DE

ATTRACTIONE CORPORVM

SPHAEROIDICO-ELLIPTICORVM.

AVCTORE

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Problema I.

Tab. XI. Fig. 1.

Sit in plano horizontali ſeu plano chartae ellipſis, cuius ſingulae particulae vniformiter attrahant in ratione reciproca duplicata diſtantiarum; Ellipſis vero huius axes ſint AB, EF centrumque C, atque in recta verticali per C ducta CO, poſitum ſit punctum O; de quo quaeritur quanta vi id ab ellipſi attrahatur.

Solutio.

Primo perſpicuum eſt, quia ſingulae ellipſis particulae aequali vi attractiua praeditae ponuntur, et punctum O, quippe in verticali CO poſitum, centro ellipſis imminet, id ab omnibus viribus coniunctim ad centrum C tractum iri. Quamobrem vires quibus ad ſingulas ellipſis particulas attrahitur, reſoluendae ſunt in laterales, quarum alterae in OC incidant, alterae directiones habeant horizontales, quae poſteriores negligi poſſunt, cum omnes ſe mutuo deſtruant, ita vt ad problema ſoluendum ſufficiat, vires eas conſiderare, quarum directiones in verticalem OC cadant.

Ponatur iam ſemiaxis $AC = a$; ſemiaxis $CF = b$; et diſtantia $CO = c$. Axi EF ducatur ordinata parallela MM, MM, eique proxima mm; eodemque interuallo ex altera parte ordinatae NN et nn vt habeantur ellipsis elementa $MmmM$, $NnnN$, ad quae quanta vi punctum O in directione OC trahatur, inuestigemus. Sit igitur $CP = CQ = x$; $Pp = dx$; erit ex natura ellipsis $PM = QN = \frac{b}{a}\sqrt{(aa-xx)}$. In elemento $MmmM$ consideretur quaeuis particula Xz existente $PX = z$ et $Xx = dz$; eritque ipsa particula $Xz = dxdz$; cuius a puncto O distantia est $\sqrt{(c^2+x^2+z^2)}$. Vis igitur qua punctum O ad hanc particulam trahetur erit vt $\frac{dxdz}{c^2+x^2+z^2}$; ex qua obtinebitur vis lateralis, qua O in directione OC trahitur si fiat vt $\sqrt{(cc+xx+zz)}$ ad c ita vis $\frac{dxdz}{c^2+x^2+z^2}$ ad quaesitum quae ergo erit $= \frac{cdxdz}{(cc+xx+zz)^{\frac{3}{2}}}$; quae integrata posito x constante dabit vim in directione OC, qua O ab elemento $PpZX$ trahitur, integrale vero est $\frac{czdx}{(cc+xx)\sqrt{(c^2+x^2+z^2)}}$. Ponatur $z = PM = \frac{b}{a}\sqrt{(a^2-x^2)}$ habebitur vis, qua O ab elemento Ppm M ad C vrgetur, eritque $= \frac{bcdx\sqrt{(aa-xx)}}{(cc+xx)\sqrt{(aacc+aabb+aaxx-bbxx)}}$. Vis ergo, qua punctum O ad C vrgebitur, ab vtroque elemento $MmmM$ et $NnnN$ coniunctim, erit $= \frac{4bcdx\sqrt{(a^2-x^2)}}{(cc+xx)\sqrt{(aa(bb+cc)+(aa-bb)xx)}}$. Huius ergo integrale ita sumtum vt euanescat posito $x = 0$, dabit vim qua punctum O ad C attrahitur ab elipsis portione MENNFM. Atque si tum ponatur $x = a$, prodibit vis attractiua ex tota ellipsi orta, quae postulatur.

Formula autem proposita differentialis ita est comparata, vt ad rationalitatem reduci et proinde integrari nequeat, nisi sit $a = b$, quo quidem casu, scilicet quando ellipsis abit in circulum, formula differentialis satis manet per-

perplexa, vt difficulter inde attractio, quae alias facile considerandis elementis circularibus ipſi C concentricis eruitur, inueniri queat. Ingens autem in hoc negotio ſubſidium adhiberi poſſe obſeruaui, ſi non integrale indefinitum formulae propoſitae, ſed ſtatim id integrale poſito $x = a$ inueſtigetur. Quod quo commode perfici queat, ſequentes integrationes ſunt praemittendae, quae ad hypotheſin $x = a$ ſunt accommodatae; vbi $\pi : 1$ denotat rationem peripheriae ad diametrum

$$\int dx\sqrt{(a^2-x^2)} = \frac{\pi a^2}{4}$$

$$\int x^2 dx\sqrt{(a^2-x^2} = \frac{1}{4}\cdot\frac{\pi a^4}{4}$$

$$\int x^4 dx\sqrt{(a^2-x^2)} = \frac{1\cdot 3}{4\cdot 6}\cdot\frac{\pi a^6}{4}$$

$$\int x^6 dx\sqrt{(a^2-x^2)} = \frac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8}\cdot\frac{\pi a^8}{4}$$

