa fractio continua Brounckeri, quam pro quadratura circuli exhibuit.

§. 5. Simili modo aliis huius generis seriebus accipiendis prodibunt sequentes formularum integralium conuersiones in fractiones continuas, posito scilicet post integrationem $x=1$:

$$\int\frac{dx}{1+x^3} = \cfrac{1}{1+\cfrac{1^2}{3+\cfrac{4^2}{3+\cfrac{7^2}{3+\cfrac{10^2}{3+\text{etc.}}}}}}\ ; \quad \int\frac{dx}{1+x^4} = \cfrac{1}{1+\cfrac{1^2}{4+\cfrac{5^2}{4+\cfrac{9^2}{4+\cfrac{13^2}{4\ \text{etc.}}}}}}$$

$\int dx$

$$\int \frac{dx}{1+x^5} = \cfrac{1}{1+\cfrac{1^2}{5+\cfrac{6^2}{5+\cfrac{11^2}{5+\cfrac{16^2}{5+\text{etc.}}}}}} \quad ; \int \frac{dx}{1+x^6} = \cfrac{1}{1+\cfrac{1^2}{6+\cfrac{7^2}{6+\cfrac{13^2}{6+\cfrac{19^2}{6\ \text{etc.}}}}}}$$

§. 6. Hinc igitur ſequitur fore generaliter:

$$\int \frac{dx}{1+x^m} = \cfrac{1}{1+\cfrac{1^2}{m+\cfrac{(m+1)^2}{m+\cfrac{(2m+1)^2}{m+\cfrac{(3m+1)^2}{m+\text{etc.}}}}}}$$

poſito poſt integrationem $x=1$. Ac ſi fuerit m numerus fractus habebitur:

$$\frac{dx}{1+x^{\frac{m}{n}}} = \cfrac{1}{1+\cfrac{n}{m+\cfrac{(m+n)^2}{m+\cfrac{(2m+n)^2}{m+\cfrac{(3m+n)^2}{m+\text{etc.}}}}}}$$

§. 7. Conſideremus nunc formulam $\int \frac{x^{n-1}dx}{1+x^m}$ quae integrata et poſt integrationem facto $x=1$ praebet hanc ſeriem: $\frac{1}{n} - \frac{1}{m+n} + \frac{1}{2m+n} - \frac{1}{3m+n} +$ etc. Hinc fiet $a = b = c = d =$ etc. $= 1$. et $p = n$; $q = m+n$; $r = 2m+n$; $s = 3m+n$; etc. Vnde habebitur

$$\int \frac{x^{n-1}dx}{1+x^m} = \cfrac{1}{n+\cfrac{n^2}{m+\cfrac{(m+n)^2}{m+\cfrac{(m+n)^2}{n+\text{etc.}}}}}$$

quae fractio continua congruit cum vltimo inuenta.

§. 8. Proponatur iam iſta formula $\int \frac{x^{n-1}dx}{(1+x^m)^{\frac{\mu}{\nu}}}$, quae integrata facto $x=1$ praebet hanc ſeriem: $\frac{1}{n} - \frac{\mu}{\nu(m+n)}$

 +

$+\frac{\mu(\mu+\nu)}{1.2\nu^2(2m+n)}-\frac{\mu(\mu+\nu)(\mu+2\nu)}{1.2.3\nu^3(3m+n)}+$ etc. quae cum generali comparata dat $a=1$; $b=\mu$; $c=\mu(\mu+\nu)$; $d=\mu(\mu+\nu)(\mu+2\nu)$; etc. $p=n$; $q=\nu(m+n)$; $r=2\nu^2(2m+n)$; $s=6\nu^3(3m+n)$; $t=24\nu^4(4m+n)$; etc. atque $aq-bp=\nu m+(\nu-\mu)n$; $br-cq=\mu\nu(3\nu-\mu)m+\mu\nu(\nu-\mu)n$; $cs-dr=2\mu\nu^2(\mu+\nu)(m(5\nu-2\mu)+n(\nu-\mu))$ $dt-es=6\mu\nu^3(\mu+\nu)(\mu+2\nu)(m(7\nu-3\mu)+n(\nu-m))$ etc. quibus substitutis, factaque reductione habebitur:

$$\int\frac{x^{n-1}dx}{(1+x^m)^{\frac{\mu}{\nu}}}=\cfrac{1}{n+\cfrac{\mu n^2}{\nu m+(\nu-\mu)n+\cfrac{\nu(\mu+\nu)(m+n)^2}{(3\nu-\mu)m+(\nu-\mu)n+\cfrac{2\nu(\mu+2\nu)(2m+n)^2}{(5\nu-2\mu)m+(\nu-\mu)n+\cfrac{3\nu(\mu+3\nu)(3m+n)^2}{(7\nu-3\mu)m+(\nu-\mu)n\text{ etc.}}}}}}$$

Sit $\mu=1$ et $\nu=2$ erit $\int\frac{x^{n-1}dx}{\sqrt{(1-x^m)}}=$

$$\cfrac{1}{n+\cfrac{n^2}{2m+n+\cfrac{6(m+n)^2}{5m+n+\cfrac{20(2m+n)^2}{8m+n+\cfrac{42(3m+n)^2}{11m+n+\cfrac{72(4m+n)^2}{14m+n+\text{ etc.}}}}}}}$$

§. 9. At si fuerit $\nu=1$ et μ numerus integer, prodibunt sequentes fractiones continuae:

$$\int\frac{x^{n-1}dx}{(1+x^m)^2}=\cfrac{1}{n+\cfrac{2n^2}{m-n+\cfrac{1.3(m+n)^2}{m-n+\cfrac{2.4(2m+n)^2}{m-n+\cfrac{3.5(3m+n)^2}{m-n+\cfrac{4.6(4m+n)^2}{m-n+\text{ etc.}}}}}}}$$

$$\int\frac{x^{n-1}dx}{(1+x^m)^3}=\cfrac{1}{n+\cfrac{3n^2}{m-2n+\cfrac{1.4(m+n)^2}{-2n+\cfrac{2.5(2m+n)^2}{-m-2n+\cfrac{3.6(3m+n)^2}{-2m-2n+\cfrac{4.7(4m+n)^2}{-3m-2n+\text{ etc.}}}}}}}$$

quae

quae expreſſio pariter ac ſequentes ob quantitates negatiuas non conuergunt ſed diuergunt.

§. 10. Conſequuntur haec omnia ex conuerſione fractionis continuae generalis §. 1 datae in ſeriem infinitam $A + \frac{B}{IP} - \frac{BD}{PQ} + \frac{BDF}{QR} - \frac{BDFH}{RS} +$ etc. Haec eadem autem ſeries addendis binis terminis transformatur in hanc $A + \frac{BE}{IQ} + \frac{BDFI}{QS} + \frac{BDFHKN}{SV} +$ etc. Eſt vero $C = P = \frac{Q-D}{E}$; $G = \frac{S-HQ}{IQ} - \frac{F(Q-D)}{EQ}$; $L = \frac{V-MS}{NS} - \frac{K(S-HQ)}{IS}$; etc. Hinc iſta ſeries infinita $A + \frac{BE}{Q} + \frac{BDFI}{QS} + \frac{BDFHKN}{SV} +$ etc. conuertetur in ſequentem fractionem continuam:

$$A + \cfrac{B}{\frac{Q-D}{E} + \cfrac{D}{E + \cfrac{F}{\frac{E(S-HQ)-FI(Q-D)}{EIQ} + \cfrac{H}{I + \cfrac{K}{\frac{I(V-MS)-KN(S-HQ)}{INS} + \text{etc.}}}}}}$$

quae a fractionibus liberata tranſit in hanc:

$$A + \cfrac{BE}{Q-D + \cfrac{D}{I + \cfrac{FIQ}{E(S-HQ)-FI(Q-D) + \cfrac{EHQ}{I + \cfrac{KNS}{I(V-MS)-KN(S-HQ) + \cfrac{IMS}{I + \text{etc.}}}}}}}$$

§. 11. Si nunc viciſſim proponatur haec ſeries infinita $\frac{a}{p} + \frac{b}{q} + \frac{c}{r} + \frac{d}{s} + \frac{e}{t} +$ etc. et comparatio cum praecedente inſtituatur erit $Q = p$; $S = \frac{q}{p}$; $V = \frac{pr}{q}$; $X = \frac{qs}{pr}$; $Z = \frac{prt}{qs}$ etc. itemque $E = \frac{a}{B}$; $I = \frac{b}{BDF}$; $N = \frac{c}{BDFHK}$; etc. quibus valoribus ſeries propoſita conuertetur in hanc fractionem continuam;

$$\cfrac{a}{p-D+\cfrac{D(1+bp:1)}{Da(\frac{q}{p}-Hp)-b(p-D)+\cfrac{DHap:1\,(1+cq:p)}{Hb(\frac{pr}{q}-\frac{Mq}{p})-c(\frac{q}{p}-Hp)+\cfrac{HMba:p\,(1+dpr:q)}{Mc(\frac{qs}{pr}-\text{etc.}}}}}$$

in quam fractionem continuam innumerabiles nouae quantitates ingrediuntur, quae in ſerie propoſita non inerant.

§. 12. Cum autem ſit ex §. 2. haec ſeries $\frac{b}{p}-\frac{bd}{pq}+\frac{bdf}{qr}-\frac{bdfh}{rs}+$ etc. aequalis iſti fractioni continuae

$$\cfrac{b}{p+\cfrac{dp}{q-d+\cfrac{fpq}{r-fp+\cfrac{hqr}{s-hq+\cfrac{krs}{\text{etc.}}}}}}$$

ſi haec ſeries ad praecedentem reducatur fiet $b=BE$; $d=\frac{-DFI}{E}$; $f=\frac{-HKN}{I}$; etc. $p=Q$; $q=S$; $r=V$, $s=X$ etc. Ex quo fractio continua §. praecedente data transmutabitur in hanc:

$$A+\cfrac{BE}{Q-\cfrac{IDFI.Q}{ES+DFI-\cfrac{EHKN.QS}{IV+HKNQ-\cfrac{IMOR.SV}{NX+MORS+\text{etc.}}}}}$$

cuius lex progreſſionis facile perſpicitur.

§. 13.

§. 13. Series autem illa $A + \frac{B}{P} - \frac{BD}{PQ} + \frac{BDF}{QR}$ - etc. quam primum ex fractione continua generali elicuimus, facile transformatur in hanc formam: $A + \frac{B}{2P} + \frac{BE}{2Q} - \frac{BDG}{2PR} + \frac{BDFI}{2QS} - \frac{BDFHL}{2RT} +$ etc. quae si litterae C, E, G, I etc. per reliquas ope aequationum datarum exprimantur, abit in hanc: $A + \frac{B}{2P} + \frac{B(Q-D)}{2PQ} - \frac{BD(R-FP)}{2PQR} + \frac{BDF(S-HQ)}{2QRS} -$ etc. cui propterea aequalis est ista fractio continua:

$$A + \cfrac{B}{P + \cfrac{DP}{Q - D + \cfrac{FPQ}{R - FP + \cfrac{HQR}{S - HQ + \text{etc.}}}}}$$

§ 14. Haec omnia igitur consequuntur ex contemplatione fractionum continuarum immediate, pluresque huius generis obseruationes iam in superiore dissertatione communicaui. Nunc ergo his relictis ad alia pergo, atque aliquot modos tam ad fractiones continuas perueniendi, quam datarum istiusmodi fractionum valores per integrationes assignandi. Primum itaque, cum hic Brounckeri expressio quadtaturae circuli sit non solum demonstrata, sed etiam quasi a priori inuenta, examini subiiciam alias similes expressiones vel ab ipso Brounckero vel a Wallisio inuentas, recensentur enim a Wallisio, nec satis clare indicatur, vtrum Brounkerus omnes inuenerit, an eam duntaxat, quae pro circuli quadratura fuit exhibita. Postmodum vero etiam reliquas illas fractiones continuas, quae altioris indaginis videntur, ex principiis maxime diuersis demonstrabo, istiusque generis multo plures eruere docebo.

§. 15.

§. 15. Quae autem apud Wallisium extant huc redeunt, vt sit productum duarum harum fractionum continuarum $= a^2 =$

$$a - 1 + \cfrac{1}{2(a-1) + \cfrac{9}{2(a-1) + \cfrac{25}{2(a-1) \text{ etc.}}}} \quad \text{et} \quad a + 1 + \cfrac{1}{2(a+1) + \cfrac{9}{2(a+1) + \cfrac{25}{2(a+1) \text{ etc.}}}}$$

Cum igitur simili modo sit $(a+2)^2 =$

$$a + 1 + \cfrac{1}{2(a+1) + \cfrac{9}{2(a+1) + \text{etc.}}} \cdot a + 3 + \cfrac{1}{2(a+3) + \cfrac{9}{2(a+3) + \text{etc.}}}$$

reperietur hoc modo infinitum progrediendo

$$a. \frac{a(a+4)(a+4)(a+8)(a+8)(a+12)(a+12)}{(a+2)(a+2)(a+6)(a+6)(a+10)(a+10)(a+14)} \text{ etc.}$$

$$= a - 1 + \cfrac{1}{2(a-1) + \cfrac{9}{2(a-1) + \cfrac{25}{2(a-1) + \text{etc.}}}}$$

§. 16. Si nunc productum istud ex infinitis factoribus constans per methodum in praecedente dissertatione traditam examinetur reperietur fore $\frac{a(a+4)(a+4)(a+8) \text{ etc.}}{(a+2)(a+2)(a+6)(a+6) \text{ etc.}}$ $= \frac{\int x^{a+1} dx : \sqrt{(1-x^4)}}{\int x^{a-1} dx : \sqrt{(1-x^4)}}$. Quocirca huius fractionis continuae valor

$$a - 1 + \cfrac{1}{2(a-1) + \cfrac{9}{2(a-1) + \cfrac{25}{2(a-1) \text{ etc.}}}}$$

aequabitur huic expressioni $a \frac{\int x^{a+1} dx : \sqrt{(1-x^4)}}{\int x^{a-1} dx : \sqrt{(1-x^4)}}$ positio post vtramque integrationem $x = 1$.

