METHODVS VNIVERSALIS
SERIES SVMMANDI
VLTERIVS PROMOTA.
AVCTORE
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§. 1.

Ethodus vniuerfalis feries fummandi, quam exeunte praeterito anno expofui, latiffime quidem patet, cum ex folo termino generali feriei dato exhibeat formulam fummae feriei aequalem; interim tamen difficulter ad eiusmodi feries, quarum termini generales algebraice exprimi non poffunt, fed vel exponentiales quantitates inuoluunt vel etiam tranfcendentes, accommodatur. Cum enim pofito termino X, cuius index eft x, fit fumma feriei a primo termino vsque ad X aequalis $\int X\, dx + \frac{X}{1.2} + \frac{dX}{1.2.3.2\,dx} - \frac{d^3 X}{1.2.3.4.5.6\,dx^5} +$ etc. facile apparet fi X faltem huiumodi quantitates a^x inuoluat, tam expreffionem $\int X\, dx$ quam differentialia ipfius X ad logarithmos deduci, vnde maxima oritur moleftia in fumma quaefita faltem proxime affignanda.

§. 2. Praeterea etfi X fuerit quantitas algebraica, tamen faepius eius differentialia, quae ad fummam obtinendam fumi debent, tam fiunt complicata, vt non folum difficulter exhiberi queant, fed etiam feriem non multum conuergentem praebeant, vti id euenit in ferie $\frac{1}{3} + \frac{1}{33} + \frac{1}{99}$ etc. circuli quadraturam continente. Cuius difficultatis ratio in eo potiffimum verfatur, quod

indi-

indices terminorum vnitate crefcentes affumfi, quas fi alio numero crefcentes fumfiffem, terminus generalis X forte tractabilior prodiiffet. Denique fi terminus generalis X ne quidem exhiberi poteft, vt id in plurimis feriebus accidit, tum data formula fummam exhibens ne vfum quidem habere poteft.

§. 3. His difficultatibus quomodo occurrere poffem diu fum meditatus, tandemque obferuaui ex eodem principio, cuius ope illam formulam inueniffem, alias quoque formulas elici pôffe ad quasque feries fummandas idoneas; quibus exhibitis pro quauis oblata ferie, ea formula fit eligenda, quae effet commodiffima. Ex quouis autem huiusmodi formularum genere conueniens vifum eft, vt binae formulae tradantur, quarum altera apta fit ad feries a termino primo ad datum vsque terminum fummandas, cuiusmodi erat formula iam ante a me communicata, altera vero ad feries a dato termino in infinitum vsque fummandas. Quamuis enim haec pofterior fummatio ex priore fluat, tamen expediet pro hoc cafu peculiarem formulam praebuiffe.

§. 4. Incipiam igitur a feriebus, quarum terminus generalis algebraice poteft exhiberi, pro quibus etiam methodus in praecedente fchediasmate data inferuit; fed indices in progreffione quacunque arithmetica progredientes affumam, quo formula inuenta latius pateat faepiusque commodiorem calculum fuppeditet. Sit igitur feries ab initio ad datum vsque terminum fummanda cum indicibus fequens:

$$a$$
$$A$$

$$\overset{a}{A} + \overset{a+b}{B} + \overset{a+2b}{C} - - - - - + \overset{x}{X} = S$$

vbi indices quantitate b crefcunt, primique termini A index eft a. Ponatur huius feriei fumma $= S_{\prime\prime}$, in qua expreffione fi loco x fubftituatur $x-b$, perfpicuum eft eam exhibituram fummam eandem demto X feu fore aequalem ipfi $S - X$. At fi in S loco x ponatur $x-b$ tum prodibit $S - \frac{bdS}{1\,dx} + \frac{b^2\,ddS}{1.2\,dx^2} - \frac{b^3\,d^3S}{1.2.3\,dx^3} +$ etc. vnde habitur fequens aequatio $X = \frac{bdS}{1.dx} - \frac{b^2\,ddS}{1.2\,dx^2} + \frac{b^3\,d^3S}{1.2.3\,dx^3} - \frac{b^4\,d^4S}{1.2.3.4\,dx^4}$ $+$ etc. Ex hac vero aequatione elicitur ifta formula