Inventio harum expreſſionum hoc nititur fundamento, quod ſit

$$\int x^{m+2}\, dx\sqrt{(aa-xx)} = \frac{-x^{m+1}(aa-xx)^{\frac{3}{2}}}{m+4} + \frac{(m+1)a^2}{m+4}\int x^m dx$$

$\sqrt{(aa-xx)}$ generaliter quidem, vti differentialia ſumenti patebit. Caſu ergo quo $x = a$ erit $\int x^{m+2} dx\sqrt{(aa-xx)} = \frac{(m+1)a^2}{m+4}\int x^m dx\sqrt{(aa-xx)}$. Vt nunc formulas has adhibere queamus, reſoluendus eſt factor ipſius $dx\sqrt{(aa-xx)}$ in ſeriem, cuius termini teneant poteſtates ipſius x parium exponentium. Atque primo quidem eſt $\frac{4bc}{cc-xx} = \frac{4b}{c} - \frac{4bxx}{c^3} + \frac{4bx^4}{c^5} - \frac{4bx^6}{c^7} + \frac{4bx^8}{c^9} -$ etc. atque $\frac{1}{\sqrt{(aa(bb+cc)+(aa-bb)xx)}} = \frac{1}{a\sqrt{(bb+cc)}} - \frac{1(aa-bb)xx}{2a^3(bb+cc)^{\frac{3}{2}}} + \frac{1\cdot 3(aa-bb)^2x^4}{2\cdot 4a^5(bb+cc)^{\frac{5}{2}}} -$ etc. His ſeriebus in ſe ductis prodibit $\frac{4bcdx\sqrt{(aa-xx)}}{(cc+xx)\sqrt{(aa(bb+cc)+(aa-bb)xx)}} =$

4bdx

$$4bdx\surd(aa-xx)\left\{\begin{array}{l}
\frac{1}{ac\surd(bb+cc)}-\frac{xx}{ac^3\surd(bb+cc)}+\frac{x^4}{ac^5\surd(bb+cc)}-\frac{x^6}{ac^7\surd(bb+cc)}+\text{etc.}\\
\qquad-\frac{1(aa-bb)xx}{2a^3c(bb+cc)^{\frac{3}{2}}}+\frac{1(aa-bb)x^4}{2a^3c^3(bb+cc)^{\frac{3}{2}}}-\frac{1(aa-bb)x^6}{2a^3c^5(bb+cc)^{\frac{3}{2}}}+\text{etc.}\\
\qquad\qquad+\frac{1.3(aa-bb)^2x^4}{2.4.a^5c(bb+cc)^{\frac{5}{2}}}-\frac{1.3.(aa-bb)^2x^6}{2.4a^5c^3(bb+cc)^{\frac{5}{2}}}\text{etc.}\\
\qquad\qquad\qquad-\frac{1.3.5(aa-bb)^3x^6}{2.4.6a^7c(bb+cc)^{\frac{7}{2}}}\text{etc.}
\end{array}\right.$$

Siue sequente forma ſuccinctiori adhibita, habebitur

$$4bdx\surd(aa-xx)\left\{\begin{array}{l}
+\frac{1}{ac\surd(bb+cc)}\left(1-\frac{xx}{cc}+\frac{x^4}{c^4}-\frac{x^6}{c^6}+\frac{x^8}{c^8}-\text{ect.}\right)\\
+\frac{1(aa-bb)c}{2a^3(bb+cc)^{\frac{3}{2}}}\left(-\frac{xx}{cc}+\frac{x^4}{c^4}-\frac{x^6}{c^6}+\frac{x^8}{c^8}-\text{etc.}\right)\\
+\frac{1.3(aa-bb)^2c^3}{2.4.a^5(bb+cc)^{\frac{5}{2}}}\left(+\frac{x^4}{c^4}-\frac{x^6}{c^6}+\frac{x^8}{c^8}-\text{etc.}\right)\\
+\frac{1.3.5(aa-bb)^3c^5}{2.4.6a^7(bb+cc)^{\frac{7}{2}}}\left(-\frac{x^6}{c^6}+\frac{x^8}{c^8}-\text{etc.}\right)\\
+\frac{1.3.5.7.(aa-bb)^4c^7}{2.4.6.8a^9(bb+cc)^{\frac{9}{2}}}\left(+\frac{x^8}{c^8}-\frac{x^{10}}{c^{10}}+\text{etc.}\right)\\
+\frac{1.3.5.7.9(aa-bb)^5c^9}{2.4.6.8.10a^{11}(bb+cc)^{\frac{11}{2}}}\left(-\frac{x^{10}}{c^{10}}+\text{etc.}\right)
\end{array}\right.$$