§. 17. Theorema hoc, quo fractionis continuae satis latae patentis valor per formulas integrales exprimitur, eo magis est notatu dignum, quo minus eius veritas est obvia. Nam quanquam ille casus quo $a = 2$, iam ante est in-

inuentus, eiusque valor per quadraturam circuli expositus, ceteri tamen casus ex eo non consequuntur. Si enim ista fractio continua modo initio praescripto conuertatur in seriem, ad tam intricatas peruenitur formulas, vt summa eius minime colligi queat; praeter casum $a = 2$. Quo circa iam pridem multam collocaui operam, vt tam veritatem istius theorematis demonstrarem, quam viam detegerem, qua a priori ad hanc ipsam fractionem continuam pertingere liceret; quae inuestigatio, quo difficilior mihi est visa, eo maiorem vtilitatem ex ea orturam esse, sum arbitratus. Quamdiu autem omne studium frustra in hoc negotio impendi, maxime dolui, methodum a Brounckero vsitatam nusquam esse expositam et forsitan omnino periisse.

§. 18. Quantum quidem ex Wallisii recensione constat, Brounckerus ad istam formam deductus est per interpolationem huius seriei: $\frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} +$ etc. cuius terminos intermedios ipsam circuli quadraturam praebere Wallisius demonstrauerat. Atque adeo indicatur initium huius interpolationis a Brounckero institutae. Sibi enim propositum fuisse perhibetur, singulas fractiones $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}$ etc. in binos factores resoluere, qui omnes inter se continuam progressionem constituant. Ita si fuerit $AB = \frac{1}{2}$; $CD = \frac{3}{4}$; $EF = \frac{5}{6}$; $GH = \frac{7}{8}$; etc. ac quantitates A, B, C, D, E, etc. continuam progressionem constituant, series illa abit in hanc; $AB + ABCD + ABCDEF +$ etc. quae in hanc formam reducta sponte interpolatur: erit enim terminus cuius index $\frac{1}{2}$ est, $= A$; et terminus indicem $\frac{3}{2}$ habens $= ABC$; et ita porro. Ex quo tota haec interpolatio ad resolutionem singulorum fractionum in binos factores reducitur.

 §. 19.

§. 19. Ex lege autem continuitatis erit $BC = \frac{2}{3}$; $DE = \frac{4}{5}$; $FG = \frac{6}{7}$; etc. Cum igitur ſit $A = \frac{1}{2B}$; $B = \frac{2}{3C}$; $C = \frac{3}{4D}$; $D = \frac{4}{5E}$; etc. ſtatim obtinetur $A = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}$ etc. quae autem eſt ipſa formula a Walliſio primum producta, qua circuli quadraturam expreſſit, atque maxime ab expreſſione Brounckeri abhorret. Quare cum iſta formula interpolationem hoc modo inueſtigando tam facile ſe praebeat, eo magis eſt mirandum Brounckerum eadem via ingreſſum ad expreſſionem tantopere differentem perueniſſe; nulla enim via ſupereſſe videtur, quae ad fractionem continuam deduceret. Neque vero exiſtimandum eſt, Brounckerum de induſtria valorem ipſius A per fractionem continuam exprimere voluiſſe; ſed potius methodum quampiam peculiarem ſecutum, quaſi inuitum in eam incidiſſe: cum eo tempore fractiones continuae omnino fuerint incognitae, atque hac occaſione primum in medium prolatae. Ex quibus ſatis colligere licet, obuiam dari methodum ad iſtiusmodi fractiones continuas deducentem, quantumuis ea nunc quidem abſcondita videatur.

§. 20. Quamuis autem diu in hac methodo reperienda irrito conatu ſim verſatus, tamen in alium incidi modum interpolationes huiusmodi ſerierum per fractiones continuas abſoluendi qui mihi autem praebuit expreſſiones a Brounckerianis maxime diuerſas. Interim tamen non ſine omni vtilitate fore ſpero, iſtam methodum exponere, cum eius ope reperiantur fractiones continuae, quarum valores iam aliunde ſint cogniti, et per quadraturas exhiberi queant. Cum enim deinde aliam methodum ſim traditurus valores quarumcunque fractionum continuarum per quadraturas exprimendi, inde egregiae orientur compara-

tiones

tiones formularum integralium, eo saltem casu quo variabili post integrationem definitus valor tribuitur, eiusmodi comparationes plures in praecedente dissertatione de productis ex infinitis factoribus constantibus exhibui.

§. 21. Vt igitur hunc a me inuentum interpolandi modum exponam proposita sit ista series latissime patens $\frac{p}{p+2q} + \frac{p(p+2r)}{(p+2q)(p+2q+2r)} + \frac{p(p+2r)(p+4r)}{(p+2q)(p+2q+2r)(p+2q+4r)} +$ etc. cuius terminus indicis $\frac{1}{2}$ sit $= A$; terminus indicis $\frac{3}{2} = AB$ C terminus indicis $\frac{5}{2} = ABCDE$, etc. Hinc igitur erit $AB = \frac{p}{p+2q}$; $CD = \frac{p+2r}{p+2q+2r}$; $EF = \frac{p+4r}{p+2q+4r}$; etc. atque ex lege continuitatis $BC = \frac{p+r}{p+2q+r}$; $DE = \frac{p+3r}{p+2q+3r}$; $FG = \frac{p+5r}{p+2q+5r}$ et ita porro.

§. 22. Ad fractiones tollendas ponatur $A = \frac{a}{p+2q-r}$; $B = \frac{b}{p+2q}$; $C = \frac{c}{p+2q+r}$; $D = \frac{d}{p+2q+2r}$ etc. eritque $ab = (p+2q-r)p$; $bc = (p+2q)(p+r)$; $cd = (p+2q+r)(p+2r)$; $de = (p+2q+2r)(p+3r)$ etc. Fiat nunc $a = m - r + \frac{1}{\alpha}$; $b = m + \frac{1}{\beta}$; $c = m + r + \frac{1}{\gamma}$; $d = m + 2r + \frac{1}{\delta}$; $e = m + 3r + \frac{1}{\varepsilon}$ etc. in quibus substitutionibus partes integrae constituunt progressionem arithmeticam, cuius differentia constans est r, id quod ipsa progressio factorum illorum postulat. His igitur valoribus substitutis prodibunt sequentes aequationes, ponendo breuitatis gratia $p^2 + 2pq - pr - m^2 + mr = P$, et $2r(p+q-m) = Q$.

$$P\alpha\beta - (m-r)\alpha = m\beta + 1$$
$$(P+Q)\beta\gamma - m\beta = (m+r)\gamma + 1$$
$$(P+2Q)\gamma\delta - (m+r)\gamma = (m+2r)\delta + 1$$
$$(P+3Q)\delta\varepsilon - (m+2r)\delta = (m+3r)\varepsilon + 1$$

etc.

§. 23. Ex his igitur aequationibus emergent sequentes litterarum $\alpha, \beta, \gamma, \delta$, etc. comparationes inter se. $\alpha = \frac{m\beta + 1}{P\beta - (m-r)} = \frac{m}{P} + \cfrac{p(p+2q-r):P^2}{-\frac{(m-r)}{P} + \beta}$, $\beta = \frac{(m+r)\gamma + 1}{(P+Q)\gamma - m} = \frac{m+r}{P+Q} + \cfrac{(p+r)(p+2q):(P+Q)^2}{-\frac{m}{P+Q} + \gamma}$ $\gamma = \frac{(m+2r)\delta + 1}{(P+2Q)\delta - (m+r)} = \frac{m+2r}{P+2Q} + \cfrac{(p+2r)(p+2q+r):(P+2Q)^2}{-\frac{(m+r)}{P+2Q} + \delta}$

Si ergo breuitatis gratia ponatur $p^2 + 2pq - mp - mq + qr = R$ et $pr + qr - mr = S$, atque valores litterarum assumtarum continuo in praecedentibus surrogentur, proueniet sequens fractio continua

$$\alpha = \frac{m}{P} + \cfrac{p(p+2q-r):P^2}{\frac{2rR}{P(P+Q)} + \cfrac{(p+r)(p+2q):(P+Q)^2}{\frac{2r(R+S)}{(P+Q)(P+2Q)} + \cfrac{(p+2r)(p+2q+r):(P+2Q)^2}{\frac{2r(R+2S)}{(P+2Q)(P+3Q)} + \text{etc.}}}}$$

§. 24. Cum igitur sit $a = m - r + \frac{1}{\alpha}$ habebitur

$$a = m - r + \cfrac{P}{m + \cfrac{p(p+2q-r)(P+Q)}{2rR + \cfrac{(p+r)(p+2q)(P+2Q)}{2r(R+S) + \cfrac{(p+2r)(p+2q+r)(P+Q)(P+3Q)}{2r(R+2S) + \text{etc.}}}}}$$

Hinc igitur seriei propositae $\frac{p}{p+2q} + \frac{p(p+2r)}{(p+2q)(p+2q+2r)} + \frac{p(p+2r)(p+4r)}{(p+2q)(p+2q+2r)(p+2q+4r)} +$ etc. terminus cuius index est $\frac{1}{2}$ erit $A = \frac{q}{p+2q-r}$. Quoniam vero huius seriei terminus generalis indicem habens n est $= \frac{\int y^{p+2q-1} dy (1-y^{2r})^{n-1}}{\int y^{p-1} dy (1-y^{2r})^{n-1}}$ erit fractio continua inuenta seu valor litterae $a = (p+2q-r) \frac{\int y^{p+2q-1} dy : \sqrt{(1-y^{2r})}}{\int y^{p-1} dy : \sqrt{(1-y^{2r})}}$ posito post vtramque integrationem $y = 1$.

§. 25.

§. 25. Cum autem in nostra fractione continua insit littera arbitraria m, innumerabiles habebuntur fractiones continuae, quarum idem est valor isque cognitus: ex quibus praecipuas contemplari iuuabit. Sit igitur primo $m-r=p$ seu $m=p+r$, erit $P=2p(q-r)$; $Q=2r(q-r)$; $R=p(q-r)$ et $S=r(p-r)$: vnde fiet

$$a=p+\cfrac{2p(q-r)}{p+r+\cfrac{(p+2q-r)(p+r)}{r+\cfrac{(p+2q)(p+2r)}{r+\cfrac{(p+2q+r)(p+2r)}{r+\text{etc.}}}}}$$

At si fuerit $r>q$, ne fractio continua fiat negatiua, erit:

$$a=\cfrac{p}{1+\cfrac{2(r-q)}{p+2q-r+\cfrac{(p+2q-r)(p+r)}{r+\cfrac{(p+2q)(p+2r)}{r+\cfrac{(p+2q+r)(p+2r)}{r+\text{etc.}}}}}}$$

§. 26. Sit nunc $m=p+q$; quo et Q et S euanescat; erit autem $P=q(r-q)$ et $R=q(r-q)$, indeque proveniet

$$a=p+q-r+\cfrac{q(r-q)}{p+q+\cfrac{p(p+2q-r)}{2r+\cfrac{(p+r)(p+2q)}{2r+\cfrac{(p+2r)(p+2q+r)}{2r+\text{etc.}}}}}$$

quae fractio continua adeo praecedentibus est aequalis, etiamsi ipsae formae sint diuersae.

§. 27 Ponatur $m=p+2q$; eritque $P=2q(r-p-2q)=-2q(p+2q-r)$; $Q=-2qr$; $R=-q(p+2q-r)$, et $S=-qr$. Ex his itaque obtinebitur sequens fractio continua:

$$a=p+2q-r-\cfrac{2q(p+2q-r)}{p+2q+\cfrac{p(p+2q)}{r+\cfrac{(p+r)(p+2q+r)}{r+\cfrac{(p+2r)(p+2q+2r)}{r+\text{etc.}}}}}$$

Ita innumerabiles prodeunt fractiones continuae quarum omnium idem est valor a, qui per formulas integrales inuentus est $=(p+2q-r)\frac{\int y^{p+2q-1}dy:\sqrt{(1-y^{2r})}}{\int y^{p-1}dy:\sqrt{(1-y^{2r})}}$ $= \frac{(p+2q-2r)\int y^{p+2q-2r-1}dy:\sqrt{(1-y^{2r})}}{\int y^{p-1}dy:\sqrt{(1-y^{2r})}}$.

§. 28. Antequam vlterius progrediamur casus nonnullos contemplemur. Sit igitur $r=2q$; eritque $a=p\frac{\int y^{p+2q-1}dy:\sqrt{(1-y^{4q})}}{\int y^{p-1}dy:\sqrt{(1-y^{4q})}}$. Cum ergo fiat $P=p^2+mq-m^2$; $Q=4q(p+q-m)$; $R=p^2+2pq+2qq-mp-mq$, et $S=2q(p+q-m)$, erit in genere

$$a=m-2q+\cfrac{P}{m+\cfrac{p^2(P+Q)}{4qR+\cfrac{(p+2q)^2P(P+Q)}{4q(R+S)+\cfrac{(p+4q)^2(P+2Q)(P+3Q)}{4q(R+2S)+\text{etc.}}}}}$$

§. 29. Si autem pro m varios illos valores substituamus, prodibunt sequentes fractiones continuae determinatae.

$$a=p-\cfrac{2pq}{p+2q+\cfrac{p(p+2q)}{2q+\cfrac{(p+2q)(p+4q)}{2q+\cfrac{(p+4q)(p+6q)}{2q+\text{etc.}}}}}$$

Siue loco huius fractionis continuae ob $r>q$

$$a=\cfrac{p}{1+\cfrac{2q}{p+\cfrac{p(p+2q)}{2q+\cfrac{(p+2q)(p+4q)}{2q+\cfrac{(p+4q)p(+6q)}{2q+\text{etc.}}}}}}$$

Deinde ex §. 26. obtinetur pro hoc casu ista fractio

$$a=p-q+\cfrac{qq}{p+q+\cfrac{pp}{4q+\cfrac{(p+2q)^2}{4q+\cfrac{(p+4q)^2}{4q+\text{etc.}}}}}$$

Tertio

Tertio vero §. 27. ſuppeditabit hanc fractionem continuam:

$$a = p - \cfrac{2pq}{p+2q+\cfrac{p(p+2q)}{2q+\cfrac{(p+2q)(p+4q)}{2q+\cfrac{(p+4q)(p+6q)}{2q+\cfrac{(p+6q)(p+8q)}{2q+\text{etc.}}}}}}$$

quae cum primo hic exhibita congruit: ita vt duae tantum fractiones continuae ſimpliciores pro hoc caſu, quo $r = 2q$, habeantur.

