
DE PRODUCTIS
EX INFINITIS FACTORIBVS ORTIS.

AUCTORE

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§. I.

Cum in Analyfi ad eiusmodi quantitates perueni-
tur, quae numeris nec rationalibus nec irratio-
nalibus exponi possunt, expressiones infinitae
ad eas quantitates denotandas adhiberi solent: quae eo
magis idoneae sunt censendae, quo citius earum ope ad
cognitionem et aestimationem quantitatum iis expressarum
peruenitur. Huiusmodi igitur expressionum maximus et
amplissimus est vsus ad valores quantitatum transcendentium,
cuiusmodi sunt logarithmi, arcus circulares, aliaeque per
quadraturas curvarum determinatae quantitates, representan-
dos earumque beneficio ad tam exactam cum logarithmo-
rum, tum arcuum circularium, tum etiam plurium alia-
rum quantitatum transcendentium cognitionem pertigimus.
Quin etiam istiusmodi expressiones infinitae insignem af-
ferunt vtilitatem ad quantitates irrationales, et radices ae-
quationum algebraicarum per numeros rationales vero pro-
xime definiendas; quae si vsus spectetur veris expressioni-
bus plerumque longe sunt anteferendae.

§. 2. Huiusmodi autem expressionum infinitarum non-
nulla genera inter se maxime diuersa sunt constituenda, quo-
rum primum in se complectitur omnes series infinitas, in-

finitis terminis signis + vel - iunctis constantes, quae doctrina nunc quidem iam tantopere est excolta, ut non solum plures habeantur methodi quasvis quantitates tam algebraicas quam transcendentes huiusmodi seriebus infinitis exprimendi, sed etiam proposita serie infinita inuestigandi, cuiusmodi quantitas ea indicetur. Duplici enim modo expressiones infinitas cuiusque generis tractari oportet, quorum alter in conuersione quantitatum vel algebraicarum vel transcendentium in expressiones infinitas consistit; alter vero in indagatione illius quantitatis, quam proposita expressio infinita designat, vicissim versatur.

§. 3. Ad alterum genus expressionum infinitarum referri conuenit eas, quae ex innumerabilibus factoribus constant, cuiusmodi expressiones, quamquam iam complures sunt inuentae ac cognitae, tamen nec modus ad eas perueniendi, nec via earum valores dignoscendi vsquam est exposita. Aequae autem dignae huius generis expressiones infinitae videntur, quae excolantur, ac priores ex infinito terminorum numero constantes, neque forte minus commodi Analyfi afferetur earum pertractatione. Praeterquam enim, quod istiusmodi expressiones naturam quantitatum quas referunt satis distincte ob oculos ponant, et saepe numero ad valores proximos inueniendos perquam sunt accommodatae, insignem praestant usum ad logarithmos ipsarum quantitatum formandos, id quod in calculo saepissime summam affert utilitatem. Sic si quantitas quaecunque X transformata fuerit in istiusmodi expressionem $\frac{a}{\alpha} \cdot \frac{b}{\beta} \cdot \frac{c}{\gamma} \cdot \frac{d}{\delta} \cdot \frac{e}{\epsilon}$, etc. statim habebitur logarithmus quantitatis $X = \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} + \frac{d}{\delta} + \frac{e}{\epsilon} + \text{etc.}$ quae series eo magis con-

vergit

vergit, quo propius factores illi ad unitatem inclinant. Hanc ob causam constitui in hac dissertatione theoriam huiusmodi expressionum infinitarum; quantum quidem observationes meae subsidii suppeditaverunt, inchoare, quo aliis facilius sit eam aliquando magis perficere.

§ 4. Primus eiusmodi expressionem infinitis factoribus contentam protulit Wallisus in Arithmetica infinitorum, ubi ostendit, si circuli diameter sit = 1. fore aream circuli $\frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11}$ etc. quam expressionem deduxit ex interpolatione seriei $\frac{2}{3} + \frac{2 \cdot 4}{3 \cdot 5} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} +$ etc. cuius terminos intermedios demonstrauerat a circuli quadratura pendere. Cum igitur istae expressiones interpolationi serierum originem suam debeant, non in congruum fore visum est tractationem hanc de productis ex infinitis factoribus constantibus ab interpolationibus incipere. Cum enim in Tomo quinto Commentariorum nostrorum methodum tradidissim interpolationes per quadraturas curvarum perficiendi, simul constabit, cuiusmodi quantitatem transcendentem producta infinita hac ratione orta exhibeant.

§. 5. Considero igitur sequentem progressionem $f+g + (f+g)^2 + (f+g)^3 + (f+g)^4 + \dots$ cuius quilibet terminus, cuius index est n , inuenitur ex praecedente hunc per $f+ng$ multiplicando: ostendi autem in dissertatione allegata huius seriei terminum, cuius index est n esse
$$\frac{g^{n+1} \int dx (-lx)^n}{(f+(n+1)g) \int x^{f:g} dx (1-x)^n}$$
 vtraque integratione ita peracta, vt integralia evanescantposito $x = 0$, tumquae facta $x = 1$. Quamobrem ista expressio simul indicabit, a quam quadratura singuli termini

mini intermedii pendeant. Quoniam enim si n fit numerus fractus, non ita facile constat, qualem quadraturam $\int dx(-lx)^n$ contineat, tamen eodem loco ostendi posito $\frac{p}{q}$ loco n formulam $\int dx(-lx)^{\frac{p}{q}}$ congruere cum $\sqrt[q]{(1.2.3. \dots . p) \cdot (\frac{2p}{q} + 1)(\frac{3p}{q} + 1)(\frac{4p}{q} + 1) \dots (\frac{qp}{q} + 1)}$ $\int dx(x-xx)^{\frac{p}{q}} \int dx(x^2-x^3)^{\frac{p}{q}} \int dx(x^3-x^4)^{\frac{p}{q}} \int dx(x^4-x^5)^{\frac{p}{q}} \dots \int dx(x^{q-1}-x^q)^{\frac{p}{q}}$ cuius reductionis ope valor ipsius $\int dx(-lx)^{\frac{p}{q}}$ per quadraturas curvarum algebraicarum exprimi potest.

§ 6. Si nunc in serie assumpta terminus, cujus index est $\frac{1}{2}$, ponatur z , ex lege seriei termini, quorum indices sunt $\frac{3}{2}, \frac{5}{2}, \frac{7}{2}$, etc. sequenti modo se habebunt:

$$z + z(f + \frac{3}{2}g) + z(f + \frac{3}{2}g)(f + \frac{5}{2}g) + z(f + \frac{3}{2}g)(f + \frac{5}{2}g)(f + \frac{7}{2}g) \text{ etc.}$$

Quoniam autem progressio assumpta tandem cum Geometrica confunditur, hi termini interpolati euadent tandem medii proportionales inter contiguos seriei terminos. Quare si singuli termini interpolati iam ab initio tanquam medii proportionales spectentur, sequentes prodibunt approximationes ad terminum z , cuius index est $\frac{1}{2}$.

$$\text{I. } z = \sqrt{f+g}$$

$$\text{II. } z = \sqrt{\frac{(f+g)(f+g)(f+2g)}{1(f+\frac{3}{2}g)(f+\frac{3}{2}g)}}$$

$$\text{III. } z = \sqrt{\frac{(f+g)(f+g)(f+2g)(f+2g)(f+3g)}{1(f+\frac{3}{2}g)(f+\frac{3}{2}g)(f+\frac{5}{2}g)(f+\frac{5}{2}g)}} \text{ etc.}$$

ex qua progressionis lege intelligitur terminum indicis $\frac{1}{2}$ vere esse $= (f+g)^{\frac{1}{2}} \sqrt{\frac{(f+g)(f+2g)(f+2g)(f+3g)}{(f+\frac{3}{2}g)(f+\frac{3}{2}g)(f+\frac{5}{2}g)(f+\frac{5}{2}g)(f+3g)}}$

$$\frac{(f+3g)(f+4g)(f+4g)(f+5g)(f+5g)(f+6g)}{(f+\frac{7}{2}g)(f+\frac{7}{2}g)(f+\frac{9}{2}g)(f+\frac{9}{2}g)(f+\frac{11}{2}g)(f+\frac{11}{2}g)} \text{ etc.}$$

§. 7. Nunc igitur non solum certum est hac expressione infinita terminum seriei assumtae

$(f+g) + (f+g)^2 + (f+g)^3 + (f+g)^4 + (f+g)^5 + (f+g)^6 + \text{etc.}$
 cuius index est $= \frac{1}{2}$, exhiberi, sed etiam eadem expressio inuenta ad quadraturas curuarum reducitur. Posito enim $n = \frac{1}{2}$, ob $p = 1$. et $q = 2$. fit $\int dx(-lx)^{\frac{1}{2}} = \sqrt{1-x}$.
 $\int dx \sqrt{x-xx}$; quae expressio debito modo integrata dat radicem quadratam ex area circuli cuius diameter est $= 1$:
 vel posita $1 : \pi$ ratione diametri ad peripheriam, erit $\int dx(-lx)^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}}$. Hinc ergo idem terminus, cuius index

$$= \frac{1}{2}, \text{ quem posuimus } z \text{ reperitur } = \frac{g\sqrt{\pi g}}{(2f+3g)x^{f+g}dx\sqrt{1-x}}$$

$= \frac{\sqrt{\pi g}}{(2f+3g)y^{f+g-1}dy\sqrt{1-y^g}}$; integrali hoc eodem tractato modo, quo ante ratione variabilis x est praescriptum. At

per reductionem formularum huius modi integralium est $\int y^{f+g-1}dy\sqrt{1-y^g} = \frac{2fg}{(2f+g)(2f+3g)} \int \frac{y^{f-1}dy}{\sqrt{1-y^g}} =$

$$\frac{2f}{2f+3g} \int y^{f-1}dy\sqrt{1-y^g}. \text{ His substitutis reperitur}$$

$$\frac{(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)(2f+7g)}{(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+6g)(2f+6g)} \text{ etc.}$$

$$= \frac{2f(2f+g)}{\pi g} \left(\int y^{f-1}dy\sqrt{1-y^g} \right)^2 = \frac{2fg}{\pi(2f+3g)} \left(\frac{\int y^{f-1}dy}{\sqrt{1-y^g}} \right)^2.$$

Per hanc igitur aequationem innumerabiles quadraturae in factores infinitos, et vicissim huiusmodi factorum infinitorum valores in quadraturas curuarum transformari possunt.

§. 8.

