CAPUT III.

De inventione curvarum maximi minimive proprietate præditarum, si in ipsa maximi minimive formula insunt quantitates indeterminatæ.

PROPOSITIO I. PROBLEMA.

1. *Invenire incrementa, quæ quantitas integralis indeterminata, in quovis abscissæ puncto, ab aucta alicubi una applicata* Nn *particula* nν, *capit.* Fig. 4.

SOLUTIO.

Sit abscissa AH $= x$, applicata respondens Hh $= y$, & proposita sit quantitas quæcunque indeterminata Π, abscissæ AH respondens, quæ sit formula integralis indefinite integrationem non admittens. Quantitas hæc Π ita sit comparata, ut ipsa, quatenus abscissæ AH seu puncto H respondet, ab aucta applicata Nn non mutetur: quod eveniet, si in Π differentialia non ultra quintum gradum assurgant; quem in finem quintam demum applicatam Nn ab Hh computando mutari ponimus. Si enim in Π differentialia altiorum graduum continerentur, tum deberet ulterior demum applicata post Nn particula infinite parva augeri. Sufficiet autem solutionem ad quinque tantum differentialium in Π contentorum gradus extendere; cum inde, si etiam altiora affuerint differentialia, solutionem ad ea accommodare liceat. Quemadmodum igitur puncto abscissæ H respondet valor Π, ita secundum nostram notandi methodum, puncto sequenti I respondebit valor Π', puncto K vero Π'', puncto L valor Π''', & ita porro. Id ergo erit investigandum, quanta incrementa ex translatione puncti n in ν singuli hi valores derivativi Π', Π'', Π''', Π^{IV}, &c. accipiant, seu definiri debent eorum differentialia, si sola applicata Nn, quæ est $= y'$ variari & particula $n\nu$ augeri ponatur: erit autem hoc sensu $d.\,\Pi = 0$, quia valorem Π puncto H respondentem inde non affici ponimus. Quoniam jam Π est formula integralis indefinita, sit ea $= \int [Z]\,dx$, & $[Z]$ sit functio ipsarum x, y, p, q, r, s & t, ita ut sit $d[Z] = [M]\,dx + [N]\,dy + [P]\,dp + [Q]\,dq + [R]\,dr + [S]\,ds + [T]\,dt$; unde simul valores derivativi ipsius $d[Z]$, nempe $d[Z']$, $d[Z'']$, $d[Z''']$, &c. per notandi modum receptum formari poterunt. His positis, erit ut sequitur

$$\Pi = \int [Z]\,dx$$
$$\Pi' = \int [Z]\,dx + [Z]\,dx$$
$$\Pi'' = \int [Z]\,dx + [Z]\,dx + [Z']\,dx$$
$$\Pi''' = \int [Z]\,dx + [Z]\,dx + [Z']\,dx + [Z'']\,dx$$
$$\Pi^{IV} = \int [Z]\,dx + [Z]\,dx + [Z']\,dx + [Z'']\,dx + [Z''']\,dx$$
&c.

Jam videamus quanta incrementa singula hæc membra $[Z]\,dx$, $[Z']\,dx$, $[Z'']\,dx$, $[Z''']\,dx$, &c. ex adjecta particula $n\nu$ ad applicatam Nn capiant; quæ obtinebuntur ex ipsorum differentialibus, ponendo loco differentialium valores §. 56 Capitis præcedentis expositos: erit itaque

$$d.\,[Z]\,dx = n\nu.\,dx.\,\frac{[T]}{dx^5}$$
$$d.\,[Z']\,dx = n\nu.\,dx\left(\frac{[S']}{dx^4} - \frac{5[T']}{dx^5}\right)$$
$$d.\,[Z'']\,dx = n\nu.\,dx\left(\frac{[R'']}{dx^3} - \frac{4[S'']}{dx^4} + \frac{10[T'']}{dx^5}\right)$$
$$d.\,[Z''']\,dx = n\nu.\,dx\left(\frac{[Q''']}{dx^2} - \frac{3[R''']}{dx^3} + \frac{6[S''']}{dx^4} - \frac{10[T''']}{dx^5}\right)$$
$$d.\,[Z^{IV}]\,dx = n\nu.\,dx\left(\frac{[P^{IV}]}{dx} - \frac{2[Q^{IV}]}{dx^2} + \frac{3[R^{IV}]}{dx^3} - \frac{4[S^{IV}]}{dx^4} + \frac{5[Z^{IV}]}{dx^5}\right)$$
$$d.\,[Z^{V}]\,dx = n\nu.\,dx\left([N^{V}] - \frac{[P^{V}]}{dx} + \frac{[Q^{V}]}{dx^2} - \frac{[R^{V}]}{dx^3} + \frac{[S^{V}]}{dx^4} - \frac{[T^{V}]}{dx^5}\right)$$
$$d.\,[Z^{VI}]\,dx = 0,$$
$d.\,[Z^{VII}]\,dx = 0.$ & reliqua sequentia omnia evanescent.

Ex

Ex his nunc colligentur incrementa valorum π, π', π'', π''', &c. quæ recipiunt ex translatione puncti n in ν; erit scilicet

$$d.\,\pi = 0$$

$$d.\,\pi' = n\nu.\,dx.\,\frac{[T]}{dx^5}$$

$$d.\,\pi'' = n\nu.\,dx\left(\frac{[S']}{dx^4} - \frac{4[T'] + d[T]}{dx^5}\right)$$

$$d.\pi''' = n\nu.\,dx\left(\frac{[R'']}{dx^3} - \frac{3[S''] + d[S']}{dx^4} + \frac{6[T''] + 4d[T'] - d[T]}{dx^5}\right)$$

$$d.\pi^{IV} = n\nu.\,dx\left(\frac{[Q''']}{dx^2} - \frac{2[R'''] + d[R'']}{dx^3} + \frac{3[S'''] + 3d[S''] - d[S']}{dx^4} - \frac{4[T'''] + 6d[T''] - 4d[T'] + [dT]}{dx^5}\right)$$

$$d.\pi^{V} = n\nu.dx\left(\frac{[P^{IV}]}{dx} - \frac{[Q^{IV}] + d[Q''']}{dx^2} + \frac{[R^{IV}] + 2d[R'''] - d[R'']}{dx^3} - \frac{[S^{IV}] + 3d[S'''] - 3d[S''] + d[S']}{dx^4} + \frac{[T^{IV}] + 4d[T'''] - 6d[T''] + 4d[T'] - d[T]}{dx^5}\right)$$

$$d.\pi^{VI} = n\nu.\,dx\left([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{d[Q^{IV}] - d[Q''']}{dx^2} - \frac{d[R^{IV}] - 2d[R'''] + d[R'']}{dx^3} + \frac{d[S^{IV}] - 3d[S'''] + 3d[S''] - d[S']}{dx^4} - \frac{d[T^{IV}] - 4d[T'''] + 6d[T''] - 4d[T'] + d[T]}{dx^5}\right)$$

Huic autem incremento æqualia sunt incrementa omnium sequentium valorum, nempe ipsorum π^{VII}, π^{VIII}, π^{IX}, &c. Atqui valoris π^{VI} & omnium sequentium incrementum idem erit $= n\nu.\,dx\left([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{dd[Q''']}{dx^2} - \frac{d^3[R'']}{dx^3} + \frac{d^4[S']}{dx^4} - \frac{d^5[T]}{dx^5}\right)$. Poterunt autem hæc incrementa ad eadem signa reduci, respectu litterarum $[P]$, $[Q]$, $[R]$, $[S]$, & $[T]$, sicque prodibit

$$d.\Pi = 0$$

$$d.\Pi' = n\nu.\, dx.\, \frac{[T]}{dx^5}$$

$$d.\Pi'' = n\nu.\, dx \left(\frac{[S']}{dx^4} - \frac{4[T] + 5d[T]}{dx^5} \right)$$

$$d.\Pi''' = n\nu.\, dx \left(\frac{[R'']}{dx^3} - \frac{3[S'] + 4d[S']}{dx^4} + \frac{6[T] + 15d[T] + 10dd[T]}{dx^5} \right)$$

$$d.\Pi^{IV} = n\nu.\, dx \left(\frac{[Q''']}{dx^2} - \frac{2[R''] + 3d[R'']}{dx^3} + \frac{3[S'] + 8d[S'] + 6dd[S']}{dx^4} - \frac{4[T] + 15d[T] + 20dd[T] + 10d^3[T]}{dx^5} \right)$$

$$d.\Pi^{V} = n\nu.\, dx \left(\frac{[P^{IV}]}{dx} - \frac{[Q'''] + 2d[Q''']}{dx^2} + \frac{[R''] + 3d[R''] + 3dd[R'']}{dx^3} - \frac{[S']4d + [S'] + 6dd[S'] + 4d^3[S']}{dx^4} + \frac{[T] + 5d[T] + 10dd[T] + 10d^3[T] + 5d^4[T]}{dx^5} \right)$$

$$d\Pi^{VI} = n\nu.\, dx \left([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{dd[Q''']}{dx^2} - \frac{d^3[R'']}{dx^3} + \frac{d^4[S']}{dx^4} - \frac{d^5[T]}{dx^5} \right)$$

cui ſequentium valorum omnium incrementa ſunt æqualia. *Q. E. I.*

C o r o l l. I.

2. Si ergo Π fuerit hujuſmodi quantitas indeterminata, ſeu formula integralis indefinite integrationem non admittens, tum ejus omnes valores poſt locum abſciſſæ, ubi una applicata augeri concipitur, mutationem patientur, & aliquot ejus etiam valores ante illum locum, quorum numerus pendet a gradu differentialium, quæ in ea formula Π inſunt.

C o r o l l. II.

3. Quod ſi ergo iſtiuſmodi quantitas inſit in maximi minimive formula $\int Z\, dx$, tum ejus valor differentialis non ſolum ab aliquot abſciſſæ elementis, verum a tota abſciſſa, cui maximum minimumve reſpondere debet, pendebit.

C o-

C o r o l l. III.

4. His igitur casibus abscissam illam, pro qua maximum minimumve quæritur, determinatam esse oportet, atque curva quæ, pro hac abscissa, maximi minimive proprietate gaudere reperta fuerit, eadem pro aliis abscissis hac proprietate non erit prædita.

S C H O L I O N.

5. Mox clarius discrimen, quod intercedit inter quæstiones, in quibus Z est quantitas vel determinata vel indeterminata, perspicietur; quando Problemata hujus generis sumus tractaturi. Pluribus modis autem tales quæstiones possunt variari, prout in maximi minimive formula $\int Z\,dx$, quantitas Z vel tantum est functio ejusmodi formulæ indeterminatæ Π, qualem contemplati sumus, vel insuper quantitates determinatas, x, y, p, q, r, s, &c. comprehendit. Deinde in Z etiam inesse poterunt plures ejusmodi formulæ integrales indefinitæ a se invicem diversæ. Ad hos autem diversos casus una regula, superioribus jam traditis addita, sufficere poterit. Præcipuum autem momentum positum est in ipsa formula indeterminata $\Pi = \int [Z]\,dx$, pro qua hîc posuimus esse $[Z]$ functionem determinatam; quod si autem hæc ipsa quantitas $[Z]$ denuo ejusmodi formulas integrales indefinitas complectatur, iterum peculiari solutione erit opus. Quin etiam ista complicatio formularum indeterminatarum in infinitum potest extendi; id quod eveniet si quantitas $[Z]$ denuo in se complectatur ipsam quantitatem Π, ita ut sit $d[Z] = [L]\,d\Pi + [M]\,dx + [N]\,dy + [P]\,dp + [Q]\,dq + [R]\,dr +$ &c. tum enim ob $d\,\Pi = [Z]\,dx$, iterum considerari oportebit valorem $d[Z] = [L]\,d\,\Pi + [M]\,dx +$ &c. hicque progressus in infinitum continuabitur. Hinc autem methodus nascetur ea resolvendi Problemata, in quibus curva quæritur maximum minimumve habens valorem formulæ $\int Z\,dx$, quando quantitas Z non datur,

ut

ut hactenus, sive determinate sive indeterminate, sed tantum per æquationem differentialem, cujus integratio omnino non potest absolvi: cujusmodi quæstio est, si quæratur curva, in qua minimum sit expressio $\int \frac{dx\sqrt{(1+pp)}}{\sqrt{v}}$, existente $dv = gdx - hv^n dx\sqrt{(1+pp)}$: atque ejusmodi quæstionum resolutionem in hoc Capite quoque tradernus.

PROPOSITIO II. PROBLEMA.

Fig. 4. 6. *Si* Z *fuerit functio quantitatis indeterminatæ* п, *ita ut sit* $dZ = L\,d\pi$, *sitque* $\pi = \int [Z]\,dx$, *existente* $d[Z] = [M]\,dx + [N]\,dy + [P]\,dp + [Q]\,dq + [R]\,dr +$ &c. *invenire curvam* az *quæ pro data abscissa* AZ *habeat valorem formulæ* $\int Z\,dx$ *maximum vel minimum.*

SOLUTIO.