§. 30. Ponatur nunc porro $q = p = 1$, vt fiat $a = \frac{\int yydy:\sqrt{(1-y^4)}}{\int dy:\sqrt{(1-y^4)}}$; erit primo:

$$a = 1 - \cfrac{2}{3+\cfrac{1\cdot 3}{2+\cfrac{3\cdot 5}{2+\cfrac{5\cdot 7}{2+\text{etc.}}}}}$$

Deinde vero habebitur:

$$a = \cfrac{1}{2+\cfrac{1}{4+\cfrac{9}{4+\cfrac{25}{4+\cfrac{49}{4+\text{etc.}}}}}}$$

Vnde ſequitur fore $- \frac{\int dy:\sqrt{(1-y^4)}}{\int yydy:\sqrt{(1-y^4)}} =$

$$2 + \cfrac{1}{4+\cfrac{9}{4+\cfrac{25}{4+\cfrac{49}{4+\text{etc.}}}}}$$

qui caſus continetur in expreſſione §. 16. data ex quo illa formula nondum ſatis demonſtrata magis confirmatur. Poſito enim ibi $a = 3$, fiet $3\frac{\int x^4dx:\sqrt{(1-x^4)}}{\int xxdx:\sqrt{(1-x^4)}} = \frac{\int dx:\sqrt{(1-x^4)}}{\int xxdx:\sqrt{(1-x^4)}}$

$$= 2 + \cfrac{1}{4+\cfrac{9}{4+\cfrac{25}{4+\cfrac{49}{4+\text{etc.}}}}}$$

ita vt nunc quidem conſtet formulam illam §. 16. exhibitam

bitam veram esse casibus quibus est tum $a=2$ tum etiam $a=3$: mox autem eius veritas in latissimo sensu euincetur.

§. 31. Sit $q=\frac{1}{2}$ et $p=1$; manente $r=2q=1$ erit $a=\frac{\int y\,dy:\sqrt{(1-y^2)}}{\int dy:\sqrt{(1-y^2)}}=\frac{2}{\pi}$ denotante π peripheriam circuli cuius diameter est $=1$. Generaliter itaque erit $P=1+m-m^2$; $Q=3-2m$; $R=\frac{5-m}{2}$ et $S=\frac{3-2m}{2}$, ideoque

$$a=m-1+\cfrac{1+m-m^2}{m+\cfrac{1^2(1-m-m^2)}{5-3m+\cfrac{2^2(1+m-m^2)(1-m-m^2)}{8-5m+\cfrac{3^2(4-m-m^2)(1-5m-m^2)}{11-7m+\text{etc.}}}}}$$

In casibus autem specialibus expositis erit

$$\frac{\pi}{2}=\cfrac{1}{1-\cfrac{1}{2+\cfrac{1\cdot 2}{1+\cfrac{2\cdot 3}{1+\cfrac{3\cdot 4}{1+\text{etc.}}}}}}=1+\cfrac{1}{1+\cfrac{1\cdot 2}{2+\cfrac{2\cdot 3}{1+\cfrac{3\cdot 4}{1+\text{etc.}}}}}$$

$$\text{et }\frac{\pi}{2}=\cfrac{1}{\frac{1}{2}+\cfrac{1:4}{\frac{3}{2}+\cfrac{1^2}{2+\cfrac{2^2}{2+\cfrac{3^2}{2+\text{etc.}}}}}}=2-\cfrac{1}{2+\cfrac{1^2}{2+\cfrac{2^2}{2+\cfrac{3^2}{2+\text{etc.}}}}}$$

§. 32. Vt vsus harum formularum in interpolationibus intelligatur, proposita sit haec series: $\frac{2}{1}+\frac{2\cdot 4}{1\cdot 3}+\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}+$ etc. cuius terminum indicis $\frac{1}{2}$ inueniri oporteat, qui sit $=A$; Erit ergo $p=2$; $r=1$; et $q=-\frac{1}{2}$. Ponatur $A=\frac{a}{p+2q-r}$ et $A=\frac{a}{0}$, vnde incommodum datarum formularum, si fiat $p+2q-r=0$ satis intelligitur. Interim tamen negotium hoc absolui potest quaerendo terminum indicis $\frac{3}{2}$, qui si fuerit $=Z$ erit $A=\frac{2}{3}Z$; At $\frac{1}{2}Z$ erit terminus indicis $\frac{1}{2}$ huius seriei $\frac{4}{3}+\frac{4\cdot 6}{3\cdot 5}+\frac{4\cdot 6\cdot 8}{3\cdot 5\cdot 7}+$ etc. quae cum generali comparata dat $p=4$; $r=1$; $q=-\frac{1}{2}$. ita vt fiat Z $=$

$= \frac{2\int y^2 dy \cdot \sqrt{(1-y^2)}}{\int y^3 dy : \sqrt{(1-y^2)}} = \frac{2\int dy : \sqrt{(1-y^2)}}{2\int y dy : \sqrt{(-y^3)}} = \frac{3}{4}\pi$. atque $A = \frac{\pi}{2}$. Cum igitur sit per §. 24. $Z = a$; et $A = \frac{2}{3}Z = \frac{2}{3}a$, erit primo generaliter ob $P = 8 + m - m^2$; $Q = 7 - 2m$; $R = \frac{23-7m}{2}$; et $S = \frac{7-2m}{2}$; $A = \frac{2}{3}a = \frac{\pi}{2} = \frac{2(m-1)}{3}$

$$+ \cfrac{2(8+m-m^2)}{5m + \cfrac{2\cdot4\cdot3(15-m-m^2)}{23-7m + \cfrac{3\cdot5(\;+m-m^2)(22-7m-m^2)}{30-9m + \cfrac{4\cdot6(5-m-m^2)(\;-5m-m^2)}{37-11m + \text{etc.}}}}}$$

§. 33. Casibus autem particularibus euoluendis erit $a = \frac{3}{4}\pi =$

$$4 - \cfrac{12}{5 + \cfrac{2\cdot5}{1 + \cfrac{3\cdot6}{1 + \cfrac{4\cdot7}{1 + \text{etc.}}}}} = \cfrac{4}{1 + \cfrac{3}{2 + \cfrac{2\cdot5}{1 + \cfrac{3\cdot6}{1 + \cfrac{4\cdot7}{1 + \text{etc.}}}}}}$$

vel etiam $\frac{3}{4}\pi = 1 + \cfrac{3}{1 + \cfrac{1\cdot4}{1 + \cfrac{2\cdot5}{1 + \cfrac{3\cdot6}{1 + \cfrac{4\cdot7}{1 + \text{etc.}}}}}}$

Simili modo per §. 26. habebitur $a = \frac{3}{4}\pi$

$$= \frac{5}{2} - \cfrac{3\cdot4}{\frac{7}{2} + \cfrac{2\cdot4}{2 + \cfrac{3\cdot5}{2 + \cfrac{4\cdot6}{2 + \cfrac{5\cdot7}{2 + \text{etc.}}}}}} = 2 + \cfrac{1}{2 + \cfrac{1\cdot3}{2 + \cfrac{2\cdot4}{2 + \cfrac{3\cdot5}{2 + \cfrac{4\cdot6}{2 + \text{etc.}}}}}}$$

Denique casus §. 27. expositus dabit $a = \frac{3}{4}\pi =$

$$2 + \cfrac{2}{3 + \cfrac{2\cdot4}{1 + \cfrac{3\cdot5}{1 + \cfrac{5\cdot5}{1 + \text{etc.}}}}}$$ seu $$\frac{\pi}{2} = 1 + \cfrac{1}{1 + \cfrac{1\cdot2}{1 + \cfrac{2\cdot3}{1 + \cfrac{3\cdot4}{1 + \cfrac{4\cdot5}{1 + \text{etc.}}}}}}$$

quae expressio conuenit cum superiore quodam in §. 31. exhibita.

§. 34. Ex hac itaque interpolandi methodo innumerabiles consecuti sumus fractiones continuas, quarum valores per quadraturas curuarum seu formulas integrales assignari possunt. Cum autem istae fractiones continuae in initio sint irregulares initia quae anomaliam continent resecentur, vt habeantur fractiones continuae vbiquae eadem lege procedentes. Ita ex §. 25, ponendo $p+2q-r=f$ et $p+r=b$; prodibit sequens aequatio:

$$r+\cfrac{fb}{r+\cfrac{(f+r)(b+r)}{r+\cfrac{(f+2r)(b+2r)}{r+\text{ e.c.}}}}=\frac{b(f-r)\int y^{b+r-1}dy:V(1-y^{2r})-f(b-r)\int y^{f+r-1}dy:V(1-y^{2r})}{f\int y^{f+r-1}dy:V(1-y^{2r})-b\int y^{b+r-1}dy:V(1-y^{2r})}$$

quae aequatio semper est realis, nisi fiat $f=b$. At casu quo $f=b$ ponatur $f=b+dw$, reperieturque

$$\frac{\int y^{b+r+dw-1}dy:V(1-y^{2r})}{\int y^{b+r-1}dy:V(1-y^{2r})}=1-r\,dw\int\frac{dx}{x^{r+1}}\int\frac{x^{b+2r-1}dx}{1+x^{2r}}$$

posito post integrationem $x=1$. Hinc ergo erit

$$r+\cfrac{bb}{r+\cfrac{(b+r)^2}{r+\cfrac{(b+2r)^2}{r+\text{ etc}}}}=\frac{r+br(b-r)\int\frac{dx}{x^{r+1}}\int\frac{x^{b+2r-1}dx}{1-x^{2r}}}{1-br\int\frac{dx}{x^{r+1}}\int\frac{x^{b+2r-1}dx}{1-x^{2r}}}$$

$$=\frac{r(b-r)^2\int\frac{dx}{x^{r+1}}\int\frac{x^{b-1}dx}{1-x^{2r}}}{1-r(b-r)\int\frac{dx}{x^{r+1}}\int\frac{x^{b-1}dx}{1-x^{2r}}}$$

Verum ex natura integralium est $\int\frac{dx}{x^{r+1}}\int\frac{x^{b+2r-1}dx}{1-x^{2r}}=\frac{-1}{rx^r}\int\frac{x^{b+2r-1}dx}{1-x^{2r}}+\frac{1}{r}\int\frac{x^{b+r-1}dx}{1-x^{2r}}+\frac{1}{r}\int\frac{x^{b+r-1}dx}{1+x^{r}}$ posito $x=1$. Quo circa habebitur

$r+$

$$r+\cfrac{bb}{r+\cfrac{(b+r)^2}{r+\text{etc.}}}=\cfrac{r+b(b-r)\int\frac{x^{b+r-1}dx}{1+x^r}}{1-b\int\frac{x^{b+r-1}dx}{1+x^r}}=$$

$$\frac{1\cdot(b-r)\int\frac{x^{b-1}dx}{1+x^r}}{\int\frac{x^{b-1}dx}{1+x^r}}$$; quae forma autem congruit cum ea, quae §. 7. est data.

§. 35. Simili modo ex §. 26. ponendo $p=f$ et $p+2q-r=b$, sequitur fore

$$2r+\cfrac{fb}{2r+\cfrac{(f+r)(b+r)}{2r+\cfrac{(f+2r)(b+2r)}{2r+\text{etc.}}}}=\frac{2(r-f)(r-b)\int\frac{y^{f-1}dy}{\sqrt{(1-y^{2r})}}-b(f+b-3r)\int\frac{y^{b+r-1}dy}{\sqrt{(1-y^{2r})}}}{2b\int\frac{y^{b+r-1}dy}{\sqrt{(1-y^{2r})}}-(f+b-r)\int\frac{y^{f-1}dy}{\sqrt{(1-y^{2r})}}}.$$

Quoniam autem formula manet immutata si f et b inter se commutentur, manifestum est esse debere
$$\frac{b\int y^{b+r-1}dy:\sqrt{(1-y^{2r})}}{\int y^{f-1}dy:\sqrt{(1-y^{2r})}}=\frac{f\int y^{f+r-1}dy:\sqrt{(1-y^{2r})}}{\int y^{b-1}dy.\sqrt{(1-y^{2r})}}$$, posito post omnes integrationes $y=1$. Hoc vero theorema iam continetur in iis, quae in praecedente dissertatione de productis ex infinitis factoribus constantibus exhibui; ibi enim plura huius generis theoremata produxi ac demonstraui.

§. 36. Hic autem pari modo casus notari meretur, quo est $f=b+r$, hoc enim tam numerator quam denominator fractionis inuentae euanescit. Posito autem vt ante $f=b+r+dw$ et calculo subducto orietur

 2r+

$$2r + \cfrac{b(b+r)}{2r + \cfrac{(b+r)(b+2r)}{2r + \cfrac{(b+2r)(b+3r)}{2r + etc.}}} = \frac{b + 2b(r-b)\int \frac{x^{b-r}\,dx}{1+x^r}}{-1 + 2b\int \frac{x^{b-r}\,dx}{1+x^r}}$$

Quare si ponatur $b = r = 1$; habebitur

$$2 + \cfrac{1 \cdot 2}{2 + \cfrac{2 \cdot 3}{2 + \cfrac{3 \cdot 4}{2 + \cfrac{4 \cdot 5}{2 + etc.}}}} = \frac{1}{2\,l\,2 - 1}.$$

Ceterum si aequatio §. 27. eodem modo tractetur, prodibit forma illi ipsi, quam ex §. 25. elicui, omnino similis.

§. 37. His expositis, quibus interpolatio serierum ad fractiones continuas reducitur, reuertor ad expressiones Brounckerianas, atque methodum tradam genuinam non solum ad eas perueniendi, sed etiam eiusmodi, quae videatur ab ipso Brounckero esse vsurpata. Maxime autem discrepant fractiones continuae hactenus inuentae a Brounckerianis, cum valores litterarum A, B, C, D, etc. methodo exposita ita a se inuicem pendeant, vt inter se comparari facile queant, methodo Brounckeri autem inter se diuersi prodierint, vt eorum mutua relatio non perspiciatur. Quod ipsum discrimen me tandem ad inuentionem alterius methodi nunc aperiendae manuduxit.