§. 8. Vt hanc aequalitatem exemplis illustremus, fit $g = 1$, eritque $\int y^{f-1} dy \sqrt{1-y} = \frac{2 \cdot 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \dots (2f-2)}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \dots (2f-1)}$. Vnde fiet $\frac{2f(2f+1)2 \cdot 2 \cdot 2 \cdot 4 \cdot 4 \dots (2f-2)(2f-2)}{\pi \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \dots (2f+1)(2f+1) \cdot (2f+2)(2f+2)(2f+4)}$ etc. quae expressio ordinata seu ad continuitatem reducta dat $\pi = 4 \cdot \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11}$ etc. quae est ipsa formula Wallisiana, prodiitque quicumque numerus integer affirmatiuus loco f substituatur. Haec eadem expressio autem prodit si ponatur $g = 2$. et $f =$ numero cuiusque impari integro.

§. 9. Cum igitur fit $\frac{f g}{\pi} \left(\frac{\int y^{f-1} dy}{\sqrt{1-y^g}} \right)^2 =$
 $\frac{(2f+g)(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)}{2f(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+6g)}$ etc.
 erit pari modo $\frac{bk}{\pi} \left(\frac{\int y^{b-1} dy}{\sqrt{1-y^k}} \right)^2 =$
 $\frac{(2b+k)(2b+k)(2b+3k)(2b+3k)(2b+5k)(2b+5k)}{2b(2b+2k)(2b+2k)(2b+4k)(2b+4k)(2b+6k)}$ etc.

Quare illa expressio per hanc diuisa obtinebitur sequens aequatio libera a peripheria circuli π
 $\frac{fg(\int y^{f-1} dy \cdot \sqrt{1-y^g})^2}{bk(\int y^{b-1} dy \cdot \sqrt{1-y^k})^2} = \frac{2b(2f+g)(2b+2k)(2f+3g)(2b+4k)(2f+5g)^2}{2f(2b+k)(2f+2g)(2b+3k)(2f+4k)(2b+5k)^2}$
 etc. Quae radice quadrata extracta praebet hanc aequationem
 $\frac{\int y^{f-1} dy \cdot \sqrt{1-y^g}}{\int y^{b-1} dy \cdot \sqrt{1-y^k}} \cdot \sqrt{\frac{g}{k}} = \frac{2b(2f+g)(2b+2k)(2f+3g)(2b+4k)(2f+5g)}{2f(2b+k)(2f+2g)(2b+3k)(2f+4k)(2b+5k)}$ etc.

§. 10. Haec autem expressio infinita valorem constantem non habet, nam etiamsi in infinitum continetur, tamen alium habet valorem, si numerus factorum capiatur par, alium si numerus impar. Quamobrem nisi fit $k=g$,
 quo

quo casu perinde est, vbi multiplicatio abrumptur, bini factores coniunctim sunt accipiendi, quo facto binæ obtinebuntur aequationes, prout numerus factorum capiatur par siue impar. Primo autem accurate euoluta expressione generali obtinebitur:

$$\frac{g \int y^{f-1} dy \cdot \mathcal{V}(1-y^g)}{k \int y^{b-1} dy \cdot \mathcal{V}(1-y^k)} = \frac{2b(2f+g)(2b+2k)(2f+3g) \dots (2b+4k)(2f+5g)}{2f(2b+k)(2f+2g)(2b+3k) \dots (2f+4g)(2b+5k)} \cdot \frac{(2b+5k)(2f+6g)}{(2f+5g)(2b+7k)} \text{ etc.}$$

Sumendis autem alteris terminorum paribus erit $\frac{\int y^{f-1} dy \cdot \mathcal{V}(1-y^g)}{h \int y^{b-1} dy \cdot \mathcal{V}(1-y^k)} = \frac{(2f+g)(2b+2k)(2f+3g)(2b+4k)}{(2b+k)(2f+2g)(2b+3k)(2f+4g)} \cdot \frac{(2f+5g)(2b+5k)}{(2b+5k)(2f+6g)} \cdot \frac{(2f+7g)(2b+7k)}{(2b+7k)(2f+8g)}$ etc. in quibus expressio- nibus loca, vbi operationem abrumpere licet, punctis sunt distincta.

§. 11. Consideremus autem attentius casum, quo est $k = g$ quippe quo expressio infinita tanquam ex simplicibus factoribus constans concipi potest; eritque

$$\frac{\int y^{f-1} dy \cdot \mathcal{V}(1-y^g)}{\int y^{b-1} dy \cdot \mathcal{V}(1-y^g)} = \frac{2b(2f+g)(2b+2g)(2f+3g)(2b+4g)}{2f(2b+g)(2f+2g)(2b+3g)(2f+4g)}$$

quae expressio, quo minus cum praecedente ob easdem litteras confundatur, ponamus hic $2f = a$ et $2b = b$ atque $y = x^2$, quo substituto prodibit

$$\frac{\int x^{a-1} dx \cdot \mathcal{V}(1-x^{2g})}{\int x^{b-1} dx \cdot \mathcal{V}(1-x^{2g})} = \frac{b(a+g)(b+2g)(a+3g)(b+4g)(a+5g)}{a(b+g)(a+2g)(b+3g)(a+4g)(b+5g)}$$

etc. quae expressio cum priori §. 9. data, quae facto pariter $y = x^2$, transit in hanc

$$\frac{4fg \left(\frac{\int x^{a-1} dx}{\mathcal{V}(1-x^{2g})} \right)^2}{\pi} = \frac{(2f+g)(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)}{2f(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+6g)}$$

etc. comparata, insignes manifestabit proprietates, quarum veritas alias vix ostendi poterit.

§. 12. Statim enim patet si ponatur $a = 2f$; et $b = 2f + g$, illam expressionem infinitam in hanc transmutari; quamobrem etiam expressiones illis aequales, quadraturas

curvarum continentes, hoc casu fient aequales, ex quo sequens emergit aequalitas: $\frac{\int x^{2f-1} dx : \sqrt{(1-x^{2g})}}{\int x^{2f+g-1} dx : \sqrt{(1-x^{2g})}} = \frac{4fg}{\pi}$

$(\int x^{2f-1} dx : \sqrt{(1-x^{2g})})^2$, si quidem ponatur post integrationem $x=1$. Hinc igitur sequitur fore $\pi = 4fg \frac{\int x^{2f-1} dx}{\sqrt{(1-x^{2g})}} \cdot \frac{\int x^{2f+g-1} dx}{\sqrt{(1-x^{2g})}}$: siue posito $2f=a$, erit π

$$= 2ag \frac{\int x^{a-1} dx}{\sqrt{(1-x^{2g})}} \cdot \frac{\int x^{a+g-1} dx}{\sqrt{(1-x^{2g})}}$$

quod sane est theorema maxime notatu dignum, cum eius beneficio productum duorum integralium, quorum saepissime neutrum exhiberi potest, assignari queat.

§. 13. Veritas huius theorematis quidem facile declaratur iis casibus, quibus altera formula integralis vel absolute integrationem admittit vel a circuli quadratura pendet. Ponamus enim $g=1$, et $a=1$; utique erit $\pi = 2 \int \frac{dx}{\sqrt{(1-x^2)}} \cdot \int \frac{xdx}{\sqrt{(1-x^2)}}$ nam $2 \int \frac{dx}{\sqrt{(1-x^2)}}$ posito post integrationem $x=1$ dat ipsam quantitatem π ; atque $\int \frac{xdx}{\sqrt{(1-xx)}}$ $= 1 - \sqrt{(-xx)}$ facto $x=1$ fit $= 1$. Simili modo si $a=2$ manente $g=1$ perspicitur fore $\pi = 4 \int \frac{xdx}{\sqrt{(1-xx)}} \cdot \int \frac{xxdx}{\sqrt{(1-xx)}}$ nam est $\int \frac{xdx}{\sqrt{(1-xx)}} = 1$, et $\int \frac{xxdx}{\sqrt{(1-xx)}} = \frac{\pi}{4}$; quibus casibus theorematis veritas aliunde cognita, confirmatur.

§. 14. Reliqui autem casus, quibus neutra quantitas integralis vel actu vel per quadraturam circuli exhiberi potest, totidem praebent theoremata maxime abstrusae indaginis. Ita posito $g=2$ et $a=1$ fiet $\pi = 4 \int \frac{dx}{\sqrt{(1-x^4)}} \cdot \int \frac{xxdx}{\sqrt{(1-x^4)}}$; vbi $\int \frac{xxdx}{\sqrt{(1-x^4)}}$ exhibet applicatam in curva elasti-

ea rectangula, $\int \frac{dx}{\sqrt{(1-x^2)}}$ vero arcum elasticae abscissae x respondentem. Quocirca rectangulum ex arcu elasticae abscissae x respondente et applicata respondente aequabitur areae circuli, cuius diameter est abscissa illa x ; quae proprietas elasticae fortasse alia methodo vix ac ne vix quidem cognosci demonstrarique poterit.

§. 15. Antequam autem hunc elasticae casum relinquam, iuuabit vtrumque integrale per seriem ordinariam exprimere casu saltem quo $x=1$. Cum enim sit $\frac{1}{\sqrt{(1-x^2)}} = \frac{(1+x^2)^{-\frac{1}{2}}}{V(1-x^2)}$ atque $(1+x^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \text{etc.}$ singula membra a circuli quadratura pendebunt. Absoluta autem vtraque integration pro casu $x=1$ erit $\int \frac{dx}{\sqrt{(1-x^2)}} = \frac{\pi}{2} (1 - \frac{1}{4} + \frac{1 \cdot 9}{4 \cdot 16} - \frac{1 \cdot 9 \cdot 25}{4 \cdot 16 \cdot 36} + \text{etc.})$ atque $\int \frac{x^2 dx}{\sqrt{(1-x^2)}} = \frac{\pi}{2} (\frac{1}{2} - \frac{1 \cdot 3}{4 \cdot 4} + \frac{1 \cdot 9 \cdot 5}{4 \cdot 16 \cdot 6} - \frac{1 \cdot 9 \cdot 25 \cdot 7}{4 \cdot 16 \cdot 36 \cdot 8} + \text{etc.})$ Hinc autem approximando prodit tam prope $\int \frac{dx}{\sqrt{(1-x^2)}} = \frac{5}{8} \frac{\pi}{2}$ et $\int \frac{x^2 dx}{\sqrt{(1-x^2)}} = \frac{3}{8}$.