Posita abscissa AH $= x$, applicata Hh $= y$, sit tota abscissa AZ, cui maximum minimumve respondere debet, $= a$, diviso igitur spatio HZ in elementa innumera infinite parva HI, IK, KL, LM, &c. debebit esse $\int Z\,dx + Z\,dx + Z'dx + Z''\,dx + Z'''\,dx +$ &c. donec ad extremum punctum Z perveniatur, maximum minimumve. Ad hoc efficiendum, quærendi sunt valores differentiales quos singuli hi termini a translatione puncti n in ν accipiunt, quorum summa, nihilo æqualis posita, dabit æquationem pro curva quæsita. Quoniam autem mutationem ab nν oriundam non ultra H versus A porrigi ponimus, erit termini $\int Z\,dx$ valor differentialis nullus. Reliquorum terminorum valores differentiales reperientur, si ii differentientur, atque in differentialibus scribantur ea incrementa, quæ in Propositione præcedente invenimus, ex translatione puncti n in ν oriri. Erit autem

$$d.\ Z dx = L dx.\ d\pi$$
$$d.\ Z' dx = L' dx.\ d\pi'$$
$$d.\ Z'' dx = L'' dx.\ d\pi''$$
$$d.\ Z''' dx = L''' dx.\ d\pi'''$$
$$d.\ Z^{IV} dx = L^{IV} dx.\ d\pi^{IV}$$

Quodsi jam loco differentialium $d\pi$, $d\pi'$, $d\pi''$, $d\pi'''$, &c. valores supra inventos ex translatione puncti n in ν ortos substituamus obtinebimus.

$$d.\ Z dx = 0.$$

$$d.\ Z' dx = n\nu.\ L' dx^2.\ \frac{[T]}{dx^5}$$

$$d.\ Z'' dx = n\nu.\ L'' dx^2 \left(\frac{[S']}{dx^4} - \frac{4[T] + 5d[T]}{dx^5}\right)$$

$$d.\ Z''' dx = n\nu.\ L''' dx^2 \left(\frac{[R'']}{dx^3} - \frac{3[S'] + 4d[S']}{dx^4} + \frac{6[T] + 15d[T] + 10dd[T]}{dx^5}\right)$$

$$d.\ Z^{IV} dx = n\nu.\ L^{IV} dx^2 \left(\frac{[Q''']}{dx^2} - \frac{2[R''] + 3d[R'']}{dx^3} + \frac{3[S'] + 8d[S'] + 6dd[S']}{dx^4} - \frac{4[T] + 15d[T] + 20dd[T] + 10d^3[T]}{dx^5}\right)$$

$$d.\ Z^{V} dx = n\nu.\ L^{V} dx^2 \left(\frac{[P^{IV}]}{dx} - \frac{[Q'''] + 2d[Q''']}{dx^2} + \frac{[R''] + 3d[R''] + 3dd[R'']}{dx^3} - \frac{[S'] + 4d[S'] + 6dd[S'] + 4d^3[S']}{dx^4} + \frac{[T] + 5d[T] + 10dd[T] + 10d^3[T] + 5d^4[T]}{dx^5}\right)$$

$$d.\ Z^{VI} dx = n\nu.\ L^{VI} dx^2 \left([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{dd[Q''']}{dx^2} - \frac{d^3[R'']}{dx^3} + \frac{d^4[S']}{dx^4} - \frac{d^5[T]}{dx^5}\right)$$

$$d.\ Z^{VII} dx = n\nu.\ L^{VII} dx^2 \left([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{dd[Q''']}{dx^2} - \frac{d^3[R'']}{dx^3} + \frac{d^4[S']}{dx^4} - \frac{d^5[T]}{dx^5}\right)$$

&c.

Sequentium scilicet terminorum incrementa eadem hac lege progrediuntur. Addantur jam senorum priorum terminorum incrementa, prodibit terminorum $Z dx + Z' dx + Z'' dx + Z''' dx + Z^{IV} dx + Z^{V} dx$ incrementum totale =

$$n\nu.dx^2\left(\frac{L^{V}[P^{IV}]}{dx} - \frac{[Q''']dL^{IV} + 2L^{IV}d[Q''']}{dx^2} + \frac{[R'']ddL''' + 3d[R'']dL''' + 3L'''dd[R'']}{dx^3} - \frac{[S']d^3L'' + 4d[S']ddL'' + 6dL''dd[S'] + 4L''d^3[S']}{dx^4} \ldots\ldots\right.$$

$$+ \frac{[T]d^4L' + 5d[T]d^3L' + 10dd[T]ddL' + 10dL'd^3[T] + 5L'd^4[T]}{dx^5})$$

in qua expressione, quia omnes termini inter se sunt homogenei, jam indices numerici negligi poterunt. Sequentium autem terminorum $L^{VI}dx + L^{VII}dx +$ &c. omnium incrementum erit $=$

$$nv.\,dx\,([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{dd[Q''']}{dx^2} - \frac{d^3[R'']}{dx^3} + \frac{d^4[S']}{dx^4} - \frac{d^5[T]}{dx^5})$$

$(L^{VI}dx + L^{VII}dx + L^{VIII}dx + L^{IX}dx +$ &c. usque in Z). Hic autem posterior factor definietur per integrationem formulæ $\int L\,dx$, quæ respondet abscissæ indefinitæ AH $= x$; ponatur in hac formula post integrationem $x = a$, abeatque ea in H, erit H valor formulæ $\int L\,dx$ abscissæ toti propositæ AZ respondens; a qua ergo si auferatur $\int L\,dx$, remanebit $H - \int L dx$ valor portioni HZ vel NZ respondens, qui ergo loco $L^{VI}dx + L^{VII}dx + L^{VIII}dx +$ &c. substitui potest. Quamobrem tandem formulæ $\int Z\,dx$ valor differentialis toti abscissæ AZ respondens erit $=$

$$nv.\,dx\,(H - \int L dx)([N] - \frac{d[P]}{dx} + \frac{dd[Q]}{dx^2} - \frac{d^3[R]}{dx^3} + \frac{d^4[S]}{dx^4} - \frac{d^5[T]}{dx^5})$$

$$+ nv.\,dx\,(L[P] - \frac{[Q]dL + 2Ld[Q]}{dx} + \frac{[R]ddL + 3d[R]dL + 3Ldd[R]}{dx^2}$$

$$- \frac{[S]d^3L + 4d[S]ddL + 6dLdd[S] + 4Ld^3[S]}{dx^3} \ldots$$

$$+ \frac{[T]d^4L + 5d[T]d^3L + 10dd[T]ddL + 10dLd^3[T] + 5Ld^4[T]}{dx^4})$$

qui ad hanc formam commodiorem reduci potest, ut sit $=$

$$nv.\,dx\,([N](H - \int L dx) - \frac{d.[P](H - \int L dx)}{dx} + \frac{dd.[Q](H - \int L dx)}{dx^2}$$

$$- \frac{d^3.[R](H - \int L dx)}{dx^3} + \frac{d^4.[S](H - \int L dx)}{dx^4} - \frac{d^5[T](H - \int L dx)}{dx^5});$$

qui valor differentialis, quousque occasio postulabit, ulterius continuari poterit: is autem, nihilo æqualis positus, dabit æquationem pro curva quæsita. Q. E. I.

Co-

COROLL. I.

7. Quoniam $H - \int L dx$ eſt valor formulæ $\int L dx$ reſpondens abſciſſæ portioni $AZ = a - x$, ſi ponatur $AZ = a - x = u$, erit $\int L du$ ille ipſe valor $H - \int L dx$, quo opus eſt; ſiquidem $\int L du$ ita integretur, ut evaneſcat poſito $u = 0$.

COROLL. II.

8. Quodſi igitur abſciſſarum initium capiatur in puncto Z; ita ut abſciſſa ZH ponatur $= u$, utque ubique ponatur $x = a - u$, prodibit æquatio pro curva inter coordinatas u & y; hujuſque curvæ ea portio quæſito ſatisfaciet, quæ reſpondet abſciſſæ $AZ = a$. Interim notandum eſt cum in ipſa maximi minimive formula $\int Z dx$, tum in $\int [Z] dx$, abſciſſarum initium in puncto A capi debere.

COROLL. III.

9. Si ergo quæratur curva ad datam abſciſſam AZ relata; in qua maximum minimumve debeat eſſe $\int Z dx$; ſitque Z functio quæcunque ipſius $\Pi = \int [Z] dx$, exiſtente $dZ = L d\Pi$ & $d[Z] = [M] dx + [N] dy + [P] dp + [Q] dq + [R] dr + \&c.$ habebitur pro curva quæſita iſta æquatio:

$$0 = [N] \int L du - \frac{d.[P] \int L du}{dx} + \frac{dd.[Q] \int L du}{dx^2} - \frac{d^3.[R] \int L du}{dx^3} + \&c.$$

ubi eſt $u = a - x$, & $\int L du$ denotat valorem formulæ $\int L dx$ portioni abſciſſæ $HZ = u$ reſpondentem.

COROLL. IV.

10. Poſſunt ergo vel bina abſciſſarum initia A & Z, binæque abſciſſæ $AH = x$, & $ZH = u$ conſiderari, quarum illa in integrali $\int [Z] dx$ ſeu Π, hæc vero in integrali $\int L dx$ ſpectari debet, vel unica tantum abſciſſa $AH = x$; quo caſu, loco

$\int L\,du$ scribi debet $H - \int L\,dx$; denotante H valorem, quem præbet formula $\int L\,dx$, posito $x = AH = a$.

Coroll. V.

11. Quia Z est functio ipsius Π tantum, ita ut nullas alias quantitates variabiles in se complectatur, ob $dZ = L\,d\Pi$, erit etiam L functio ipsius Π tantum.

Coroll. VI.

12. Si $[Z]$ esset functio ipsius x tantum; tum foret $\Pi = \int [Z]\,dx$ quantitas determinata, atque functio ipsius x, hincque etiam Z; ex quo maximum minimumve non inveniet locum. Idem ostendit solutio; fiet enim $[N] = 0$, $[R] = 0$ &c. atque æquatio abit in identicam $0 = 0$.

Scholion I.

13. Occurrunt hic nonnulli primarii casus considerandi, quorum primus est, si fuerit $[Z]$ functio ipsarum x & y tantum; ita ut sit $d[Z] = [M]\,dx + [N]\,dy$. Quod si nunc quæratur curva in qua maximum minimumve sit formula $\int Z\,dx$ pro data abscissa $AZ = a$, existente Z functione quacunque ipsius $\int [Z]\,dx = \Pi$, ita ut sit $dZ = L\,d\Pi$; habebitur pro curva quæsita ista æquatio $0 = [N]\,(H - \int L\,dx)$; erit ergo vel $[N] = 0$ vel $H = \int L\,dx$, seu $L = 0$; quarum æquationum si vel altera vel utraque præbeat lineam curvam, ea non solum satisfaciet Problemati pro abscissa $AZ = a$, sed etiam pro alia quacunque abscissa indefinita x: id quod inde colligitur, quod ex æquatione, quantitas H, quæ pendet ab abscissa determinata a, ex calculo excesserit. Quod autem speciatim ad æquationem $L = 0$ attinet: quia L est functio ipsius Π seu $\int [Z]\,dx$, fiet $\int [Z]\,dx =$ const. determinatæ, quod nisi sit $[Z] = 0$, fieri nequit: binæ igitur æquationes hoc casu satisfacientes, erunt $[N] = 0$, atque $[Z] = 0$.

SCHO-

SCHOLION II.

14. Deinde vero conſiderari meretur caſus quo $[N]$ evaneſcit; id quod evenit, ſi $[Z]$ fuerit functio ipſarum x, p, q, r, &c. non involvens y. Ponamus eſſe $[Z]$ functionem ipſarum x & p, atque $d[Z] = [M]dx + [P]dp$. Si igitur ponatur $\int[Z]dx = \Pi$, atque curva quæratur, in qua, pro abſciſſa definita $AZ = a$, maximum minimumve ſit formula $\int Z dx$, exiſtente Z functione ipſius Π, ita ut ſit $dZ = L d\Pi$; orietur pro curva quæſita iſta æquatio $o = -\frac{d.[P](H - \int L dx)}{dx}$; ideoque $Conſt. = [P](H - \int L dx)$. Hæc vero conſtans, per integrationem ingreſſa, non eſt arbitraria; nam eam ita comparatam eſſe oportet, ut poſito $x = a$, quo caſu fit $\int L dx = H$, fiat $\frac{Conſt.}{[P]} = o$. Hoc autem evenire non poteſt, niſi vel hæc conſtans ponatur $= o$, vel quantitas $[P]$ ita comparata ſit ut fiat $= \infty$, poſito $x = a$. Priori caſu habetur vel $[P] = o$, vel $\int L dx = H$, hoc eſt $L = o$, ſeu $\int[Z]dx = Conſt.$ ſeu potius $[Z] = o$; poſteriori caſu autem, conſtans tamen pro arbitrio non accipi poteſt, nam determinabitur, ponendo $x = a - dx$, eo modo, quo expreſſiones quæ certis caſibus indeterminatæ videntur definiri ſolent. Atque hinc perſpicitur in hujuſmodi Problematis numerum conſtantium arbitrariarum in ſolutionem ingredientium, cui æqualis ſumi debet numerus punctorum, per quæ curvæ ſatisfacienti tranſeundum eſt, non ex gradu differentialium judicari poſſe. Pervenietur enim ſæpe, tollendo per differentiationem omnes formulas integrales, ad æquationem differentialem altioris gradus, a quo nequaquam Problematis determinatio per aliquot puncta pendebit.

EXEMPLUM I.

15. *Si denotet* Π *aream curvæ* $\int y dx$, *atque* Z *ſit functio quæcunque ipſius* Π, *invenire curvam quæ, pro data abſciſſa* $= a$, *habeat valorem formulæ* $\int Z dx$ *maximum vel minimum.*

 Quia

Quia est Z functio ipsius Π; sit $dZ = L\,d\Pi$, erit L functio ipsius $\Pi = \int y\,dx$. Deinde cum sit $d\Pi = y\,dx$; erit $[Z] = y$, & ob $d[Z] = [M]dx + [N]dy + [P]dp +$ &c. fiet $[M] = 0$, $[N] = 1$, $[P] = 0$, $[Q] = 0$, &c. unde pro curva quæsita hæc habebitur æquatio $0 = H - \int L\,dx$; ideoque $L = 0$. Hinc erit $\Pi = \int y\,dx =$ constanti cuidam, porroque $y = 0$. Satisfacit ergo sola linea recta in ipsum axem incidens; idque pro quacunque abscissa æque ac pro definita $= a$.

