DE PROGRESSIONIBVS HARMONICIS OBSERVATIONES.

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§. 1.

PRogressionum harmonicarum nomine intelliguntur omnes series fractionum, quarum numeratores sunt aequales inter se, denominatores vero progressionem arithmeticam constituunt. Huiusmodi ergo forma generalis est $\frac{c}{a}$, $\frac{c}{a+b}$, $\frac{c}{a+2b}$, $\frac{c}{a+3b}$, etc. Quique enim tres termini contigui vt $\frac{c}{a+b}$, $\frac{c}{a+2b}$, $\frac{c}{a+3b}$ hanc habent proprietatem, vt differentiae extremorum a medio sint ipsis extremis proportionales. Scilicet est $\frac{c}{a+b} - \frac{c}{a+2b} : \frac{c}{a+2b} - \frac{c}{a+3b} = \frac{c}{a+b} : \frac{c}{a+3b}$. Cum autem haec sit proprietas proportionis harmonicae; vocatae sunt istiusmodi fractionum series progressiones harmonicae. Vocari etiam possent reciprocae primi ordinis, quia in termino generali $\frac{c}{a+(n-1)b}$ index n vnicam eamque negatiuam habet dimensionem.

§. 2. Quanquam in his seriebus termini perpetuo decrescunt; tamen summa huiusmodi seriei in infinitum continuatae semper est infinita. Ad hoc demonstrandum non opus est methodo hasce series summandi; sed veritas facile ex sequente principio elucebit. Series quae in infinitum continuata summam habet finitam, etiamsi ea si ea duplo longius continuetur nullum accipiet augmentum, sed id quod post infinitum adiicitur cogitatione, re vera erit infinite paruum. Nisi enim hoc ita se haberet, summa seriei etsi in infinitum continuatae non esset determinata et propterea non finita. Ex quo consequitur, si id, quod ex continuatione vltra terminum infinitesimum oritur, sit finitae magnitudinis, summam seriei necessario infinitam esse debere. Ex hoc ergo principio iudicare poterimus, vtrum seriei cuiusque propositae summa sit infinita an finita.

§. 3. Sit itaque series $\frac{c}{a}$, $\frac{c}{a+b}$, $\frac{c}{a+2b}$ etc. in infinitum continuata, terminusque infinitesimus $\frac{c}{a+(i-1)b}$, denotante i numerum infinitum, qui sit index huius termini. Iam haec series vlterius continuetur a termino $\frac{c}{a+ib}$ vsque ad terminum $\frac{c}{a+(ni-1)b}$ cuius exponens est ni. Horum terminorum igitur insuper adiectorum numerus est $(n-1)i$; Summa eorum vero minor erit quam $\frac{(n-1)ic}{a+ib}$ maior vero quam $\frac{(n-1)ic}{a+(ni-1)b}$. Sed quia i est infinite magnum, evanescet a in vtroque denominatore. Quare summa maior erit quam $\frac{(n-1)c}{nb}$ at minor quam $\frac{(n-1)c}{b}$, Ex quo perspicitur hanc summam esse finitam, atque consequenter seriei propositae $\frac{c}{a}$, $\frac{c}{a+b}$, etc. in infinitum continuatae summam infinite magnam.

§. 4. Huius autem summae terminorum ab i ad ni limites propiores ex sequentibus proportionis harmonicae proprietatibus eliciuntur. Scilicet omnis proportio harmonica ita est comparata, vt terminus medius minor sit

ſit quam pars tertia ſummae terminorum omnium. Hanc ob rem terminus medius inter $\frac{c}{a+ib}$ et $\frac{c}{a+(ni-1)b}$, qui eſt $\frac{c}{a+\frac{(ni+i-1)}{2}b}$, ductus in terminorum numerum $(n-1)i$, ſeu $\frac{(n-1)ic}{a+\frac{(ni+i-1)}{2}b}$ minor erit quam ſumma terminorum Siue terminorum ſumma hinc maior erit quam $\frac{2(n-1)c}{(n+1)b}$ ob i infinitum. Praeterea medium arithmeticum inter terminos extremos maius eſt parte tertia ſummae terminorum. Ex hoc ſequitur fore etiam in ſerie harmonica terminorum ſummam minorem quam $(n-1)i$ in medium arithmeticum terminorum extremorum, quod eſt $\frac{(2a+(ni+i-1)b)c}{2(a+ib)(a+(ni-1)b)}$ ſeu $\frac{(n+1)c}{2nib}$, ductum. Quare ſumma erit minor quam $\frac{(n^2-1)c}{2nb}$, ita vt hi duo limites ſint $\frac{2(n-1)c}{(n+1)b}$ et $\frac{(n^2-1)c}{2nb}$, adeoque ſumma proxime $=\frac{(n-1)c}{b\sqrt{n}}$ quod eſt medium proportionale inter limites.