§. 16. Si fuerit $a=1$ erit $\pi = 2g \int \frac{dx}{\sqrt{(1-x^{2g})}} \cdot \int \frac{x^g dx}{\sqrt{(1-x^{2g})}}$ quae duae expressiones integrales ita sunt comparatae, vt si fuerit $\int \frac{x^g dx}{\sqrt{(1-x^{2g})}}$ applicata curuae cuiusdam abscissae x respondens, futura sit $\int \frac{dx}{\sqrt{(1-x^{2g})}}$ ipsa [eiusdem] curuae longitudo. Quamobrem si in hac curua sumatur abscissa $x=1$, erit productum seu rectangulum ex applicata in longitudinem curuae ad aream circuli, cuius diameter est abscissa $x=1$, vti se habet 2 ad numerum g ; quae propositio

positio locum habet, dummodo g fuerit numerus affirmatiuus; valores negatiui enim sponte excipiuntur.

§. 17. Si $a-1$ minor accipiatur quam g , ita ut numeri a et g sint primi inter se, sequentia habebuntur theoremata notatu digna; nam si $a+g-1 > 2g$ tum integratio ad formulam simpliciozem reduci posset.

$\pi = 2 \int \frac{dx}{\sqrt{(1-x^2)}} \cdot \int \frac{x dx}{\sqrt{(1-x^2)}}$	$\pi = 30 \int \frac{x^2 dx}{\sqrt{(1-x^{10})}} \cdot \int \frac{x^7 dx}{\sqrt{(1-x^{10})}}$
$\pi = 4 \int \frac{dx}{\sqrt{(1-x^4)}} \cdot \int \frac{x^2 dx}{\sqrt{(1-x^4)}}$	$\pi = 40 \int \frac{x^5 dx}{\sqrt{(1-x^{10})}} \cdot \int \frac{x^5 dx}{\sqrt{(1-x^{10})}}$
$\pi = 6 \int \frac{dx}{\sqrt{(1-x^6)}} \cdot \int \frac{x^3 dx}{\sqrt{(1-x^6)}}$	$\pi = 12 \int \frac{dx}{\sqrt{(1-x^{12})}} \cdot \int \frac{x^6 dx}{\sqrt{(1-x^{12})}}$
$\pi = 12 \int \frac{x dx}{\sqrt{(1-x^6)}} \cdot \int \frac{x^4 dx}{\sqrt{(1-x^6)}}$	$\pi = 60 \int \frac{x^4 dx}{\sqrt{(1-x^{12})}} \cdot \int \frac{x^{10} dx}{\sqrt{(1-x^{12})}}$
$\pi = 8 \int \frac{dx}{\sqrt{(1-x^8)}} \cdot \int \frac{x^4 dx}{\sqrt{(1-x^8)}}$	$\pi = 14 \int \frac{dx}{\sqrt{(1-x^{14})}} \cdot \int \frac{x^7 dx}{\sqrt{(1-x^{14})}}$
$\pi = 24 \int \frac{x^2 dx}{\sqrt{(1-x^8)}} \cdot \int \frac{x^6 dx}{\sqrt{(1-x^8)}}$	$\pi = 28 \int \frac{x dx}{\sqrt{(1-x^{14})}} \cdot \int \frac{x^8 dx}{\sqrt{(1-x^{14})}}$
$\pi = 10 \int \frac{dx}{\sqrt{(1-x^{10})}} \cdot \int \frac{x^5 dx}{\sqrt{(1-x^{10})}}$	$\pi = 42 \int \frac{x^2 dx}{\sqrt{(1-x^{14})}} \cdot \int \frac{x^9 dx}{\sqrt{(1-x^{14})}}$
$\pi = 20 \int \frac{x dx}{\sqrt{(1-x^{10})}} \cdot \int \frac{x^6 dx}{\sqrt{(1-x^{10})}}$	$\pi = 56 \int \frac{x^3 dx}{\sqrt{(1-x^{14})}} \cdot \int \frac{x^{10} dx}{\sqrt{(1-x^{14})}}$
	$\pi = 70 \int \frac{x^4 dx}{\sqrt{(1-x^{14})}} \cdot \int \frac{x^{11} dx}{\sqrt{(1-x^{14})}}$

§. 18. Hoc ipso igitur inuento reductio etiam formularum integralium ad simpliciores insigniter est promotata. Cum enim adhuc duae istae formulae $\int \frac{x^m dx}{\sqrt{(1-x^{2g})}}$ et

$\int \frac{x^{m+n} dx}{\sqrt{(1-x^{2g})}}$ ad se inuicem tantum reduci potuissent, si n erat multipulum exponentis $2g$; ita nunc reductio etiam succedit, si n tantum ipsius g fuerit multipulum: casu intellige, quo fit $x=1$. Quemadmodum autem si n est productum exponentis g per numerum parem, quotus, qui resultat ex diuisione alterius formulae per alteram, facile assignatur, ita e contrario, si n sit factum ex g in numerum

merum imparem, tum productum formularum facillime assignatur.

§. 19. Haec omnia ergo huc redeunt, ut si cognitum fuerit integrale formulae $\frac{\int x^m dx}{\sqrt{(1-x^{2g})}}$ casu quo $x=1$, eodem casu etiam huius formulae $\frac{\int x^{m+n} dx}{\sqrt{(1-x^{2g})}}$ integrale, si fit n multipum ipsius g , exhiberi queat. Sit enim A integrale formulae $\frac{\int x^m dx}{\sqrt{(1-x^{2g})}}$ casu, quo est $x=1$; integralia alterius formulae, ponendo $g, 2g, 3g,$ etc. successive loco n sequenti modo se habebunt.

$$\begin{aligned} \int \frac{x^m dx}{\sqrt{(1-x^{2g})}} &= A \\ \int \frac{x^{m+g} dx}{\sqrt{(1-x^{2g})}} &= \frac{\pi}{2(m+1)gA} \\ \int \frac{x^{m+2g} dx}{\sqrt{(1-x^{2g})}} &= \frac{(m+1)A}{m+g+1} \\ \int \frac{x^{m+3g} dx}{\sqrt{(1-x^{2g})}} &= \frac{(m+g+1)\pi}{2(m+1)(m+2g+1)gA} \\ \int \frac{x^{m+4g} dx}{\sqrt{(1-x^{2g})}} &= \frac{(m+1)(m+g+1)A}{(m+g+1)(m+3g+1)} \\ \int \frac{x^{m+5g} dx}{\sqrt{(1-x^{2g})}} &= \frac{(m+g+1)(m+2g+1)\pi}{2(m+1)(m+2g+1)(m+4g+1)gA} \text{ etc.} \end{aligned}$$

§. 20. Cum deinde haec formula generalis $\int x^{m+ig} dx (1-x^{2g})^{k-\frac{i}{2}}$ denotantibus i et k numeros integros quoscunque, reduci queat ad hanc formulam $\int \frac{x^{m+ig} dx}{\sqrt{(1-x^{2g})}}$, intel-

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igitur illius formulae latissime patentis $\int x^{m+ig} dx (1-x^{2g})^{\frac{p-1}{2}}$ integrale assignari posse ex integrali $\int \frac{x^m dx}{\sqrt{(1-x^{2g})}}$ cognito, casu faltem, quo post integrationem fit $x=1$. Casus autem, quibus i est numerus impar, praeter hoc integrale etiam circuli quadraturam π requirunt.

§. 21. Quemadmodum igitur per terminum indicis $\frac{p}{q}$ seriei supra §. 5. assumtae ad istas formularum integralium comparationes sum deductus, ita operae pretium forte erit alios terminos intermedios simili modo inuestigare. Quaeratur igitur terminus cuius index est $\frac{p}{q}$ qui ponatur $= z$, ex quo sequentes ita se habebunt

$$z + \frac{\frac{p}{q}}{z(fg + g(p+q))} + \frac{\frac{p+2q}{q}}{z(fg + (p+q)g)(fg + (p+2q)g)}, + \text{etc.}$$

Considerando nunc pari modo, quod haec progressio tandem in geometricam abeat, sequentes orientur approximationes ad terminum z .

$$\text{I. } z = 1 (f+g)^{\frac{p}{q}}$$

$$\text{II. } \frac{z(fg + (p+q)g)}{q} = (f+g)^{\frac{q-p}{q}} (f+g)^{\frac{p}{q}} (f+2g)^{\frac{p}{q}}$$

$$\text{III. } z(f + \frac{(p+q)g}{q})(f + \frac{(p+2q)g}{q}) = (f+g)^{\frac{q-p}{q}} (f+g)^{\frac{p}{q}}$$

$$(f+2g)^{\frac{q-p}{q}} (f+2g)^{\frac{p}{q}} (f+3g)^{\frac{p}{q}} \text{ Hinc igitur eli}$$

$$\text{cietur verus valor ipsius } z = \frac{(f+g)^{\frac{p}{q}} (f+g)^{\frac{q-p}{q}}}{1 (f + \frac{(p+q)g}{q})^{\frac{p}{q}}}$$

$$1 (f + \frac{(p+q)g}{q})^{\frac{p}{q}}$$

(f +

$(f+2g)^{\frac{p}{q}} \cdot (f+2g)^{\frac{q-p}{q}} \cdot (f+3g)^{\frac{p}{q}} \cdot (f+3g)^{\frac{q-p}{q}}$
 $(f+\frac{(p+q)}{q}g)^{\frac{(q-p)}{q}} \cdot (f+\frac{(p+q)}{q}g)^{\frac{p}{q}} \cdot f+\frac{(p+q)}{q}g)^{\frac{q-p}{q}} \cdot (f+\frac{(p+q)}{q}g)^{\frac{p}{q}}$
 etc. Vel paucis mutatis, vt factores infinitesimi fiant
 $= 1$, et expressio vbi libuerit abrumpi queat

erit
$$\frac{z}{(f+\frac{p}{q}g)^{\frac{p}{q}}} = \frac{(f+g)^{\frac{p}{q}} (f+g)^{\frac{q-p}{q}}}{(f+\frac{p}{q}g)^{\frac{p}{q}} (f+\frac{(p+q)}{q}g)^{\frac{q-p}{q}}}$$

$(f+2g)^{\frac{p}{q}} \cdot (f+2g)^{\frac{q-p}{q}} \cdot (f+3g)^{\frac{p}{q}}$
 $(f+\frac{(p+q)}{q}g)^{\frac{p}{q}} \cdot (f+\frac{(p+q)}{q}g)^{\frac{q-p}{q}} \cdot (f+\frac{(p+q)}{q}g)^{\frac{p}{q}}$ etc. cuius
 expressionis lex, qua factores progrediuntur, sponte elucet.