§. 38. Antequam autem ipsum interpolandi modum exponam, sequens lemma latissime patens praemitti conueniet. Si fuerint innumerabiles quantitates α, β, γ, δ, ε, etc. quae ita a se inuicem pendeant vt sit:

$$\alpha\beta - m\alpha - n\beta - \kappa = 0$$
$$\beta\gamma - (m+s)\beta - (n+s)\gamma - \kappa = 0$$

$\gamma\delta$

$$\gamma\delta-(m+2s)\gamma-(n+2s)\delta-\varkappa=0$$
$$\delta\varepsilon-(m+3s)\delta-(n+3s)\varepsilon-\varkappa=0$$
etc.

ac tribuantur litteris α, β, γ, δ, etc. ſequentes valores:

$$\alpha=m+n-s+\frac{ss-ms+ns+\varkappa}{a}$$
$$\beta=m+n+s+\frac{ss-ms+ns+\varkappa}{b}$$
$$\gamma=m+n+3s+\frac{ss-ms+ns+\varkappa}{c}$$
$$\delta=m+n+5s+\frac{ss-ms+ns+\varkappa}{d}$$
etc.

ſuperiores aequationes transformabuntur in ſequentes ſimiles:

$$ab-(m-s)a-(n+s)b-ss+ms-ns-\varkappa=0$$
$$bc-mb-(n+2s)c-ss+ms-ns-\varkappa=0$$
$$cd-(m+s)c-(n+3s)d-ss+ms-ns-\varkappa=0$$
$$de-(m+2s)d-(n+4s)e-ss+ms-ns-\varkappa=0$$
etc.

Atque ex hoc ipſo vt iſtiusmodi formae ſimiles prodeant, ſubſtitutiones illae ſunt ortae.

§. 39. Si nunc ſimili modo hae vltimae aequationes ope idonearum ſubſtitutionum in ſui ſimiles transmutentur, reperientur loco a, b, c, d, etc. ſequentes ſubſtitutiones

$$a=m+n-s+\frac{4ss-2ms+2ns+\varkappa}{a1}$$
$$b=m+n+s+\frac{4ss-2ms+2ns+\varkappa}{b1}$$
$$c=m+n+3s+\frac{4ss-2ms+2ns+\varkappa}{c1}$$
$$d=m+n+5s+\frac{4ss-2ms+2ns+\varkappa}{d1}$$
etc.

quibus factis ſequentes prouenient aequationes:

$$a1b1-(m-2s)a1-(n+2s)b1-4ss+2ms-2ns-\varkappa=0$$
$$b1c1-(m-s)b1-(n+3s)c1-4ss+2ms-2ns-\varkappa=0$$

 c1d1

$$c1\,d1 - m\,c1 - (n+4s)\,d1 - 4ss + 2ms - 2ns - \varkappa = 0$$
$$d1\,e1 - (m+s)\,d1 - (n+5s)\,e1 - 4ss + 2ms - 2ns - \varkappa = 0$$
etc.

§. 40. Vlterius igitur pergendo poni debebit:

$$a1 = m + n - s + \frac{9ss - 3ms + 3ns + \varkappa}{a2}$$
$$b1 = m + n + s + \frac{9ss - 3ms + 3ns + \varkappa}{b2}$$
$$c1 = m + n + 3s + \frac{9ss - 3ms + 3ns + \varkappa}{c2}$$
etc.

Atque ex his fubftitutionibus emergent hae aequationes:

$$a2\,b2 - (m-3s)\,a2 - (n+3s)\,b2 - 9ss + 3ms - 3ns - \varkappa = 0$$
$$b2\,c2 - (m-2s)\,b2 - (n+4s)\,c2 - 9ss + 3ms - 3ns - \varkappa = 0$$
$$c2\,d2 - (m-s)\,c2 - (n+5s)\,d2 - 9ss + 3ms - 3ns - \varkappa = 0$$
etc.

§. 41. Si nunc hae fubftitutiones continuentur in infinitum, atque perpetuo fequentes valores in praecedentibus fubftituantur, litterarum α, β, γ, δ, etc. valores exprimentur fractionibus continuis fequentibus:

$$\alpha = m + n - s + \cfrac{ss - ms + ns + \varkappa}{m + n - s + \cfrac{4ss - 2ms + 2ns + \varkappa}{m + n - s + \cfrac{9ss - 3ms + 3ns + \varkappa}{m + n - s + \cfrac{16ss - 4ms + 4ns + \varkappa}{m + n + s + \text{etc.}}}}}$$

$$\beta = m + n + s + \cfrac{ss - ms + ns + \varkappa}{m + n + s + \cfrac{4ss - 2ms + 2ns + \varkappa}{m + n + s + \cfrac{9ss - 3ms + 3ns + \varkappa}{m + n + s + \cfrac{16ss - 4ms + 4ns + \varkappa}{m + n + s + \text{etc.}}}}}$$

$$\gamma = m + n + 3s + \cfrac{ss - ms + ns + \varkappa}{m + n + 3s + \cfrac{4ss - 2ms + 2ns + \varkappa}{m + n + 3s + \cfrac{9ss - 3ms + 3ns + \varkappa}{m + n + 3s + \text{etc.}}}}$$

quae fractiones continuae fatis funt fimiles iis, quas Brounckerus dedit, cum fequentes in praecedentibus non contineantur.

§. 42.

§. 42. Quo autem vſus harum formularum in interpolationibus pateat, propoſita ſit haec ſeries: $\frac{p}{p+2q}+\frac{p(p+2r)}{(p+2q)(p+2q+2r)}+\frac{p(p+2r)(p+4r)}{(p+2q)(p+2q+2r)(p+2q+4r)}+$ etc. cuius terminus indicis $\frac{1}{2}$ ſit $=A$; terminus indicis $\frac{3}{2}=ABC$; terminus indicis $\frac{5}{2}=ABCDE$; et ita porro. His poſitis erit $AB=\frac{p}{p+2q}$; $CD=\frac{p+2r}{p+2q+2r}$; $EF=\frac{p+4r}{p+2q+4r}$; etc. Ponatur nunc $A=\frac{a}{p+2q-r}$; $B=\frac{b}{p+2q}$; $C=\frac{c}{p+2q+r}$; $D=\frac{d}{p+2q+2r}$; etc. eritque $ab=p(p+2q-r)$; $bc=(p+r)(p+2q)$; $cd=(p+2r)(p+2q+r)$; $de=(p+3r)(p+2q+2r)$; etc. Iam fiat vlterius $a=p+q-r+\frac{g}{\alpha}$; $b=p+q+\frac{g}{\beta}$; $c=p+q+r+\frac{g}{\gamma}$; $d=p+q+2r+\frac{g}{\delta}$; etc. quibus valoribus ſubſtitutis emergent ſequentes aequationes, facto $g=q(r-q)$:

$$\alpha\beta-(p+q-r)\alpha-(p+q)\beta-q(r-q)=0$$
$$\beta\gamma-(p+q)\beta-(p+q+r)\gamma-q(r-q)=0$$
$$\gamma\delta-(p+q+r)\gamma-(p+q+2r)\delta-q(r-q)=0$$
$$\delta\varepsilon-(p+q+2r)\delta-(p+q+3r)\varepsilon-q(r-q)=0$$

etc.

§. 43. Comparatis his aequationibus cum iis, quas §. 38. aſſumſimus, reperietur:

$m=p+q-r$; $n=p+q$; $\varkappa=qr-qq$; et $s=r$ vnde fiet $ss-ms+ns+\varkappa=2rr+qr-qq$; $4ss-2ms+2ns+\varkappa=6rr+qr-qq$; $9ss-3ms+3ns+\varkappa=12rr+qr-qq$; etc. Quibus valoribus omnibus ſubſtitutis obtinebuntur ſequentes fractiones continuae, quibus litterae a, b, c, d, etc. exprimentur.

$$a=p+q-r+\cfrac{qr-qq}{2(p+q-r)+\cfrac{2rr+qr-qq}{2(p+q-r)+\cfrac{5rr+qr-qq)}{2(p+q-r)+\cfrac{12rr+qr-qq}{2(p+q-r)+\text{etc.}}}}}$$

$b=$

$$b = p+q+\cfrac{qr-qq}{2(p+q)+\cfrac{2rr+qr-qq}{2(p+q)+\cfrac{6rr+qr-qq}{2(p+q)+\cfrac{12rr+qr-qq}{2(p+q)+\text{etc.}}}}}$$

$$c = p+q+r+\cfrac{qr-qq}{2(p+q+r)+\cfrac{2rr+qr-qq}{2(p+q+r)+\cfrac{6rr+qr-qq}{2(p+q+r)+\cfrac{12rr+qr-qq}{2(p+q+r)+\text{etc.}}}}}$$

etc.

§. 44. Cum autem feriei propofitae terminus qui indicem habet n fit $= \frac{\int y^{p+2q-1}dy(1-y^{2r})^{n-1}}{\int y^{p-1}dy(1-y^{2r})^{n-1}}$; erit $A = \frac{a}{p+2q-r} = \frac{\int y^{p+2q-1}dy:\sqrt{(1-y^{2r})}}{\int y^{p-1}dy:\sqrt{(1-y^{2r})}}$; feu $a = (p+2q-r)\frac{\int y^{p+2q-1}dy:\sqrt{(1-y^{2r})}}{\int y^{p-1}dy:\sqrt{(1-y^{2r})}}$. Deinde ob $ab = p(p+2q-r)$ erit $b = \frac{p\int y^{p-1}dy:\sqrt{(1-y^{2r})}}{\int y^{p+2q-1}dy:\sqrt{(1-y^{2r})}}$. Quoniam autem eft per theorema in praecedente differtatione expofitum $\frac{p\int y^{p-1}dy:\sqrt{(1-y^{2r})}}{\int y^{p+r-1}dy:\sqrt{(1-y^{2r})}} = \frac{f\int y^{f-1}dy:\sqrt{(1-y^{2r})}}{\int y^{p+r-1}dy:\sqrt{(1-y^{2r})}} = \frac{(f+r)\int y^{f+2r-1}dy:\sqrt{(1-y^{2r})}}{\int y^{p+r-1}dy:\sqrt{(1-y^{2r})}}$ ponatur $f = p+2q-r$; quo facto erit $b = \frac{(p+2q)\int y^{p+2q+r-1}dy:\sqrt{(1-y^{2r})}}{\int y^{p+r-1}dy:\sqrt{(1-y^{2r})}}$. Simili vero modo progrediendo erit $c = \frac{(p+2q+r)\int y^{p+2q+2r-1}dy:\sqrt{(1-y^{2r})}}{\int y^{p+2r-1}dy:\sqrt{(1-y^{2r})}}$ et $d = \frac{(p+2q+2r)\int y^{p+2q+3r-1}dy:\sqrt{(1-y^{2r})}}{\int y^{p+3r-1}dy:\sqrt{(1-y^{2r})}}$ etc.

§. 45.

§. 45. Cum igitur lex progressionis harum formularum integralium constet, colligetur huius fractionis continuae generalis

$$p+q+mr+\cfrac{qr-qq}{2(p+q+mr)+\cfrac{2rr+qr-qq}{2(p+q+mr)+\cfrac{6rr+qr-qq}{2(p+q+mr)+etc.}}}$$

valor esse $= (p+2q+mr)\dfrac{\int y^{p+2q+(m+1)r-1}dy:\sqrt{(1-y^{2r})}}{\int y^{p+(m+1)r-1}dy:\sqrt{(1-y^{2r})}}$

Quare si ponatur $p+q+mr=s$, ita vt sit $p=s-q-mr$, proueniet sequens fractio continua:

$$s+\cfrac{qr-qq}{2s+\cfrac{2rr+qr-qq}{2s+\cfrac{6rr+qr-qq}{2s+\cfrac{12rr+qr-qq}{2s+\cfrac{20rr+qr-qq}{2s+etc.}}}}}$$

cuius propterea valor erit ista expressio

$$(q+s)\frac{\int y^{q+r+s-1}dy:\sqrt{(1-y^{2r})}}{\int y^{r+s-q-1}dy:\sqrt{(1-y^{2r})}}.$$

§. 46. Simili modo cum huius fractionis continuae

$$s+r+\cfrac{qr-qq}{2(s+r)+\cfrac{2rr+qr-qq}{2(s+r)+\cfrac{6rr+qr-qq}{2(s+r)+etc.}}}$$

valor sit $= (q+r+s)\dfrac{\int y^{s+2r+q-1}dy:\sqrt{(1-y^{2r})}}{\int y^{s+2r-q-1}dy:\sqrt{(1-y^{2r})}}.$

Harum duarum itaque fractionum continuarum productum erit $=(s+q)(s+r-q)$ quemadmodum productum formularum integralium declarat. Est enim per theorema in praecedente dissertatione datum:

$\frac{f}{a} = \dfrac{\int x^{a-1}dx:\sqrt{(1-x^{2r})}\,.\int x^{a+r-1}dx:\sqrt{(1-x^{2r})}}{\int x^{f-1}dx:\sqrt{(1-x^{2r})}\,.\int x^{f+r-1}dx:\sqrt{(1-x^{2r})}}$ ad quam formam productum formularum integralium sponte reducitur.

§. 47. Fractio continua inuenta in aliam commodiorem formam potest transmutari eo quod singuli numeratores in factores resolui possunt: ita habebitur ista fractio continua

$$s+\cfrac{q(r-q)}{2s+\cfrac{(r+q)(2r-q)}{2s+\cfrac{(2r+q)(3r-q)}{2s+\cfrac{(3r+q)(4r-q)}{2s+\text{etc}}}}}$$

cuius adeo valor erit $=(q+s)\dfrac{\int y^{r+s+q-1}\,dy:\sqrt{(1-y^{2r})}}{\int y^{r+s-q-1}\,dy:\sqrt{(1-y^{2r})}}$

Quocirca si ad fractionem continuam addatur s vt vbique eadem sit progressionis lex, erit

$$\frac{(q+s)\int y^{r+s+q-1}\,dy:\sqrt{(1-y^{2r})}+s\int y^{r+s-q-1}\,dy:\sqrt{(1-y^{2r})}}{\int y^{r+s-q-1}\,dy:\sqrt{(1-y^{2r})}}$$

$$=2s+\cfrac{q(r-q)}{2s+\cfrac{(r+q)(2r-q)}{2s+\cfrac{(2r+q)(3r-q)}{2s+\cfrac{(3r+q)(4r-q)}{2s+\text{etc.}}}}}$$

§. 48. Si nunc ponatur $r=2$ et $q=1$, prodibunt coniunctim omnes fractiones continuae a Brounckero exhibitae, quae omnes continebuntur in hac fractione continua:

$$s+\cfrac{1}{2s+\cfrac{9}{2s+\cfrac{25}{2s+\cfrac{49}{2s+\cfrac{81}{2s+\text{etc.}}}}}}$$

cuius propterea valor erit $=(s+1)\dfrac{\int y^{s+2}\,dy:\sqrt{(1-y^{4})}}{\int y^{s}\,dy:\sqrt{(1-y^{4})}}$

quae expressio apprime congruit cum ea, quam supra, ante-

antequam veritas omnino constaret, assignauimus, vide §. 16.