$$S = \int \frac{Xdx}{b} + \frac{X}{1.2} + \frac{bdX}{1.2.3.2dx} - \frac{b^3\,d^3X}{1.2.3\,4.5.6\,d\,dx^3} + \frac{b^5\,d^5X}{1.2.3\,-\,-\,-\,7.6dx^5}$$
$$- \frac{3b^7\,d^7X}{1.2.3\,-\,-\,-\,9.10dx^7} + \frac{5b^9\,d^9X}{1.2.3\,-\,-\,-\,11.6dx^9} - \frac{691b^{11}\,d^{11}X}{1.2.3\,-\,-\,-\,13.210\,d\,dx^{11}}$$
$$+ \frac{35b^{13}\,d^{13}X}{1.2.3\,-\,-\,-\,15.2dx^{13}} - \frac{3617b^{15}\,d^{15}X}{1.2.3\,-\,-\,-\,17.30dx^{15}} + \frac{2423279b^{17}\,d^{17}X}{1.2.3\,-\,-\,-\,19.1890dx^{17}}$$

etc. cui expreffioni tanta quantitas conftans eft addenda vt pofito $x = a$ fiat $S = A$, vel pofito $x = a - b$ fiat $S = 0$.

§. 5. Si ponatur $X = x^n$, feu fi inuenienda fit fumma huius feriei $a^n + (a+b)^n + (a+2b)^n + - -$ $+ x^n$, erit $\int X dx = \frac{x^{n+1}}{n+1}$; $\frac{dX}{dx} = nx^{n-1}$, $\frac{d^3X}{dx^3} = n$ $(n-1)(n-2)x^{n-3}$ etc. Hinc ergo erit fumma quaefita $S = \frac{x^{n+1}}{(n+1)b} + \frac{x^n}{1.2} + \frac{nbx^{n-1}}{1.2.3.2} - \frac{n(n-1)(n-2)b^3x^{n-3}}{1.2.3.4.5.6}$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)b^5\,x^{n-5}}{1.2.3.4.5.6.7.6} - \text{etc.} - \frac{a^{n+1}}{(n+1)b}$$

$$+ \frac{a^n}{1.2} - \frac{nba^{n-1}}{1.2.3.2} + \frac{n(n-1)(n-2)b^3\,a^{n-3}}{1.2.3.4.5.6} \underline{\qquad\qquad}$$

$$n(n-1)$$

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$$\frac{n(n-1)(n-2)(n-3)(n-4)\,b^5 a^{n-6}}{1.2.3.4.5.6.7.6} + \text{etc.}$$ adiecta debita

constante. Summa ergo seriei $a+(a+b)+(a+2b)+--$
$+x$ erit $=\frac{x^2}{2b}+\frac{x}{2}+\frac{b}{12}-\frac{a^2}{2b}+\frac{a}{2}-\frac{b}{12}=\frac{x^2-a^2+bx+ab}{2b}$,
atque summa seriei $a^2+(a+b)^2+(a+2b)^2+----$
$+x^2=\frac{x^3}{3b}+\frac{x^2}{2}+\frac{bx}{6}-\frac{a^3}{3b}+\frac{a^2}{2}-\frac{ab}{6}=\frac{2x^3-2a^3+3bx^2+3a^2b+b^2x-ab^2}{6b}$
Quae expressiones similes sunt eis, quas pro summis po-
testatum numerorum naturalium in superiori differtatio-
ne dedi, atque ex iis quoque facile formantur.

§. 6. Sit nunc ad alteram huius generis formulam
inueniendam series a dato termino X, cuius index sit x
in infinitum vsque summanda, haec scilicet