Exemplum II.

16. *Si* Π *denotet arcum curvæ* $= \int dx\sqrt{(1+pp)}$ *ejusque functio quæcunque fuerit* Z; *invenire curvam, quæ, pro data abscissa* $AZ = a$, *habeat valorem formulæ* $\int Z\,dx$ *maximum vel minimum.*

Ob $dZ = L\,d\Pi$, erit L functio ipsius arcus Π; & ob $d\Pi = dx\sqrt{(1+pp)}$, erit $[Z] = \sqrt{(1+pp)}$ & $[M] = 0$, $[N] = 0$, $[P] = \frac{p}{\sqrt{(1+pp)}}$, $[Q] = 0$, &c. unde pro curva quæsita ista habebitur æquatio: $0 = -d.\frac{p}{dx\sqrt{(1+pp)}}$ $(H - \int L\,dx)$; hincque $C = \frac{p}{\sqrt{(1+pp)}}(H - \int L\,dx)$: ubi constans C ita determinari debet, ut, posito $x = a$, fiat $C = \frac{p}{\sqrt{(1+pp)}} \times 0$; quare quia $\frac{p}{\sqrt{(1+pp)}}$ infinitum fieri nequit, necesse est ut sit $C = 0$; ideoque vel $\frac{p}{\sqrt{(1+pp)}} = 0$; vel $\int L\,dx = H$. Fiet ergo, ex posteriore æquatione, $L = 0$, & $\Pi =$ constanti cuidam: ex quo porro deducitur $d\Pi = dx\sqrt{(1+pp)} = 0$, cui conditioni nullo modo satisfieri potest. Ex priore æquatione autem deducitur $p = 0$, seu $dy = 0$, quæ est æquatio pro linea recta axi AZ parallela, quæ quæstioni pro abscissa quacunque satisfacit.

Exem-

EXEMPLUM III.

17. *Denotet* Π *superficiem solidi rotundi ex conversione curvæ* ah *circa axem* AZ *orti, quæ est ut* $\int y\,dx\sqrt{(1+pp)}$, *hujusque superficiei functio sit quæcunque* Z, *invenire curvam, in qua pro data abscissa* AZ = a, *maximum minimumve sit* $\int Z\,dx$.

Ob $dZ = L\,d\Pi$, erit L functio ipsius $\Pi = \int y\,dx\sqrt{(1+pp)}$; & ob $d\Pi = y\,dx\sqrt{(1+pp)}$ fiet $[Z] = y\sqrt{(1+pp)}$, & $d[Z] = dy\sqrt{(1+pp)} + \frac{yp\,dp}{\sqrt{(1+pp)}}$: unde erit $[M] = 0$, $[N] = \sqrt{(1+pp)}$; $[P] = \frac{yp}{\sqrt{(1+pp)}}$, reliqui valores $[Q]$, $[R]$, $[S]$, &c. omnes erunt $= 0$. Quocirca pro curva quæsita ista habebitur æquatio: $0 = (H - \int L\,dx)\sqrt{(1+pp)} - \frac{1}{dx}\,d.\frac{yp}{\sqrt{(1+pp)}}(H - \int L\,dx)$. Ponatur, brevitatis gratia, $H - \int L\,dx = V$; erit $V\,dx\sqrt{(1+pp)} = d.\frac{ypV}{\sqrt{(1+pp)}} = \frac{Vpp\,dx}{\sqrt{(1+pp)}} + \frac{Vy\,dp}{(1+pp)^{3:2}} + \frac{yp\,dV}{\sqrt{(1+pp)}}$, seu $V\,dx = \frac{Vy\,dp}{1+pp} + yp\,dV = \frac{Vy\,dp}{1+pp} - yp\,L\,dx$, ob $dV = -L\,dx$. Ponamus esse $Z = \Pi$, ita ut maximum esse debeat $\int dx \int y\,dx\sqrt{(1+pp)}$, erit $L = 1$ & $\int L\,dx = x$, atque $V = a - x$, ob $H = a$. Erit $(a-x)\,dx = \frac{(a-x)y\,dp}{1+pp} - yp\,dx$. Sit $a - x = u$; erit $dx = -du$, & $dy = -p\,du$, atque habebitur ista æquatio, $0 = u\,du - y\,dy + \frac{uy\,dp}{1+pp}$, seu $u\,du - y\,dy - \frac{uy\,du\,ddy}{du^2+dy^2} = 0$. Ponatur $u = e^t$ & $y = e^t z$: erit $du = e^t dt$, & $ddu = 0 = e^t(ddt + dt^2)$, seu $ddt = -dt^2$; porro $dy = e^t(dz + z\,dt)$ & $ddy = e^t(ddz + 2\,dt\,dz)$; quibus substitutis, oritur

dt.—

$dt - zdz - zzdt = \frac{zdt(ddz + 2dtdz)}{dt^2 + (dz + zdt)^2}$. Sit porro $dt = sdz$, erit $ddt = -s^2dz^2 = sddz + dsdz$, hincque $ddz = -sdz^2 - \frac{dsdz}{s}$. Habebitur ergo hæc æquatio; $sdz - zdz - szzdz = \frac{zs^2dz - zds}{ss + (1 + sz)^2}$; quæ quidem est differentialis primi gradus inter duas variabiles s & z tantum; verumtamen ultra integrationem non admittit. Multo minus igitur quicquam effici poterit, si in genere quæstionem consideremus.

SCHOLION III.

18. Hujus exempli casus, quo curvam investigavimus, in qua maximum minimumve sit $\int dx \int y dx \surd(1 + pp)$, etsi inest duplex signum integrale, tamen etiam per methodum præcedentis Capitis potest resolvi; id quod ideo operæ pretium est ostendere, ut consensus utriusque methodi declaretur. Præcipue autem hoc opere nova via patefiet resolvendi plurima alia Problemata circa maxima & minima, quæ adhuc, quantum constat, non est tacta. Quæstio scilicet est, ut pro data abscissa AZ $= a$, fiat maximum minimumve hæc expressio $\int dx \int y dx \surd(1 + pp)$, quæ transmutatur in hanc $x \int y dx \surd(1 + pp) - \int xy dx \surd(1 + pp)$. Ut hæc forma reddatur maximum minimumve, oportet ut ejus valor, pro abscissa AZ $= a$, idem sit pro ipsa curva quæsita az & pro eadem puncto n in ν translato. Ponamus ergo fieri $\int y dx \surd(1 + pp) = A$, si ponatur $x = a$, atque eodem casu $\int xy dx \surd(1 + pp) = B$. Jam, elementis mno in mνo transmutatis, valor A augebitur suo valore differentiali, qui, per Caput præcedens, est $=$ nν. $dx(\surd(1 + pp) - \frac{1}{dx} d. \frac{yp}{\surd(1 + pp)})$; per eadem præcepta autem quantitatis B valor differentialis prodit $=$ nν. $dx(x\surd(1 + pp) - \frac{1}{dx} d. \frac{xyp}{\surd(1 + pp)})$. Quamobrem formulæ propositæ $\int dx \int y dx \surd(1 + pp)$, translato puncto n in ν, pro abscissa AZ

$= a$,

$= a$, valor erit $= a\left(A + n\nu\, dx\left(\surd(1+pp) - d.\frac{yp}{\surd(1+pp)}\right)\right.$ $\left. - B - n\nu\left(x\,dx\surd(1+pp) - d.\frac{xyp}{\surd(1+pp)}\right)\right)$, qui æqualis esse debet ejusdem formulæ valori naturali pro abscissa $= a$, non mutato puncto n, qui est $aA - B$. Hinc proveniet ista æquatio $(a - x)\,dx\surd(1+pp) - d.\frac{(a-x)yp}{\surd(1+pp)} = 0$; quæ omnino congruit cum æquatione in solutione Exempli inventa.

Propositio III. Problema.

19. *Existente* п *functione integrali indeterminata* $\int[Z]\,dx$, *ita ut sit* $d[Z] = [M]\,dx + [N]\,dy + [P]\,dp + [Q]\,dq + [R]\,dr +$ &c. *sit* Z *functio quæcunque cum hujus quantitatis* п, *tum quantitatum determinatarum* x, y, p, q, r, s, &c. *ita ut sit* $dZ = L\,d\Pi + M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr +$ &c. *invenire curvam* az, *quæ pro data abscissa* AZ $= a$, *habeat maximum minimumve valorem formulæ* $\int Z\,dx$.

Solutio.

Augmentum nν, quod uni applicatæ Nn accedere concipitur, ita remotum a prima applicata Hh capiatur, ut nullam mutationem inferat in valorem formulæ $\int Z\,dx$ abscissæ AH respondentem, atque tantum hujus formulæ valores sequentibus post H abscissæ elementis respondentes mutationes patiantur, qui sunt $Z\,dx$, $Z'\,dx$, $Z''\,dx$, $Z'''\,dx$, &c. usque ad ultimum abscissæ elementum in Z. Horum igitur valorum incrementa a translatione puncti n in ν orta, si in unam summam conjiciantur, & nihilo æquales ponantur, dabunt æquationem pro curva quæsita. Incrementa autem horum valorum obtinebuntur eos differentiando, & loco differentialium eos valores scribendo, quos supra, tam in ultima Propositione præcedentis Capitis quam prima hujus, ex translatione n in ν oriri invenimus: ita erit

$$d.\,Z\,dx = dx\,(L\,d\pi + M\,dx + N\,dy + P\,dp + \&c.)$$
$$d.\,Z'dx = dx\,(L'd\pi' + M'dx + N'dy' + P'dp' + \&c.)$$
$$d.\,Z''dx = dx\,(L''d\pi'' + M''dx + N''dy'' + P''dp'' + \&c.)$$
&c.

Quod si nunc loco differentialium $d\pi$, $d\pi'$, $d\pi''$ &c. dy, dy', dy'', &c. dp, dp', dp'', &c. dq, dq', dq'', &c. valores supra inventi substituantur, & eodem modo, quo ante usi sumus, in unam summam conferantur, prodibit formulæ $\int Z\,dx$ pro abscissa AZ $= a$ valor differentialis $=$

$$nv.\,dx\left([N](H-\int L\,dx) - \frac{d.[P](H-\int L\,dx)}{dx} + \frac{dd.[Q](H-\int L\,dx)}{dx^2} - \frac{d^3.[R](H-\int L\,dx)}{dx^3} + \frac{d^4.[S](H-\int L\,dx)}{dx^4} - \&c.\right)$$
$$+ nv.\,dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \&c.\right).$$

Atque ex hoc resultabit æquatio pro curva quæsita hæc:

$$o = [N](H-\int L\,dx) - \frac{d.[P](H-\int L\,dx)}{dx} + \frac{dd.[Q](H-\int L\,dx)}{dx^2} - \frac{d^3.[R](H-\int L\,dx)}{dx^3} + \&c.$$
$$+ N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \&c.$$

ubi notandum esse H valorem formulæ $\int L\,dx$, qui oritur posito $x = a$. Q. E. I.

COROLL. I.

20. Regula igitur Capite præcedente inventa amplior est reddita; nunc enim curvam definire possumus, maximum minimumve habentem valorem formulæ $\int Z\,dx$, si Z non solum est functio quantitatum determinatarum x, y, p, q, r, &c. sed etiam unam quantitatem integralem indefinitam $\int [Z]\,dx$ in se complectitur: dummodo $[Z]$ sit functio determinata.

Co-

COROLL. II.

21. Quin etiam si plures hujusmodi quantitates integrales indefinitæ fuerint in Z; solutio usurpari poterit. Nam qualis expressio ex una ejusmodi formula indefinita in valorem differentialem est ingressa, tales ex singulis, si plures affuerint, nascentur & ad valorem differentialem accedent.

COROLL. III.

22. Quoniam Z hîc ponitur functio non solum quantitatum definitarum x, y, p, q, r &c. sed etiam quantitatis indefinitæ $\Pi = \int [Z] dx$, ob $dZ = L d\Pi + M dx + N dy + P dp + Q dq +$ &c. etiam quantitates M, N, P, Q &c. hanc formulam integralem $\Pi = \int [Z] dx$ involvent; atque etiam ipsa quantitas L, nisi forte Π in Z unicam habeat dimensionem.

COROLL. IV.

23. Hanc ob rem, in æquatione pro curva inventa, inerunt quantitates integrales duplicis generis, scilicet $\int L dx$, atque $\int [Z] dx$: ex quo, si æquatio inventa per differentiationem ab his formulis liberari debeat, ad multo altiorem differentialium gradum assurget, quam quidem ipsa forma ostendit.

COROLL. V.

24. Pervenietur autem, eliminando has formulas integrales, ad æquationem differentialem duobus gradibus altiorem. Quod si enim æquatio resultans, si evolvatur, sit differentialis n gradus; tum primo ex ea definiatur valor formulæ $\int L dx$, & differentiatione instituta, devenietur ad æquationem differentialem $n + 1$ graduum, in qua adhuc inerit formula $\int [Z] dx$, quæ ulterius reducta, & a formula $\int [Z] dx$ per differentiationem liberata, fiet differentialis gradus $n + 2$.

SCHOLION I.