§. 49. Cum igitur hactenus plurimas dederim fractiones continuas, quarum valores per formulas integrales assignari possunt, methodum nunc directam exponam, cuius ope ex formulis integralibus vicissim ad fractiones continuas peruenire liceat. Nititur autem haec methodus reductione vnius formulae integralis ad duas alias, quae reductio non multum dissimilis est illi solitae, qua formulae cuiusdam differentialis integratio ad integrationem alius reducitur. Sint igitur huiusmodi formulae integrales infinitae $\int P\,dx$; $\int PR\,dx$; $\int PR^2dx$; $\int PR^3dx$; $\int PR^4dx$ etc. quae ita sint comparatae, vt si singulae ita integrentur, vt euanescant posito $x = 0$, tumque ponatur $x = 1$ fit vt sequitur:

$$a\int P\,dx = b\int PR\,dx + c\int PR^2dx$$
$$(a+\alpha)\int PR\,dx = (b+\beta)\int PR^2dx + (c+\gamma)\int PR^3dx$$
$$(a+2\alpha)\int PR^2dx = (b+2\beta)\int PR^3dx + (c+2\gamma)\int PR^4dx$$
$$(a+3\alpha)\int PR^3dx = (b+3\beta)\int PR^4dx + (c+3\gamma)\int PR^5dx$$

et generaliter

$$(a+n\alpha)\int PR^ndx = (b+n\beta)\int PR^{n+1}dx + (c+n\gamma)\int PR^{n+2}dx$$

§. 50. Si igitur huiusmodi habeantur formulae integrales, facili negotio ex iis fractiones continuae formabuntur. Cum enim sit

$$\frac{\int P\,dx}{\int PR\,dx} = \frac{b}{a} + \frac{c\int PR^2dx}{a\int PR\,dx}$$
$$\frac{\int PR\,dx}{\int PR^2dx} = \frac{b+\beta}{a+\alpha} + \frac{(c+\gamma)\int PR^3dx}{(a+\alpha)\int PR^2dx}$$
$$\frac{\int PR^2dx}{\int PR^3dx} = \frac{b+2\beta}{a+2\alpha} + \frac{(c+2\gamma)\int PR^4dx}{(a+2\alpha)\int PR^3dx}$$
$$\frac{\int PR^3dx}{\int PR^4dx} = \frac{b+3\beta}{a+3\alpha} + \frac{(c+3\gamma)\int PR^5dx}{(a+3\alpha)\int PR^4dx}$$

etc.

erit

erit ſubſtituendo quemque valorem in praecedente aequatione

$$\frac{\int P\,dx}{\int PR\,dx} = \frac{b}{a} + \cfrac{c:\alpha}{\cfrac{b+\beta}{a+2\alpha} + \cfrac{(c+\gamma):(a+\alpha)}{\cfrac{b+2\beta}{a+2\alpha} + \cfrac{(c+2\gamma):(a+2\alpha)}{\cfrac{b+3\beta}{a+3\alpha} + \cfrac{(c+3\gamma):(a+3\alpha)}{\cfrac{b+4\beta}{a+4\alpha} + \text{etc.}}}}}$$

Haec vero expreſſio inuerſa et a fractionibus partialibus liberata abit in hanc:

$$\frac{\int PR\,dx}{\int P\,dx} = \cfrac{a}{b + \cfrac{(a+\alpha)c}{b+\beta + \cfrac{(a+2\alpha)(c+\gamma)}{b+2\beta + \cfrac{(a+3\alpha)(c+2\gamma)}{b+3\beta + \cfrac{(a+4\alpha)(c+3\gamma)}{b+4\beta + \text{etc.}}}}}}$$

§. 51. Si fuerit etiam deſignante n numerum negativum $(a+n\alpha)\int PR^n\,dx = (b+n\beta)\int PR^{n+1}\,dx + (c+n\gamma)\int PR^{n+2}\,dx$, ſequentes habebuntur aequationes.

$$(a-\alpha)\int\frac{P\,dx}{R} = (b-\beta)\int P\,dx + (c-\gamma)\int PR\,dx$$

$$(a-2\alpha)\int\frac{P\,dx}{R^2} = (b-2\beta)\int\frac{P\,dx}{R} + (c-2\gamma)\int P\,dx$$

$$(a-3\alpha)\int\frac{P\,dx}{R^3} = (b-3\beta)\int\frac{P\,dx}{R^2} + (c-3\gamma)\int\frac{P\,dx}{R}$$

$$(a-4\alpha)\int\frac{P\,dx}{R^4} = (b-4\beta)\int\frac{P\,dx}{R^3} + (c-4\gamma)\int\frac{P\,dx}{R^2}$$

etc.

Hinc igitur pari modo conficietur:

$$\frac{\int PR\,dx}{\int P\,dx} = \frac{-(b-\beta)}{c-\gamma} + \frac{(a-\alpha)\int P\,dx:R}{(c-\gamma)\int P\,dx}$$

$$\frac{\int P\,dx}{\int P\,dx:R} = \frac{-(b-2\beta)}{c-2\gamma} + \frac{(a-2\alpha)\int P\,dx:R^2}{(c-2\gamma)\int P\,dx:R}$$

$$\frac{\int P\,dx:R}{\int P\,dx:R^2)} = \frac{-(b-3\beta)}{c-3\gamma} + \frac{(a-3\alpha)\int P\,dx:R^3}{(c-3\gamma)\int P\,dx:R^2}$$

etc.

Ex his autem aequationibus producetur

$\frac{\int PR\,dx}{\int P\,dx}$

$$\frac{\int PR\,dx}{\int P\,dx} = \frac{-(b-\beta)}{c-\gamma} + \cfrac{(a-\alpha):(c-\gamma)}{\cfrac{-(b-2\beta)}{c-2\gamma} + \cfrac{(a-2\alpha):(c-2\gamma)}{\cfrac{-(b-3\beta)}{c-3\gamma} + \cfrac{(a-3\alpha):(c-3\gamma)}{\cfrac{-(b-4\beta)}{c-4\gamma} + \text{etc.}}}}$$

ſiue fractionibus partialibus ſublatis

$$\frac{(c-\gamma)\int PR\,dx}{\int P\,dx} = -(b-\beta) + \cfrac{(a-\alpha)(c-2\gamma)}{-(b-2\beta) + \cfrac{(a-2\alpha)(c-3\gamma)}{-(b-3\beta) + \cfrac{(a-3\alpha)(c-4\gamma)}{-(b-4\beta) + \text{etc.}}}}$$

Duplex igitur habetur fractio continua, cuius vtriusque idem eſt valor $\frac{\int PR\,dx}{\int P\,dx}$.

§. 52. Praecipuum autem eſt in hoc negotio, vt definiantur idoneae functiones ipſius x loco P et R ſubſtituendae, quo fiat $(a+n\alpha)\int PR^n dx = (b+n\beta)\int PR^{n+1} dx + (c+n\gamma)\int PR^{n+2} dx$ eo ſaltem caſu, quo poſt ſingulas integrationes ponitur $x = 1$. Ponamus igitur eſſe generaliter $(a+n\alpha)\int PR^n dx + R^{n+1} S = (b+n\beta)\int PR^{n+1} dx + (c+n\gamma)\int PR^{n+2} dx$, atque $R^{n+1} S$ eiusmodi eſſe functionem ipſius x, quae euaneſcat poſito tam $x = 0$, quam $x = 1$. Sumtis ergo differentialibus, et facta per R^n diuiſione, erit: $(a+n\alpha)\, P\,dx + R\,dS + (n+1)\, S\,dR = (b+n\beta)\, PR\,dx + (c+n\gamma)\, PR^2 dx$; quae aequatio, cum ſemper locum habere debeat, quicquid ſit n, in duas reſoluitur aequationes has:

$$a P\,dx + R\,dS + S\,dR = b PR\,dx + c PR^2 dx \quad \text{et}$$

$$\alpha P\,dx + S\,dR = \beta PR\,dx + \gamma PR^2 dx$$

Ex his aequationibus elicitur duplici modo $P\,dx = \frac{R\,dS + S\,dR}{bR + cR^2 - a} = \frac{S\,dR}{\beta R + \gamma R^2 - \alpha}$, vnde fit $\frac{dS}{S} = \frac{(b-\beta)R\,dR + (c-\gamma)R^2 dR - (a-\alpha)dR}{\beta R^2 + \gamma R^3 - \alpha R} = \frac{(a-\alpha)dR}{\alpha R} + \frac{(\alpha b - \beta a)dR + (\alpha c - \gamma a)R\,dR}{\alpha(\beta R + \gamma R^2 - \alpha)}$. Ex hac ergo aequatione definitur S per R; inuento autem S erit $P = \frac{S\,dR}{(\beta R + \gamma R^2 - \alpha)dx}$; indeque cognitae erunt formulae $\int P\,dx$ et $\int PR\,dx$, quibus valor fractionum continuarum ſuperiorum determinatur.

§. 53. Quoniam igitur quantitas R per x non definitur, pro ea functio quaecunque ipsius x accipi poterit. At cum conditio quaestionis postulet vt $R^{n+1}S$ euanescat posito tam $x = 0$, quam $x = 1$, eo ipso natura functionis loco R accipiendae determinatur. Deinde vero etiam ad hoc est respiciendum vt integralia $\int PR^n dx$ posito post integrationem $x = 1$, finitum obtineant valorem, si enim integralia ista hoc casu fierent vel o vel ∞, tum difficulter valor $\frac{\int PR dx}{\int P dx}$ colligeretur. Prius incommodum tutissime euitatur, tribuendo ipsi R eiusmodi valorem, vt PR^n nunquam negatiuum induat valorem, quamdiu x intra limiter o et 1 consistit. Ne autem $\int PR^n dx$ posito $x = 1$ fiat infinitum, difficilius saepenumero obtinetur. Conueniet autem casus, quibus n est numerus vel affirmatiuus vel negatiuus a se inuicem discernere; cum saepissime, si his conditionibus satisfiat, existente n numero affirmatiuo, simul reliquis casibus satisfieri nequeat. Sin autem conditiones praescriptae tantum impleantur casibus, quibus n est numerus affirmatiuus, tum prioris fractionis continuae tantum valor exhiberi potest; posterioris vero tantum, si conditionibus fuerit satisfactum, existente n numero negatiuo.

§. 54. Incipiamus euolutionem huius methodi valores fractionum continuarum inueniendi ab exemplis iam ante tractatis, et primo quidem proposita sit ista fractio continua:

$$r + \cfrac{fb}{r + \cfrac{(f+r)(b+r)}{r + \cfrac{(f+2r)(b+2r)}{r + \text{etc.}}}}$$

cuius valor supra §. 34. assignatus est iste

$$\frac{b(f-r)\int y^{b+r-1}dy : \sqrt{(1-y^{2r})} - f(b-r)\int y^{f+r-1}dy : \sqrt{(1-y^{2r})}}{f\int y^{f+r-1}dy : \sqrt{(1-y^{2r})} - b\int y^{b+r-1}dy : \sqrt{(1-y^{2r})}}$$

Com-

Comparetur ergo haec fractio continua cum ista generali:

$$\frac{a\int Pdx}{\int PRdx} = b + \cfrac{(a+\alpha)c}{b+\beta+\cfrac{(a+2\alpha)(c+\gamma)}{b+2\beta+\cfrac{(a+3\alpha)(c+2\gamma)}{b+3\beta+\text{etc.}}}}$$

eritque $b = r$; $\beta = 0$; $\alpha = r$; $\gamma = r$; $a = f - r$; $c = b$. His valoribus substitutis orietur $\frac{dS}{S} = \frac{rRdR+(b-r)R^2dR-(f-2r)dR}{rR^3-rR}$ $= \frac{(f-2r)dR}{rR} + \frac{rdR+(b-f+r)RdR}{r(R^2-1)}$: atque integrando $lS = \frac{f-2r}{r}lR + \frac{b-f}{2r}l(R+1) + \frac{b-f+2r}{2r}l(R-1) + lC$ seu $S = CR^{\frac{f-2r}{r}}(R^2-1)^{\frac{b-f}{2r}}(R-1)$. Hinc itaque erit $R^{n+1}S = R^{\frac{f+(n-1)r}{r}}(R^2-1)^{\frac{b-f}{2r}}(R-1)$, atque $Pdx = \frac{CR^{\frac{f-2r}{r}}(R^2-1)^{\frac{b-f}{2r}}dR}{r(R+1)}$.

§. 55. Cum autem $R^{n+1}S$ duobus casibus euanescere debeat posito tam $x = 0$ quam $x = 1$; idque quicunque numerus affirmatiuus loco n substituatur; ad negatiuos enim valores ipsius n respicere non est opus. Ponamus vero f, b, et r esse numeros affirmatiuos atque $b > f$, quod tuto assumere licet nisi sit $f = b$, deinde sit etiam $f > r$. His positis manifestum est formulam $R^{n+1}S$ duobus casibus euanescere scilicet si $R = 0$ et $R = 1$: hocque etiam locum habet si sit $f = b$. Dummodo ergo sit $f > r$ poni poterit $R = x$, eritque $Pdx = \frac{x^{\frac{f-2r}{r}}(1-x^2)^{\frac{b-f}{2r}}dx}{1+x}$ determinata constante C. Ex his itaque valor fractionis continuae propositae erit

$$= \frac{(f-r)\int \frac{x^{\frac{f-2r}{r}}(1-x^2)^{\frac{b-f}{2r}}dx}{1+x}}{\int \frac{x^{\frac{f-r}{r}}(1-x^2)^{\frac{b-f}{2r}}dx}{1+x}}.$$

Posito

Posito autem $x = y^r$ erit valor quaesitus $=$

$$\frac{(f-r)\int y^{f-r-1}(1-y^{2r})^{\frac{h-f}{2r}}dy:(1+y^r)}{\int y^{f-1}(1-y^{2r})^{\frac{h-f}{2r}}dy:(1+y^r)}.$$

§. 56. Aliam igitur nacti sumus expressionem huius fractionis continuae

$$r+\cfrac{fh}{r+\cfrac{(f+r)(h+r)}{r+\text{etc.}}}$$

valorem continentem, quae etsi formulas integrales in se complectitur, tamen discrepat ab expressione ante inuenta. Haec enim posterior expressio locum non habet nisi sit $f > r$, pro h autem accipi oportet maiorem quantitatum binarum f et h, siquidem fuerint inaequales. Attamen si etiam f fuerit minus quam r, valor fractionis continuae exhiberi potest considerando hanc

$$r+\cfrac{(f+r)(h+r)}{r+\cfrac{(f+2r)(h+2r)}{r+\text{etc.}}}$$

cuius valor erit $= \dfrac{f\int y^{f-1}(1-y^{2r})^{\frac{h-f}{2r}}dy:(1+y^r)}{\int y^{f+r-1}(1-y^{2r})^{\frac{h-f}{2r}}dy:(1+y^r)}$ quae nulla indiget restrictione. Posito enim hoc valore $= V$ erit fractionis continuae propositae valor $= r + \frac{fh}{V}$.