Si nunc huius expreſſionis ſinguli termini ſeorſim integrentur per formulas datas pro hypotheſi $x=a$, attractio quaeſita habebitur; at commode hic accidit, vt poſt integrationem ſingulae ſeries ſummationem admittant; quod quo melius pateat conſideremus cuiusque ſeriei integrale ſeorſim, eritque integrale huius $4bdx\surd(aa-xx)\left(1-\frac{xx}{cc}+\frac{x^4}{c^4}-\frac{x^6}{c^6}+\frac{x^8}{c^8}-\text{etc.}\right)$ quod reperietur $=\pi\left(a^2b-\frac{1}{4}\frac{a^4b}{cc}+\frac{1.3}{4.6}.\frac{a^6b}{c^4}-\frac{1.3.5a^8b}{4.6.8c^6}+\frac{1.3.5.7a^{10}b}{4.6.8.10c^8}-\text{etc.}\right)$ Simili modo erit $\int 4bdx\surd(aa-xx).\ \left(-\frac{xx}{cc}+\frac{x^4}{c^4}-\frac{x^6}{c^6}+\text{etc.}\right)$

 =

$=\pi(-\frac{1}{4}\frac{a^4b}{c^2}+\frac{1.3.a^6b}{4.6.c^4}-\frac{1.3.5a^7b}{4.6.8c^6}+$ etc.) atque $\int 4bdx\sqrt{(aa-xx)}.$ $(\frac{x^4}{c^4}-\frac{x^6}{c^6}+\frac{x^8}{c^8}-$etc.$)=\pi(\frac{1.3a^6b}{4.6c^4}-\frac{1.3.5a^7b}{4.6.8.c^6}+$ etc.), etc. Vnde intelligitur ſi vnica ſerierum harum ſit ſummabilis etiam omnium ſummas exhiberi poſſe. Conſidero igitur primam, cui hanc do formam $\pi bcc(\frac{aa}{cc}-\frac{1a^4}{4c^4}+\frac{1.3a^6}{4.6c^6}-\frac{1.3.5a^8}{4.6.8c^8}+$etc.) poſitoque $\frac{a}{c}=r$, pono $r^2-\frac{1}{4}r^4+\frac{1.3}{4.6}r^6-\frac{1.3.5}{4.6.8}r^8+$etc.$=s$. hinc erit differentiando $\frac{ds}{dr}=2r-1r^3+\frac{1.3}{4}r^5-\frac{1.3.5}{4.6.}r^7+$etc. atque $\frac{ds}{r^3}=\frac{2dr}{r^2}-1dr+\frac{1.3}{4}r^2dr-\frac{1.3.5}{4.6}r^4dr+$etc. et integrando $\int\frac{ds}{r^3}=-\frac{2}{r}-r+\frac{1}{4}r^3-\frac{1.3}{4.6}r^5+$etc. $=-\frac{2}{r}-\frac{s}{r}$ Ex aequatione ergo $\int\frac{ds}{r^3}+\frac{2+s}{r}=0$ oritur $\frac{ds}{r}+rds-2dr-sdr=0$ ſeu $ds-\frac{srdr}{1+rr}=\frac{2rdr}{1+rr}$, quae per $\sqrt{(1+rr)}$ diuidendo abit in $\frac{ds}{\sqrt{(1+rr)}}-\frac{srdr}{(1+rr)^{\frac{3}{2}}}=\frac{2rdr}{(1+rr)^{\frac{3}{2}}}$, cuius integrale eſt $\frac{s}{\sqrt{(1+rr)}}=C-\frac{2}{\sqrt{(1+rr)}}$ $(=2-\frac{2}{\sqrt{(1+rr)}})$ quia facto $r=0$ fit $s=0$. Erit igitur $s=2\sqrt{(1+rr)}-2=\frac{2\sqrt{(aa+cc)}-2c}{c}=\frac{aa}{cc}-\frac{1a^4}{4c^4}+\frac{1.3a^6}{4.6c^6}-$etc. Conſequenter habebitur $\pi(aab-\frac{1a^4b}{4c^2}+\frac{1.3a^6b}{4.6c^4}-\frac{1.3.5a^8b}{4.6.8c^6}+$etc.$)=2\pi bc$ $(\sqrt{(aa+cc)}-c)$. Quocirca ſuperiores integrationes erunt