§. 22. Eiusdem autem termini intermedii z valor ope termini generalis huius seriei exprimi potest, fiet enim $z =$

$$\frac{g^{\frac{p+q}{q}} \int dx (-lx)^{\frac{p}{q}}}{(f+\frac{(p+q)}{q}g) \int x^{f+g} dx (1-x)^{\frac{p}{q}}}$$

Quare si ponatur $\int dx (-lx)^{\frac{p}{q}} =$

$$= \sqrt[q]{(1 \cdot 2 \cdot 3 \dots p) (\frac{2p}{q} + 1) (\frac{3p}{q} + 1) (\frac{4p}{q} + 1) \dots (\frac{qp}{q} + 1)}$$

$$\int dx (x-x^2)^{\frac{p}{q}} \cdot \int dx (x^2-x^3)^{\frac{p}{q}} \cdot \int dx (x^3-x^4)^{\frac{p}{q}} \dots \int dx (x^{q-1}-x^q)^{\frac{p}{q}}$$

$= \sqrt[q]{P}$: atque $x = y^g$, quo fit $\int x^{f+g} dx (1-x)^{\frac{p}{q}} = g \int y^{f+g-1} dy$

$$(1-y^g)^{\frac{p}{q}} = \frac{g g^{\frac{p}{q}}}{g + (p+q)g} \int y^{f+g-1} dy = \frac{p f g}{q (f+\frac{p}{q}g) (f+\frac{(p+q)}{q}g)} (1-y^g)^{\frac{p}{q}}$$

fy

$$\frac{\int y^{f-1} dy}{(1-y^g)^{\frac{q-p}{q}}}. \quad \text{Ponatur porro } \frac{\int y^{f-1} dy}{(1-y^g)^{\frac{q-p}{q}}} = Q, \text{ erit } z =$$

$$\frac{q(f + \frac{p}{q}g)^p P^{\frac{x}{q}}}{pfg^{\frac{q-p}{q}} Q}$$

§. 23. Substituta nunc loco z superiore expressione infinita, sumtisque potestatibus exponentis q , prodibit ista

$$\text{aequatio: } \frac{q^q P}{p^q f^p g^{q-p} Q^q} = \frac{f^{q-p}}{(f + \frac{p}{q}g)^{q-p}} \cdot \frac{(f+g)^p}{(f + \frac{p}{q}g)^p} \\ \frac{(f+g)^{q-p}}{(f + \frac{p+q}{q}g)^{q-p}} \cdot \frac{(f+2g)^p}{(f + \frac{p+q}{q}g)^p} \cdot \frac{(f+2g)^{q-p}}{(f + \frac{p+q}{q}g)^{q-p}} \text{ etc.}$$

$$\text{Si igitur pari modo ponatur } \frac{\int y^{b-1} dy}{(1-y^g)^{\frac{q-p}{q}}} = R, \text{ erit}$$

$$\frac{p^q b^p g^{q-p} R^q}{q^q P} = \frac{(b + \frac{p}{q}g)^{q-p}}{b^{q-p}} \cdot \frac{(b + \frac{p}{q}g)^p}{(b+g)^p} \cdot \frac{(b + \frac{p+q}{q}g)^{q-p}}{(b+g)^{q-p}} \text{ etc.}$$

quae duae expressiones in se mutuo ductae dabunt $\frac{b^p R^q}{f^p Q^q}$

$$= \frac{f^{q-p} (b + \frac{p}{q}g)^q (f+g)^q (b + \frac{p+q}{q}g)^q (f+2g)^q}{b^{q-p} (f + \frac{p}{q}g)^q (b+g)^q (f + \frac{p+q}{q}g)^q (b+2g)^q}$$

$$\frac{(b + \frac{p+q}{q}g)^q}{(f + \frac{p+q}{q}g)^q} \text{ etc.}$$

$$\frac{(f + \frac{p+q}{q}g)^q}{(f + \frac{p+q}{q}g)^q}$$

§. 24. Si ergo vtrunque multiplicetur per $\frac{f^p}{b^p}$ atque radix potestatis q extrahatur reperietur $\frac{R}{Q} = \frac{f(b + \frac{p}{q}g)}{b(f + \frac{p}{q}g)}$

$$\frac{(f+g)(b + \frac{(p+q)}{q}g)(f+2g)(b + \frac{(p+2q)}{q}g)}{(b+g)(f + \frac{(p+q)}{q}g)(b+2g)(f + \frac{(p+q)}{q}g)} \text{ etc.} =$$

$$\frac{\int y^{b-1} dy (1-y^{\frac{p-q}{q}})^{\frac{p-q}{q}}}{\int y^{f-1} dy (1-y^{\frac{p-q}{q}})^{\frac{p-q}{q}}}, \text{ in quibus integralibus cum ita fue-}$$

rint accepta, vt euanescant posito $y=0$, fieri debet $y=1$, quo facto habebitur per quadraturas valor expressionis infinitae propositae. Ope huius igitur expressionis infinitae altera quadratura ad alteram, siquidem ponatur $y=1$, reduci poterit.

§. 25. Vt autem hinc eiusmodi integralium comparationes deducamus, sicuti ex priori casu, quo erat $p=1$ et $q=2$, ponamus hic $p=1$ et $q=3$; fietque $P = \frac{1}{2}$

$$\int dx(x-x^2)^{\frac{1}{2}} \cdot \int dx(x^2-x^3)^{\frac{1}{2}} \text{ et } Q = \frac{\int y^{f-1} dy}{(1-y^{\frac{2}{3}})^{\frac{2}{3}}} \text{ atque } R =$$

$$\frac{\int y^{b-1} dy}{(2-y^{\frac{2}{3}})^{\frac{2}{3}}}. \text{ Erit ergo } \frac{27P}{f^2 Q^2} = \frac{f \cdot (f + \frac{1}{3}g)}{(f + \frac{1}{3}g)(f + \frac{2}{3}g)(f + \frac{1}{3}g)}$$

$$\frac{(f+g)(f+g)(f+2g)}{(f + \frac{1}{3}g)(f + \frac{2}{3}g)(f + \frac{1}{3}g)} \text{ etc. atque } \frac{R}{Q} = \frac{f(b + \frac{1}{3}g)(f+g)}{b(f + \frac{1}{3}g)(b+g)}$$

$$\frac{(b + \frac{1}{3}g)(f+2g)(b + \frac{2}{3}g)}{(f + \frac{1}{3}g)(b+2g)(f + \frac{2}{3}g)} \text{ etc. quae duae expressiones, cum}$$

in illa vna reuolutio ex tribus hic autem ex duobus factoribus constat, in se mutuo transformari nequeunt; quicquid etiam loco b substituatur.

§. 26. Sit igitur $S = \frac{\int y^{x-1} dy}{(1-y^g)^{\frac{2}{3}}}$ erit $\frac{S}{Q} = \frac{f(k+\frac{1}{3}g)}{k \cdot j + \frac{1}{3}g}$
 $\frac{(f+g)(k+\frac{1}{3}g)(f+2g)(k+\frac{2}{3}g)}{(k+g)(j+\frac{1}{3}g)(k+2g)(j+\frac{2}{3}g)}$ etc. quae expressio cum
 praecedente coniuncta dabit $\frac{RS}{Q^2} = \frac{f \cdot f \cdot (b+\frac{1}{3}g)(k+\frac{1}{3}g)}{b \cdot k \cdot (j+\frac{1}{3}g)(j+\frac{2}{3}g)}$
 $\frac{(f+g)(f+g)(b+\frac{1}{3}g)}{(b+g)(k+g)(j+\frac{1}{3}g)}$ etc. quae expressio in illam ipsi
 $\frac{27 P}{jg^2 Q^3}$ aequalem conuertetur, ponendo $b = f + \frac{1}{3}g$, et $k =$

$f + \frac{2}{3}g$. Quamobrem habebitur ista aequatio $\frac{27 P}{jg^2} = Q$
 RS, seu substitutis veris valoribus erit $90 \int dx (x-x^2)^{\frac{1}{3}}$.
 $\int dx (x^3-x^2)^{\frac{1}{3}} = fg^2 \int \frac{y^{f-1} dy}{(1-y^g)^{\frac{2}{3}}} \cdot \int \frac{y^{f+\frac{1}{3}g-1} dy}{(1-y^g)^{\frac{2}{3}}} \cdot \int \frac{y^{f+\frac{2}{3}g-1} dy}{(1-y^g)^{\frac{2}{3}}}$

§. 27. Antequam autem haec vterius profsequamur
 conveniet valori ipsius P commodiorem formam generali-
 ter tribui. Facto autem $x = z^q$, cum sit $\int dx (x^n - x^{n+1})^{\frac{p}{q}} =$
 $\frac{n \cdot p \cdot q}{(n+1)(n+1)p+q} \int \frac{z^{np-1} dz}{(1-z^q)^{\frac{q-p}{q}}}$, post substitutionem prodibit

$P = 1 \cdot 2 \cdot 3 \dots p \cdot \frac{p^{q-1}}{q} \cdot \int \frac{z^{p-1} dz}{(1-z^q)^{\frac{q-p}{q}}} \cdot \int \frac{z^{2p-1} dz}{(1-z^q)^{\frac{q-p}{q}}}$
 $\cdot \int \frac{z^{3p-1} dz}{(1-z^q)^{\frac{q-p}{q}}} \dots \int \frac{z^{(q-1)p-1} dz}{(1-z^q)^{\frac{q-p}{q}}}$. Ex qua expressione si
 extrahatur radix potestatis q , prodibit valor ipsius $\int dx$
 $(-1x)^{\frac{p}{q}}$.

§. 28.

§. 28. Posito nunc $p = 1$ et $q = 3$ prodibit $P = \frac{1}{3}$
 $\int \frac{dz}{(1-z^2)^{\frac{2}{3}}} \cdot \int \frac{z dz}{(1-z^2)^{\frac{2}{3}}}$. Facto autem $y = z^2$; obtinebi-
 tur sequens aequatio: $\int \frac{dz}{(1-z^2)^{\frac{2}{3}}} \cdot \int \frac{z dz}{(1-z^2)^{\frac{2}{3}}} = 3fg^2$
 $\int \frac{z^{2f-1} dz}{(1-z^{2g})^{\frac{2}{3}}} \cdot \int \frac{z^{2f+g-1} dz}{(1-z^{2g})^{\frac{2}{3}}} \cdot \int \frac{z^{2f+2g-1} dz}{(1-z^{2g})^{\frac{2}{3}}}$. Si nunc ponatur $3f$
 $= a$ oriatur sequens aequatio notatu digna; $\int \frac{dz}{(1-z^2)^{\frac{2}{3}}} \cdot \int \frac{z dz}{(1-z^2)^{\frac{2}{3}}}$
 $= ag^2 \int \frac{z^{a-1} dz}{(1-z^{2g})^{\frac{2}{3}}} \cdot \int \frac{z^{a+g-1} dz}{(1-z^{2g})^{\frac{2}{3}}} \cdot \int \frac{z^{a+2g-1} dz}{(1-z^{2g})^{\frac{2}{3}}}$. Quae cum
 superiore $\int \frac{dz}{\sqrt{(1-z^2)}} = ag \int \frac{z^{a-1} dz}{\sqrt{(1-z^{2g})}} \cdot \int \frac{z^{a+g-1} dz}{\sqrt{(1-z^{2g})}}$ compa-
 rata iam quodammodo indicat, quo modo sequentes huius
 generis aequationes se sint habiturae.