§. 5. Ex his colligere licet, quibus caſibus haec ſeries magis vniuerſalis $\frac{c}{a}$, $\frac{c}{a+b}$, $\frac{c}{a+2^{\alpha}b}$ etc. in infinitum vsque ad $\frac{c}{a+i^{\alpha}b}$ habeat ſummam finitam vel infinitam. Sequantur enim terminum vltimum termini $(n-1)i$, eritque horum ſumma minor quam $\frac{(n-1)c}{i^{\alpha-1}b}$, at maior quam $\frac{(n-1)c}{n^{\alpha}i^{\alpha-1}c}$. Quare ſi fuerit α numerus vnitate

maior

maior, ſumma horum terminorum ſequentium erit $=0$, et propterea ſumma progreſſionis finita. At ſi ſit $\alpha < 1$, ſumma terminorum ſequentium erit infinita; quocirca ipſius progreſſiones ſumma in infinities maiore gradu erit infinita. Inter has igitur progreſſiones ſola harmonica, in qua $\alpha = 1$, hanc habet proprietatem, vt ſumma eius in infinitum continuatae ſit infinite magna, terminorum vero ſequentium poſt terminum infiniteſimum ſumma finita.

§. 6. Quanta vero ſit ſumma terminorum a termino indicis i ad terminum indicis ni ſequenti modo inueſtigo. Ponatur ſumma ſeriei $\frac{c}{a}, \frac{c}{a+b}$ - - - - - - - - $\frac{c}{a+(i-1)b}$ ad terminum indicis i vsque $=s$, quae eſt quantitas ex a, b, c et i determinanda. Creſcat i vnitate, habebitque s pro augmento terminum ſequentem $\frac{c}{a+ib}$. Quare erit $di : ds = 1 : \frac{c}{a+ib}$ ſeu $ds = \frac{c\,di}{a+ib}$. Vnde inuenitur $s = C + \frac{c}{b} l(a+ib)$, denotante C quantitatem quandam conſtantem. Apparet quoque ex hac forma ſummam eiusdem ſeriei ab initio ad terminum indicis ni continuatae fore $= C + \frac{c}{b} l(a+nib)$. Harum igitur ſummarum differentia $\frac{c}{b} l \frac{a+nb}{a+ib} = \frac{c}{b} ln$ (euaneſcente a) dabit ſummam terminorum ab $\frac{c}{a+ib}$ vsque ad $\frac{c}{a+nib}$. Quia autem huius ſummae limites ſupra aſſignauimus erit $\frac{c}{b} ln$ maior quam $\frac{2(n-1)c}{(n+1)b}$ atque minor quam $\frac{(n^2-1)c}{2nb}$, ſeu $ln > \frac{2(n-1)}{n+1}$ atque $ln < \frac{n^2-1}{2n}$.

§. 7. Infra oſtendemus quantitatem illam conſtantem C eſſe finitam, eamque definire conabimur. Euaneſcet ergo C in ſumma, fietque progreſſionis $\frac{c}{a}, \frac{c}{a+b}, - - - - \frac{c}{a+(i-1)b}$ exiſtente terminorum numero infinito $=i$, ſumma $=\frac{c}{b}l(a+ib)=\frac{c}{b}li$. Quamobrem ſumma erit vt logarithmus numeri terminorum, hincque infinities minor quam radix quantumuis magnae poteſtatis ex numero terminorum; nihilo tamen minus eſt infinite magna.

§. 8. Ex hac conſideratione innumerabiles oriuntur ſeries ad logarithmos quorumuis numerorum deſignandos. Sumamus primo hanc progreſſionem harmonicam $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ etc. pro qua fit $a=1$, $b=1$, $c=1$. Differentia igitur inter hanc ſeriem $1+\frac{1}{2}+\frac{1}{3} - - - - \frac{1}{i}$ ad terminum indicis i continuatam, et eandem $1+\frac{1}{2}+\frac{1}{3} - - - - \frac{1}{ni}$ ad terminum indicis ni continuatam, erit $=ln$. Quare illa ſeries ab hac ſubtracta relinquit ln. Quia autem huius ſeriei numerus terminorum eſt n vicibus maior quam illius, ab n terminis ſeriei $1+\frac{1}{2} - - - - \frac{1}{ni}$ ſubtrahi oportet vnicum alterius ſeriei $1+\frac{1}{2} - - - - \frac{1}{i}$, quo ſubtractio in infinitum eodem modo poſſit perfici. Quare erit

$$ln = 1+\frac{1}{2} - - - - \underset{-1}{\frac{1}{n}} + \frac{1}{n+1} - - - - \underset{-\frac{1}{2}}{\frac{1}{2n}} + \frac{1}{2n+1} - - - - \underset{-\frac{1}{3}}{\frac{1}{3n}} + \text{etc.}$$