$\int 4bdx\sqrt{(aa-xx)}.\ (1-\frac{xx}{cc}+\frac{x^4}{c^4}-$etc.$)=\pi(2bc\sqrt{(aa+cc)}-2bcc)$

$\int 4bdx\sqrt{(aa-xx)}.(-\frac{xx}{cc}+\frac{x^4}{c^4}-$etc.$)=\pi(2bc\sqrt{(aa+cc)}-2bcc-aab)$

$\int 4bdx\sqrt{(aa-xx)}\ (\frac{x^4}{c^4}-\frac{x^6}{c^6}+$etc.$)=\pi(2bc\sqrt{(aa+cc)}-2bcc-aab+\frac{1a^4b}{4cc})$

$\int 4bdx\sqrt{(aa-xx)}\ (-\frac{x^6}{c^6}+\frac{x^8}{c^8}-$ etc.$)=\pi(2bc\sqrt{(aa+cc)}-2bcc-aab+\frac{1a^4b}{4cc}-\frac{1.3a^6b}{4.6c^4})$

etc.

Pro ſingularum ergo ſerierum ſuperioris formulae differentialis integralibus nacti ſumus expreſſiones finitas. Tantum igitur ſuper eſt, vt eas ſubſtituamus, quo facto pro integrali

grali formula $\frac{4bcdx\sqrt{(aa-xx)}}{(cc+xx)\sqrt{(aa(bb+cc)+(aa-bb)xx)}}$ atque ideo pro quantitate attractionis quaesitae puncti O ad centrum ellipsis AEBFA sequens orietur valor. $\pi\left(\frac{2b\sqrt{(aa+cc)}}{a\sqrt{(bb+cc)}}-\frac{2bc}{a\sqrt{(bb+cc)}}+\frac{bcc(aa-bb)\sqrt{(aa+cc)}}{a^3(bb+cc)^{\frac{3}{2}}}-\frac{bc^3(aa-bb)}{a^3(bb+cc)^{\frac{3}{2}}}-\frac{bc(aa-bb)}{2a(bb+cc)^{\frac{3}{2}}}+\text{etc.}\right.$

Vel cum ad applicationem ad computum expediat ipsas series retinere, quo singulorum terminorum integralia algebraice exhiberi queant, praecipue casibus quibus a et b non multum a se inuicem differunt, pono $\sqrt{(aa+cc)} = \sqrt{(bb+cc+aa-bb)}$ eritque $\sqrt{(aa+cc)} = \sqrt{(bb+cc)} + \frac{1(aa-bb)}{2\sqrt{(bb+cc)}} - \frac{1.1(aa-bb)^2}{2.4(bb+cc)^{\frac{3}{2}}} + \frac{1.1.3(aa-bb)^3}{2.4.6(bb+cc)^{\frac{5}{2}}}$ &c. Quo substituto prodibit attractio quaesita =

$$\pi\left\{\begin{aligned}
&\frac{2b}{a}+\frac{b(aa-bb)}{a(bb+cc)}-\frac{1b(aa-bb)^2}{4a(bb+cc)^2}+\frac{1.3b(aa-bb)^3}{4.6a(bb+cc)^3}\\
&\frac{-2bc}{a\sqrt{(bb+cc)}}+\frac{bcc(aa-bb)}{a^3(bb+cc)}+\frac{1bc^2(aa-bb)^2}{2a^3(bb+cc)^2}-\frac{1.1bc^2(aa-bb)^3}{2.4a^3(bb+cc)^3}\\
&\qquad-\frac{bc^3(aa-bb)}{a^3(bb+cc)^{\frac{3}{2}}}+\frac{3bc^4(aa-bb)^2}{4a^5(bb+cc)^2}+\frac{1.3bc^4(aa-bb)^3}{2.4a^5(bb+cc)^3}\\
&\qquad-\frac{bc(aa-bb)}{2a(bb+cc)^{\frac{3}{2}}}-\frac{3bc^5(aa-bb)^2}{4a^5(bb+cc)^{\frac{5}{2}}}+\frac{3.5bc^6(aa-bb)^3}{4.6a^7(bb+cc)^3}\\
&\qquad\qquad-\frac{1.3bc^3(aa-bb)^2}{2.4a^3(bb+cc)^{\frac{5}{2}}}-\frac{3.5bc^7(aa-bb)^3}{4.6a^7(bb+cc)^{\frac{7}{2}}}\\
&\qquad\qquad+\frac{1.1.3bc(aa-bb)^2}{4.2.4a(bb+cc)^{\frac{5}{2}}}-\frac{1.3.5bc^5(aa-bb)^3}{2.4.6a^5(bb+cc)^{\frac{7}{2}}}\\
&\qquad\qquad\qquad+\frac{1.1.3.5bc^3(aa-bb)^3}{4.2.4.6a^3(bb+cc)^{\frac{7}{2}}}\\
&\qquad\qquad\qquad-\frac{1.3.1.3.5bc(aa-bb)^3}{4.6.2.4.6a(bb+cc)^{\frac{7}{2}}}
\end{aligned}\right.$$