§. 29. Antequam autem per inductionem quicquam
 concludendi periculum faciam, casus nonnullos actu euoluam.
 Sit igitur $p = 2$ et $q = 3$ hincque reperietur $P = \frac{2}{3}$
 $\int \frac{z dz}{(1-z^2)^{\frac{1}{3}}} \cdot \int \frac{z^2 dz}{(1-z^2)^{\frac{1}{3}}} = \frac{2}{3} \int \frac{dz}{(1-z^2)^{\frac{1}{3}}} \cdot \int \frac{z dz}{(1-z^2)^{\frac{1}{3}}}$; $Q =$
 $\int \frac{y^{f-1} dy}{(1-y^g)^{\frac{1}{3}}}$; $R = \int \frac{y^{b-1} dy}{(1-y^g)^{\frac{1}{3}}}$. Expressiones autem infinitae
 ita se habebunt; $\frac{27 \cdot P}{8j^2 g Q^{\frac{2}{3}}} = \frac{f \cdot (f+g) \cdot (f+g)}{(f+\frac{2}{3}g) \cdot (f+\frac{2}{3}g) \cdot (f+\frac{2}{3}g)}$
 $\frac{(f+g)(f+2g)(f+2g)}{(f+\frac{5}{3}g)(f+\frac{5}{3}g)(f+\frac{5}{3}g)}$ etc. et $\frac{R}{Q} = \frac{f(b+\frac{2}{3}g)(f+g)}{b(f+\frac{2}{3}g)(b+g)}$
C 2 (b+

$$\frac{(b+\frac{1}{3}g)(f+2g)(b+\frac{2}{3}g)}{(j+\frac{1}{3}g)(b+2g)(j+\frac{2}{3}g)} \quad \text{Sit praeterea } S = \int \frac{y^{m-1} dy}{(1-y^g)^{\frac{1}{3}}} \quad \text{et}$$

$$T = \int \frac{y^{n-1} dy}{(1-y^g)^{\frac{1}{3}}} \quad \text{erit } \frac{T}{S} = \frac{m(n+\frac{2}{3}g)(m+g)(n+\frac{1}{3}g)(m+2g)}{n(m+\frac{1}{3}g)(n+g)(m+\frac{2}{3}g)(n+2g)}$$

$$\text{etc. quae duae expressiones in se ductae dant } \frac{RT}{QS} =$$

$$\frac{fm(b+\frac{2}{3}g)(n+\frac{2}{3}g)(f+g)(m+g)(b+\frac{1}{3}g)(n+\frac{1}{3}g)}{bn(j+\frac{1}{3}g)(m+\frac{1}{3}g)(b+g)(n+g)(j+\frac{2}{3}g)(m+\frac{2}{3}g)} \quad \text{etc.}$$

§. 30. Haec autem expressio ad illam, cui $\frac{27 P}{8f^2 g Q^3}$ aequale est inuentum, reduci non potest, nisi illa multiplicetur per $\frac{f}{f-\frac{1}{3}g}$, ita ut sit $\frac{27 P}{8fg(f-\frac{1}{3}g)Q^3} = \frac{f}{(f-\frac{1}{3}g)}$

$$\frac{f \cdot (f+g)(f+g)(f+g)(f+\frac{1}{2}g)}{(j+\frac{2}{3}g)(j+\frac{2}{3}g)(j+\frac{2}{3}g)(j+\frac{2}{3}g)(f+\frac{2}{3}g)} \quad \text{etc. nunc enim}$$

fiet reductio ponendo $m = f$; $b = f - \frac{1}{3}g$, et $n = f + \frac{1}{3}g$.

$$\text{His igitur valoribus substitutis erit } \frac{27 P}{8fg(f-\frac{1}{3}g)Q^3} = \frac{RT}{QS}$$

$$\text{Cum vero sit } S = Q \text{ et } R = \int \frac{y^{f-\frac{1}{3}g-1} dy}{(1-y^g)^{\frac{1}{3}}} = \frac{f+\frac{1}{3}g}{f-\frac{1}{3}g} \int \frac{y^{f+\frac{2}{3}g-1} dy}{(1-y^g)^{\frac{1}{3}}}$$

$$\text{et } T = \int \frac{y^{f+\frac{1}{3}g-1} dy}{(1-y^g)^{\frac{1}{3}}}. \text{ obtinebitur haec aequatio; posito } y = z^{\frac{1}{3}};$$

$$\int \frac{dz}{(1-z^3)^{\frac{1}{3}}} \cdot \int \frac{z dz}{(1-z^3)^{\frac{1}{3}}} = 3fg(3f+g) \int \frac{z^{2f-1} dz}{(1-z^{3g})^{\frac{1}{3}}} \cdot \int \frac{z^{3f+g-1} dz}{(1-z^{3g})^{\frac{1}{3}}}$$

$$\int \frac{z^{2f+2g-1} dz}{(1-z^{3g})^{\frac{1}{3}}}. \quad \text{Ac si ponatur } 3f = a \text{ erit } \int \frac{dz}{(1-z^3)^{\frac{1}{3}}}$$

[zdz

$$\int \frac{z dz}{(1-z^2)^{\frac{1}{2}}} = ag(a+g) \int \frac{z^{a-1} dz}{(1-z^{2g})^{\frac{1}{2}}} \cdot \int \frac{z^{a+g-1} dz}{(1-z^{2g})^{\frac{1}{2}}}$$

$$\int \frac{z^{a+2g-1} dz}{(1-z^{2g})^{\frac{1}{2}}}$$

§. 31. Ponamus $p = 1$ et $q = 4$; habebiturque

$$\frac{4^4 P}{fg^3 Q^4} = \frac{f \cdot f \cdot f \cdot (f+g)(f+g)(f+g)}{(f+\frac{1}{4}g)(f+\frac{1}{4}g)(f+\frac{1}{4}g)(f+\frac{1}{4}g)(f+\frac{2}{4}g)(f+\frac{2}{4}g)}$$

etc. et $\frac{R}{Q} = \frac{f(b+\frac{1}{4}g)(f+g)(b+\frac{2}{4}g)(f+2g)}{b(f+\frac{1}{4}g)(b+g)(f+\frac{2}{4}g)(b+2g)}$ etc. Sit

vero ut ante $S = \frac{\int y^{m-1} dy}{(1-y^g)^{\frac{q-p}{q}}}$; $T = \frac{\int y^{n-1} dy}{(1-y^g)^{\frac{q-p}{q}}}$; erit

$$\frac{RST}{Q^3} = \frac{f \cdot f \cdot f(b+\frac{1}{4}g)(m+\frac{1}{4}g)(n+\frac{1}{4}g) \cdot (f+g)}{b \cdot m \cdot n (f+\frac{1}{4}g)(f+\frac{1}{4}g)(f+\frac{1}{4}g)(b+g)}$$
 etc.

cuius expressionis 6 factores in illius quatuor sunt transmutandi, quod fiet ponendo $b=f+\frac{1}{4}g$; $m=f+\frac{2}{4}g$; et $n=f+\frac{3}{4}g$; quo facto habebitur $4^4 P = fg^3 QRST$. Quare cum sit

$$P = \frac{1}{4} \int \frac{dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z^2 dz}{(1-z^4)^{\frac{1}{4}}}$$
; si

ponatur $y = z^4$ et $4f = a$ orietur ista aequatio: $\int \frac{dz}{(1-z^4)^{\frac{1}{4}}}$

$$\int \frac{z dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z dz}{(1-z^4)^{\frac{1}{4}}} = ag^3 \int \frac{z^{a-1} dz}{(1-z^{2g})^{\frac{1}{4}}} \cdot \int \frac{z^{a+g-1} dz}{(1-z^{2g})^{\frac{1}{4}}}$$

$$\int \frac{z^{a+2g-1} dz}{(1-z^{2g})^{\frac{1}{4}}} \cdot \int \frac{z^{a+3g-1} dz}{(1-z^{2g})^{\frac{1}{4}}}$$
 cuius cum praecedentibus casibus, quibus erat $p=1, q=2$; et $p=1, q=3$, connexio facile perspicitur.

§. 32. Ex his igitur licebit omnes istius modi aequationes, quae oriuntur si ponatur $p = 1$, et $q =$ numero cuicumque affirmatio integro, formare; erit scilicet.

$$\begin{aligned} \text{I. } \int \frac{dz}{\sqrt{1-z^2}} &= a g \int \frac{z^{a-1} dz}{\sqrt{1-z^{2g}}} \cdot \int \frac{z^{a+g-1} dz}{\sqrt{1-z^{2g}}} \\ \text{II. } \int \frac{dz}{(1-z^2)^{\frac{1}{2}}} \int \frac{z dz}{(1-z^2)^{\frac{1}{2}}} &= a g^2 \int \frac{z^{a-1} dz}{(1-z^{2g})^{\frac{1}{2}}} \cdot \int \frac{z^{a+g-1} dz}{(1-z^{2g})^{\frac{1}{2}}} \\ &\quad \int \frac{z^{a+2g-1} dz}{(1-z^{2g})^{\frac{1}{2}}} \\ \text{III. } \int \frac{dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z^2 dz}{(1-z^4)^{\frac{1}{4}}} &= a g^3 \int \frac{z^{a-1} dz}{(1-z^{4g})^{\frac{1}{4}}} \\ &\quad \int \frac{z^{a+g-1} dz}{(1-z^{4g})^{\frac{1}{4}}} \cdot \int \frac{z^{a+2g-1} dz}{(1-z^{4g})^{\frac{1}{4}}} \cdot \int \frac{z^{a+3g-1} dz}{(1-z^{4g})^{\frac{1}{4}}} \\ \text{III. } \int \frac{dz}{(1-z^5)^{\frac{1}{5}}} \cdot \int \frac{z dz}{(1-z^5)^{\frac{1}{5}}} \cdot \int \frac{z^2 dz}{(1-z^5)^{\frac{1}{5}}} \int \frac{z^3 dz}{(1-z^5)^{\frac{1}{5}}} &= a g^4 \\ &\quad \int \frac{z^{a-1} dz}{(1-z^{5g})^{\frac{1}{5}}} \cdot \int \frac{z^{a+g-1} dz}{(1-z^{5g})^{\frac{1}{5}}} \cdot \int \frac{z^{a+2g-1} dz}{(1-z^{5g})^{\frac{1}{5}}} \cdot \int \frac{z^{a+3g-1} dz}{(1-z^{5g})^{\frac{1}{5}}} \\ &\quad \int \frac{z^{a+4g-1} dz}{(1-z^{5g})^{\frac{1}{5}}} \text{ etc.} \end{aligned}$$