§. 57. Casus ille quo $f = h$, qui ante peculiari modo erat erutus, eiusque valor in §. 34. inuentus $=$

$$\frac{1-(h-r)\int x^{h-1}dx:(1+x^r)}{\int x^{h-1}dx:(1+x^r)} = \frac{(h-r)\int x^{h-r-1}dx:(1+x^r)}{\int x^{h-1}dx:(1+x^r)}$$

ex hac posteriore expressione sponte fluit; facto enim $f = h$, expressio §. 55. inuenta abibit in hanc $\dfrac{(h-r)\int y^{h-r-1}dy:(1+y^r)}{\int y^{f-1}dy:(1+y^r)}$

om-

omnino eandem ex quo consensus ambarum expressionum generalium satis perspicitur. Hic autem tuto accipere licet esse $h > r$, cum ii casus, quibus hoc secus accidit, facillime ad hos reducantur, vti modo est monstratum.

§. 58. Quo autem consensus ambarum expressionum omni casu intelligatur, praemittendum nobis est hoc lemma, quod ab aliis iam est demonstratum. Si fuerit series $1 + \frac{p}{q+s} + \frac{p(p+s)}{(q+s)(q+2s)} + \frac{p(p+s)(p+2s)}{(q+s)(q+2s)(q+3s)} +$ etc. in qua sint quantitatis p, q, et s affirmatiuae atque $q > p$; huius seriei in infinitum continuatae summa erit $= \frac{q}{q-p}$. Huius autem lemmatis veritas per methodum meam generalem series summandi sequenti modo euinci potest. Consideretur enim haec series $x^q + \frac{p}{q+s} x^{q+s} + \frac{p(p+s)}{(q+s)(q+2s)} x^{q+2s} +$ etc. cuius summa dicatur z; eritque differentiando $\frac{dz}{dx} = q x^{q-1} + p x^{q+s-1} + \frac{p(p+s)}{(q+s)} x^{q+2s-1} +$ etc. atque $x^{p-q-s} dz = q x^{p-s-1} dx + p x^{p-1} dx + \frac{p(p+s)}{q+s} x^{p+s-1} dx +$ etc. quae aequatio integrata dat $\int x^{p-q-s} dz = \frac{q x^{p-s}}{p-s} + x^p + \frac{p x^{p+s}}{q+s} +$ etc. $= \frac{q x^{p-s}}{p-s} + x^{p-q} z$. Ex hac aequatione differentiata prodibit ista $x^{p-q-s} dz = q x^{p-s-1} dx + x^{p-q} dz + (p-q) x^{p-q-1} z dx$ seu $dz (1-x^s) + (q-p) x^{s-1} z dx = q x^{q-1} dx$ siue $dz + \frac{(q-p) x^{s-1} z dx}{1-x^s} = \frac{q x^{q-1} dx}{1-x^s}$, cuius integralis est $\frac{z}{(1-x^s)^{\frac{q-p}{s}}}$

$$= q \int \frac{x^{q-1} dx}{(1-x^s)^{\frac{q-p+s}{s}}} = \frac{q x^q}{(q-p)(1-x^s)^{\frac{q-p}{s}}} - \frac{pq}{q-p} \int \frac{x^{q-1} dx}{(1-x^s)^{\frac{q-p}{s}}}:$$

vnde erit $z = \frac{q x^q}{q-p} - \frac{pq(1-x^s)^{\frac{q-p}{s}}}{q-p} \int \frac{x^{q-1} dx}{(1-x^s)^{\frac{q-p}{s}}}$. Quare facto $x = 1$, erit $z = \frac{q}{q-p} = 1 + \frac{p}{q+s} + \frac{p(p+s)}{(q+s)(q+2s)}$ + etc. quae est demonstratio lemmatis dati, ex qua simul intelligitur lemmatis veritatem non consistere nisi sit $q > p$.

§. 59. Cum igitur valorem huius fractionis continuae

$$r + \cfrac{fb}{r + \cfrac{(f+r)(b+r)}{r + \cfrac{(f+2r)(b+2r)}{r + \text{etc.}}}}$$

duplici modo habeamus expressum, quorum alter est $=$

$$\frac{b(f-r)\int y^{b+r-1}dy : \sqrt{(1-y^{2r})} - f(b-r)\int y^{f+r-1}dy : \sqrt{(1-y^{2r})}}{f\int y^{f+r-1}dy : \sqrt{(1-y^{2r})} - b\int y^{b+r-1}dy : \sqrt{(1-y^{2r})}}$$

alter vero, qui in §. 56. est erutus $= r +$

$$\frac{b\int y^{f+r-1}dy\,(1-y^{2r})^{\frac{b-f}{2r}} : (1+y^r)}{\int y^{f-1}dy(1-y^{2r})^{\frac{b-f}{2r}} : (1+y^r)},$$ operae pretium erit harum expressionum consensum declarare. Cum igitur sit $\frac{1}{1+y^r} = \frac{1-y^r}{1-y^{2r}}$ erit $\int y^{f-1}\,dy(1-y^{2r})^{\frac{b-f}{2r}} : (1+y^r) = \int y^{f-1}$ $dy\,(1-y^{2r})^{\frac{b-f-2r}{2r}} - \int y^{f+r-1}\,dy(1-y^{2r})^{\frac{b-f-2r}{2r}}$, atque $\int y^{f+r-1}$ $dy(1-y^{2r})^{\frac{b-f}{2r}} : (1+y^r) = \int y^{f+r-1}\,dy(1-y^{2r})^{\frac{b-f-2r}{2r}} - \int y^{f+2r-1}$ $dy(1-y^{2r})^{\frac{b-f-2r}{2r}} = \int y^{f+r-1}\,dy(1-y^{2r})^{\frac{b-f-2r}{2r}} - \frac{f}{b}\int y^{f-1}dy(1-y^{2r})^{\frac{b-f-2r}{2r}}$.

Ponatur $\frac{\int y^{f+r-1}dy(1-y^{2r})^{\frac{b-f-2r}{2r}}}{\int y^{f-1}\,dy\,(1-y^{2r})^{\frac{b-f-2r}{2r}}} = V$, erit valor po-

sterior

sterior fractionis continuae $= r + \frac{bV - f}{1 - V}$. Ponatur praeterea $\frac{\int y^{b+r-1} dy : \sqrt{(1-y^{2r})}}{\int y^{f+r-1} dy : \sqrt{(1-y^{2r})}} = W$ erit prior valor $= \frac{b(f-r)W - f(b-r)}{f - bW}$, ex quorum aequalitate sequitur fore $V = \frac{f}{bW}$ ita vt sit $\int y^{f+r-1} dy (1-y^{2r})^{\frac{b-f-2r}{2r}} = \frac{f \int y^{f+r-1} dy : \sqrt{(1-y^{2r})}}{b \int y^{b+r-1} dy : \sqrt{(1-y^{2r})}}$, cuius aequalitatis ratio per Theoremata in praecedente dissertatione exhibita constat: est enim per vnum ex illis theorematis $\frac{\int y^{f+r-1} dy (1-y^{2r})^{\frac{b-f-2r}{2r}}}{\int y^{f+2r-1} dy (1-y^{2r})^{\frac{b-f-2r}{2r}}}$

$$= \frac{\int y^{f+r-1} dy : \sqrt{(1-y^{2r})}}{\int y^{b+r-1} dy : \sqrt{(1-y^{2r})}}.$$

§. 60. Consideremus nunc hanc fractionem continuam

$$2r + \cfrac{fb}{2r + \cfrac{(f+r)(b+r)}{2r + \cfrac{(f+2r)(b+2r)}{2r + \text{etc.}}}}$$

cuius valor supra §. 35. inuentus est $=$

$$\frac{2(f-r)(b-r)\int y^{f-1} dy : \sqrt{(1-y^{2r})} - b(f+b-3r)\int y^{b+r-1} dy : \sqrt{(1-y^{2r})}}{2b \int y^{b+r-1} dy : \sqrt{(1-y^{2r})} - (f+b-r)\int y^{f-1} dy : \sqrt{(1-y^{2r})}}$$

si nunc haec fractio continua comparetur cum hac $\frac{a \int P dx}{\int PR dx}$

 $= b$

$$= b + \cfrac{(a+\alpha)c}{b + \beta + \cfrac{(a+2\alpha)(c+\gamma)}{b+2\beta + \cfrac{(a+3\alpha)(c+2\gamma)}{b+3\beta+\text{etc.}}}}$$

erit $b = 2r$; $\beta = 0$; $\alpha = r$; $\gamma = r$; $a = f - r$ et $c = b$. Hinc igitur ex §. 52. habebitur $\frac{dS}{S} = \frac{(f-2r)dR}{rR} + \frac{2cdR + (b-f+r)RdR}{r(R^2-1)}$ et integrando $S = CR^{\frac{f-2r}{r}}(R^2-1)^{\frac{b-f-r}{2r}}(R-1)^2$ vnde fit $Pdx = CR^{\frac{f-2r}{r}}(R^2-1)^{\frac{b-f-3r}{2r}}(R-1)^2 dR$ et $R^{n+1}S = CR^{\frac{f+(n-1)r}{r}}(R^2-1)^{\frac{b-f-r}{2r}}(R-1)^2$, quae expreſſio duobus caſibus euaneſcit, ponendo tum $R = 0$ tum $R = 1$, modo ſit $f > r$ et $b + 3r > f$, quibus conditionibus ſemper ſatisfieri poteſt.

§. 61. Sit igitur $R = x$ et conſtante C determinata erit $Pdx = x^{\frac{f-2r}{r}}dx(1-x^2)^{\frac{b-f-3r}{2r}}(1-x)^2$: vel poſito $R = x = y^r$; erit $Pdx = y^{f-r-1}dy(1-y^{2r})^{\frac{b-f-3r}{2r}}(1-y^r)^2$, ex quibus erit valor fractionis continuae propoſitae $\frac{a\int Pdx}{\int PRdx} = \frac{(f-r)\int y^{f-r-1}dy(1-y^{2r})^{\frac{b-f-3r}{2r}}(1-y^r)^2}{\int y^{f-1}dy(1-y^{2r})^{\frac{b-f-3r}{2r}}(1-y^r)^2}$ quae per theoremata ſuperioris diſſertationis ad priorem formam reducetur, euoluendo quadratum $(1-y^r)^2$, quo facto vtraque formula integralis in binas ſimpliciores reſoluetur. Ipſam autem reductionem in exemplo ſequente latius patente declarabo.

§. 62.

§. 62. Si habeatur haec formula integralis $\int y^{m-1}dy (1-y^{2r})^{\varkappa}(1-y^r)^n$, atque $(1-y^r)^n$ resoluatur in seriem $1 - ny^r + \frac{n(n-1)}{1.2}y^{2r} -$ etc. cuius alternis terminis sumendis formula integralis proposita reducetur ad binas sequentes:

$$\int y^{m-1}dy(1-y^{2r})^{\varkappa}\left(1+\frac{n(n-1)}{1.2}\cdot\frac{m}{p}+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{m(m+2r)}{p(p+2r)}+\text{etc.}\right.$$

$$-\int y^{m+r-1}dy(1-y^{2r})^{\varkappa}\left(n+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\frac{(m+r)}{(p+r)}+\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{(m+r)(m+3r)}{(p+r)(p+3r)}\ \text{etc.}\right.$$

posito breuitatis gratia $m+2\varkappa r+2r=p$. Quare si fuerit vt in casu praecedente $n=2$ erit $\int y^{m-1}dy(1-y^{2r})^{\varkappa}(1-y^r)^2 = \frac{m+p}{p}\int y^{m-1}dy(1-y^{2r})^{\varkappa} - 2\int y^{m+r-1}dy(1-y^{2r})^{\varkappa}$. Ex quo habebitur $\frac{a\int Pdx}{\int PRdx}$

$$= \frac{\frac{(f-r)(f+b-3r)}{b-2r}\int y^{f-r-1}dy(1-y^{2r})^{\frac{b-f-3r}{2r}} - 2(f-r)\int y^{f-1}dy(1-y^{2r})^{\frac{b-f-3r}{2r}}}{\frac{f+b-r}{b-r}\int y^{f-1}dy(1-y^{2r})^{\frac{b-f-3r}{br}} - 2\int y^{f+r-1}dy(1-y^{2r})^{\frac{b-f-3r}{2r}}}$$

$$= \frac{b(f+b-3r)\int y^{f+r-1}dy(1-y^{2r})^{\frac{b-f-3r}{2r}} - 2(f-r)(b-r)\int y^{f-1}dy(1-y^{2r})^{\frac{b-f-r}{2r}}}{(f+b-r)\int y^{f-1}dy(1-y^{2r})^{\frac{b-f-r}{2r}} - 2b\int y^{f+r-1}dy(1-y^{2r})^{\frac{b-f-r}{2r}}}$$

quae expressio cum aequalis esse debeat illi, quae supra §. 35. est inuenta, praebebit hanc aequationem:

$$\frac{\int y^{f+r-1}dy(1-y^{2r})^{\frac{b-f-r}{2r}}}{\int y^{f-1}dy(1-y^{2r})^{\frac{b-f-r}{2r}}} = \frac{\int y^{b+r-1}dy:\sqrt{(1-y^{2r})}}{\int y^{f-1}dy:\sqrt{(1-y^{2r})}}$$

cuius quidem ratio iam in theorematis superioris dissertationis continetur.