 Huius

Huius progreſſionis altera terminorum pars eſt rationalis altera irrationalis, et quemadmodum ſinguli progrediantur exceptis vnciis facile patet. Vnciae autem rationalium ita formantur quaeque ex ſuperiore, vt factor ad dextram reiiciatur, et ad ſiniſtram praecedens apponatur, ſic ex $\frac{1.3}{4.6}$ reiiciendo $\frac{3}{6}$ fit $\frac{1}{4}$ et praecedens, qui prodit minuendo tam numeratorem quam denominatorem binario, et eſt $-\frac{1}{2}$ appoſitus, dat $-\frac{1.1}{2.4}$ vnciam ſequentem. Vnciae vero irrationalium ex ſuperioribus formantur ſola appoſitione ad ſiniſtram. Qua lege obſeruata, quousque libuerit hanc expreſſionem continuare licebit; termini autem, quos hic appoſuimus abunde ſufficiunt, ſi differentia inter axes ſit ſatis exigua, ſeu ellipſis circulo propinqua; pro quibus caſibus potiſſimum hoc problemate vtemur. Q. E. I.

Corollarium.

Caſu ergo, quo ellipſis abit in circulum, cuius ſemidiameter eſt $= b$, ſeu quo fit $a = b$, ob omnes terminos praeter primos euaneſcentes, attractio puncti O ad centrum circuli C erit $= \pi \left(2 - \frac{2c}{\sqrt{(bb+cc)}}\right)$, prorſus vt altero modo, quo elementa circularia conſiderantur, facilius reperitur.

Problema II.

Fig. 2. Si ſphaeroides generetur conuerſione ellipſis AEBF circa ſuum axem minorem EF eiusque particulae omnes aequali vi attractiua praeditae fuerint, quae diſtantiarum quadratis reciproce ſit proportionalis, inuenire vim attractiuam corporis in polo E ſiti, verſus centrum ſphaeroidis.

So-

Solutio.

Posito semiaxe minore $CE = n$, maiore $AC = m$; et abscissa $EP = x$, erit $PM = \frac{m}{n}\sqrt{(2nx - xx)}$. Concipiatur sectio huius sphaeroidis ad axem EP normalis et per P transiens, erit ea circulus cuius radius erit $PM = \frac{m}{n}\sqrt{(2nx - xx)}$; a quo corpusculum in E distat interuallo $EP = x$. Facta ergo coroll. praec. applicatione erit ob $c = x$ et $b = \frac{m}{n}\sqrt{(2nx - xx)}$, attractio corporis in E ad hunc circulum $= \pi\left(2 - \frac{2nx}{\sqrt{(2mmnx + nnxx - mmxx)}}\right)$. Atque ad discum rotundum elemento $Ppm M$ genitum $= 2\pi\left(dx - \frac{nx\,dx}{\sqrt{(2mmnx - (mm-nn)xx)}}\right)$, cuius integrale dabit attractionem portionis sphaeroidis ab EPM genitae; quod vt commode exprimatur pono $(2m^2nx - (mm-nn)xx)^{-\frac{1}{2}} = \frac{1}{m\sqrt{2nx}} + \frac{1(mm-nn)\sqrt{x}}{4m^3n\sqrt{2n}} + \frac{1.3(m^2-n^2)^2x\sqrt{x}}{4.8m^5n^2\sqrt{2n}} + \frac{1.3.5(m^2-n^2)^3x^2\sqrt{x}}{4.8.12m^7n^3\sqrt{2n}} - \text{etc.}$