§. 33. Quo etiam eas aequationes, quae oriuntur si p non $= 1$, colligere queamus, ponamus $p = 3$ et $q = 4$; quo posito, et reliquis manentibus vt supra, erit $\frac{4^4 P}{3^4 J^3 g Q^4} =$

$\frac{f}{(f+\frac{1}{4}g)(f+\frac{2}{4}g)(f+\frac{3}{4}g)}$ etc. vbi reliqua membra ex quaternis factoribus constantia ex his formantur singulos factores quantitate g augendo. Simili vero modo erit

erit $\frac{RST}{Q^3} = \frac{f \cdot f \cdot f \cdot (b + \frac{3}{4}g)(m + \frac{3}{4}g)(n + \frac{3}{4}g)}{b \cdot m \cdot n (f + \frac{3}{4}g)(j + \frac{3}{4}g)(f + \frac{3}{4}g)}$ etc. vbi

seni factores vnam reuolutionem seu periodum constituunt. Ad comparationem autem instituendam necesse est vtram-

que seriem ita contemplari: $\frac{4^P}{3^+ j^2 g (f - \frac{1}{4}g) Q^+} = \frac{f}{(f - \frac{1}{4}g)}$

$\frac{f}{(j + \frac{3}{4}g)(j + \frac{3}{4}g)(f + \frac{3}{4}g)}$ etc. $\frac{bRST}{fQ^3} = \frac{f \cdot f \cdot (b + \frac{3}{4}g)}{m \cdot n \cdot (f + \frac{3}{4}g)}$

$\frac{(m + \frac{3}{4}g)(n + \frac{3}{4}g)(f + g)}{(j + \frac{3}{4}g)(j + \frac{3}{4}g)(b + g)}$ etc. quarum haec transmutatur in

illam, ita vt fiat $\frac{4^+ P}{3^+ j g b (f - \frac{1}{4}g)} = QRST$, si fiat $b = f + \frac{3}{4}g$; $m = f - \frac{1}{4}g$; et $n = f + \frac{3}{4}g$.

§. 34. Cum igitur sit $P = \frac{1}{2} \cdot \int \frac{z^2 dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z^5 dz}{(1-z^4)^{\frac{1}{4}}}$

$\int \frac{z^2 dz}{(1-z^4)^{\frac{1}{4}}} = \frac{3^+}{32} \int \frac{dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z z dz}{(1-z^4)^{\frac{1}{4}}}$ et Q

$= \int \frac{y^{f-1} dy}{(1-y^4)^{\frac{1}{4}}}$; $R = \int \frac{y^{f+\frac{3}{4}g-1} dy}{(1-y^4)^{\frac{1}{4}}}$; $S = \int \frac{y^{f-\frac{1}{4}g-1} dy}{(1-y^4)^{\frac{1}{4}}} =$

$\frac{f + \frac{3}{4}g}{f - \frac{1}{4}g} \int \frac{y^{f+\frac{3}{4}g-1} dy}{(1-y^4)^{\frac{1}{4}}}$ atque $T = \int \frac{y^{f+\frac{3}{4}g-1} dy}{(1-y^4)^{\frac{1}{4}}}$. Ex quibus

posito $y = z^4$ et $4f = a$ sequens conficitur aequatio:

$\frac{dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z z dz}{(1-z^4)^{\frac{1}{4}}} = a g \frac{(a+\frac{3}{4}g)(a+\frac{3}{4}g)}{1 \cdot 2} \int \frac{z^{a-1} dz}{(1-z^4)^{\frac{1}{4}}}$

$\int \frac{z^{a+\frac{3}{4}g-1} dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z^{a+\frac{3}{4}g-1} dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z^{a+\frac{3}{4}g-1} dz}{(1-z^4)^{\frac{1}{4}}}$

§. 35. Hoc modo progrediendo reperientur sequentes aequationes, quando p non est $= 1$ et quidem si $p = 2$ inuenietur.

$$I. \int \frac{dz}{(1-z^3)^{\frac{1}{3}}} \cdot \int \frac{zdz}{(1-z^3)^{\frac{1}{3}}} = ag(a+g) \int \frac{z^{a-1}dz}{(1-z^{3g})^{\frac{1}{3}}}$$

$$\int \frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{1}{3}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{1}{3}}}$$

$$II. \int \frac{dz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{zdz}{(1-z^4)^{\frac{1}{4}}} \cdot \int \frac{z^2dz}{(1-z^4)^{\frac{1}{4}}} = ag^2(a+g) \int \frac{z^{a-1}dz}{(1-z^{4g})^{\frac{1}{4}}}$$

$$\int \frac{z^{a+g-1}dz}{(1-z^{4g})^{\frac{1}{4}}} \cdot \int \frac{z^{a+2g-1}dz}{(1-z^{4g})^{\frac{1}{4}}} \cdot \int \frac{z^{a+3g-1}dz}{(1-z^{4g})^{\frac{1}{4}}} \quad \text{Generaliter}$$

autem quicquid fit q , si ponatur $\frac{dz}{(1-z^q)^{\frac{q-2}{q}}} = X dz$ et

$$\frac{z^{a-1}dz}{(1-z^{qg})^{\frac{q-2}{q}}} = Y dz \quad \text{erit } \int X dz \cdot \int z X dz \cdot \int z^2 X dz \dots$$

$$\int z^{q-2} X dz = ag^{q-2}(a+g) \int Y dz \cdot \int z^g Y dz \cdot \int z^{2g} Y dz \dots$$

$$\int z^{(q-1)g} Y dz.$$

§. 36. Simili modo si fit $p = 3$, ac ponatur

$$\frac{dz}{(1-z^q)^{\frac{q-3}{q}}} = X dz, \quad \text{et} \quad \frac{z^{a-1}dz}{(1-z^{qg})^{\frac{q-3}{q}}} = Y dz \quad \text{prodibit sequens}$$

aequatio generalis, $\int X dz \cdot \int z X dz \cdot \int z^2 X dz \dots \int z^{q-2} X dz = ag^{q-3} \frac{(a+g)(1+2g)}{1 \cdot 2} \int Y dz \cdot \int z^g Y dz \cdot \int z^{2g} Y dz \dots$

$\int z^{(q-1)g} Y dz.$ Atque hinc omnes has formulas in vnam latissime patentem colligi licet. Sint enim p et q numeri quicumque integri affirmatiui, ac ponatur $\frac{dz}{(1-z^q)^{\frac{q-p}{q}}} = X dz$

$$X dz \text{ et } \frac{z^{q-1} dz}{(1-z^{qg})^{\frac{q-p}{q}}} = Y dz; \text{ habebitur } \int X dz. \int z X dz. \\ \int z^2 X dz. \dots \int z^{q-2} X dz = a g^{q-p} \frac{(a+g)(a+2g)(a+3g) \dots (a+(p-1)g)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (p-1)} \\ \int Y dz. \int z^g Y dz. \int z^{2g} Y dz. \dots \int z^{(q-1)g} Y dz.$$

§. 37. Cum autem fit $\int z^{q-1} X dz = \frac{1}{p}$, si per hunc factorem vtrinq̄ multiplicetur proueniet sequens aequatio

$$\text{fatis elegans: } \frac{a(a+g)(a+2g)(a+3g) \dots (a+(p-1)g)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot p} z^{q-p} = \frac{\int X dz}{\int Y dz}$$

$$\frac{\int z X dz}{\int z^g Y dz} \cdot \frac{\int z^2 X dz}{\int z^{2g} Y dz} \cdot \frac{\int z^3 X dz}{\int z^{3g} Y dz} \dots \frac{\int z^{q-1} X dz}{\int z^{(q-1)g} Y dz} \text{ quae ex-}$$

pressio omnes haecenus inuentas in se complectitur; atque ob insignem ordinem est notatu digna.

§. 38. Progrediar nunc ad aliam methodum, cuius ope ad huiusmodi expressiones ex factoribus innumerabilibus constantes peruenire licet, quae magis ad analysin est accommodata. Obseruavi enim ex reductione formularum integralium ad alias istiusmodi expressiones obtineri posse.

Sit enim proposita ista formula integralis $\int x^{m-1} dx (1-x^{nq})^{\frac{p}{q}}$, quae non difficulter transmutatur in hanc expressionem

$$\frac{x^m (1-x^{nq})^{\frac{p+q}{q}}}{m} + \frac{m+(p+1)n}{m} \int x^{m+nq-1} dx (1-x^{nq})^{\frac{p}{q}}. \text{ Si ergo}$$

m et $\frac{p+q}{q}$ fuerint numeri affirmatiui, atque integralia ita capiuntur, vt euanescant, posito $x=0$, tumque ponatur $x=1$, fiet $\int x^{m-1} dx (1-x^{nq})^{\frac{p}{q}} = \frac{m+(p+1)n}{m} \int x^{m+nq-1} dx (1-x^{nq})^{\frac{p}{q}}$.

§. 39. Cum deinde simili modo fit $\int x^{m+nq-1} dx (1-x^{nq})^{\frac{p}{q}} = \frac{m+(p+2)n}{m+nq} \int x^{m+2nq-1} dx (1-x^{nq})^{\frac{p}{q}}$ erit quoque $\int x^{m-1} dx (1-x^{nq})^{\frac{p}{q}} = \frac{(m+(p+1)n)(m+(p+2)n)}{m(m+nq)} \int x^{m+nq-1} dx (1-x^{nq})^{\frac{p}{q}}$. Hac

ergo reductione in infinitum continuata prodibit: $\int x^{m-1} dx$
 $(1-x^{nq})^{\frac{p}{q}} = \frac{(m+(p+q)n)(m+(p+q)2n)\dots(m+(p+q)\infty n)}{m \cdot (m+nq) \cdot (m+2nq) \cdot \dots \cdot (m+\infty nq)}$
 $\int x^{m+\infty nq-1} dx (1-x^{nq})^{\frac{p}{q}}$. Ac simili modo est $\int x^{\mu-1} dx (1-x^{nq})^{\frac{p}{q}}$
 $= \frac{(\mu+(p+q)n)(\mu+(p+q)2n)\dots(\mu+(p+q)\infty n)}{\mu \cdot (\mu+nq) \cdot (\mu+2nq) \cdot \dots \cdot (\mu+\infty nq)} \int x^{\mu+\infty nq-1}$
 $dx (1-x^{nq})^{\frac{p}{q}}$; dummodo m , et nq et $\frac{p-1}{q}$ sint numeri affir-
 matiui, seu nihilo maiores.