25. Etsi autem numerus punctorum, per quæ curva quæsita transire debet, a gradu *differentialitatis* pendet, tamen hoc casu non per numerum $n+2$ definiri potest. Æquatio enim hæc differentialis $n+2$ graduum, potestate quidem involvit $n+2$ constantes, verum eæ non omnes sunt arbitrariæ. Una namque constans ex eo determinatur, quod integrale $\int[Z]\,dx$ obtinere debeat valorem, non vagum, sed talem qualem in quantitate Z obtinet, hoc est, qui evanescat posito $x=0$, siquidem hæc conditio fuerit in formula $\int Z\,dx$ assumta. Deinde pari modo una constans definitur formula $\int L\,dx$, quæ, uti posuimus, evanescere debet posito $x=0$. Quocirca tantum n supererunt constantes mere arbitrariæ, quæ totidem præbebunt puncta, quibus Problema determinabitur. Similiter igitur, uti in præcedente Capite, Problema, ut sit determinatum, ita erit proponendum, ut inter omnes curvas per data n puncta transeuntes ea determinetur, quæ pro data abscissa $x=a$ contineat valorem formulæ $\int Z\,dx$ maximum minimumve. Ad hanc igitur dijudicationem instituendam, æquatio inventa debebit evolvi; hoc est, omnes differentiationes indicatæ actu perfici debebunt; quo facto, patebit quanti gradus differentialia insint, ex hocque gradu habebitur numerus n. Quantum autem insuper circa hunc numerum n observare liceat, in Exemplis sequentibus videbimus.

EXEMPLUM I.

26. *Invenire curvam, quæ, pro data abscissa* $AZ=a$, *habeat valorem formulæ* $\int yx\,dx\int y\,dx$ *maximum vel minimum, integrali* $\int y\,dx$ *ita accipiendo, ut evanescat posito* $x=0$.

Erit igitur $\Pi=\int y\,dx$, & $[Z]=y$; unde fiet $[N]=1$, reliquis litteris $[M]$, $[P]$, $[Q]$, &c. existentibus $=0$. Porro erit $Z=yx\Pi$ & $dZ=yx\,d\Pi+y\Pi\,dx+x\Pi\,dy$; ex quo habebitur $L=yx$; $M=y\Pi$ & $N=x\Pi$, $P=Q=R$, &c. $=0$.

$= 0$. Ex his formabitur pro curva quæsita ista æquatio; $0 = (H - \int yx\,dx) + x\Pi$ seu $\int yx\,dx = H + x\int y\,dx$, ubi H est valor formulæ $\int yx\,dx$, qui prodit posito $x = a$. Perspicuum autem est hinc nullam pro aliqua linea curva æquationem oriri: differentiatione enim instituta, fit $dx\int y\,dx = 0$, porroque $y = 0$, quæ est æquatio pro linea recta in axem A Z incidente.

EXEMPLUM II.

27. *Invenire curvam, quæ, pro data abscissa* A Z $= a$, *habeat valorem formulæ* $\int y\,dx \int dx\sqrt{(1+pp)}$ *maximum vel minimum.*

Quoniam igitur est $\Pi = \int dx\sqrt{(1+pp)}$, erit $[Z] = \sqrt{(1+pp)}$ & $[P] = \frac{p}{\sqrt{(1+pp)}}$: Porro erit $Z = y\Pi$ & $L = y$; & $N = \Pi$; reliquæ litteræ omnes evanescunt. Hinc ergo resultabit ista æquatio pro curva quæsita: $0 = \frac{1}{dx} \times d.\frac{p(H - \int y\,dx)}{\sqrt{(1+pp)}} + \Pi$ seu $\Pi dx = d.\frac{(H - \int y dx)p}{\sqrt{(1+pp)}} = \frac{(H - \int y dx)dp}{(1+pp)^{\frac{3}{2}}} - \frac{ypdx}{\sqrt{(1+pp)}}$; ergo $dx\int dx\sqrt{(1+pp)} = \frac{(H - \int y\,dx)\,dp}{(1+pp)^{\frac{3}{2}}} - \frac{ypdx}{\sqrt{(1+pp)}}$. Quia igitur fit $\int y\,dx = H$, posito $x = a$, eodem casu fiet $\int dx\sqrt{(1+pp)} = \frac{-yp}{\sqrt{(1+pp)}} =$ arcui curvæ abscissæ a respondenti. Quæ conditio adimpleri debet per determinationem unius constantis, quæ per integrationem ingredietur. Est autem actu hæc æquatio differentialis secundi gradus, quæ vero bis debet differentiari, antequam a formulis integralibus $\int y\,dx$ & $\int dx\sqrt{(1+pp)}$ liberetur: hocque modo ad gradum sextum assurget, & potestate sex constantes involvet; quarum duæ inde determinabuntur, quod facto $x = 0$ evanescere debent formulæ $\int y\,dx$ & $\int dx\sqrt{(1+pp)}$. Ipsa autem æquatio ita fiet intricata, ut ejus tractatio suscipi non mereatur.

EXEMPLUM III.

28. *Invenire curvam, in qua pro data abſciſſa ſit* $\int \frac{dx}{p} \int y\,dx$ *maximum vel minimum.*

Hic erit $\Pi = \int y\,dx$, & $[Z] = y$ & $[N] = 1$; deinde cum ſit $Z = \frac{\Pi}{p}$, erit $L = \frac{1}{p}$ & $P = -\frac{\Pi}{pp}$; reliquæ litteræ omnes evaneſcunt. Hinc ergo prodit iſta æquatio. $0 = H - \int \frac{dx}{p} + \frac{1}{dx} d. \frac{\Pi}{pp}$; ſeu $0 = H - \int \frac{dx}{p} + \frac{y}{pp} - \frac{2\Pi\,dp}{p^3\,dx}$ Poſito ergo $x = a$, quo caſu ſit $\int \frac{dx}{p} = H$; erit $y\,dx = \frac{2\Pi\,dp}{p}$. Differentietur ea æquatio, eritque $0 = -\frac{dx}{p} + \frac{dx}{p} - \frac{2y\,dp}{p^3} - \frac{2y\,dp}{p^3} + \frac{6\Pi\,dp^2}{p^4\,dx} - \frac{2\Pi\,ddp}{p^3\,dx}$. Seu $0 = 3\Pi\,dp^2 - 2ypdxdp - \Pi p\,ddp$; quæ æquatio commode fit integrabilis, ſi dividatur per $\Pi p\,dp$, prodit enim $0 = \frac{3\,dp}{p} - \frac{2y\,dx}{\Pi} - \frac{ddp}{dp}$, cujus integrale eſt $C = 3\,lp = 2\,l\Pi - l\,\frac{dp}{dx}$. Seu $C\,\Pi^2\,dp = p^3\,dx$; poſito ergo $x = a$, cum eſſe debeat $y\,dx = \frac{2\Pi\,dp}{p}$; erit ex hac æquatione $C\Pi y = 2p^2$, qua una conſtans definietur. Erit ergo $\Pi = \sqrt{\frac{p^3\,dx}{C\,dp}} = \frac{2y\,p\,dx\,dp}{3\,dp^2 - pddp}$, ſeu $3dp^2 - pddp = \frac{2\,y\,dp\sqrt{dx\,dp}}{b\sqrt{bp}}$, quæ æquatio eſt differentialis tertii gradus, & propterea præter conſtantem b (poſuimus autem $\frac{1}{b^3}$ loco C), tres novas conſtantes involvit. Harum una determinabitur, eo quod, poſito $x = a$, fieri debeat $\frac{\Pi y}{b^3} = 2pp$; alia vero inde quod, poſito $x = 0$, eſſe debeat $\Pi = 0$, ſeu $\frac{p^3\,dx}{dp} = 0$. Re-

liquæ

liquæ binæ constantes manent arbitrariæ, ac propterea curva quæsita per duo data puncta per quæ transeat, debet determinari.

EXEMPLUM IV.

29. *Invenire curvam* az *ad abscissam* AZ = a *relatam, in qua sit* $\int dx \frac{\int yxdx}{\int ydx}$ *maximum vel minimum.*

Hoc exemplum ideo afferre visum est, ut appareat quomodo quæstiones ejusmodi sint resolvendæ, si duæ pluresve formulæ integrales indefinitæ adsint. Sit igitur $\int yxdx = \Pi$ & $\int ydx = \pi$: & posito $d\Pi = [Z]dx$, & $d\pi = [z]dx$, erit $[Z] = yx$, & $[z] = y$. Quod si nunc littera minuscula $[z]$ simili modo tractetur quo majuscula $[Z]$, ita ut sit $d[z] = [m]dx + [n]dy + [p]dp +$ &c. erit $[M] = y$ & $[N] = x$, itemque $[n] = 1$. Deinde cum sit $Z = \frac{\Pi}{\pi}$, erit $dZ = \frac{d\Pi}{\pi} - \frac{\Pi d\pi}{\pi^2}$. Ponatur $\frac{1}{\pi} = L$ & $\frac{\Pi}{\pi^2} = l$; atque habebitur ob N & P, Q, R, &c. $= 0$, ista pro curva quæsita æquatio, $0 = x(H - \int \frac{dx}{\pi}) - 1\,(h - \int \frac{\Pi dx}{\pi^2})$, ubi sit $\int \frac{dx}{\pi} = H$ & $\int \frac{\Pi dx}{\pi^2} = h$, si ponatur $x = a$. Cum igitur sit $Hx - x\int \frac{dx}{\pi} = h - \int \frac{\Pi dx}{\pi^2}$ erit differentiando $H - \int \frac{dx}{\pi} - \frac{x}{\pi} = - \frac{\Pi}{\pi^2}$. Posito ergo $x = a$, fieri debet $\Pi = \pi x$. Differentietur denuo, prodibitque $-\frac{2}{\pi} + \frac{xy}{\pi^2} = -\frac{yx}{\pi^2} + \frac{2\Pi y}{\pi^3}$, seu $xy - \pi = \frac{\Pi y}{x}$; hincque $\Pi = \pi x - \frac{\pi\pi}{y}$. Si porro differentiatio instituatur, habebitur $yxdx = \pi dx + yxdx - 2\pi dx + \frac{\pi\pi dy}{yy}$, seu $yydx = \pi dy$, vel $\frac{ydx}{\pi} = \frac{dy}{y}$. Quoniam vero, posito

posito $x = 0$, fit $\pi = 0$, fiet hoc casu $\frac{yy\,dx}{dy} = 0$. Æquatio autem $\frac{y\,dx}{\pi} = \frac{dy}{y}$, ob $y\,dx = d\pi$, integrata dat $\pi = by$; ideoque facto $x = 0$ evanescere debet y. Ex æquatione $\pi = by$ autem sequitur $y\,dx = b\,dy$; hincque $x = bly - bl0$, siquidem $\pi = by$ evanescere debeat, posito $x = 0$; quo casu fieret $y = 0$, & curva abiret in rectam in axem AZ incidentem. Sin autem ponamus, posito $x = 0$ valorem $\pi = \int y\,dx$ non evanescere oportere, sed fieri $= bc$, erit $x = bl\frac{y}{c}$, quæ est æquatio pro Curva logarithmica. Ad hanc penitus determinandam, quæratur valor $\Pi = \int yx\,dx$; quia est $y\,dx = b\,dy$, erit $yx\,dx = bx\,dy$, & $\Pi = bxy - b\pi + Const.$ seu $\Pi = bbyl\frac{y}{c} - bby + C$. Oporteat autem Π esse $= 0$, posito $x = 0$, seu $y = c$, erit $\Pi = bbyl\frac{y}{c} + bb(c - y)$. Jam ponatur $x = a$, erit $l\frac{y}{c} = \frac{a}{b}$, & $y = ce^{a:b}$: hoc vero casu, necesse est ut sit $\Pi = \pi x$, seu $abce^{a:b} + bbc - bbce^{a:b} = abce^{a:b}$, hincque $e^{a:b} = 1$, unde erit, vel $a = 0$, vel $b = \infty$. Incommodum hoc inde oritur, quod posuimus fieri $\Pi = 0$, facto $x = 0$. Ponamus igitur, posito $y = g$, tum eo casu Π evanescere, erit $\Pi = bbyl\frac{y}{c} - bby + bbg - bbgl\frac{g}{c}$. Jam posito $x = a$, quo casu fieri debet $\Pi = \pi x = a\pi$; erit $abce^{a:b} - bbce^{a:b} + bbg - bbgl\frac{g}{c} = abce^{a:b}$, hincque $e^{a:b} = \frac{g}{c}(1 - l\frac{g}{c})$, seu $b = \frac{a}{l\frac{g}{c}(1 - l\frac{g}{c})}$ ideoque $x = \frac{a(ly - lc)}{lg(1 - l\frac{g}{c}) - lc}$. Quæ est æquatio curvam

penitus

penitus determinans, ita ut nullum curvæ punctum pro arbitrio accipi liceat.

SCHOLION II.

30. Per hoc igitur Problema, non solum illæ quæstiones curvam pro data abscissa maximum minimumve habentem formulam $\int Z dx$ desiderantes resolvi possunt, in quibus Z præter quantitates determinatas $x, y, p, q, r, s,$ &c. unam formulam integralem $\Pi = \int [Z] dx$ complectitur; sed etiamsi plures ejusmodi formulæ affuerint. Interim tamen notandum est has formulas integrales $\Pi = \int [Z] dx$ in functione Z contentas, ita comparatas esse debere, ut $[Z]$ sit functio determinata, hoc est functio quantitatum $x, y, p, q, r,$ &c. nullas ultra formulas integrales involvens. Hinc ob rem, nunc investigemus methodum resolvendi ejusmodi Problemata, quando ista functio $[Z]$ non est determinata, sed præter $x, y, p, q,$ &c. formulam integralem novam $\pi = \int [z] dx$ involvit. Ne autem solutio nimium fiat prolixa, non ultra differentialia secundi gradus considerabimus. Jam enim intelligitur si solutio fuerit adornata usque ad differentialia secundi gradus, tum per inductionem, solutionem ad quosque ulteriores gradus extendi posse. Hunc in finem nobis erit Ll prima applicata designanda per y, a qua tertia quæ sequitur $Nn = y''$ particula $n\nu$ augeri concipiatur: Ex hoc augmento nascentur sequentia quantitatum $y, p,$ & q, cum suis derivativis incrementa

$d.\,y = 0$	$d.\,p = 0$	$d.\,q = +\dfrac{n\nu}{dx^2}$
$d.\,y' = 0$	$d.\,p' = +\dfrac{n\nu}{dx}$	$d.\,q' = -\dfrac{2n\nu}{dx^2}$
$d.\,y'' = +n\nu$	$d.\,p'' = -\dfrac{n\nu}{dx}$	$d.\,q'' = +\dfrac{n\nu}{dx^2}$

quæ Tabella sufficiet ad Problemata quæcunque resolvenda, uti ex sequente Propositione intelligetur.