Vnde integrale erit $2\pi\left(x - \frac{2nx^{\frac{3}{2}}}{3m\sqrt{2n}} - \frac{1(mm-nn)x^2\sqrt{2nx}}{4.5m^3n} - \frac{1.3(m^2-n^2)^2x^3\sqrt{2nx}}{4.8.7m^5n^2} - \text{etc.}\right)$ Hincque ponendo $x = 2n$ prodibit totalis attractio ad sphaeroides $2\pi\left(2n - \frac{4nn}{3m} - \frac{1.(mm-nn)8n^3}{4.5m^3} - \frac{1.3(m^2-n^2)^2 16n^4}{4.8.7m^5} - \frac{1.3.5(mm-nn)^3 32n^5}{4.8.12.9.m^7} - \text{etc.}\right) = 4\pi n - 8\pi nn\left(\frac{1}{3m} + \frac{1(mm-nn)}{2.5m^3} + \frac{1.3(mm-nn)^2}{2.4.7m^5} + \frac{1.3.5(mm-nn)^3}{2.4.6.9m^7} + \text{etc.}\right)$ quae series vehementer conuergit et cito verum valorum exhibet nisi sphaeroides multum a sphaera discrepet. Q. E. I.

Corollarium 1.

Si sphaeroides abeat in globum cuius radius $= n$, fiet $m = n$ atque attractio in quoque eius superficiei puncto erit $= \frac{4\pi n}{3}$ euanescentibus reliquis terminis omnibus;

Attractiones ergo diuerſarum ſphaerarum homogenearum in ipſis ſuperficiebus ſunt vt diametri.

Corollarium 2.

Si ſphaeroides ſit admodum propinquum ſphaerae vt differentia $m-n$ ſit quaſi infinite parua, ponatur $m=n+dz$; erit attractio in polis huius ſphaeroidis $=4\pi\left(\frac{n}{3}+\frac{4dz}{15}-\frac{4dz^2}{42n}+\frac{199dz^3}{945n^2}-\text{etc.}\right)$. Qui termini abunde ſufficiunt, ſi dz reſpectu n fuerit valde paruum.

Problema III.

Fig. 3. In praecedente ſphaeroide, quae generatur conuerſione ellipſis AEBF circa axem EF, inuenire attractionem, qua corpus ſub aequatore in A ſitum ad centrum C vrgetur.

Solutio.

Maneat vt ante $AC=m$; $CE=n$ ſitque abſciſſa $AP=x$ erit $PM=\frac{n}{m}\sqrt{(2mx-xx)}$. Per P concipiatur ſectio facta ad AB normalis quae erit ellipſis, cuius alter ſemiaxis erit $=\frac{n}{m}\sqrt{(2mx-xx)}$ alter vero $=\sqrt{(2mx-xx)}$; ad hanc ergo ellipſin quanta vi corpus in A poſitum attrahatur, determinari oportet; id quod ſubſidio primi problematis efficietur. Erit autem facta applicatione $c=x$; $b=\sqrt{(2mx-xx)}$ et $a=\frac{n}{m}\sqrt{(2mx-xx)}$ atque $\sqrt{(bb+cc)}=\sqrt{2mx}$; et $aa-bb=-\frac{(mm-nn)(2mx-xx)}{mm}$, quibus valoribus ſubſtitutis ſequens prodibit expreſſio pro attractione quaeſita in A. a ſectione elliptica per MM facta. =

π

$$\pi\begin{cases}\frac{2m}{n}-\frac{\sqrt{2mx}}{n}-\frac{(mm-nn)}{mn}-\frac{(mm-nn)^2x}{2mmn^3}\\ +\frac{(mm-nn)}{4m^2n^3\sqrt{2m}}((2mm-nn)x\sqrt{x}+2mnn\sqrt{x})\\ +\frac{(mm-nn)^2}{16m^5n^5}((3m^4-2mmnn-n^4)x^2+4mn^2(m^2+n^2)x-4m^2n^4)\\ -\frac{3(mm-nn)^2}{128m^5n^5\sqrt{2m}}((8m^4-4mmnn-n^4)x^2\sqrt{x}+4mn^2(2m^2+n^2)x\sqrt{x}-\\ \qquad 4m^2n^4\sqrt{x})\ \text{etc.}\end{cases}$$