$$\overset{x}{X}+\overset{x+b,}{Y}+\overset{x+2b}{Z}+\text{etc. infinitum}=S.$$

In summa ergo S si pro x scribatur $x+b$ prodibit $S-X$,
erit adeo $X=\frac{-bdS}{1.dx}-\frac{b^2d.dS}{1.2dx^2}-\frac{b^3d^3S}{1.2.3dx^3}-$ etc. vnde vt su-
pra reperietur $S=-\int\frac{Xdx}{b}+\frac{X}{1.2}-\frac{bdX}{1.2.3.2dx}+\frac{b^3d^3X}{1.2.3.4.5.6dx^3}$
$-\frac{b^5d^5X}{1.2.3---7.6dx^5}+\frac{2b^7d^7X}{1.2.3.---9.10dx^7}-\frac{5b^9d^9X}{1.2.3-11.6dx^9}$
$+\frac{691\,b^{11}d^{11}X}{1.2.3---13.210dx^{11}}-\frac{35\,b^{13}d^{13}X}{1.2.3----15.2dx^{13}}+\frac{3617\,b^{15}d^{15}X}{1.2.3----17.30dx^{15}}$
$-\frac{2423279\,b^{17}d^{17}X}{1.2.3----19.1890dx^{17}}+$ etc. Cui formulae tanta con-
stans est adiicienda vt fiat $S=0$, si ponatur $x=\infty$;
si enim terminus X iam fuerit infinitesimus seu vltimus
in serie, summa debet esse euanescens, si quidem series
finitam habeat summam, pro quo casu haec formula
est accommodata.

§. 7. Quo vsus huius formulae appareat, sit $X=\frac{1}{x^2}$
seu ista series $\frac{1}{x^2}+\frac{1}{(x+b)^2}+\frac{1}{(x+2b)^2}+$ etc. in infinitum
<div align="right">sum-</div>

fummanda erit ob $\int X dx = -\frac{1}{x}$, $\frac{dX}{dx} = \frac{-2}{x^3}$, $\frac{d^3X}{dx^3} = \frac{-2.3.4}{x^5}$

etc. $S = \frac{1}{bx} + \frac{1}{2x^2} + \frac{b}{6x^3} - \frac{b^3}{30x^5} + \frac{b^5}{42x^7} - \frac{b^7}{30x^9} + \frac{5b^9}{66x^{11}}$

$- \frac{691 b^{11}}{2730 x^{13}} + \frac{7 b^{13}}{6 x^{15}} - \frac{3617 b^{15}}{510 x^{17}} + \frac{2423279 b^{17}}{33910 x^{19}} -$ etc. quae expref-

fio conftantem non requirit, cum per fe euanefcat po-
fito $x = \infty$. Eo magis autem conuergit haec feries,
quo maior fuerit x refpectu ipfius b. Quare fi datae
feriei aliquot termini initiales addantur actu, reliquorum
hac methodo fumma inuenta illi aggregato addita dabit
fummam propofitae feriei in infinitum continuatae.

§. 8. Sed miffis his, quae priorem regulam tan-
tum commodiorem reddunt, progredior ad feries fum-
mandas, ad quas illa formula non fufficit. Sit nimi-
rum feries fummanda, in qua figna terminorum alter-
nantur, vti

$$\overset{a,}{A} - \overset{a+b,}{B} + \overset{a+2b,}{C} - \overset{a+3b,}{D} - - - - - + \overset{x,}{X} - \overset{x+b}{Y} = S$$

in qua ferie cum fit Y talis functio ipfius $x+b$ qualis
X eft ipfius x, erit $Y = X + \frac{b dx}{1 dx} + \frac{b^2 ddx}{1.2.dx^2} + \frac{b^3 d^3 x}{1.2.3 dx^3}$
$+$ etc. Deinde fi in S loco x ponatur $x-2b$ pro-
dibit $S - X + Y$; erit ergo $-Y + X = \frac{2b dS}{1 dx} - \frac{4b^2 ddS}{1.2.dx^2} +$
$+ \frac{8 b^3 d^3 S}{1.2.3 dx^3} - \frac{16 b^4 d^4 S}{1.2.3.4.dx^4} + \frac{32 b^5 d^5 S}{1.2.3.4.5 dx^5} -$ etc. $= - \frac{b dX}{1 dx} - \frac{b^2 d^2 X}{1.2 dx^2}$
$- \frac{b^3 d^3 X}{1.2.3 dx^3} - \frac{b^4 d^4 X}{1.2.3.4 dx^4} -$ etc. Ponatur $\frac{dS}{dx} = \frac{\alpha dX}{dx} + \frac{\beta d^2 X}{dx^2}$
$+ \frac{\gamma d^3 X}{dx^3} + \frac{\delta d^4 X}{dx^4}$ etc. et comparandis terminis homologis
prodibit $S = C - \frac{X}{2} - \frac{3 b dX}{4 dx} - \frac{b^2 ddX}{2 dx^2} - \frac{3 b^3 d^3 X}{16 dx^3} - \frac{b^4 d^4 X}{24 dx^4} - \frac{9 b^5 d^5 X}{1440 dx^5}$
$-$ etc. et introducendo Y erit $S = C + Y + \frac{X}{2} + \frac{b dX}{1.2.2 dx}$
$- \frac{b^3 d^3 X}{1.2.3.4.2 dx^3} + \frac{7 b^5 d^5 X}{1.2.3.4.5.6.2 dx^5}$ etc. Atque ideo erit $A - B$