PROPOSITIO IV. PROBLEMA.

31. *Sit* $\pi = \int [z] dx$ *&* $d[z] = [m]dx + [n]dy + [p]dp + [q]dq$, *atque quantitas* $[Z]$ *ita involvat formulam integralem* π, *ut sit* $d[Z] = Zd\pi + [M]dx + [N]dy + [P]dp + [Q]dq$. *Jam posito* $\Pi = \int [Z] dx$, *sit* Z *functio ipsarum* x, y, p, q, *itemque ipsius* Π, *ita ut sit* $dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq$. *His positis, oporteat definiri curvam* az, *quæ, pro data abscissa* $AZ = a$, *habeat valorem formulæ* $\int Zdx$ *maximum vel minimum.*

SOLUTIO.

Ut in Scholio præcedente monuimus, est nobis abscissa AL $= x$, & applicata $Ll = y$; abscissæ autem $AL = x$ respondeat valor $\int Zdx$ qui a particula $n\nu$ non afficietur. Ex quo valor differentialis ex sequentibus abscissæ elementis determinari debebit, quibus respondebunt valores Zdx, $Z'dx$, $Z''dx$, $Z'''dx$, $Z^{IV}dx$, &c. usque ad ultimum abscissæ totius propositæ AZ elementum in Z. Invenientur autem singulorum horum terminorum valores differentiales per differentiationem; substituendo loco differentialium dy, dp, dq, valores paragrapho præcedenti indicatos. Erit igitur

$$d.Zdx = dx\left(Ld\Pi + \frac{Q.n\nu}{dx^2}\right)$$
$$d.Z'dx = dx\left(L'd\Pi' + \frac{P'n\nu}{dx} - \frac{2Q'.n\nu}{dx^2}\right)$$
$$d.Z''dx = dx\left(L''d\Pi'' + N''.n\nu - \frac{P''.n\nu}{dx} + \frac{Q''.n\nu}{dx^2}\right)$$
$$d.Z'''dx = dx.L'''d\Pi'''$$
$$d.Z^{IV}dx = dx.L^{IV}d\Pi^{IV}$$
&c.

Superest igitur ut per $n\nu$ definiamus differentialia $d\Pi$, $d\Pi'$, $d\Pi''$, $d\Pi'''$ &c. hoc est valores differentiales quantitatum Π, Π', Π'', Π''' &c. Est vero $\Pi =$

$$\Pi = \int [Z] dx$$
$$\Pi' = \int [Z] dx + [Z] dx$$
$$\Pi'' = \int [Z] dx + [Z] dx + [Z'] dx$$
$$\Pi''' = \int [Z] dx + [Z] dx + [Z'] dx + [Z''] dx$$
$$\Pi^{IV} = \int [Z] dx + [Z] dx + [Z'] dx + [Z''] dx + [Z'''] dx$$
&c.

Ubi notandum eſt quantitatis $\int [Z] dx$ valorem differentialem eſſe $= 0$, eo quod particula $n\nu$ nullam mutationem infert in abſciſſam A L ad quam $\int [Z] dx$ refertur, Tantum igitur terminorum differentialium $[Z] dx$, $[Z'] dx$, $[Z''] dx$ &c. valores differentiales inveſtigari oportebit. Erit autem.

$$d. [Z] dx = dx\left([L]\, d\pi + \frac{[Q.]\, n\nu}{dx^2} \right)$$
$$d. [Z'] dx = dx\left([L']\, d\pi' + \frac{[P']\, n\nu}{dx} - \frac{2\,[Q'.]\, n\nu}{dx^2} \right)$$
$$d. [Z''] dx = dx\left([L'']\, d\pi'' + [N''.]\, n\nu - \frac{[P''.]\, n\nu}{dx} + \frac{[Q''.]\, n\nu}{dx^2} \right)$$
$$d. [Z'''] dx = dx\, [L''']\, d\pi'''$$
$$d. [Z^{IV}] dx = dx.\, [L^{IV}]\, d\pi^{IV}.$$
&c.

Nunc porro definiendi ſunt valores differentiales quantitatum π, π', π'', π''', &c. per $n\nu$, quos loco $d\pi$, $d\pi'$, $d\pi''$, &c. ſubſtitui oportet. Cum autem ſit $\pi = \int [z] dx$, & in $[z]$ differentialia ſecundum gradum ſuperantia non ineſſe ponantur, fiet valor differentialis ipſius π, ſeu $d\pi = 0$, ad ſequentium autem quantitatum π', π'', π''' &c. valores differentiales inveniendos, notaſſe conveniet eſſe

$$\pi = \int [z] dx$$
$$\pi' = \int [z] dx + [z] dx$$
$$\pi'' = \int [z] dx + [z[dx + [z'] dx$$
$$\pi''' = \int [z] dx + [z] dx + [z'] dx + [z''] dx$$
$$\pi^{IV} = \int [z] dx + [z] dx + [z'] dx + [z''] dx + [z'''] dx$$
&c.

Erit autem

$$d.[z]dx = nv.dx\,\frac{[q]}{dx^2}$$
$$d.[z']dx = nv.dx\left(\frac{[p']}{dx} - \frac{2[q']}{dx^2}\right)$$
$$d.[z'']dx = nv.dx\left([n''] - \frac{[p'']}{dx} + \frac{[q'']}{dx^2}\right)$$
$$d.[z''']dx = 0$$
$$d.[z^{IV}]dx = 0$$
&c.

Ex his itaque obtinebitur

$$d.\pi = 0$$
$$d.\pi' = nv.dx.\frac{[q]}{dx^2}$$
$$d.\pi'' = nv.dx\left(\frac{[p']}{dx} - \frac{[q]}{dx^2} - \frac{2d[q]}{dx^2}\right)$$
$$d.\pi''' = nv.dx\left([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\right)$$
$$d.\pi^{IV} = nv.dx\left([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\right)$$
$$d.\pi^{V} = nv.dx\left([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\right)$$

omnesque sequentes valores inter se erunt æquales. Quod si jam hi valores inventi substituantur, erit

$$d.[Z]dx = nv.dx.\frac{[Q]}{dx^2}$$
$$d.[Z']dx = nv.dx\left(\frac{[L'][q]}{dx} + \frac{[P']}{dx} - \frac{2[Q]}{dx^2}\right)$$
$$d.[Z'']dx = nv.dx\left([L'']dx\left(\frac{[p']}{dx} - \frac{[q] - 2d[q]}{dx^2}\right) + [N''] - \frac{[P'']}{dx} + \frac{[Q'']}{dx^2}\right)$$

d.

$$d.[Z'''] dx = nv.dx.[L'''] dx\left([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\right)$$
$$d.[Z^{IV}] dx = nv.dx.[L^{IV}] dx\left([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\right)$$
$$d.[Z^{V}] dx = nv.dx\,[L^{V}] dx\left([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\right)$$

&c.

Hinc porro deducitur:

$$d\Pi = 0$$
$$d.\Pi' = nv.dx.\frac{[Q]}{dx^2}$$
$$d.\Pi'' = nv.dx\left([L']dx.\frac{[q]}{dx^2} + \frac{[P']}{dx} - \frac{[Q] + 2d[Q]}{dx^2}\right)$$
$$d.\Pi''' = nv.dx\left([L''][p'] - \frac{[q]d[L'] + 2[L']d[q]}{dx} + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\right)$$
$$d.\Pi^{IV} = nv.dx\left([L''']dx\left([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\right) + [L''][p] - \frac{[q]d[L'] + 2[L']d[q]}{dx} + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\right)$$
$$d.\Pi^{V} = nv.dx\left(([L''']dx + [L^{IV}]dx)\left([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\right) + [L''][p] - \frac{[q]d[L'] + 2[L']d[q]}{dx} + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\right)$$

&c.

Ex his jam orientur sequentes determinationes:

$$d.Z dx = nv.dx.\frac{Q}{dx^2}$$
$$d.Z' dx = nv.dx\left(L'dx.\frac{[Q]}{dx^2} + \frac{P'}{dx} - \frac{2Q'}{dx^2}\right)$$
$$d.Z'' dx = nv.dx\left(L''dx\left([L']dx.\frac{[q]}{dx^2} + \frac{[P']}{dx} - \frac{[Q] + 2d[Q]}{dx^2}\right) + N'' - \frac{P''}{dx} + \frac{Q''}{dx^2}\right)$$

$$d.\,Z''dx = nv.\,dx.\,L'''dx\left([L''][p'] - \frac{[q]d[L'] + 2[L']d[q]}{dx} + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\right)$$

$$d.\,Z^{IV}dx = nv.\,dx\,L^{IV}dx\left([L''']dx\left([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\right) + [L''][p'] - \frac{[q]d[L] + 2[L]d[q]}{dx} + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\right)$$

$$d.Z^{V}dx = nv.dx.L^{V}dx\left(([L''']dx + [L^{IV}]dx)\left([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\right) + [L''][p'] - \frac{[q]d[L'] + 2[L']d[q]}{dx} + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\right)$$

$$d.Z^{VI}dx = nv.dx.L^{VI}dx\left(([L''']dx + [L^{IV}]dx + [L^{V}]dx)\left([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\right) + [L''][p'] - \frac{[q]d[L'] + 2[L']d[q]}{dx} + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\right)$$

&c.

Ut hi valores omnes eo commodius ad se invicem addi queant, ponamus brevitatis gratia $[h] = [n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2} = [n] - \frac{d[p]}{dx} + \frac{dd[q]}{dx^2}$; & $[H] = [L][p] - \frac{[q]d[L] + 2[L]d[q]}{dx} + [N] - \frac{d[P]}{dx} + \frac{dd[Q]}{dx^2}$: eritque summa omnium, hoc est valor differentialis formulæ propositæ $\int Z\,dx$, ut sequitur.

$$nv.\,dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2}\right) + nv.\,dx\left(L[P] - \frac{[Q]dL - 2Ld[Q]}{dx}\right) + nv.dx.\,L[L][q] + nv.dx.[H](L'''dx + L^{IV}dx + L^{V}dx + \&c.\text{ in } Z) + nv.dx.[h](L^{IV}dx.[L''']dx + L^{V}dx([L''']dx + [L^{IV}]dx) + L^{VI}dx([L''']dx + [L^{IV}]dx + [L^{V}]dx) + L^{VII}dx([L''']dx + [L^{IV}]dx + [L^{V}]dx + [L^{VI}]dx) + \&c.)$$

Binæ igitur hic habentur series infinitæ, a termino Ll usque ad Zz progredientes, quarum illius $L'''dx + L^{IV}dx + L^{V}dx + \&c.$ summa exprimi potest per $H - \int L\,dx$, denotante H valorem ipsius $\int L\,dx$, posito $x = a$. Quo autem valorem alterius seriei investigemus, ponatur ejus summa $= S$, ita ut sit $S = L^{IV}dx.$

$[L''']$

$[L'''] dx + L^{V} dx ([L'''] dx + [L^{IV}] dx) +$&c. Sumatur valor sequens $S' = S + dS$, erit $S + dS = L^{V} dx. [L^{IV}] dx + L^{VI} dx ([L^{IV}] dx + [L^{V}] dx) +$ &c. qui ab illo subtractus relinquet, $- dS = L^{IV} [L'''] dx^2 + L^{V} [L'''] dx^2 + L^{VI} [L'''] dx^2 +$&c. seu $- dS = [L'''] dx (L^{IV} dx + L^{V} dx + L^{VI} dx +$ &c.) ideoque $- dS = [L'''] dx (H - \int L dx)$, & integrando $S = G - \int [L] dx (H - \int L dx)$, constante G ita assumta, ut fiat $S = 0$ si ponatur $x = a$. His inventis fiet valor differentialis formulæ propositæ $\int Z dx = nv. dx (N - \frac{dP}{dx} + \frac{ddQ}{dx^2} + L[P] - \frac{[Q] dL + 2 L d[Q]}{dx} + L[L][q] + (H - \int L dx)([L][p] - \frac{[q] d[L] + 2[L] d[q]}{dx} + [N] - \frac{d[P]}{dx} + \frac{dd[Q]}{dx^2}) + (G - \int [L] dx (H - \int L dx))([n] - \frac{d[p]}{dx} + \frac{dd[q]}{dx^2}))$. Hæc expressio autem in sequentem formam transmutari potest, ex qua facilius valor differentialis formari poterit, si differentialia altiorum graduum quam secundi, tam in Z quam in $[Z]$ & $[z]$ insint. Erit scilicet formulæ $\int Z dx$ valor differentialis abscissæ $AZ = a$ respondens

$$= nv. dx (N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3 R}{dx^3} + \frac{d^4 S}{dx^4} - \text{\&c.})$$

$$+ nv. dx ([N](H - \int L dx) - \frac{d.[P](H - \int L dx)}{dx} + \frac{dd.[Q](H - \int L dx)}{dx^2} - \frac{d^3.[R](H - \int L dx)}{dx^3} + \frac{d^4.[S](H - \int L dx)}{dx^4} - \text{\&c.}) + nv. dx$$

$$([n](G - \int [L] dx (H - \int L dx)) - \frac{d.[p](G - \int [L] dx (H - \int L dx))}{dx} + \frac{dd.[q](G - \int [L] dx (H - \int L dx))}{dx^2} - \frac{d.^3 [r](G - \int [L] dx (H - \int L dx))}{dx^3}$$

+ &c.). Invento autem valore differentiali, si is ponatur $= 0$; habebitur æquatio pro curva quæsita. *Q. E. I.*

Co-

COROLL. I.