Si igitur inferioris ſeriei ſinguli termini a ſupra ſcriptis terminis ſuperioris ſeriei actu ſubtrahantur, et pro n numeri integri ſcribantur 2, 3, 4 - - - etc. ſucceſſiue ſequentes logarithmorum ſeries obtinebimus.

$$l2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \tfrac{1}{9} - \tfrac{1}{10} + \tfrac{1}{11} - \tfrac{1}{12} \text{ etc.}$$
$$l3 = 1 + \tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{2}{6} + \tfrac{1}{7} + \tfrac{1}{8} - \tfrac{2}{9} + \tfrac{1}{10} + \tfrac{1}{11} - \tfrac{2}{12} \text{ etc.}$$
$$l4 = 1 + \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{3}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{3}{8} + \tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} - \tfrac{3}{12} \text{ etc.}$$
$$l5 = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{4}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} - \tfrac{4}{10} + \tfrac{1}{11} + \tfrac{1}{12} \text{ etc.}$$
$$l6 = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{5}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} - \tfrac{5}{12} \text{ etc.}$$

etc. etc.

Vnde pro cuiusuis numeri logarithmo facile series conuergens inuenitur.

§. 9. Ex his seriebus aliae eiusdem formae, quae summam habeant rationalem, possunt deriuari, Nam, quia seriei $= l2$ duplum aequale est $l4$, si series $1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4}$ etc. subtrahatur ab hac $2 - \frac{2}{2} + \frac{2}{3} - \frac{2}{4}$ etc. residuum, nempe haec series $1 - \frac{3}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{3}{6}$ etc erit $= 0$, seu $\frac{1}{2} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{3}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{3}{10}$ etc. Similiter si series $l6$ exhibens subtrahatur a summa serierum $l2$ et $l3$ exhibentium, residuum, nempe $1 - \frac{1}{2} - \frac{2}{3} - \frac{1}{4} + \frac{1}{5} + \frac{2}{6} + \frac{1}{7} - \frac{1}{8} - \frac{2}{9} - \frac{1}{10}$ etc. erit $= 0$ seu $1 = \frac{1}{2} + \frac{2}{3} + \frac{1}{4} - \frac{1}{5} - \frac{2}{6} - \frac{1}{7} + \frac{1}{8} + \frac{2}{9} + \frac{1}{10}$ etc. Pari modo huiusmodi series innumerabiles poterunt inueniri.

§. 10. Series illae logarithmos exprimentes conuergunt quidem, sed admodum tarde, quare, quo earum ope logarithmi commode erui queant, requiritur aliquod subsidium. Ad quod inueniendum notari oportet eas series non aequabiliter progredi, sed certas habere reuolutiones, quae tot terminis absoluuntur, quot n habet vnitates, tot igitur terminos simul sumtos vnum seriei membrum vocabo. Ita in seriei pro $l2$ duo termini constituent vnum membrum, in serie pro $l3$, tres, in serie pro $l4$ quatuor et ita porro. Membra igitur ista

 aequa-

aequabilem conſtituent ſeriem, et ad logarithmos inueniendos oportet aliquot membra addi. Ponamus iam m membra eſſe addita ad logarithmum binarii inueniendum; poteritque loco omnium ſequentium addi $\frac{1}{4m}$, id quod eo propius accedet, quo maior fuerit numerus m. Ad $l3$ ininueniendum ad m membra iam addita loco omnium ſequentium addatur $\frac{1}{9m}$. Similiter pro $l4$ addi debet $\frac{1}{16m}$ et ita porro. Fluunt haec ex modo ſummandi (§. 6) adhibito, in quo cum m debeat eſſe quantitas valde magna, neglexi in differentiali numeros ipſi m adiectos, ne integratio a logarithmis pendeat.