Quae expreſſio multiplicata per dx dabit attractionem corpuſculi in A ſiti ad diſcum ellipticum craſſitiei dx. Huius ergo integrale ſi ponatur $x=2m$ dabit attractionem puncti A ad totam ſphaeroidem; quae attractio ſequentem habebit valorem

$$\pi\begin{cases}\frac{4mm}{n}-\frac{2(mm-nn)}{n}\\ \frac{-9mm}{3n}-\frac{(mm-nn)^2}{n^3}\\ \qquad +\frac{4(mm-nn)}{n^3}\left(\frac{1}{5}mm+\frac{1}{15}nn\right)\\ \qquad +\frac{(m^2-n^2)^2}{2m^2n^5}\left(m^4+\frac{1}{3}m^2n^2-\frac{1}{3}n^4\right)\\ \qquad -\frac{3(m^2-n^2)^2}{m^2n^5}\left(\frac{1}{7}m^4+\frac{1}{7.5}m^2n^2-\frac{1}{7.5.3}\ n^4\right)\end{cases}$$

qui reductus in ſequentem formam abit

$$\pi\left(\frac{4mm}{3n}-\frac{(mm-nn)}{5n^3}(mm+\frac{11}{3}nn)+\frac{(mm-nn)^2}{14m^2n^5}(m^4+\frac{17}{15}m^2n^2-\frac{29}{15}n^4)\text{etc.}\right)$$

Q. E. I.

Corollarium

Si differentia inter m et n ſit minima ita vt ſit $m=n+dz$ erit grauitas ſub aequatore in noſtra ſphaeroide $=4\pi(\frac{n}{3}+\frac{dz}{5}-\frac{3dz^2}{35n})$, cum contra grauitas ſub polo inuenta ſit $=4\pi(\frac{n}{3}+\frac{4dz}{15}-\frac{2dz^2}{21n})$, ita vt grauitas ſub polo maior ſit parte $4\pi(\frac{dz}{15}-\frac{dz^2}{105n})$. Si ſit $n:m=100:101$ fiet grauitas ſub polo ad grauitatem ſub aequatore vt 509 ad 508.

Proble-

Problema IV.

Si planeta conſtans ex materia vniformi, cuius ſingulae particulae attrahant in ratione reciproca duplicata diſtantiarum, habeat motum vertiginis circa axem, indeque grauitas vera ſub aequatore a vi centrifuga diminuatur parte ſua $\frac{1}{k}$; inuenire eius planetae rationem inter axes per polos et aequatorem ductos.

Solutio.

Ponamus decrementum grauitatis a vi centrifuga ortum tam eſſe exiguum ratione ipſius grauitatis, vt figura planetae non multum a ſphaera diſcrepet. Si enim nulla eſſet vis centrifuga dubium non eſt, quin planeta ipſum ſphaericam figuram induere debeat. Cum ergo figura planetae tantillum a ſphaerica diſcrepet, ea pro ſphaeroide elliptica tuto haberi poterit, cuius ellipſis generantis axes non multum a ſe inuicem diſcrepent. Erit vero planeta ſolidum rotundum circa axem per polos ductum; ita vt eius figura concipi queat, tanquam ſphaerois elliptica, cuius poli cum polis planetae congruant.

Fig. 4. Sit ergo planetae figura quam quaerimus ſphaerois elliptica genita conuerſione ellipſis APBQ circa axem PQ ita vt P et Q futuri ſunt poli planetae et PC ſemiaxis planetae; et AC ſemidiameter aequatoris. Ponatur $PC = n$ et $AC = m$; et quia differentia inter hos ſemiaxes eſt valde parua ſit $m = n + dz$. Sit grauitas ſub polo $P = g$, et ſub aequatore ſi nullum haberet motum vertiginis $= \gamma$, erit per coroll. praec. $g : \gamma = \frac{n}{3} + \frac{4dz}{15} - \frac{2dz^2}{21n} : \frac{n}{3} + \frac{dz}{5} - \frac{3dz^2}{35n} = n + \frac{4dz}{5} - \frac{2dz^2}{7n} : n + \frac{3dz}{5} - \frac{9dz^2}{35n}$. Figura autem planetae ita debet eſſe comparata, vt eam planeta con-