$+ C$

$$+ C - D + E - F - - - - - + X = \text{Conft.} + \frac{X}{2} + \frac{b\,dX}{1.2.2d\overline{x}}$$

$$\frac{b^3 d^3 X}{1.2.3.4.2 dx^3} + \frac{3 b^5 d^5 X}{1.2.3.4.5.6.2 dx^5} \text{ etc. vbi conftans vt ante}$$

ita debet effe comparata, vt fiat haec fumma $= A$ pofito $x = a$.

§. 9. Vlterius autem huius formulae terminis continuatis prodibit pofita fumma huius feriei

$$\overset{a,\quad a+b,\quad a+2b,\quad a+3b,\qquad\quad x}{A - B + C - D - - - - - + X = S}$$

$$S = \text{Conft.} + \frac{X}{1.2} + \frac{b\,dX}{1.2.2dx} - \frac{b^3 d^3 X}{1.2.3.4.2 dx^3} + \frac{3 b^5 d^5 X}{1.2.3.4.5.6.2 dx^5} -$$

$$\frac{17 b^7 d^7 X}{1.2.3.4.5.6.7.8.2 dx^7} + \frac{155 b^9 d^9 X}{1.2.3.4.5.6.7.8.9.10.2 dx^9} - \frac{2073 b^{11} d^{11} X}{1.2.3 - - - - - 12.2 dx^{11}}$$

$$+ \frac{38227 6 b^{13} d^{13} X}{1.2.3. - - - - - - 14.2 dx^{13}} - \text{ etc. Si ergo quaerendus fit}$$

valor huius progreffionis $a^2 - (a+b)^2 + (a+2b)^2 - (a+3b)^2 + \text{etc.} - - - - - + x^2$, qui fit S: erit ob $\frac{dX}{dx}$ $= 2x$, $S = \text{Conft.} + \frac{x^2}{2} + \frac{bx}{2}$. Conftans vero C inuenietur ponendo $x = a$ eritque $S = \frac{a^2 - ab + x^2 + bx}{2}$. Exempli gratia erit $1 - 4 + 9 - 16 + 25 - - - - + 121 = 66$.

§. 10. Confideremus nunc huiusmodi feriem in infinitum productam, fcilicet

$$S = \overset{x,\quad x+b,\quad x+2b}{X - Y + Z} - \text{ etc. in infinitum.}$$

Erit ergo $S + \frac{2 b\,dS}{1.dx} + \frac{4 l^2 ddS}{1.2 dx^2} + \frac{8 b^3 d^3 S}{1.2.3.dx^3} + \text{etcet.}$ $= S - X + Y$, feu $X - Y = - \frac{2 b\,dS}{1.dx} - \frac{4 b^2 ddS}{1.2 dx^2}$ etc. vnde cum fit $Y = X + \frac{b\,dX}{1.dx} + \frac{b^2 ddX}{1.2 dx^2} + \frac{b^3 d^3 X}{1.2.3 dx^3} + \text{etc.}$ inuenietur $S = \frac{X}{1.2} - \frac{b\,dX}{1.2.2dx} + \frac{b^3 d^3 X}{1.2.3.4.2dx^3} - \frac{3 b^5 d^5 X}{1.2.3.4.5.6.2dx^5} +$

$$\frac{17 b^7 d^7 X}{1.2.3 - - - - 8.2 dx^7} \quad \frac{155 b^9 d^9 X}{1.2.3 - - - 10.2 dx^9} + \frac{2073 b^{11} d^{11} X}{1.2.3 - - - - 12.2 dx^{11}} \quad \frac{38227 l^{13} d^{13} X}{1.2.3 - 14.2 dx^{13}}$$

$$+ \text{etc.}$$

$+$ etc. $+$ Conſt. Quae conſtans ita debet eſſe com-
parata vt fiat $S = 0$ poſito $x = \infty$. Huius igitur for-
mulae ope plurimae ſeries lente conuergentes, in quibus
terminorum ſigna alternantur, valde prope ſummari po-
terunt.