32. Inventus igitur eſt valor differentialis pro formula $\int Z\,dx$ latius patente, quam quidem in Propoſitione eſt aſſumta: ſcilicet ſi fuerit $dZ = L\,d\Pi + M\,dx + N\,dy + P\,dp + Q\,dq + R\,dR$ + &c. atque exiſtente $d\Pi = [Z]\,dx$, ſi ſit $d[Z] = [L]\,d\pi + [M]\,dx + [N]\,dy + [P]\,dp + [Q]\,dq + [R]\,dr$ + &c. itemque ſi poſito $d\pi = [z]\,dx$ fuerit $d[z] = [m]\,dx + [n]\,dy + [p]\,dp + [q]\,dq + [r]\,dr$ + &c. Quoticunque nimirum gradus differentialia inſint in quantitatibus Z, $[Z]$, & $[z]$ ſolutio data inſerviet.

COROLL. II.

33. Quod ſi ponatur $H - \int L\,dx = T$, & $G - \int [L]\,dx\,(H - \int L\,dx) = V$, erit valor differentialis

$$= nv.\,dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \&c.\right)$$
$$+ nv.\,dx\left([N]\,T - \frac{d.[P]\,T}{dx} + \frac{dd.[Q]\,T}{dx^2} - \frac{d^3\,[R]\,T}{dx^3} + \&c.\right)$$
$$+ nv.\,dx\left([n]\,V - \frac{d.[p]\,V}{dx} + \frac{dd.[q]\,V}{dx^2} - \frac{d^3\,[r]\,V}{dx^3} + \&c.\right)$$

COROLL. III.

34. Hinc igitur æquatio pro curva quæſita erit hæc, $0 = N + [N]\,T + [n]\,V - \frac{d(P + [P]T + [p]\,V)}{dx} + \frac{dd(Q + [Q]T + [q]\,V)}{dx^2} - \frac{d^3(R + [R]\,T + [r]\,V)}{dx^3}$ + &c. cujus progreſſionis lex, ſi forte opus ſit pluribus terminis, ſponte patet.

COROLL. IV.

35. Quin etiam hinc reſolvi poterunt ejuſmodi Problemata, in quibus Z non unam, ſed plures iſtiuſmodi formulas integrales

les indefinitas π in se complectitur; vel etiam si $[Z]$ plures ejusmodi formulas $\pi = \int [z] dx$ in se contineat.

C O R O L L. V.

36. Denique, etsi posuimus esse $[z]$ functionem determinatam, tamen per inductionem hinc modus patet valorem differentialem formandi, si ulterius $[z]$ in se contineat formulam integralem indefinitam.

S C H O L I O N.

37. Latissime igitur solutio hujus Problematis patet, quia non solum precedentia Problemata omnia in se complectitur, atque ipsi casui proposito satisfacit, verum etiam per inductionem ad casus qualescunque magis intricatos accommodari potest. Quod ut facilius percipiatur, ponamus in $[z]$ insuper inesse formulam integralem $\pi = \int \zeta dx$, ita ut sit $\zeta = [l] d\pi + [m] dx + [n] dy + [p] dp + [q] dq +$ &c. existente $d\zeta = \mu dx + \nu dy + \Phi dp + \chi dq +$ &c. Jam ad valorem differentialem determinandum, præter quantitates integrales binas T & V, tertia debet definiri W, ita comparata ut sit $W = F - \int [l] dx\, (G - \int [L] dx\, (H - \int L dx))$ quæ evanescat posito $x = a$. Hocque facto, erit valor differentialis
$$= n\nu . dx \left(N + [N] T + [n] V + \nu W - \frac{d(P + [P] T + [p] V + \Phi W)}{dx} + \frac{dd. (Q + [Q] T + [q] V + \chi W)}{dx^2} - \text{\&c.}\right)$$
Quamobrem nequidem maximi minimive formula excogitari poterit, quæ non in solutione esset contenta, aut ex talibus formulis composita, ad quas ista solutio patet. Quinetiam liceret hanc expressionem in infinitum extendere, si quælibet formula indeterminata aliam novam formulam integralem indefinitam in se complectatur; neque difficultas ulla adesset, nisi in characterum sufficienti numero suppeditando. Quæ cum ulterius prosequi non sit necesse, unicum casum principalem evolvere conveniet, quo

in formula $\int [Z] dx$, quæ valorem ipsius Π præbet, ipsa quantitas $[Z]$ denuo Π involvit. Hoc enim casu complexio istiusmodi formularum integralium actu in infinitum progreditur; namque si sit $d[Z] = [L] d\Pi + [M] dx + [N] dy + [P] dp + [Q] dq +$ &c. erit hic iterum $d\Pi$, quod ante fuerat $d\pi$, & quoniam est $d\Pi = [Z] dx$, denuo eadem æquatio $d[Z] = [L] d\Pi + [M] dx + [N] dy +$ &c. recurrit, atque ita tractatio formularum integralium nusquam abrumpetur. Casum igitur hunc, cum quia insignem nobis afferet usum, tum quia concinnam admittit solutionem, pertractabimus.

PROPOSITIO V. PROBLEMA.

38. *Si* Π *aliter non detur nisi per æquationem differentialem* $d\Pi = [Z] dx$, *in qua* $[Z]$, *præter quantitates ad curvam pertinentes* x, y, p, q, r, *&c. ipsam quantitatem* Π *complectatur, ita ut sit* $d[Z] = [L] d\Pi + [M] dx + [N] dy + [P] dp$
Fig. 4. $+ [Q] dq +$ &c. *Sit* Z *functio quæcunque ipsius* Π *& ipsarum* x, y, p, q, *&c. ita ut sit* $dZ = L d\Pi + M dx + N dy + P dp + Q dq +$ *&c. invenire curvam, in qua, pro data abscissa* $AZ = a$, *maximum minimumve sit formula* $\int Z dx$.

SOLUTIO.

Ponamus differentialia, quæ tam in Z quam in $[Z]$ insunt, secundum gradum non excedere, ita ut particula $n\nu$ ultra abscissæ punctum L versus initium nullam mutationem inferat. Solutio enim nihilominus hinc poterit maxime generalis confici. Sit igitur abscissa $AZ = x$, & applicata $LI = y$, patietur $\int Z dx$ ab adjecta particula $n\nu$ applicatæ $Nn = y''$ nullam mutationem, ejusque valor differentialis erit $= 0$. Quamobrem valor differentialis formulæ $\int Z dx$, quatenus ad totam abscissam AZ extenditur, colligi debebit ex elementis $Z dx$, $Z' dx$, $Z'' dx$, $Z''' dx$, &c. Singulorum autem horum elementorum valores differentiales invenientur, si ea differentientur, & loco differentialium dy, dy', dy'', dp, dp', dp'', & dq, dq', dq'' valores

valores §. 30 indicati substituantur. Quoniam autem insuper in hæc differentialia ingrediuntur $d\pi$, $d\pi'$, $d\pi''$, &c. ponamus eorum valores ex $n\nu$ oriundos tantisper, donec eos inveniamus esse hos :

$$\begin{array}{l|l|l} d\pi = n\nu.\,\alpha & d\pi''' = n\nu.\,\delta & d\pi^{VI} = n\nu.\,\eta \\ d\pi' = n\nu.\,\beta & d\pi^{IV} = n\nu.\,\varepsilon & d\pi^{VII} = n\nu.\,\theta \\ d\pi'' = n\nu.\,\gamma & d\pi^{V} = n\nu.\,\zeta & \&c. \end{array}$$

Hinc itaque erunt valores differentiales

$$d.\,Z\,dx = n\nu.\,dx\,(L\alpha + \frac{Q}{dx^2})$$
$$d.\,Z'\,dx = n\nu.\,dx\,(L'\beta + \frac{P'}{dx} - \frac{2Q'}{dx^2})$$
$$d.\,Z''\,dx = n\nu.\,dx\,(L''\gamma + N'' - \frac{P''}{dx} + \frac{Q''}{dx^2})$$
$$d.\,Z'''\,dx = n\nu.\,dx\,L'''\delta$$
$$d.\,Z^{IV}\,dx = n\nu.\,dx\,L^{IV}\varepsilon$$
$$d.\,Z^{V}\,dx = n\nu.\,dx.\,L^{V}\zeta$$
&c.

Ut nunc valores litterarum α, β, γ, δ, ε, &c. definiamus, notandum est esse $d\pi$, $d\pi'$, $d\pi''$, &c. valores differentiales quantitatum π, π', π'', &c. Est vero

$$\pi = \int [Z]\,dx$$
$$\pi' = \int [Z]\,dx + [Z]\,dx$$
$$\pi'' = \int [Z]\,dx + [Z]\,dx + [Z']\,dx$$
$$\pi''' = \int [Z]\,dx + [Z]\,dx + [Z']\,dx + [Z'']\,dx$$
&c.

ubi $\int [Z]\,dx$, per hypothesin, a particula $n\nu$ non afficitur. Valores igitur differentiales formularum $[Z]\,dx$, $[Z']\,dx$, $[Z'']\,dx$ &c. sunt investigandi, qui erunt

 $d.\,[Z]\,dx$

$$d.[Z]\,dx = n\nu.\,dx\left([L]\alpha + \frac{[Q]}{dx^2}\right)$$
$$d.[Z']\,dx = n\nu.\,dx\left([L']\beta + \frac{[P]}{dx} - \frac{2[Q']}{dx^2}\right)$$
$$d.[Z'']\,dx = n\nu.\,dx\left([L'']\gamma + [N'] - \frac{[P'']}{dx} + \frac{[Q'']}{dx^2}\right)$$
$$d.[Z''']\,dx = n\nu.\,dx\,[L''']\delta$$
$$d.[Z^{IV}]\,dx = n\nu.\,dx.\,[L^{IV}]\varepsilon$$
$$d.[Z^{V}]\,dx = n\nu.\,dx.\,[L^{V}]\zeta$$

&c.

Ex his igitur erit ut sequitur

$$d\Pi = \alpha$$
$$d\Pi' = n\nu.\,dx\left([L]\alpha + \frac{[Q]}{dx^2}\right)$$
$$d\Pi'' = n\nu.\,dx.\left([L]\alpha + [L']\beta + \frac{[P']}{dx} - \frac{[Q]+2d[Q]}{dx^2}\right)$$
$$d\Pi''' = n\nu.\,dx\left([L]\alpha + [L']\beta + [L'']\gamma + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\right)$$
$$d\Pi^{IV} = n\nu.\,dx\left([L]\alpha + [L']\beta + [L'']\gamma + [L''']\delta + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\right)$$
$$d\Pi^{V} = n\nu.\,dx\left([L]\alpha + [L']\beta + [L'']\gamma + [L''']\delta + [L^{IV}]\varepsilon + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\right)$$

&c.

His comparatis cum valoribus assumtis, erit

$$\alpha = 0$$
$$\beta = [L]\alpha\,dx + \frac{[Q]}{dx}$$
$$\gamma = dx\left([L]\alpha + [L']\beta + \frac{[P']}{dx} - \frac{[Q]+2d[Q]}{dx^2}\right)$$

$\delta =$

$$\delta = dx\,([L]\alpha + [L']\beta + [L'']\gamma + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2})\,.$$

$$\varepsilon = dx\,([L]\alpha + [L']\beta + [L'']\gamma + [L''']\delta + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2})$$

&c.

Ex hiſque æquationibus elicitur:

$$\alpha = 0$$

$$\beta = \frac{[Q]}{dx}$$

$$\gamma = [L'][Q] + [P'] - \frac{[Q] + 2d[Q]}{dx}$$

$$\delta = [L'][Q] + [L''][L'][Q]dx + [L''][P']dx - [L''][Q] - 2[L'']d[Q] + [N'']dx - d[P'] + \frac{dd[Q]}{dx}$$

ſeu $\delta = [L''][L'][Q]dx + [L''][P']dx - [Q]d[L'] - 2[L'']d[Q] + [N'']dx - d[P'] + \frac{dd[Q]}{dx}$

qui valor ipſius δ notetur, eritque porro

$$\varepsilon = \delta(1 + [L''']dx)$$

$$\zeta = \delta(1 + [L''']dx)(1 + [L^{IV}]dx)$$

$$\eta = \delta(1 + [L''']dx)(1 + [L^{IV}]dx)(1 + [L^{V}]dx)$$

&c.

Cognitis his valoribus, erit valor differentialis elementis $Zdx + Z'dx + Z''dx$ reſpondens

$= nv.\, dx\,(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} + L[L][Q] + L[P] - \frac{[Q]dL + 2Ld[Q]}{dx})$. Sequentium autem elementorum omnium uſque ad Z valor differentialis, ſi ponatur $V = [L^2][Q] + [L]$

$+ [L][P] - \frac{[Q]d[L] + 2[L]d[Q]}{dx} + N - \frac{d[P]}{dx} + \frac{dd[Q]}{dx}$, seu $\delta = V dx$, erit sequens: $nv. dx(L'''dx + L^{IV} dx (1 + [L''']dx) + L^{V} dx (1 + [L''']dx)(1 + [L^{IV}]dx) + L^{VI}dx (1 + [L''']dx)(1 + [L^{IV}]dx)(1 + [L^{V}]dx) +$ &c. $)V$.