§. 40. Quoniam autem si m est infinitum fit $\int x^m dx$
 $(1-x^{nq})^{\frac{p}{q}} = \int x^{m+\alpha} dx (1-x^{nq})^{\frac{p}{q}}$, quicumque numerus finitus
 loco α accipiatur, vti ex paragr. 38 colligitur, erit quo-
 que $\int x^{m+\infty nq-1} dx (1-x^{nq})^{\frac{p}{q}} = \int x^{\mu+\infty nq-1} dx (1-x^{nq})^{\frac{p}{q}}$. Quam-
 obrem si praecedentium expressionum altera per alteram
 diuidatur, proueniet ista aequatio: $\frac{\int x^{m-1} dx (1-x^{nq})^{\frac{p}{q}}}{\int x^{\mu-1} dx (1-x^{nq})^{\frac{p}{q}}} =$

$\frac{\mu(m+(p+q)n)(\mu+nq)(m+(p+q)n)(\mu+2nq)(m+(p+q)2n)(\mu+3nq)}{\mu(\mu+(p+q)n)(m+nq)(\mu+(p+q)n)(m+2nq)(\mu+(p+q)2n)(m+3nq)}$ etc. in
 infinitum, cuius expressionis ope innumerabilia producta
 ex infinitis factoribus constantia exhiberi possunt, quorum
 valores per quadraturas curuarum assignari poterunt.

§. 41. Si altera formula integralis admittat integra-
 tionem, tum commoda expressio infinita pro altero inte-
 grali habebitur. Sit enim $\mu = nq$, erit $\int x^{\mu-1} dx (1-x^{nq})^{\frac{p}{q}}$
 $= \frac{1}{(p+q)n}$, quo valore substituto prodibit $\int x^{m-1} dx (1-x^{nq})^{\frac{p}{q}}$
 $= \frac{1}{(p+q)n} \cdot \frac{nq(m+(p+q)n)2nq(m+(p+q)2n)3nq}{m(p+2q)n(m+nq)(p+3q)n(m+2nq)}$ etc. cuius ope pro
 innumerabilibus integralibus expressiones per continuos facto-
 res in infinitum excurrentes inueniri possunt; eo saltem
 casu quo $x = 1$; quippe qui plerumque potissimum desi-
 deratur.

§. 42.

§. 42. Ponatur n loco nq , et prodibit: $\int x^{m-1} dx$
 $(1-x^n)^{\frac{p}{q}} = \frac{q}{(p+q)n} \cdot \frac{n(mq+(p+q)n)2n(mq+(p+2q)n)3n(mq+(p+3q)n)}{m(p+2q)n(m+n)(p+3q)n(m+2n)(p+4q)n}$
 etc. quae in binos factores resoluta fit simplicior euaditque
 $\int x^{m-1} dx (1-x^n)^{\frac{p}{q}} = \frac{q}{(p+q)n} \cdot \frac{1(mq+(p+q)n)}{m(p+2q)} \cdot \frac{2(mq+(p+2q)n)}{(m+n)(p+3q)}$
 $\frac{3(mq+(p+3q)n)}{(m+2n)(p+4q)}$ etc. vnde sequentia exempla notabiliora deducuntur.

$$\int \frac{dx}{\sqrt{(1-xx)}} = I. \frac{1 \cdot 4}{1 \cdot 3} \cdot \frac{2 \cdot 5}{3 \cdot 5} \cdot \frac{3 \cdot 12}{5 \cdot 7} \text{ etc.} = \frac{2 \cdot 2 \cdot 4 \cdot 6 \cdot 6}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7} \text{ etc.}$$

$$\int \frac{x dx}{\sqrt{(1-xx)}} = I. \frac{1 \cdot 6}{2 \cdot 3} \cdot \frac{2 \cdot 10}{4 \cdot 5} \cdot \frac{3 \cdot 14}{6 \cdot 7} \text{ etc.} = I$$

$$\int \frac{x^2 dx}{\sqrt{(1-xx)}} = I. \frac{1 \cdot 8}{3 \cdot 3} \cdot \frac{2 \cdot 12}{5 \cdot 5} \cdot \frac{3 \cdot 16}{7 \cdot 7} \text{ etc.} = \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7} \text{ etc.}$$

$$\int \frac{dx}{\sqrt{(1-xx^3)}} = \frac{2}{3} \cdot \frac{1 \cdot 5 \cdot 2 \cdot 11 \cdot 3 \cdot 17 \cdot 4 \cdot 23 \cdot 5 \cdot 29}{1 \cdot 3 \cdot 4 \cdot 5 \cdot 7 \cdot 7 \cdot 10 \cdot 9 \cdot 13 \cdot 11} \text{ etc.}$$

$$\int \frac{x dx}{\sqrt{(1-xx^3)}} = \frac{2}{3} \cdot \frac{1 \cdot 7 \cdot 2 \cdot 13 \cdot 3 \cdot 19 \cdot 4 \cdot 25 \cdot 5 \cdot 31}{2 \cdot 3 \cdot 5 \cdot 5 \cdot 8 \cdot 7 \cdot 11 \cdot 9 \cdot 14 \cdot 11} \text{ etc.}$$

$$\int \frac{dx}{\sqrt{(1-xx^4)}} = \frac{1}{2} \cdot \frac{1 \cdot 6 \cdot 2 \cdot 14 \cdot 3 \cdot 22 \cdot 4 \cdot 30}{1 \cdot 3 \cdot 5 \cdot 5 \cdot 9 \cdot 7 \cdot 13 \cdot 9} \text{ etc.} = \frac{1}{2} \cdot \frac{2 \cdot 3 \cdot 4 \cdot 7 \cdot 6 \cdot 11 \cdot 3 \cdot 15}{1 \cdot 3 \cdot 5 \cdot 3 \cdot 9 \cdot 7 \cdot 13 \cdot 9} \text{ etc.}$$

$$\int \frac{xx dx}{\sqrt{(1-xx^4)}} = \frac{1}{2} \cdot \frac{1 \cdot 10 \cdot 2 \cdot 15 \cdot 3 \cdot 26 \cdot 4 \cdot 34}{3 \cdot 3 \cdot 7 \cdot 5 \cdot 11 \cdot 7 \cdot 15 \cdot 9} \text{ etc.}$$

$$\int \frac{dx}{\sqrt[3]{(1-xx^3)}} = \frac{1}{2} \cdot \frac{3 \cdot 3 \cdot 6 \cdot 6 \cdot 9 \cdot 9 \cdot 12 \cdot 12}{1 \cdot 5 \cdot 4 \cdot 8 \cdot 7 \cdot 11 \cdot 10 \cdot 14} \text{ etc.}$$

$$\int \frac{dx}{\sqrt[4]{(1-xx^4)}} = \frac{1}{3} \cdot \frac{4 \cdot 4 \cdot 8 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdot 16}{1 \cdot 7 \cdot 5 \cdot 11 \cdot 9 \cdot 15 \cdot 13 \cdot 19} \text{ etc.}$$

Praeterea hae expressiones notari merentur:

$$\int x^{m-1} dx (1-x^n)^{-\frac{m}{n}} = \frac{1}{n-m} \cdot \frac{n \cdot n \cdot 2n \cdot 2n \cdot 3n \cdot 3n}{m(2n-m)(m+n)(3n-m)(m+2n)(4n-m)} \text{ etc.}$$

$$\int x^{m-1} dx (1-x^n)^{\frac{m-n}{n}} = \frac{1}{m} \cdot \frac{n \cdot 2n \cdot 2n \cdot (2m+n) \cdot 3n \cdot (2m+2n) \cdot 4n}{m(m+n)(m+n)(m+2n)(m+2n)(m+3n)(m+3n)}$$

$$\frac{(2m+3n)}{(m+4n)} \text{ etc.}$$

§. 43. Cum autem pari modo fit $\int x^{\mu-1} dx (1-x^\nu)^{\frac{r}{s}} =$
 $\frac{s}{(r+s)\nu} \cdot \frac{1(\mu s+(r+s)\nu)2(\mu s+(r+s)\nu)3(\mu s+(r+s)\nu)}{\mu(r+2s)(\mu+\nu)(r+3s)(\mu+2\nu)(r+4s)}$ etc. erit priorem
 expressionem per hanc diuidendo $\frac{\int x^{m-1} dx (1-x^n)^{\frac{p}{q}}}{\int x^{\mu-1} dx (1-x^\nu)^{\frac{r}{s}}} =$

$$\frac{(r+s)qv}{(p+q)sn} \cdot \frac{u(r+2s)(mq+(p+q)n)}{m(p+2q)(us+(r+s)v)} \cdot \frac{(u+v)(r+3s)(mq+(p+2q)n)}{(m+n)(p+3q)(us+(r+2s)v)} \text{ etc.}$$

Haec igitur expressio infinita, quoties habet valorem finitum, toties summatio alterius integralis ad alterum reduci poterit. Huiusmodi autem casus existunt, quando factores numeratoris destruunt factores denominatoris, ita ut post destructionem finitus factorum numerus supersit. Continentur enim in hac expressione omnes omnino reductiones formularum integralium ad alias.

§. 44. Quo autem plures istiusmodi expressiones inter se comparari queant, eam hoc modo accipere visum est:

$$\frac{\int x^{a-1} dx (1-x^b)^c}{\int x^{f-1} dx (1-x^g)^h} = \frac{(b+1)g}{(c+1)b} \cdot \frac{f(b+2)(a+(c+1)b)}{a(c+2)(f+(b+1)g)} \cdot \frac{(f+g)(b+3)(a+(c+2)b)}{(a+b)(c+3)(f+(b+2)g)}$$

$$\text{etc. Simili modo erit } \frac{\int x^{\alpha-1} dx (1-x^\beta)^\gamma}{\int x^{\zeta-1} dx (1-x^\eta)^\theta} = \frac{(\theta+1)\eta}{(\gamma+1)\xi} \cdot \frac{\xi(\theta+2)(\alpha+(\gamma+1)\xi)}{\alpha(\gamma+2)(\zeta+(\theta+1)\eta)}$$

$\frac{(\xi+\eta)(\theta+3)(\alpha+(\gamma+2)\xi)}{(\alpha+\xi)(\gamma+3)(\zeta+(\theta+2)\eta)}$ etc. quae expressiones, etsi re non inter se differunt, tamen quoniam habent formam diuersam, inter se comparari poterunt.