§. 11. Sit $X = \frac{1}{x}$, ita vt inueniri debeat ſumma
huius ſeriei $\frac{1}{x} - \frac{1}{x+b} + \frac{1}{x+2b} - \frac{1}{x+3b} +$ etc. in infini-
tum. Cum ergo ſit $X = \frac{1}{x}$ erit $\frac{dX}{dx} = \frac{-1}{x^2}$, $\frac{d^3 X}{dx^3} = \frac{-2.3}{x^4}$,
$\frac{d^5 X}{dx^5} = \frac{-2.7.4.5}{x^6}$ hincque fiet $S = \frac{1}{2x} + \frac{b}{2.2x^2} - \frac{b^3}{4.2x^4} + \frac{3b^5}{6.2x^6}$
$- \frac{17b^7}{8.2x^8} + \frac{155b^9}{10.2x^{10}} - \frac{2073b^{11}}{12.2x^{12}} + \frac{38227b^{13}}{14.2x^{14}}$ etc. vbi conſtantis ad-
ditione non eſt opus. Vt ſit $b = 2$ atque $x = 25$, erit
ſeriei $\frac{1}{25} - \frac{1}{27} + \frac{1}{29} - \frac{1}{31} + \frac{1}{33} -$ etc. ſumma $= \frac{2}{100} + \frac{8}{100^2} -$
$\frac{256}{100^4} + \frac{8.4^6}{100^6} - \frac{17.4^8}{100^8} + \frac{31.128.4^{10}}{100^{10}} - \frac{691.256.4^{12}}{100^{12}} + \frac{5461.2048.4^{14}}{100^{14}}$ etc.
$= 0, 02079747191 8$ q. pr. ad quam ſi praecedentium
terminorum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ etc. $- - - - - \frac{1}{23}$ actu inuenta
ſumma addatur, prodibit quarta peripheriae pars radio
exiſtente $= 1$.

§. 12. Hae iam formulae hactenus traditae, quo
facile expediri queant calculo, requirunt, vt X ſit fun-
ctio algebraica ipſius x, cuius differentialia cuiusque gra-
dus commode exhiberi queant. Vix enim vel ne vix
quidem iſtae formulae in vſum vocari poſſent, ſi hu-
iusmodi quantitas exponentialis n^x in termino generali
progreſſionis ineſſet. Pro huiusmodi ergo progreſſioni-
bus, hoc termino generali $X n^x$, vbi X vt ante deno-
tat functionem algebraicam ipſius x, contentis peculia-

res formulas fummatrices erui conueniet. Sit itaque ista Series

$$\mathrm{A}\,\overset{a,}{n^a} + \mathrm{B}\,\overset{a+b.}{n^{a+b}} + \mathrm{C}\,\overset{a+2b}{n^{a+2b}} + \cdots \cdots \mathrm{X}\,\overset{\infty}{n^{\infty}}$$

ad fummandum propofita; ponaturque fumma $= \mathrm{S}\,n^{\infty}$. Haec formula autem pofito $x-b$ loco x abibit in hanc $n^{x-b}\left(\mathrm{S} - \frac{b\,d\mathrm{S}}{1\,dx} + \frac{b^2\,d^2\mathrm{S}}{1.2\,dx^2} - \frac{b^3\,d^3\mathrm{S}}{1.2.3\,dx^3} + \text{etc.}\right)$ quae aequalis effe debet priori fummae $\mathrm{S}\,n^{\infty}$ demto vltimo termino $\mathrm{X}\,n^{\infty}$. Habebitur ergo ifta aequatio $\mathrm{S}\,n^b - \mathrm{X}\,n^b = \mathrm{S} - \frac{b\,d\mathrm{S}}{1.\,dx} + \frac{b^2\,d^2\mathrm{S}}{1.2\,dx^2} - \frac{b^3\,d^3\mathrm{S}}{1.2.3\,dx^2} + \frac{b^4\,d^4\mathrm{S}}{1.2.3.4\,dx^4} - \text{etc.}$ Ex qua aequatione valor ipfius S erui debet.