Quamobrem hujus seriei summa est indaganda; hunc in finem, scribamus L loco L''', & $[L]$ loco $[L''']$, sitque summa, quam quærimus, $= S$: erit $S = Ldx + L'dx(1 + [L]dx) + L''dx(1 + [L]dx)(1 + [L']dx) + L'''dx (1 + [L]dx)(1 + [L']dx)(1 + [L'']dx) +$ &c. Jam ipsius S sumatur valor sequens $S' = S + dS$ erit $S + dS = L'dx + L''dx(1 + [L']dx) + L'''dx(1 + [L']dx)(1 + [L'']dx) +$ &c. Hincque $- dS = Ldx + L'[L]dx^2 + [L]dx. L''dx (1 + [L'']dx) + [L]dx. L'''dx (1 + [L']dx)(1 + [L'']dx) +$ &c. quæ series cum ad priorem reduci queat, erit $- dS = Ldx + S'[L]dx$, sive ob $S' = S$, $dS + S[L]dx = - Ldx$; quæ integrata dat $e^{\int [L]dx} S = C - \int e^{\int [L]dx} Ldx$, quæ constans C ita debet accipi, ut posito $x = a$ fiat $S = 0$. Hanc ob rem erit valor illius seriei $S = e^{-\int [L]dx} (C - \int e^{\int [L]dx} Ldx)$. Ex his igitur formulæ propositæ $\int Z dx$ orietur sequens valor differentialis: $nv. dx (N - \frac{dP}{dx} + \frac{ddQ}{dx^2} + L[L][Q] + L[P] - \frac{[Q]dL + 2Ld[Q]}{dx} + S([L^2][Q] + [L][P] - \frac{[Q]d[L] + 2[L]d[Q]}{dx} + [N] - \frac{d[P]}{dx} + \frac{dd[Q]}{dx^2}))$, qui transmutatur in hanc formam commodiorem, $nv. dx (N - \frac{dP}{dx} + \frac{ddQ}{dx^2} + [N]S - \frac{d.[P]S}{dx} + \frac{dd.[Q]S}{dx^2})$. Hinc autem formari potest valor differentialis formulæ $\int Z dx$, si tam in Z quam in $[Z]$ differentialia

tialia ad gradum quemcunque aſſurgant. Ad hoc efficiendum, ſit valor formulæ integralis $\int e^{\int [L]\, dx} L\, dx$, quem obtinet, ſi $x = a$ ponatur, $= H$, ac ſcribatur, brevitatis ergo, V loco hujus expreſſionis $e^{-\int [L]\, dx} (H - \int e^{\int [L]\, dx} L\, dx)$, eritque valor differentialis $= n\nu . dx \left(N + [N] V - \frac{d.(P + [P] V)}{dx} + \frac{dd(Q + [Q] V)}{dx^2} - \frac{d^3 (R + [R] V)}{dx^3} + \&c. \right)$ Atque hinc pro curva quæſita orietur iſta æquatio, $o = N + [N] V - \frac{d(P + [P] V)}{dx} + \frac{dd(Q + [Q] V)}{dx^2} - \frac{d^3 (R + [R] V)}{dx^3} + \frac{d^4 (S + [S] V)}{dx^4} - \&c.$ *Q. E. I.*

COROLL. I.

39. Inſervit igitur iſta propoſitio ejuſmodi Problematibus reſolvendis, in quibus maximi minimive formula $\int Z dx$ talem in ſe continet quantitatem Π, quæ nequidem formula integrali ex quantitatibus ad curvam pertinentibus x, y, p, q, r, &c. exhiberi poteſt, ſed cujus determinatio pendet a reſolutione æquationis differentialis cujuſcunque. Habetur enim $d\Pi = [Z] dx$, atque $[Z]$ ipſam quantitatem Π utcunque in ſe complecti ponitur.

COROLL. II.

40. Caſus hic notari meretur, quo eſt $L = [L]$, quippe quo fit formula $\int e^{\int [L]\, dx} L\, dx$ integrabilis, integrali exiſtente $= e^{\int [L]\, dx}$. Quod ſi ergo, poſito $x = a$, abeat $e^{\int [L]\, dx}$ in H, fiet $V = H e^{-\int [L]\, dx} - 1$.

CO-

COROLL. III.

41. Casus hic potissimum locum habet, quando curva quæritur, in qua sit ipsa formula $\Pi = \int [Z] dx$ maximum vel minimum. Tum enim fit $Z = [Z]$, & hinc $L = [L]$, $M = [M]$, $N = [N]$ &c. Hinc itaque erit valor differentialis $= nv.\, dx \left(H[N] e^{-\int [L] dx} - \frac{d.\, H[P] e^{-\int [L] dx}}{dx} + \frac{dd.\, H[Q] e^{-\int [L] dx}}{dx^2} - \text{\&c.} \right.$ Atque æquatio pro curva erit $0 = [N] e^{-\int [L] dx} - \frac{d.[P] e^{-\int [L] dx}}{dx} + \frac{dd.[Q] e^{-\int [L] dx}}{dx^2}$ — &c.

COROLL. IV.

42. Quia ex hac æquatione quantitas H a data abscissa AZ $= a$ pendens per divisionem est egressa; patet his casibus curvam uni abscissæ satisfacientem, eandem pro omni alia abscissa esse satisfacturam: ita ut hæc Problemata similia sint iis, in quibus quantitas Z est functio determinata.

COROLL. V.

43. Si ergo quantitas $\Pi = \int [Z] dx$ debeat esse maximum vel minimum, existente $d[Z] = [L] d\Pi + [M] dx + [N] dy + [P] dp + [Q] dq +$ &c. curva poterit exhiberi, quæ una pro quacunque abscissa ista proprietate gaudeat; ejusque natura exprimetur hac æquatione: $0 = [N] e^{-\int [L] dx} - \frac{d.[P] e^{-\int [L] dx}}{dx} + \frac{dd.[Q] e^{-\int [L] dx}}{dx^2}$ — &c. Ex qua insuper, evolutis singulis terminis, quantitas exponentialis $e^{-\int [L] dx}$, atque adeo ipsa formula integralis $\int [L] dx$ excedent.

SCHOLION I.

44. Usus hujus Propositionis eximius est in quæstionibus ita comparatis, ut quantitates indefinitæ in iis contentæ per formulas integrales exhiberi nequeant, verum constructionem æquationum differentialium postulent. Atque hæc solutio perinde valet, sive una hujusmodi quantitas π insit in formula maximi minimive $\int Z dx$ sive plures; quod si enim plures insint ejusmodi quantitates π, plures etiam habebuntur valores litterarum L, $[L]$, $[M]$, $[N]$, $[P]$, $[Q]$, &c. atque etiam litteræ $V = e^{-\int [L] dx} (H - \int e^{\int [L] dx} L dx)$; qui omnes æqualiter, eo modo quem invenimus, in valorem differentialem formulæ $\int Z dx$ introducti præbebunt æquationem pro curva; similisque omnino tractatio erit, ac si unica tantum adesset. Quoniam autem littera ista π, cujus valor absolutus per quantitates ad curvam pertinentes exhiberi non potest, in omnibus fere terminis manet; æquatio pro curva, quæ invenitur, non solum ex litteris x, y, p, q, r, &c. constabit, sed etiam ipsam eam quantitatem π, aliasque formulas integrales plerumque ab ea pendentes, uti $\int [L] dx$ & $\int L dx$, involvet. Quare ut æquatio pro curva pura, quæ tantum litteris x, y, p, q, &c. contineatur, prodeat, oportet cum æquatione inventa, postquam a formulis integralibus $\int [L] dx$ & $\int L dx$ est liberata, conjungi æquationem $d\pi = [Z] dx$, ejusque ope valorem π eliminari. Quanquam autem hoc modo ad differentialia altiorum ordinum pervenitur, tamen non totidem inesse censendæ sunt constantes arbitrariæ. Nam tam ipsa æquatio $d\pi = [Z] dx$, quam reliquæ anteriores æquationes, certam requirunt determinationem, unde plures constantes determinabuntur. Cæterum notandum est veritatem hujus Methodi comprobari posse per præcedentes, quando æquatio $d\pi = [Z] dx$ ita est comparata ut integrationem admittat: tum enim eædem quæstiones per Methodos ante traditas resolvi poterunt, indeque consensum observare licebit. Ita si $[Z]$ tantum ex x & π constet, tum certum erit π esse functio-

nem quamdam ipsius x determinatam, atque solutionem ad Caput præcedens pertinere. Idem vero hæc solutio patefaciet, cum enim sit hoc casu $[N]=0$, $[P]=0$, $[Q]=0$, &c. æquatio pro curva erit $0=N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-$ &c. quæ eadem per Methodum priorem obtinetur. Usus autem hujus solutionis clarius per aliquot Exempla declarabitur.

EXEMPLUM I.

45. *Invenire curvam, in qua sit maximus valor ipsius* Π, *existente* $d\Pi = g\,dx - \alpha\Pi^n\,dx\sqrt{(1+pp)}$.

Quæstio hæc occurrit quando quæritur curva, super qua grave in medio resistente secundum celeritatum rationem $2n$ *plicatam* descendens maximam obtinet celeritatem: denotat enim Π quadratum celeritatis, & g vim gravitatis secundum directionem axis AZ exertam. Pertinet itaque hæc quæstio ad casum Coroll. 3 4, & 5 expositum, quo erat $Z=[Z]=g-\alpha\Pi^n\sqrt{(1+pp)}$; atque adeo curva uni abscissæ satisfaciens pro omni abscissa æque valebit. Cum igitur sit $dZ=-\alpha n\Pi^{n-1}d\Pi\sqrt{(1+pp)}-\frac{\alpha\Pi^n p\,dp}{\sqrt{(1+pp)}}$, erit $[L]=-\alpha n\Pi^{n-1}\sqrt{(1+pp)}$, $[M]=0$, $[N]=0$, $[P]=-\frac{\alpha\Pi^n p}{\sqrt{(1+pp)}}$; $[Q]=0$, &c. Unde pro curva quæsita ista invenitur æquatio: $0=-d.[P]e^{-\int[L]dx}$, seu $[P]e^{-\int[L]dx}=C$: hincque $-\int[L]dx=lC-l[P]$, & $[L]dx=\frac{d[P]}{[P]}$. Substitutis ergo loco $[L]$ & $[P]$ debitis valoribus, erit $\int\alpha n\Pi^{n-1}dx\sqrt{(1+pp)}=+lC-l-\alpha-l\Pi^n-lp+l\sqrt{(1+pp)}$; hincque $\alpha n\Pi^{n-1}dx\sqrt{(1+pp)}=-\frac{nd\Pi}{\Pi}-\frac{dp}{p}+\frac{p\,dp}{(1+pp)}$

$=$

$= - \frac{dp}{p(1+pp)} - \frac{nd\Pi}{\Pi}$; seu $o = nd\Pi + \alpha n\Pi^n dx\sqrt{(1+pp)} + \frac{\Pi dp}{p(1+pp)}$. Quæ æquatio, ut eliminetur Π, conjungenda est cum hac $d\Pi + \alpha\Pi^n dx\sqrt{(1+pp)} = gdx$; unde statim fit $o = ngdx + \frac{\Pi dp}{p(1+pp)}$, & $\Pi = -\frac{ngpdx(1+pp)}{dp}$. Cum igitur curva fuerit inventa, hæc æquatio statim præbet celeritatem corporis in quovis curvæ loco. Ponatur $dx = -\frac{tdp}{ng}$, erit $\Pi = pt(1+pp)$ & $d\Pi = pdt(1+pp) + tdp(1+3pp)$; hincque obinebitur ista æquatio, $pdt(1+pp) + tdp(1+3pp) - \frac{\alpha p^n t^{n+1}(1+pp)^{n+\frac{1}{2}}dp}{ng} + \frac{tdp}{n} = o$, quæ transmutatur in hanc $\frac{npdt(1+pp) + tdp(n+1+3npp)}{nt^{n+1}p^{n+1}(1+pp)^{n+\frac{1}{2}}} = \frac{\alpha dp}{ngp^2}$; cujus integralis est $\frac{1}{nt^n p^{n+1}(1+pp)^{n-\frac{1}{2}}} = \frac{\alpha}{ngp} + \frac{\beta}{ng}$, seu $g = (\alpha + \beta p)t^n p^n (1+pp)^{n-\frac{1}{2}}$; hincque $t = \frac{\sqrt[n]{g}}{p(1+pp)^{1-1:2n}\sqrt[n]{(\alpha+\beta p)}}$. Erit igitur $dx = \frac{-dp}{np(1+pp)^{1-1:2n}\sqrt[n]{g^{n-1}(\alpha+\beta p)}}$, & $dy = \frac{-dp}{n(1+pp)^{1-1:2n}\sqrt[n]{g^{n-1}(\alpha+\beta p)}}$; hincque $\Pi = \sqrt[n]{\frac{g\sqrt{(1+pp)}}{\alpha+\beta p}}$. Erit ergo $x = -\frac{1}{ng}\int\frac{dp}{p(1+pp)}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{\alpha+\beta p}}$, atque $y = -\frac{1}{ng}\int\frac{dp}{1+pp}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{\alpha+\beta p}}$.