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ſeruare poſſet, etiamſi totus fluidus foret; quod eueniet ſi omnes preſſiones verſus centrum tam ex vi grauitatis quam vi centrifuga ortae ſeſe in aequilibrio teneant. Si ergo concipiatur tubus reflexus ACP a polo per centrum ad aequatorem pertingens; atque aqua repletus, pondus aquae in tubo AC contentae aequale eſſe debebit ponderi aquae in tubo PC contentae, ſiquidem tubus vbique eandem habeat amplitudinem. Nam ſi preſſio aquae in altero crure praeualeret, tum aqua ex altero efflueret, figuramque planetae immutaret. Preſſio vero aquae in vtroque tubo contentae habebitur, ſi ſingularum particularum pondera colligantur; quo facto vtrobique eadem ſumma emergere debebit. Ad quod praeſtandum notari debet in eodem crure grauitates ſeu niſus ad centrum eſſe diſtantiis a centro proportionales, ſi quidem figura a ſphaerica non multum differat; et ſimili modo vim centrifugam eandem retinere rationem. Quamobrem aquae in canali AC vera preſſio obtinebitur ſi eius pondus a grauitate ortum, diminuatur ſui parte $\frac{1}{k}$. Ad vtramque ergo preſſionem inueniendam conſideretur in tubo AC aquae particula $Xx = dx$, poſita $CX = x$; cuius pondus ſi in A eſſet poſita foret γdx; quod ergo ſe habebit ad pondus eiusdem particulae in ſuo loco X verſantis vt CA (m) ad CX(x). ita vt pondus huius particulae futurum ſit $= \frac{\gamma x dx}{m}$; cuius integrale $\frac{\gamma xx}{2m}$ dabit pondus columnae CX et poſito $x = m$ habebitur pondus totius columnae $AC = \frac{\gamma m}{2}$ a ſola grauitate ortum. Simili modo pondus columnae PC erit $= \frac{gn}{2}$. At inuentum pondus columnae AC ob vim centrifugam minuendum eſt ſui parte $\frac{1}{k}$, ita vt vera preſſio columnae AC deorſum futura ſit $= \frac{\gamma m}{2} \left(\frac{k-1}{k} \right)$ quae aequalis eſſe debet

bet preſſioni columnae $PC = \frac{gn}{2}$; vnde ſequens obtinetur aequatio $\gamma mk - \gamma m = gnk$. Cum vero ſit $m = n + dz$; et $g : \gamma = n + \frac{4dz}{5} - \frac{2dz^2}{7n} : n + \frac{3dz}{5} - \frac{9dz^2}{35n}$ erit ſubſtituendo $nnk + \frac{4nkdz}{5} - \frac{2kdz^2}{7} = (n + dz)(k - 1)(n + \frac{3dz}{5} - \frac{9dz^2}{35n} = n^2k + \frac{8nkdz}{5} - \frac{12kdz^2}{35} - n^2 - \frac{8ndz}{5} + \frac{12dz^2}{35}$ vnde ſequens concinnatur aequatio $\frac{4nkdz}{5} - \frac{2kdz^2}{35} = n^2 - \frac{8ndz}{5} + \frac{12dz^2}{35}$ ex qua eruitur $dz = \frac{5n}{4(k+2)} + \frac{25n(k+6)}{8(k+2)(26k^2+107k+32)}$, qui poſterior terminus ob k valde magnum facile negligitur. Erit itaque $AB : PQ = 4k + 13 : 4k + 8$. Q. E. I.

Corollarium 1.

In terra ergo noſtra, vbi vis centrifuga ſub aequatore eſt pars $\frac{1}{289}$ ipſius grauitatis, fiet $K = 289$; ideoque per regulam inuentam erit diameter aequatoris ad axem terrae vt 1169 ad 1164 hoc eſt, proxime vt 234 ad 233. pro qua ratione Newton inuenit ut 230 ad 229.

Corollarium 2.

In ſuperficie ſolis ponit Newton pondus corporis cuiusque $\frac{1000}{41}$ quod in terra eſſet 1. Deinde cum ſol vertatur circa axem ſuum diebus circiter 25. eiusque diameter ſit ad diametrum terrae 10000 ad 104 erit vis centrifuga ſub aequatore ſolis $= \frac{10000}{104} \cdot \frac{1}{625} \cdot \frac{1}{289}$ ideoque erit $K = \frac{104 \cdot 625 \cdot 289}{410} = 17666$. In ſole ergo erit diameter aequatoris ad eius axem vt 14136 ad 14135.

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Corollarium 3.

In Ioue ponit Newton pondus corporis, quod in terra 1 esset $= \frac{167}{82}$. Deinde cum Iupiter conuertatur horis 9. 56′ eiusque diameter sit ad diametrum terrae vt 1077 ad 104 erit vis centrifuga sub aequatore Iouis $= \frac{1077}{104} \cdot \frac{29}{5} \cdot \frac{1}{289}$. Ergo pro Ioue fit $K = \frac{167 \cdot 104 \cdot 5 \cdot 289}{82 \cdot 29 \cdot 1077} = 9\frac{4}{5}$. Vnde inuenitur diameter aequatoris Iouis ad eius axem vt 261 ad 236 hoc est proxime vt 10 : 9.

In his autem ponuntur corpora solis et planetarum horum ex materia uniformi composita.