§. 13. Ponatur igitur $n^b = m$, eritque $\mathrm{S} = \frac{m\mathrm{X}}{m-1} - \frac{\alpha b\,d\mathrm{X}}{a(m-1)^2\,dx} + \frac{\mathcal{E}b^2\,dd\mathrm{X}}{1.2(m-1)^3\,dx^2} - \frac{\gamma b^3\,d^3\mathrm{X}}{1.2.3(m-1)^4\,dx^3} + \frac{\delta b^4\,d^4\mathrm{X}}{1.2.3.4(m-1)^5\,dx^4} - \frac{\mathcal{E}b^5\,d^5\mathrm{X}}{1.2.3.4.5(m-1)^6\,dx^5} + \text{etc.}$ Hinc terminis homologis comparandis pofito breuitatis ergo $m-1 = p$ vt fequitur

$$\alpha = m$$
$$\mathcal{E} = 2\,\alpha + mp$$
$$\gamma = 3\,\mathcal{E} + 3\,\alpha p + mp^2$$
$$\delta = 4\,\gamma + 6\,\mathcal{E}p + 4\,\alpha p^2 + mp^3$$
$$\varepsilon = 5\,\delta + 10\,\gamma p + 10\,\mathcal{E}p^2 + 5\,\alpha p^3 + mp^4 \quad \text{etc.}$$

vnde pro litteris α, $\mathcal{E}$, γ, δ etc. fequentes obtinentur valores:

$$\alpha = m$$
$$\mathcal{E} = 2\,m + mp$$
$$\gamma = 6\,m + 6\,mp + mp^2$$
$$\delta = 24\,m + 36\,mp + 14\,mp^2 + mp^3$$
$$\varepsilon = 120\,m + 240\,mp + 150\,mp^2 + 30\,mp^3 + mp^4 \quad \text{etc.}$$

hic

hic quilibet numerus eft multiplum fuprafcripti cum praecedente iuncti, vti

$$30 = 2(1 + 14)$$
$$150 = 3(14 + 36)$$
$$240 = 4(36 + 24)$$
$$120 = 5(24 + 0)$$

vel reftituto $m-1$ loco p, vti fequitur:

1. $\alpha = m$

2. $\beta = m + m^2$

3. $\gamma = m + 4m^2 + m^3$

4. $\delta = m + 11m^2 + 11m^3 + m^4$

5. $\varepsilon = m + 26m^2 + 66m^3 + 26m^4 + m^5$

6. $\zeta = m + 57m^2 + 302m^3 + 302m^4 + 57m^5 + m^6$ etc.

qui valores ita progrediuntur, vt fit coefficiens ψ, cuius index eft k, $= m + (2^k - \frac{(k+1)}{1})m^2 + (3^k - \frac{(k+1)}{1}2^k + \frac{(k+1)k}{1.2})m^3 + (4^k - \frac{(k+1)}{1}3^k + \frac{(k+1)k}{1.2}2^k - \frac{(k+1)k(k-1)}{1.2.3})m^4 + - - - - - - + m^k$.

§. 14. Ex his ergo colligitur feriei propofitae

$$\overset{a,}{A}n^a + \overset{a+b,}{B}n^{a+b} + \overset{a+2b}{C}n^{a+2b} - - - - - + \overset{x}{X}n^x$$

fumma $= n^x \left(\frac{n^b X}{n^b - 1} - \frac{n^b b\, dX}{1(n^b-1)^2 dx} + \frac{(n^{2b}+n^b)b^2 ddX}{1.2(n^b-1)^3 dx^2} - \frac{(n^{3b}+4n^{2b}+n^b)b^3 d^3 X}{1.2.3(n^b-1)^4 dx^3} + \frac{(n^{4b}+11n^{3b}+11n^{2b}+n^b)b^4 d^4 X}{1.2.3.4(n^b-1)^5 dx^4} - \text{etc.} \right)$

$+$ Conft.