§. 63. Sumamus nunc vicissim pro P et R datos valores, ex iisque fractiones continuas formemus; atque ponamus

namus $P = x^{m-1}(1-x^r)^n(p+qx^r)^{\varkappa}$, et $R = x^r$. Cum autem esse debeat $(a+\nu\alpha)\int PR^{\nu}dx = x(b+\nu\beta)\int PR^{\nu+1}dx + (c+\nu\gamma)\int PR^{\nu+2}dx$, hincque ob P et R datas fiat ex §. 52. $S = \frac{1}{r}x^{m-r}(1-x^r)^n(p+qx^r)^{\varkappa}(\gamma x^{2r}+\beta x^r-\alpha)$ erit $\frac{dS}{S} = \frac{(m-r)dx}{x} + \frac{nrx^{r-1}dx}{-1+x^r} + \frac{\varkappa qrx^{r-1}dx}{p+qx^r} + \frac{2\gamma rx^{2r-1}dx+\beta rx^{r-1}dx}{\gamma x^{2r}+\beta xr^r-\alpha} = \frac{(a-\alpha)rdx}{\alpha x} + \frac{(\alpha b-\beta a)rx^{r-1}dx+(\alpha c-\gamma a)rx^{2r-1}dx}{\alpha(\gamma x^{2r}+\beta x^r-\alpha)}$. Sit nunc $(p+qx^r)(x^r-1) = \gamma x^{2r}+\beta x^r-\alpha$, erit $\gamma = q$, $\beta = p-q$ et $\alpha = p$. Sit praeterea $\frac{(a-\alpha)r}{\alpha} = m-r$, erit $a = \frac{mp}{r}$. Vnde debebit porro esse $nqr+\varkappa qr+2qr = \frac{cpr-mpq}{p}$ seu $c = \frac{mq}{r}+nq+(\varkappa+2)q$, et tandem $b = \frac{m(p-q)}{r}+(n+1)p-(\varkappa+1)q$. Dummodo ergo m et $n+1$ fuerint numeri affirmatiui, quo $R^{\nu+1}S$ euanescat posito tam $x=0$, quam $x=1$, prodibit sequens expressio

$$\frac{\int x^{m+r-1}dx(1-x^r)^n(p+qx^r)^{\varkappa}}{\int x^{m-1}dx(1-x^r)^n(p+qx^r)^{\varkappa}} = \frac{\int PRdx}{\int Pdx}$$

quae propterea aequalis erit huic fractioni continuae

$$\cfrac{mp}{m(p-q)+(n+1)pr-(\varkappa+1)qr+\cfrac{pq(m+r)(m+nr+(\varkappa+2)r)}{m(p-q)+(n+2)pr-(\varkappa+2)qr+\cfrac{pq(m+2r)(m+(n+1)r+(\varkappa+2)r)}{m(p-q)+(n+3)pr-(\varkappa+3)qr+\text{etc.}}}}$$

§. 64. Quo fractio continua simpliciorem induat formam, ponatur $m+nr+r=a$; $m+\varkappa r+r=b$; et $m+nr+\varkappa r+r=c$, fiet $\varkappa = \frac{c-a}{r}$; $n = \frac{c-b}{r}$ et $m = a+b-c-r$; ideoque erit

$$\cfrac{p(a+b-c-r)}{ap-bq+\cfrac{pq(a+b-c)(c+r)}{(a+r)p-(b+r)q+\cfrac{pq(a+b-c+r)(c+2r)}{(a+2r)p-(b+2r)q+\cfrac{pq(a+b-c+2r)(c+3r)}{(a+3r)p-(b+3r)q+\text{etc.}}}}}$$

$$= \frac{\int x^{a+b-c-1}dx(1-x^r)^{\frac{c-b}{r}}(p+qx^r)^{\frac{c-a}{r}}}{\int x^{a+b-c-r-1}dx(1-x^r)^{\frac{c-b}{r}}(p+qx^r)^{\frac{c-a}{r}}}$$ posito post vtramque integrationem $x = 1$. Requiritur autem vt sint $a+b-c-r$ et $c-b+r$ numeri affirmatiui. Sin autem ponatur breuitatis causa $a+b-c-r = g$ erit

$$\frac{\int x^{g+r-1}dx(1-x^r)^{\frac{c-b}{r}}(p+qx^r)^{\frac{c-a}{r}}}{\int x^{g-1}dx(1-x^r)^{\frac{c-b}{r}}(p+qx^r)^{\frac{c-a}{r}}}$$

$$= \cfrac{pg}{ap-bq+\cfrac{pq(c+r)(g+r)}{(a+r)p-(b+r)q+\cfrac{pq(c+2r)(g+2r)}{(a+2r)p-(b+2r)q+\text{etc.}}}}$$

quae aequatio latissime patet, et omnes hactenus erutas fractiones continuas sub se comprehendit.

§. 65. Si quantitates c et g inter se commutentur, prodibit sequens fractio continua

$$\cfrac{pc}{ap-bq+\cfrac{pq(c+r)(g+r)}{(a+r)p-(b+r)q+\cfrac{pq(c+2r)(g+2r)}{(a+2r)p-(b+2r)q+\text{etc.}}}}$$

cuius adeo valor erit $\frac{\int x^{c+r-1}dx(1-x^r)^{\frac{g-b}{r}}(p+qx^r)^{\frac{g-a}{r}}}{\int x^{c-1}dx(1-x^r)^{\frac{g-b}{r}}(p+qx^r)^{\frac{g-a}{r}}}$.

Quare cum fractiones hae continuae datam inter se teneant rationem, scilicet g ad c hinc sequens orietur Theorema restituto loco g suo valore $\frac{c\int x^{a+b-c-1}dx(1-x^r)^{\frac{c-b}{r}}(p+qx^r)^{\frac{c-a}{r}}}{\int x^{a+b-c-r-1}dx(1-x^r)^{\frac{c-b}{r}}(p+qx^r)^{\frac{c-a}{r}}}$

$$= \frac{(a+b-c-r)\int x^{c+r-1}dx(1-x^r)^{\frac{a-c-r}{r}}(p+qx^r)^{\frac{b-c-r}{r}}}{\int x^{c-1}dx(1-x^r)^{\frac{a-c-r}{r}}(p+qx^r)^{\frac{b-c-r}{r}}}$$

Sub

Sub qua amplissima forma plurimae egregiae reductiones particulares continentur. Sit verbi gratiae $b = c + r$ erit

$$\frac{c\int x^{a+r-1}dx(p+qx^r)^{\frac{c-a}{r}} : (1-x^r)}{\int x^{a-1}dx(p+qx^r)^{\frac{c-a}{r}} : (1-x^r)} = \frac{a\int x^{c+r-1}dx(1-x^r)^{\frac{a-c-r}{r}}}{\int x^{c-1}dx(1-x^r)^{\frac{a-c-r}{r}}}$$

$= c$, vnde sequitur fore $\int \frac{x^{a+r-1}dx(p+qx^r)^{\frac{c-a}{r}}}{1-x^r} = \frac{\int x^{a-1}dx(p+qx^r)^{\frac{c-a}{r}}}{1-x^r}$

Habebitur ergo hinc istud theorema latius patens

$$\frac{\int x^{m-1}dx(p+qx^r)^{\kappa}}{1-x^r} = \frac{\int x^{n-1}dx(p+qx^r)^{\kappa}}{1-x^r},$$

vbi semper integrationibus ita institutis vt euanescant integralia posito $x = 0$, fieri intelligitur $x = 1$. Excipitur autem solus ille casus quo est $q + p = 0$; quo incommodum accidit.

§. 66. Fractiones continuae, quas hactenus eruimus ope interpolationum, huc redeunt vt denominatores partiales sint constantes. Quo igitur formam generalem nunc inuentam ad eas transferamus, ponatur $p = q = 1$; prodibitque haec fractio coutinua

$$\cfrac{cg}{a-b+\cfrac{(c+r)(g+r)}{a-b+\cfrac{(c+2r)(g+2r)}{a-b+\cfrac{(c+3r)(g+3r)}{a-b+\text{etc.}}}}} = \frac{c\int x^{g+r-1}dx(1-x^r)^{\frac{c-b}{r}}(1+x^r)^{\frac{c-a}{r}}}{\int x^{g-1}dx(1-x^r)^{\frac{c-b}{r}}(1+x^r)^{\frac{c-a}{r}}}$$

vel eiusdem valor erit quoque $= \dfrac{g\int x^{c+r-1}dx(1-x^r)^{\frac{g-b}{r}}(1+x^r)^{\frac{g-a}{r}}}{\int x^{c-1}dx(1-x^r)^{\frac{g-b}{r}}(1+x^r)^{\frac{g-a}{r}}}$

existente $g = a + b - c - r$. Ponatur $a - b = s$ ob $a + b = c + g + r$ erit $a = \frac{c+g+r+s}{2}$ et $b = \frac{c+g+r-s}{2}$,

vnde

vnde fiet

$$\cfrac{cg}{s+\cfrac{(c+r)(g+r)}{s+\cfrac{(c+2r)(g+2r)}{s+\text{etc.}}}}$$

$$=\frac{c\int x^{g+r-1}dx(1-x^{2r})^{\frac{c-g-r-s}{2r}}(1-x^{r})^{\frac{s}{r}}}{\int x^{g-1}dx(1-x^{2r})^{\frac{c-g-r-s}{2r}}(1-x^{r})^{\frac{s}{r}}}=$$

$$\frac{g\int x^{c+r-1}dx(1-x^{2r})^{\frac{g-c-r-s}{2r}}(1-x^{r})^{\frac{s}{r}}}{\int x^{c-1}dx(1-x^{2r})^{\frac{g-c-r-s}{2r}}(1-x^{r})^{\frac{s}{r}}}$$

§. 67. Ponamus vt ad formam §. 47. perueniatur $2s$ loco s, sitque $c=q$ et $g=r-q$, habebitur haec fractio continua

$$\cfrac{q(r-q)}{2s+\cfrac{(q+r(2r-q)}{2s+\cfrac{(q+2r)(3r-q)}{2s+\text{etc.}}}}$$

cuius valor adeo erit vel $=\frac{q\int x^{2r-q-1}dx(1-x^{2r})^{\frac{q-r-s}{r}}(1-x^{r})^{\frac{2s}{r}}}{\int x^{r-q-1}dx(1-x^{2r})^{\frac{q-r-s}{r}}(1-x^{r})^{\frac{2s}{r}}}$

vel $=\frac{(r-q)\int x^{q+r-1}dx(1-x^{2r})^{\frac{-q-s}{r}}(1-x^{r})^{\frac{2s}{r}}}{\int x^{q-1}dx(1-x^{2r})^{\frac{-q-s}{r}}(1-x^{r})^{\frac{2s}{r}}}$. Eiusdem autem fractionis continuae valor ante est inuentus $=\frac{(q+s)\int y^{r+s+q-1}dy:\sqrt{(1-y^{2r})}}{\int y^{r+s-q-1}dy:\sqrt{(1-y^{2r})}}-s$. Quamobrem istae

 for-

formulae integrales interſe erunt aequales; quod eſt theorema minime contemnendum.

§. 68. Sit vti §. 48. poſuimus $r = 2$, et $q = 1$ erit

$$\frac{(1+s)\int y^{s+2}dy : V(1-y^4)}{\int y^s dy : V(1-y^4)} - s = \frac{\int x^2 dx (1-x^4)^{\frac{-s-1}{2}} (1-x^2)^s}{\int dx (1-x^4)^{\frac{-s-1}{2}} (1-x^2)^s}$$

quae aequalitas conſpicua eſt ſi $s = 0$; caſibus autem quibus s eſt numerus integer impar, aequalitas non difficulter oſtenditur. Vt ſi fuerit $s = 1$, erit poſterior formula $\frac{\int xxdx : (1+xx)}{\int dx : (1+xx)} = \frac{x - \int dx : (1+xx)}{\int dx : (1+xx)} = \frac{4-\pi}{\pi}$ poſito $x = 1$. Prior vero formula dabit in $\frac{2\int y^3 dy : V(1-y^4)}{\int ydy : V(1-y^4)} - 1 = \frac{4}{\pi} - 1 = \frac{4-\pi}{\pi}$ prorſus vti praecedens. At ſi s numerus par, per euolutionem poteſtatis $(1-xx)^s$ conſenſus ambarum expreſſionum facile perſpicietur.

§. 69. Praeter fractiones autem continuas hactenus erutas forma generalis inuenta innumerabiles alias ſub ſe complectitur; ex quibus nonnullas euoluere expediet. Sit igitur $g = c$, eritque huius fractionis continuae

$$\cfrac{c^2}{s + \cfrac{(c+r)^2}{s + \cfrac{(c+2r)^2}{s + \text{etc.}}}}$$

$$\text{valor} = \frac{c\int x^{c+r-1} dx (1-x^r)^{\frac{s}{r}} : (1-x^{2r})^{\frac{r+s}{2r}}}{\int x^{c-1} dx (1-x^r)^{\frac{s}{r}} : (1-x^{2r})^{\frac{r+s}{2r}}}.$$ Ponatur

$c = 1$, et $r = 1$, eritque

$$\cfrac{1}{s + \cfrac{4}{s + \cfrac{9}{s + \cfrac{16}{s + \text{etc.}}}}} =$$

$$= \frac{\int x\,dx(1-x)^s:(1-xx)^{\frac{s+1}{2}}}{\int dx(1-x)^s:(1-xx)^{\frac{s+1}{2}}}$$, cuius expressionis valores, quos pro variis ipsius s significationibus induit, investigemus. Posito igitur huius expressionis valore $= V$, erit vt sequitur:

$$\text{si } s=0;\ V=\frac{\int xdx:\sqrt{(1-xx)}}{\int dx:\sqrt{(1-xx)}}=\frac{1}{2\int dy:(1+yy)}$$

$$\text{si } s=2;\ V=\frac{2\int dx:\sqrt{(1-xx)}-3\int xdx:\sqrt{(1-xx)}}{2\int xdx:\sqrt{(1-xx)}-\int dx:\sqrt{(1-xx)}}=\frac{1}{2\int y^2dy:(1+yy)}-2$$

$$\text{si } s=4;\ V=\frac{19\int xdx:\sqrt{(1-xx)}-12\int dx:\sqrt{(1-xx)}}{3\int dx:\sqrt{(1-xx)}-4\int xdx:\sqrt{(1-xx)}}=\frac{1}{2\int y^4dy:(1+yy)}-4$$

Generaliter autem erit

$$V=\frac{1}{2\int y^s dy:(1+yy)}-s,$$ ex qua forma apparet, si fuerit s numerus integer par, quadraturam circuli inuolui, contra autem si s impar, logarithmos.