§. 45. Vt nunc ex his expressiōibus eadem theoremata eliciamus, quae supra inuenimus, sit $\theta = \gamma = b =$

$$c; \quad \eta = \xi = g = b; \quad \text{erit } \frac{\int x^{a-1} dx (1-x^b)^c}{\int x^{f-1} dx (1-x^b)^c} = \frac{f(a+(c+1)b)(f+b)}{a(f+(c+1)b)(a+b)}$$

etc. atque altera formula

$$\frac{\int x^{a-1} dx (1-x^b)^c}{\int x^{\zeta-1} dx (1-x^b)^c} = \frac{\xi(a+(c+1)b)(\xi+b)(a+(c+2)b)(\xi+2b)(a+(c+3)b)}{\alpha(\xi+(c+1)b)(\alpha+b)(\xi+(c+2)b)(\alpha+2b)(\xi+(c+3)b)}$$

etc. Harum expressiōnum productum si ponatur $= \frac{f}{a}$ oportet esse

$$\frac{(a+(c+1)b)(f+b)\xi(a+(c+1)b)}{(f+(c+1)b)(a+b)\alpha(\xi+(c+1)b)} = 1, \text{ hoc enim si fuerit, totarum expressiōnum infinitarum productum fiet } = \frac{f}{a}. \text{ At hoc ob-$$

time-

tinebitur faciendo $\alpha = a + (c + 1)b$; $\zeta = f + (c + 1)b$; fietque $c = \frac{1}{2}$, ita ut sit $\alpha = a + \frac{1}{2}b$; $\zeta = f + \frac{1}{2}b$, eritque

$$\text{ideo } \int \frac{x^{a-1} dx}{\sqrt{(1-x^b)}} \cdot \int \frac{x^{a+\frac{1}{2}b-1} dx}{\sqrt{(1-x^b)}} = \frac{f}{a} \int \frac{x^{f-1} dx}{\sqrt{(1-x^b)}} \cdot \int \frac{x^{f+\frac{1}{2}b-1} dx}{\sqrt{(1-x^b)}} \quad \text{feu}$$

$$\text{fi ponatur } x = z^2; \text{ erit } \int \frac{z^{a-1} dz}{\sqrt{(1-z^{2b})}} \cdot \int \frac{z^{a+\frac{1}{2}b-1} dz}{\sqrt{(1-z^{2b})}} = \frac{f}{a} \cdot$$

$$\int \frac{z^{f-1} dz}{\sqrt{(1-z^{2b})}} \cdot \int \frac{z^{f+\frac{1}{2}b-1} dz}{\sqrt{(1-z^{2b})}} \quad \text{positis } a \text{ et } f \text{ loco } 2a \text{ et } 2f. \text{ Haec}$$

autem aequatio nil aliud est nisi Theorema supra inuen-

tum §. 12. facto enim $f = b$ fit $\int \frac{z^{2b-1} dz}{\sqrt{(1-z^{2b})}} = \frac{1}{b}$ et $\int \frac{z^{b-1} dz}{\sqrt{(1-z^{2b})}}$

$$= \frac{\pi}{2b}; \text{ vnde fiet } \pi = 2ab \int \frac{z^{a-1} dz}{\sqrt{(1-z^{2b})}} \cdot \int \frac{z^{a+\frac{1}{2}b-1} dz}{\sqrt{(1-z^{2b})}}.$$

§. 46. Simili modo alia huius generis theoremata inveniri possunt; sit enim $g = b$; $h = c$; $\eta = \xi = b$ et $\theta = \gamma$, quaeraturque casus, quo productura ambarum expressio- num fiat = 1. Hoc autem obtinebitur si fit $\frac{f(\alpha+(c+1)b)\xi(\alpha+(\gamma+1)b)}{a(f+(c+1)b)\alpha(\xi+(\gamma+1)b)} = 1$; id quod fiet capiendo $\alpha = a + (c + 1)b$; $f = a + (\gamma + 1)b$; $\xi = a$. His igitur valoribus substitutis orietur

$$\text{sequens theorema non inelegans } \frac{\int x^{a-1} dx (1-x^b)^c}{\int x^{a-1} dx (1-x^b)^\gamma}.$$

$$\frac{\int x^{a+(c+1)b-1} dx (1-x^b)^\gamma}{\int x^{a+(\gamma+1)b-1} dx (1-x^b)^c} = 1; \text{ siue si ponatur } c + 1 = m \text{ et}$$

$$\gamma + 1 = n \text{ habebitur } \int \frac{x^{a-1} dx}{(1-x^b)^{1-m}} \cdot \int \frac{x^{a+mb-1} dx}{(1-x^b)^{1-n}} =$$

$$\frac{\int x^{a-1} dx}{(1-x^b)^{1-n}} \cdot \frac{\int x^{a+nb-1} dx}{(1-x^b)^{1-m}}.$$

§. 47. Alio insuper modo concinnum theorema elici poterit ponendo $\gamma = b$ et $\theta = c$, manente $\eta = \xi = g = b$; atque efficiendo vt productum expressionum integralium fiat $= \frac{f}{a}$, quod, quo eueniat, oportet esse $\frac{(a+(c+1)b)(f+b)\xi}{(f+(b+1)b)(a+b)a} \frac{(\alpha+(b+1)b)}{(\xi+(c+1)b)} = 1$. Hoc vero efficietur capiendo $\alpha = a + (c+1)b$; $\xi = f + (b+1)b$, ex quo reperietur $c + b + 1 = 0$ seu $b = -1 - c$; quare sumatur $c = -\frac{1}{2} + n$; et $b = -\frac{1}{2} - n$, atque sequens prodibit theorema: $\frac{f}{a} = \frac{\int x^{a-1} dx (1-x^b)^{-\frac{1}{2}+n} \int x^{a+(1-n)b-1} dx (1-x^b)^{-\frac{1}{2}-n}}{\int x^{f-1} dx (1-x^b)^{-1-n} \cdot \int x^{f+(1-n)b-1} dx (1-x^b)^{-\frac{1}{2}+n}}$.

§. 48. Sint nunc omnes exponentes c , b , γ et θ inaequales, at $g = \xi = \eta = b$, quaeranturque casus quibus productum ambarum expressionum fiat $= \frac{(b+1)(\theta+1)}{(c+1)(\gamma+1)}$. Hoc autem eueniet si reddatur haec forma $\frac{f(bb+2b)(a+(c+1)b)\xi(b\theta+2b)}{a(bc+2b)(f+(b+1)b)\alpha(b\gamma+2b)} \frac{(\alpha+(\gamma+1)b)}{(\xi+(\theta+1)b)} = 1$ quos factores ita expressi, vt singuli in sequentibus membris quantitate b crescant. Ponatur iam $\xi + (\theta+1)b = bb + 2b$, seu $\xi = b(1+b-\theta)$ et $\alpha + (\gamma+1)b = bc + 2b$, seu $\alpha = b(1+c-\gamma)$. Porro fiat $f + (b+1)b = b\theta + 2b$, seu $f = b(1+\theta-b)$ et $a + (c+1)b = b\gamma + 2b$ seu $a = b(1+\gamma-c)$. Denique debeat esse $a = f$ et $\xi = a$, quae duae aequationes requirunt vt sit $c - \gamma = \theta - b$, siue $c + b = \gamma + \theta$. Unde sequens oriatur Theorema: $\frac{(b+1)(\theta+1)}{(c+1)(\gamma+1)} = \frac{\int x^{b(1+\gamma-c)-1} dx (1-x^b)^c \cdot \int x^{b(1+c-\gamma)-1} dx (1-x^b)^\gamma}{\int x^{b(1+\theta-b)-1} dx (1-x^b)^b \cdot \int x^{b(1+b-\theta)-1} dx (1-x^b)^\theta}$ dummodo sit $c + b = \gamma + \theta$.

§. 49. Alio autem insuper modo expressio illa effici potest $= 1$, ponendo $\alpha = a + (c+1)b$ et $\xi = f + (b+1)$

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$+1)b$; $f=b(\gamma+2)$; $a=b(\theta+2)$; ita vt fit $a=b(3+c+\theta)$ et $\zeta=b(3+\gamma+b)$. Porro autem debet esse $\zeta+(\theta+1)b=bb+2b$, et $a+(\gamma+1)b=bc+2b$; quibus postulatur vt fit $\gamma+\theta+2=0$. Ponatur ergo $\gamma=-1+n$ et $\theta=-1-n$. At si requiratur, vt productum ambarum expressionum fit $=\frac{f(b+1)(\theta+1)}{a(c+1)(\gamma+1)}$, id obtinebitur ponendo $a=a+(c+1)b$, $\zeta=f+(b+1)b$; $f=b(\gamma+1)$; $a=b(\theta+1)$ vnde erit $a=b(2+c+\theta)$ et $\zeta=b(2+b+\gamma)$. Tandem vero debebit esse $\gamma+\theta+1=0$. Ponatur $\gamma=-\frac{1}{2}+n$ et $\theta=-\frac{1}{2}-n$; at-

que habebitur hoc theorema $\frac{b+1}{c+1} = \frac{\int x^{b(\frac{1}{2}-n)-1} dx (1-x^b)^c}{\int x^{b(\frac{1}{2}+n)-1} dx (1-x^b)^b}$

$\frac{\int x^{b(\frac{1}{2}+c-n)-1} dx (1-x^b)^{-\frac{1}{2}+n}}{\int x^{b(\frac{1}{2}+b+n)-1} dx (1-x^b)^{-\frac{1}{2}-n}}$: in quo notandum est, exponen-

tes $c, b, -\frac{1}{2}+n, -\frac{1}{2}-n$ numeros negativos quidem esse posse, sed tales vt cum vnitare ad affirmatiuos transeant; alioquin enim integralia valorem finitum non obtinerent casu $x=1$.

§. 50. Quemadmodum igitur non solum theorema supra inuentum circa duarum formularum integralium producta detexi hac methodo magis directa, sed etiam alia noua elicui non minus notatu digna, ita, si pari modo tres eiusmodi expressiones in se inuicem ducantur, theoremata complura circa producta trium formularum integralium prodibunt; atque ultra ad quotcunque factorum numerum progredi licebit; sed cum haec inquisitio adeo prolixum calculum requirat, vt etiam litterae vix sufficiant, cum ipsius theorematis praecipuis indicatis, tum via monstrata contentus ero.

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