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—|—Conſt. quae ita debet eſſe comparata vt poſito $x = a$ fiat ſumma $= A\,n^a$. Si ponatur $n^b = -1$, abibit ſeries in pure algebraicam, in qua ſigna terminorum alternantur; hincque eadem formula ponendo -1 loco n^b, reſultat, quam pro eodem caſu iam ſupra §. 9. inuenimus. Seriei vero in infinitum excurrentis

$$\overset{x}{X}n^x + \overset{x+b}{Y}n^{x+b} + \overset{x+2b}{Z}n^{x+2b} + \text{etc.}$$

in infinitum ſumma eſt

$$= n^x \left(\frac{-X}{n^b-1} + \frac{n^b\,b\,dX}{1(n^b-1)^2\,dx} - \frac{(n^{2b}+n^b)b^2\,dd\,X}{1.2(n^b-1)^2\,dx^2} + \right.$$

$$\frac{(n^{3b}+4n^{2b}+n^b)b^3\,d^3X}{1.2.3(n^b-1)^4\,dx^3} \cdot \frac{(n^{4b}+11n^{3b}+11n^{2b}+n^b)b^4\,d^4X}{1.2.3.4(n^b-1)^5\,dx^4}$$

$$\left. + \text{etc.}\right) + \text{Conſt.}$$

quae conſtans ita debet eſſe comparata vt poſito $x = \infty$ ſumma fiat $= 0$: quod quidem ſemper accidere ſolet per ſe, ita vt conſtante non ſit opus, ſi quidem ſeries conuergit, ſummamque habet finitam.

§. 15. Ex priore formula pro huiusmodi ſeriebus ad datum vsque terminum ſummandis intelligitur, ſi X fuerit functio algebraica talis ipſius x, vt tandem altiora eius differentialia euaneſcant, tum terminum ſummatorium reuera exhiberi poſſe. Quamobrem ſi ſeries cuius terminus generalis eſt X fuerit ſummabilis, tum quoque ſeries, cuius terminus generalis eſt $X\,n^x$, erit ſummabilis. Sic propoſita ſerie $a^2\,n^a + (a+b)^2\,n^{a+b} + (a+2b)^2\,n^{a+2b} ------ + x^2\,n^x$ erit eius ſumma $= x^n$
$(n^b\,x^2$

$$\left(\frac{n^b x^2}{n^b-1} - \frac{2n^b bx}{(n^b-1)^2} + \frac{(n^{2b}+n^b)b^2}{(n^b-1)^3}\right) + \text{Conft. quae con-}$$

ftans facto $x=a$ et fumma $=a^2 n^a$ prodibit $=n^a$

$$\left(a^2 - \frac{n^b a^2}{n^b-1} + \frac{2n^b ab}{(n^b-1)^2} - \frac{(n^{2b}+n^b)b^2}{(n^b-1)^3}\right).$$

§. 16. Altera praeterea formula ingentem praeftat vtilitatem in feriebus infinitis fummandis, quod vt clarius pateat propofita fit feries $\dfrac{n^x}{x}+\dfrac{n^{x+2}}{x+2}+\dfrac{n^{x+4}}{x+4}+\dfrac{n^{x+6}}{x+6}+$ etc. cuius fumma fit $=S$. Erit ergo $b=2$, et $X=\frac{1}{x}$, vnde fiet fumma $S=n^x\Big(\dfrac{-1}{(n^2-1)x} - \dfrac{2n^2}{(n^2-1)^2 x^2} - \dfrac{4(n^4+n^2)}{(n^2-1)^3 x^3} - \dfrac{8(n^6+4n^4+n^2)}{(n^2-1)^4 x^4} - \dfrac{16(n^8+11n^6+11n^4+n^2)}{(n^2-1)^5 x^5} -$ etcet.$\Big)$