Hinc apparet quantitatem Π super curva nusquam esse posse $= o$; hanc ob rem, in curvæ initio Π jam habebit certum quemdam valorem. Ut autem indoles curvæ magis percipiatur, ex æquatione $\Pi = -\frac{ngp\,dx(1+pp)}{dp}$ patet valorem ipsius dp ubique negativum esse oportere, ex quo curva versus axem erit concava. Quia igitur valores ipsius p recedendo a curvæ initio decrescunt, in ipso curvæ initio p maximum habebit valorem. Hinc ponamus initium curvæ ibi, ubi est $p = \infty$. Sit ergo AP axis curvæ verticalis, in cujus directione vis gravitatis g corpus deorsum trahat, atque in initio curvæ A sit tangens horizontalis Aa: ibique corpus motum super curva incipiat, celeritate, cujus quadratum sit $= b$. Erit igitur, posito $p = \infty$, $b = \sqrt[n]{\frac{g}{\mathfrak{G}}}$, atque $\mathfrak{G}b^n = g$, seu $\mathfrak{G} = \frac{g}{b^n}$. Porro ad uniformitatem

Fig. 6.

conservandam sit $\alpha = \frac{1}{k^n}$. Quod si jam curva quæsita sit AM, & ponatur AP $= x$, PM $= y$, & $dy = p\,dx$; erit in M celeritatis quadratum $\Pi = bk\sqrt[n]{\frac{g\sqrt{(1+pp)}}{b^n + gk^n p}}$; atque ubi tangens curvæ fiet verticalis, ibi erit celeritatis quadratum $= k\sqrt[n]{g}$. Curvæ autem constructio ita conficietur, ut sit

$$x = -\frac{bk}{ng}\int\frac{dp}{p(1+pp)}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{b^n + gk^n p}} \;\&$$

$$y = -\frac{bk}{ng}\int\frac{dp}{1+pp}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{b^n + gk^n p}}.$$

Deinde commemorari meretur singularis proprietas, seu relatio inter corporis descendentis vim centrifugam, quæ est $\frac{2\Pi}{\text{rad. osculi}}$, & vim normalem quæ est $\frac{qp}{\sqrt{(1+pp)}}$. Quod si enim vis centrifuga $\frac{2\Pi}{\text{rad. osc.}} = \frac{-2\Pi\,dp}{dx(1+pp)^{3:2}}$ ponatur $= F$, & vis normalis $\frac{gp}{\sqrt{(1+pp)}} = G$; erit, ex æquatione $\Pi = -\frac{ngp\,dx(1+pp)}{dp}$, seu $\frac{-2\Pi\,dp}{dx(1+pp)^{3:2}} = \frac{2ngp}{\sqrt{(1+pp)}}$, hæc relatio

tio inter vim centrifugam F & vim normalem G, ut sit $F = 2nG$: nempe vis normalis se habebit ad vim centrifugam ut 1 ad $2n$. Corpus in A data celeritate motum inchoans descendendo super curva AM, in quovis loco M abscissæ AP respondente majorem habebit celeritatem, quam si super alia quacunque curva eadem celeritate initiali descendisset. Evolvamus autem binos casus principales;

Sitque 1°. resistentia quadratis celeritatum proportionalis, erit $n = 1$, & $F = 2G$. Pro curva autem habebitur:

$$x = -bk\int \frac{dp}{p(b+gkp)\sqrt{(1+pp)}}$$

$$\& \ y = -bk\int \frac{dp}{(b+gkp)\sqrt{(1+pp)}};$$

itemque arcus curvæ AM $= -bk\int \frac{dp}{p(b+gkp)} = C + kl\frac{b+gkp}{p}$. Ponatur arcus AM $= s$, qui cum evanescere debeat posito $p = \infty$, erit $s = kl\frac{b+gkp}{gkp}$, hincque $e^{s:k}gkp = b + gkp$, & $p = \frac{b}{gk(e^{s:k}-1)} = \frac{dy}{dx}$. Unde oritur $bdx + gkdy = gke^{s:k}dy$.

Erit autem porro ex æquatione $y = -bk\int \frac{dp}{(b+gkp)\sqrt{(1+pp)}}$ integrata $y = \frac{bk}{\sqrt{(bb+ggkk)}} \, l\frac{(b+gkp)(b+\sqrt{(bb+ggkk)})}{gk\,bp - gk + \sqrt{(bb+ggkk)(1+pp)})}$.

2°. Sit resistentia ipsis celeritatibus proportionalis, fiet $n = \frac{1}{2}$ & $F = G$, hoc est vis centrifuga vi normali erit æqualis. Quæ binæ vires cum sint contrariæ, quæsito satisfaciet ea curva, quæ a corpore super ea descendente omnino non premitur. Erit autem

$$x = -2gbk\int \frac{dp}{p(\sqrt{b}+gp\sqrt{k})^2}$$

$$\& \ y = -2gbk\int \frac{dp}{(\sqrt{b}+gp\sqrt{k})^2} = \frac{2b\sqrt{k}}{\sqrt{b}+gp\sqrt{k}};$$

hincque $ydx\sqrt{b} + gydy\sqrt{k} = 2bdx\sqrt{k}$, & $dx = \frac{gydy\sqrt{k}}{2b\sqrt{k} - y\sqrt{b}}$; hincque integrando $x = -gy\sqrt{\frac{k}{b}} + 2gkl\frac{2b\sqrt{k}}{2b\sqrt{k} - y\sqrt{b}}$. Hæc ergo go curva non solum per Logarithmicam construi potest, verum est portio ipsius Logarithmicæ obliquangulæ. Erit scilicet ipsa curva projectoria, quam corpus in hac resistentiæ hypothesi projectum libere describit. Hæc convenientia ex eo patet, quod curva a corpore moto nullam sustinet pressionem, quæ est proprietas curvarum libere descriptarum.

EXEMPLUM II.

46. *Invenire curvam in qua, pro data abscissa* $x = a$, *minimum sit ista formula* $\int \frac{dx\sqrt{(1+pp)}}{\sqrt{\Pi}}$, *existente* $d\Pi = g\,dx - \alpha\Pi^n dx \sqrt{(1+pp)}$.

Quæstio hæc congruit cum illa, in qua requiritur curva, super qua corpus descendens, in medio resistente cujus resistentia est ut potestas exponentis $2n$ celeritatis, citissime arcum abscissæ a respondentem absolvit. Denotat enim hîc g vim gravitatis secundum directionem axis sollicitantem, $\sqrt{\Pi}$ celeritatem corporis in quocunque loco, & $\alpha\Pi^n$ resistentiam medii ipsam. Erit itaque $Z = \frac{\sqrt{(1+pp)}}{\sqrt{\Pi}}$, & hinc $dZ = \ldots$ $\frac{-d\Pi\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}} + \frac{p\,dp}{\sqrt{\Pi(1+pp)}}$, unde erit $L = \frac{-\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}}$; $M = 0$, $N = 0$, $P = \frac{p}{\sqrt{\Pi(1+pp)}}$ Porro erit $[Z] = g - \alpha\Pi^n.\sqrt{(1+pp)}$, & $d[Z] = -\alpha n\Pi^{n-1} d\Pi\sqrt{(1+pp)} - \frac{\alpha\Pi^n p\,dp}{\sqrt{(1+pp)}}$; unde erit $[L] = -\alpha n\Pi^{n-1}\sqrt{(1+pp)}$; $[M] = 0$, $[N] = 0$, & $[P] = \frac{-\alpha\Pi^n p}{\sqrt{(1+pp)}}$. Habebitur ergo $V = e^{\alpha n\int\Pi^{n-1}dx\sqrt{(1+pp)}} \ldots\ldots$ $\times\left(\int e^{-\alpha n\int\Pi^{n-1}dx\sqrt{(1+pp)}}\frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}} - H\right)$;

deno-

denotante H eum valorem formulæ . . . .
$\int e^{-\alpha n \int \Pi^{n-1} dx \surd(1+pp)} \frac{dx \surd(1+pp)}{2\Pi \surd \Pi}$ quem obtinet si fit $x = a$. Namque V evanescere debet posito $x = a$, estque $dV = \alpha n V \Pi^{n-1} dx \surd(1+pp) + \frac{dx \surd(1+pp)}{2\Pi \surd \Pi}$.
Ex his pro curva quæsita obtinebitur ista æquatio: $d.(P + [P]V) = 0$, & $P + [P]V = C$, seu $V = \frac{C - P}{[P]}$. Substitutis ergo valoribus debitis, erit $e^{\alpha n \int \Pi^{n-1} dx \surd(1+pp)}$ . . . . . .
$\times(\int e^{-\alpha n \int \Pi^{n-1} dx \surd(1+pp)} \frac{dx \surd(1+pp)}{2\Pi \surd \Pi} - H)$
$= \frac{p - C \surd \Pi(1+pp)}{\alpha \Pi^n p \surd \Pi}$. Quare constantem C ita determinari oportet, ut posito $x = a$, fiat $C = \frac{p}{\surd \Pi(1+pp)}$. Cum autem sit $V = \frac{1}{\alpha \Pi^n \surd \Pi} - \frac{C \surd(1+pp)}{\alpha \Pi^n p}$, erit $dV =$
$\frac{-(n+\frac{1}{2}) d\Pi}{\alpha \Pi^{n+1} \surd \Pi} + \frac{nC d\Pi \surd(1+pp)}{\alpha \Pi^{n+1} p} + \frac{C dp}{\alpha \Pi^n p^2 \surd(1+pp)}$
$= \frac{dx \surd(1+pp)}{2\Pi \surd \Pi} + \frac{n dx \surd(1+pp)}{\Pi \surd \Pi} - \frac{nC(1+pp)dx}{p\Pi}$,
in subsidium vocata æquatione $dV = \alpha n V \Pi^{n-1} dx \surd(1+pp)$
$+ \frac{dx \surd(1+pp)}{2\Pi \surd \Pi}$. Cum autem sit $d\Pi = g dx - \alpha \Pi^n dx$.
$\times \surd(1+pp)$ erit $- \frac{(n+\frac{1}{2}) g dx}{\alpha \Pi^{n+1} \surd \Pi} + \frac{nCg dx(1+pp)}{\alpha \Pi^{n+1} p}$ . . .
$+ \frac{C dp}{\alpha \Pi^n p^2 \surd(1+pp)} = 0$, seu $\frac{C dp}{p^2 \surd(1+pp)} = \frac{(n+\frac{1}{2}) g dx}{\Pi \surd \Pi}$
$- \frac{nCg dx \surd(1+pp)}{\Pi p}$. Quod si jam hæc æquatio cum illa $d\Pi = g dx - \alpha \Pi^n dx \surd(1+pp)$ conjungatur, poterit eliminari

minari quantitas Π, hocque pacto inveniri æquatio pro curva quæsita. Hoc autem modo calculus fieret maxime tædiosus, ac minime tractabilis. Adminiculum vero summum afferet ultima æquatio in hanc formam transmutata: $\frac{C\,dp}{gp^2} = \frac{(n+\frac{1}{2})dx\sqrt{(1+pp)}}{\Pi\sqrt{\Pi}} - \frac{nC\,dx(1+pp)}{\Pi\,p}$, cui expressioni ante æqualis esse inventus est valor ipsius dV; erit ergo $dV = \frac{Cdp}{gpp}$ & $V = D - \frac{C}{gp} = \frac{1}{\alpha\,\Pi^n\sqrt{\Pi}} - \frac{C\sqrt{(1+pp)}}{\alpha\,\Pi^n\,p}$. Jam igitur habemus duas æquationes has $\frac{C\,dp}{gp^2} = \frac{(n+\frac{1}{2})\,dx\sqrt{(1+pp)}}{\Pi\sqrt{\Pi}}$ . . . . . . $- \frac{nCdx(1+pp)}{\Pi p}$, & $\alpha D - \frac{\alpha C}{gp} = \frac{1}{\Pi^n\sqrt{\Pi}} - \frac{C\sqrt{(1+pp)}}{\Pi^n\,p}$.

Ex quibus si eliminetur Π, habebitur æquatio inter p & x ejusmodi, ut nusquam x sed ubique tantum dx occurrat, ex quo illa æquatio poterit construi atque adeo ipsa curva. Vel facilius ex posteriori æquatione determinetur p per Π, hicque valor in æquatione fundamentali $dx = \frac{d\Pi}{g - \alpha\Pi^n\sqrt{(1+pp)}}$ substitutus, dabit valorem ipsius x per Π, erit scilicet $x = \int \frac{d\Pi}{g - \alpha\,\Pi^n\sqrt{(1+pp)}}$ atque $y = \int \frac{p\,d\Pi}{g - \alpha\,\Pi^n\sqrt{(1+pp)}}$. Constans autem D ita debet accipi, ut posito $x = a$, quo casu fit $C = \frac{p}{\sqrt{\Pi(1+pp)}}$, fiat $D = \frac{1}{g\sqrt{\Pi(+pp)}}$, seu tum esse debet $\frac{C}{D} = gp$.

SCHOLION II.

47. In his igitur duobus Capitibus, Methodum exposuimus inveniendi lineam curvam, in qua, pro datæ magnitudinis abscissa $= a$, maximum minimumve sit formula $\int Z\,dx$, existente Z func-

functione ipsarum x, y, p, q, r, &c. sive determinata sive indeterminata. Functio autem determinata nobis est, quæ, si alicubi dentur valores litterarum x, y, p, q, r &c. ipsa assignari potest, sive algebraïce sive transcendenter. Functio autem indeterminata est, quæ per datos istarum litterarum valores, quos uno in loco obtinent, assignari nequit, sed omnes valores præcedentes simul involvit, quemadmodum hoc evenit, si signa integralia occurrant. In Capite igitur secundo Methodum tradidimus omnia Problemata resolvendi, in quibus Z est functio determinata; in tertio vero hoc Capite persecuti sumus eas formulas, in quibus Z, vel ipsa est functio indefinita, vel talium unam pluresve involvit; simulque Methodum exhibuimus pro iis casibus, quibus functio illa indefinita nequidem per formulas integrales repræsentari potest, verum resolutionem æquationis differentialis requirit. Nunc igitur eos casus evolvamus, in quibus expressio, quæ maximum minimumve esse debet, non simplex est formula integralis, uti hactenus posuimus, sed ex pluribus ejusmodi formulis utcunque composita: atque simul Methodum aperiemus plura alia Problemata, quæ non ad coordinatas orthogonales spectant, expedite resolvendi.

CAPUT IV.

De Usu Methodi hactenus traditæ in resolutione varii generis quæstionum.

PROPOSITIO I. PROBLEMA.

1. *Invenire æquationem inter binas variabiles* x *&* y, *ita ut, pro dato ipsius* x *valore, puta posito* $x = a$, *formula* $\int Z dx$ *obtineat maximum minimumve valorem, existente* Z *functione ipsarum* x, y, p, q, r, *&c. sive determinata sive indeterminata.*

SOLUTIO.

Ex quacunque consideratione variabiles x & y sint natæ, ex

 semper