§. 70. Proposita nunc nobis sit haec fractio continua

$$1+\cfrac{1}{2+\cfrac{4}{3+\cfrac{9}{4+\cfrac{16}{5+\cfrac{25}{6+\text{etc.}}}}}}$$

Comparetur haec cum forma §. 64. exhibita, fietque $pqcg=1$; $pq(c+r)(g+r)=4$, $pq(c+2r)(g+2r)=9$; $ap-bq=2$, et $(p-q)r=1$, vnde erit $c=g=r$; $p=\frac{\sqrt{5}+1}{2r}$; $q=\frac{\sqrt{5}-1}{2r}$; $a=\frac{r(1+2\sqrt{5})}{2\sqrt{5}}$ et $b=\frac{r(3\sqrt{5}-1)}{2\sqrt{5}}$, quibus

bus ſubſtitutis habebitur valor propoſitae fractionis continuae

$$= 1 + \frac{(\sqrt{5}-1)\int x^{2r-1}dx(1-x^r)^{\frac{1-\sqrt{5}}{2\sqrt{5}}}(1+\sqrt{5}+(\sqrt{5}-1)x^r)^{\frac{-\sqrt{5}-1}{2\sqrt{5}}}}{2\int x^{r-1}dx(1-x^r)^{\frac{1-\sqrt{5}}{2\sqrt{5}}}(1+\sqrt{5}+(\sqrt{5}-1)x^r)^{\frac{-\sqrt{5}-1}{2\sqrt{5}}}}$$

Ex qua expreſſione ob exponentes ſurdos nihil concludi poteſt notatu dignum.

§. 71. Cum in his fractionibus continuis numeratores partiales ex duobus fractoribus ſint compoſiti, ita nunc ad eiusmodi fractiones continuas pergam, in quibus numeratores hi partiales progreſſionem arithmeticam conſtituant. Fiat igitur, ad §. 50. recurrendo, $\gamma = 0$ et $c = 1$. erit

$$\frac{\int PRdx}{\int Pdx} = \cfrac{a}{b + \cfrac{a+\alpha}{b+\beta + \cfrac{a+2\alpha}{b+2\beta + \cfrac{a+3\alpha}{b+3\beta + \text{etc.}}}}}$$

Oportet autem ſumi $\frac{dS}{S} = \frac{(a-\alpha)dR}{\alpha R} + \frac{(\alpha b-\beta a)dR + \alpha R dR}{\alpha(\beta R-\alpha)} = \frac{(a-\alpha)dR}{\alpha R} + \frac{dR}{\beta} + \frac{(\alpha^2+\alpha\beta b-\beta^2 a)dR}{\alpha 8(\beta R-\alpha)}$, vnde fit $S = Ce^{\frac{R}{\beta}} R^{\frac{a-\alpha}{\alpha}} (\beta R-\alpha)^{\frac{\alpha^2+\alpha\beta b-\beta^2 a}{\alpha\beta\beta}}$.

Ponatur $R = \frac{\alpha x}{\beta}$, erit $S = Ce^{\frac{\alpha x}{\beta\beta}} x^{\frac{a-\alpha}{\alpha}} (1-x)^{\frac{\alpha^2+\alpha\beta b-\beta^2 a}{\alpha\beta\beta}}$ ac $R^{n+1}S$ duplici caſu euaneſcit, poſito ſcilicet tam $x = 0$ quam $x = 1$, modo ſit $\alpha^2 + \alpha\beta b > \beta^2 a$. Hinc ergo erit $Pdx = e^{\frac{\alpha x}{\beta\beta}} x^{\frac{a-\alpha}{\alpha}} dx (1-x)^{\frac{\alpha^2+\alpha\beta b-\alpha\beta^2-\beta^2 a}{\alpha\beta\beta}}$ atque fractionis continuae propoſitae valor $= \frac{\int PRdx}{\int Pdx} =$

$$\frac{\alpha\int e^{\frac{\alpha x}{\beta\beta}} x^{\frac{a}{\alpha}} dx (1-x)^{\frac{\alpha^2+\alpha\beta b-\alpha\beta^2-\beta^2 a}{\alpha\beta\beta}}}{\beta\int e^{\frac{\alpha x}{\beta\beta}} x^{\frac{a-\alpha}{\alpha}} dx (1-x)^{\frac{\alpha^2+\alpha\beta b-\alpha\beta^2-\beta^2 a}{\alpha\beta\beta}}}$$

poſito poſt integrationem $x = 1$.

§. 72.

§. 72. Vt hic casus exemplo illustretur sit $a = 1$, $b = 1$, $\alpha = 1$, et $\beta = 1$, habebitur haec fractio continua

$$\cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \text{etc.}}}}}$$

cuius valor erit $= \frac{\int e^x x\,dx}{\int e^x dx} = \frac{e^x x - e^x + 1}{e^x - 1} = \frac{1}{e - 1}$ posito $x = 1$. Vnde erit $e = 2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \text{etc.}}}}}$

qua expressione satis cito ad valorem numeri e, cuius logarithmus est $= 1$, pertingitur.

§. 73. Ponamus nunc in superiori fractione continua §. 71. data, esse $\beta = 0$, vt sit

$$\frac{\int PR\,dx}{\int P\,dx} = \cfrac{a}{b + \cfrac{a + \alpha}{b + \cfrac{a + 2\alpha}{b + \cfrac{a + 3\alpha}{b + \text{etc.}}}}}$$

erit $\frac{dS}{S} = \frac{(a-\alpha)dR}{\alpha R} - \frac{b\,dR}{\alpha} - \frac{R\,dR}{\alpha}$, hincque $S = CR^{\frac{a-\alpha}{\alpha}} e^{\frac{-2bR-RR}{2\alpha}}$; Duplici nunc casu $R^{n+1}S$ euanescit, quorum alter est si $R = 0$, alter si $R = \infty$, modo sint a et α numeri affirmatiui. Ponatur ergo $R = \frac{x}{1-x}$ eritque $S = Cx^{\frac{a-\alpha}{\alpha}} : (1-x)^{\frac{a-\alpha}{\alpha}} e^{\frac{2bx-(2b-1)xx}{2\alpha(1-x)^2}}$. Ob $dR = \frac{dx}{(1-x)^2}$ erit $\int P\,dx$ =

$$=\int\frac{x^{\frac{a-\alpha}{\alpha}}dx}{(1-x)^{\frac{a+\alpha}{\alpha}}e^{\frac{2bx-(2b-1)xx}{2\alpha(1-x)^2}}}$$ atque $\int PR\,dx =$

$$\int\frac{x^{\frac{a}{\alpha}}dx}{(1-x)^{\frac{a+2\alpha}{\alpha}}e^{\frac{2bx-(2b-1)xx}{2\alpha(1-x)^2}}}.$$

§. 74. Sit denique in §. 50, $a=1$; $c=1$ $\alpha=0$, $\gamma=0$, erit $\frac{\int PR\,dx}{\int P\,dx}=\cfrac{1}{b+\cfrac{1}{b+\beta+\cfrac{1}{b+2\beta+\cfrac{1}{b+3\beta+\text{etc.}}}}}$

atque $\frac{dS}{S}=\frac{R^2dR+(b-\beta)RdR-dR}{\beta R^2}$; vnde fiet $S=e^{\frac{RR+1}{\beta R}}R^{\frac{b-\beta}{\beta}}$, et $P\,dx=e^{\frac{RR+1}{\beta R}}R^{\frac{b-2\beta}{\beta}}dR$ atque $PR\,dx=e^{\frac{RR+1}{\beta R}}R^{\frac{b-\beta}{\beta}}dR$. Oportet autem R talem esse functionem ipsius x, vt R^{n+1} euanescat posito tam $x=0$, quam $x=1$. Eiusmodi autem functionem assignare, opus est multo difficilius, quam pro reliquis casibus. Neque igitur hunc casum eadem methodo resoluere conabor sed eum alii methodo nunc exponendae reseruabo.

§. 75. Huius quidem methodi ad fractiones continuas perueniendi iam ante aliquod tempus feci mentionem, sed quoniam tum casum tantum particularem tractaui, hic eam fusius exponere conueniet. Continetur ea autem non vti praecedens formulis integralibus, sed resolutione aequationis differentialis similis illi, quam quondam Comes Riccati pro-

propoſuit. Conſidero ſcilicet hanc aequationem $ax^m dx + bx^{m+1} y dx + cy^2 dx + dy = 0$, quae ponendo $x^{m+3} = t$ et $y = \frac{1}{cx} + \frac{1}{xxz}$ tranſit in hanc: $\frac{-c}{m+3} t^{\frac{-m-4}{m+3}} dt - \frac{b}{m+3} t^{\frac{-1}{m+3}} z dt - \frac{(ac+b)}{(m+3)c} z^2 dt + dz = 0$, quae ſimilis eſt priori. Quare ſi conſtaret valor ipſius z per t, ſimul y per x innoteſceret. Reducatur autem eodem modo haec aequatio ad aliam ſui ſimilem ponendo $t^{\frac{2m+5}{m+3}} = u$, et $z = \frac{-(m+3)c}{(ac+b)t} + \frac{1}{ttv}$, ac iſtiusmodi reductiones continuentur in infinitum, quo facto ſi omnes valores poſteriores in praecedentibus ſubſtituantur, exprimetur y ſequenti modo.

$$y = Ax^{-1} + \cfrac{1}{-Bx^{-m-1} + \cfrac{1}{Cx^{-1} + \cfrac{1}{-Dx^{-m-1} + \cfrac{1}{Ex^{-1} + \cfrac{1}{-Fx^{-m-1} + \text{etc.}}}}}}$$

litterae vero A, B, C, D, etc. ſequentes obtinebunt valores

$$A = \frac{1}{c}$$

$$B = \frac{(m+3)c}{ac+b}$$

$$C = \frac{(2m+5)(ac+b)}{c(ac-(m+2)b)}$$

$$D = \frac{(3m+7)c(ac-(m+2)b)}{(ac+b)(ac+(m+3)b)}$$

$$E = \frac{(4m+9)(ac+b)(ac+(m+3)b)}{c(ac-(m+2)b)(ac-(2m+4)b)}$$

$$F = \frac{(5m+11)c(ac-(m+2)b)(ac-(2m+4)b)}{(ac+b)(ac+(m+3)b)(ac+(2m+5)b)}$$

etc.

quae determinationes ſimplicius ſequentibus aequationibus comprehenduntur:

AB=

$AB = \frac{m+3}{ac+b}$	$DE = \frac{(3m+7)(4m+9)}{ac-(2m+4)b}$
$BC = \frac{(m+3)(2m+5)}{ac-(m+2)b}$	$EF = \frac{(4m+9)(5m+11)}{ac+(2m+5)b}$
$CD = \frac{(2m+5)(3m+7)}{ac+(m+3)b}$	$FG = \frac{(5m+11)(6m+13)}{ac-(3m+6)b}$
etc.	etc.

76. Si nunc hi valores in fractione continua inuenta substituantur reperietur:

$$cxy = 1 + \cfrac{(ac+b)x^{m+2}}{-(m+3) + \cfrac{(ac-(m+2)b)x^{m+2}}{(2m+5) + \cfrac{(ac+(m+3)b)x^{m+2}}{-(3m+7) + \cfrac{(ac-(2m+4)b)x^{m+2}}{(4m+9)+\text{etc.}}}}}$$

Ex hac expressione patet aequationem propositam absolute esse integrabilem casibus quibus b aequatur termino cuipiam huius progressionis $-ac$; $-\frac{ac}{m+3}$; $-\frac{ac}{2m+5}$; $-\frac{ac}{3m+7}$; etc. $-\frac{ac}{im+2i+1}$ deinde etiam casibus quibus b est terminus huius progressionis: $\frac{ac}{m+2}$; $\frac{ac}{2(m+2)}$; $\frac{ac}{3(m+2)}$; etc. $\frac{ac}{im+2i}$ Fractio autem haec continua aequationis propositae exhibet integrale huius conditionis, vt posito $x=0$, fiat $cxy = 1$, siquidem $m+2>0$; at si $m+2<0$, tum integrale hanc tenet legem vt posito $x=\infty$ fiat $cxy=1$.

§. 77. Ponamus esse $b=0$; atque $a=nc$, ac post integrationem poni $x=1$; proueniet ex hac aequatione $ncx^m dx + cy^2 dx + dy = 0$ sequens fractio continua, qua valor ipsius y definietur, casu quo ponitur $x=1$:

$$y = \frac{1}{c} + \cfrac{n}{-\frac{(m+3)}{c} + \cfrac{n}{\frac{(2m+5)}{c} + \cfrac{n}{-\frac{(3m+7)}{c} + \cfrac{n}{\frac{(4m+9)}{c} + \text{etc.}}}}}$$

siue

siue ponatur $c = \frac{1}{\varkappa}$, ex aequatione $nx^m dx + y^2 dx + \varkappa dy = 0$, valor ipsius y casu quo $x = 1$, ita se hebebit

$$y = \varkappa + \cfrac{n}{-(m\varkappa + 3\varkappa) + \cfrac{n}{(2m\varkappa + 5\varkappa) + \cfrac{n}{-(3m\varkappa + 7\varkappa) + \text{etc.}}}}$$

seu

$$y = \varkappa - \cfrac{n}{m\varkappa + 3\varkappa - \cfrac{n}{2m\varkappa + 5\varkappa - \cfrac{n}{3m\varkappa + 7\varkappa - \cfrac{n}{4m\varkappa + 9\varkappa - \text{etc.}}}}}$$

§. 78. Si ergo proposita sit ista fractio continua

$$b + \cfrac{1}{b + 6 + \cfrac{1}{b + 26 + \cfrac{1}{b + 36 + \cfrac{1}{\text{etc.}}}}}$$

erit $\varkappa = b$; $n = -1$; $(m + 2)b = 6$ seu $m = \frac{6}{b} - 2$. Quare huius fractionis continuae valor erit valor ipsius y casu quo $x = 1$, ex hac aequatione $x^{\frac{6-2b}{b}} dx = y^2 dx + b dy$, integratione ita instituta, vt posito $x = 0$, fiat $xy = b$. cum sit $m + 2 > 0$, si quidem $\frac{6}{b}$ sit numerus affirmatiuus.