Sit nunc $x=25$ et $n^2=-\frac{1}{3}$ feu $n=\frac{1}{\sqrt{-3}}$, erit $\frac{1}{\sqrt{-3}})\frac{1}{25.3^{12}}$ $— \frac{1}{27.3^{13}} + \frac{1}{29.3^{14}} - \frac{1}{31.3^{15}} +$ etcet.$)$ $= \frac{1}{3^{12}\sqrt{-3}}\Big(\frac{3}{4.25}$ $+ \frac{3}{8.25^2} - \frac{3}{8.25^3} - \frac{3}{16.25^4} + \frac{15}{8.25^5}$ etc.$\Big)$. Cum igitur in circulo radii $=1$ fit arcus triginta graduum $=\frac{1}{\sqrt{3}}\big(1-\frac{1}{3.3}+\frac{1}{5.3^2}-\frac{1}{7.3^3}+\frac{1}{9.3^4}-$ etc.$\big)$, fi horum terminorum actu 12 addantur, erunt fequentes reliqui $\frac{1}{\sqrt{3}}\big(\frac{1}{25.3^{12}} - \frac{1}{27.3^{13}} +$ $\frac{1}{29.3^{14}} -$ etc.$\big) = \frac{\sqrt{3}}{3^{12}}\big(\frac{1}{4.25} + \frac{1}{8.25^2} - \frac{1}{8.25^3} - \frac{1}{16.25^4} + \frac{5}{8.25^5} +$ etc.

§. 17. Si termini feriei fummandae ex factoribus fuerint compofiti ita vt feries huiusmodi habeat formam.

$$\overset{a}{A}+\overset{a+b}{AB}+\overset{a+2b}{ABC}+\cdots+\overset{x}{ABC}\ldots VX$$

V 3 pona-

ponatur fumma $= S. ABC ----- VX$. Abibit ergo haec fumma pofito $x - b$ loco x in hanc $ABC -- V$ $(S - \frac{b \, dS}{1 . dx} + \frac{b^2 \, ddS}{1 . 2 . dx^2} - \frac{b^3 \, d^3 S}{1 . 2 . 3 . dx^3}$ etc.) quae aequalis effe debet priori fummae demto termino vltimo, id eft huic quantitati $AB ---- V(SX - X)$. Hanc ob rem habebitur ifta aequatio $SX - X = S - \frac{b \, dS}{1 . dx} + \frac{b^2 \, ddS}{1 . 2 dx^2} - \frac{b^3 \, d^3 S}{1 . 2 . 3 \, dx^3}$ $\frac{b^4 \, d^4 S}{1 . 2 . 3 . 4 \, dx^4}$ — etc.

Ex hac aequatione valor ipfius S erutus erit ifte $S = \frac{X}{X - 1} + E + F + G +$ etc. qui termini ita progrediuntur, vt pofito $\frac{X}{X - 1} = D$ fit $E = \frac{-b \, dD}{(X - 1) \, dx}$; $F =$ $\frac{-b \, dE}{1 (X - 1) \, dx} + \frac{b^2 \, ddD}{1 . 2 (X - 1) \, dx^2}$; $G = \frac{-b \, dF}{1 (X - 1) \, dx} + \frac{b^2 \, dd E}{1 . 2 (X - 1) \, dx^2}$ ——— $\frac{b^3 \, d^3 D}{1 . 2 . 3 (X - 1) \, dx^3}$; atque ita porro. Adeo vt fumma feriei propofitae fit $= ABC ----- VX (D + E + F + G$ $----$ etc.). Seriei vero in infinitum continuatae

$$A\overset{x}{B}C --- VX + A\overset{x+b}{B}CD ---- VXY +\text{ etc.}$$

fumma erit $= ABC ----- VX (1 - D - E - F - G$ $----$ etc.) $+$ Conft. Vt fi quaeratur fumma huius feriei $\frac{1}{1 . 2 . 3 ----- x} + \frac{1}{1 . 2 . 3 ---- x(x + 1)} +$ etc. in infinitum erit $b = 1$, $X = \frac{1}{x}$, atque $D = \frac{1}{1 - x}$. Tum vero $E = \frac{-x}{(1 - x)^3}$; $F = \frac{x(2 + x)}{(1 - x)^5}$ etc. fumma ergo feriei propofitae ob conftantem euanefcentem erit $= \frac{1}{1 . 2 . 3 -- x} (1 + \frac{1}{x - 1} - \frac{x}{(x - 1)^3}$ $+ \frac{x(x + 2)}{(x - 1)^5}$ etc.). Ex his vero traditis facile intelligitur, cuiusmodi fummae formam affumere oporteat in quouis cafu oblato, quo fumma minimo labore inueniatur.