§. 11. Quo autem ſeriei $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-----\frac{1}{i}$ ſumma, etiamſi infinita accurate determinetur, ſingulos terminos ſequenti modo exprimo.

$$\text{Eſt } 1 = l2 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \tfrac{1}{6} - \tfrac{1}{7} \text{ etc.}$$

$$\text{atque } \tfrac{1}{2} = l\tfrac{3}{2} + \frac{1}{2.4} - \frac{1}{3.8} + \frac{1}{4.16} - \frac{1}{5.32} \text{ etc.}$$

$$\tfrac{1}{3} = l\tfrac{4}{3} + \frac{1}{2.9} - \frac{1}{3.27} + \frac{1}{4.81} - \frac{1}{5.243} \text{ etc.}$$

$$\tfrac{1}{4} = l\tfrac{5}{4} + \frac{1}{2.16} - \frac{1}{3.64} + \frac{1}{4.256} - \frac{1}{5.1024} \text{ etc.}$$

$$\vdots$$

$$\tfrac{1}{i} = l\frac{i+1}{i} + \frac{1}{2.i^2} - \frac{1}{3.i^3} + \frac{1}{4.i^4} - \frac{1}{5.i^5} \text{ etc.}$$

His ſeriebus additis prodibit

$$1+\tfrac{1}{2}+\tfrac{1}{3}-----\tfrac{1}{i} = l(i+1) + \tfrac{1}{2}(1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\text{etc.}) \text{ in infin.}$$
$$-\tfrac{1}{3}(1+\tfrac{1}{8}+\tfrac{1}{27}+\tfrac{1}{64}+\text{etc.})$$
$$+\tfrac{1}{4}(1+\tfrac{1}{16}+\tfrac{1}{81}-+\tfrac{1}{256}+\text{etc.}$$
$$\text{etc.}$$

Quae

Quae ſeries, cum ſint conuergentes, ſi proxime ſummentur prodibit $1+\frac{1}{2}+\frac{1}{3}---\frac{1}{i}=l(i+1)+0{,}577218$. Si ſumma dicatur s, foret, vt ſupra fecimus, $ds=\frac{di}{i+1}$, ideoque $s=l(i+1)+C$. Huius igitur quantitatis conſtantis c valorem deteximus, quippe eſt $C=0{,}577218$.

§. 12. Si ſeries $1+\frac{1}{2}+\frac{1}{3}-----\frac{1}{i}$ vlterius in infinitum continuetur, et in membra diuidatur, quorum quodvis vt ipſa ſeries i terminos contineat; erit membrum inter $\frac{1}{i}$ et $\frac{1}{2i}$ contentum $=l\,2$, ſequens $=l\frac{3}{2}$, tertium $=l\frac{4}{3}$, etc. Atque cum ipſius ſeriei ſumma ſit log. infiniti, poterit ad analogiam poni $l\frac{1}{0}$. Hocque modo ſequens ſchema obtinebimus non parum curioſum

Series.	$1+\frac{1}{2}---\frac{1}{i}$	$----\frac{1}{2i}$	$----\frac{1}{3i}$	$---\frac{1}{4i}$	$---\frac{1}{5i}$	etc.
Summae	$l\frac{1}{0}$	$l\frac{2}{1}$	$l\frac{3}{2}$	$l\frac{4}{3}$	$l\frac{5}{4}$	

§. 13. Difficile quidem videatur has easdem proprietates progreſſionum harmonicarum et logarithmorum expreſſiones analytice, eoque modo, quem alibi ad ſeries ſummandas tradidi, inuenire. At rem attentius perpendenti hoc non ſolum fieri, ſed multo generalius etiam fieri poſſe deprehenſum eſt. Confidero enim non ſimplicem progreſſionem harmonicam, ſed cum geometrica coniunctam, cuiusmodi eſt $\frac{cx}{a}+\frac{cx^2}{a+b}+\frac{cx^3}{a+2b}+\frac{cx^4}{a+3b}$ etc. Huius ſummam pono s, et vtroque per $bx^{\frac{a-b}{b}}$ multiplicato erit $bx^{\frac{a-b}{b}}s=\frac{bcx^{\frac{a}{b}}}{a}+\frac{bcx^{\frac{a+b}{b}}}{a+b}+\frac{bcx^{\frac{a+2b}{b}}}{a+2b}$ Sumtisque differentialibus habebitur $b\mathrm{D}.x^{\frac{a-b}{b}}s=dx(cx^{\frac{a-b}{b}}$

$+cx^{\frac{a}{b}}+cx^{\frac{a+b}{b}}+$etc.$)=\frac{cx^{\frac{a-b}{b}}dx}{1-x}$ Sumtisq; iterum integralibus erit $bx^{\frac{a-b}{b}}s=c\int\frac{x^{\frac{a-b}{b}}dx}{1-x}$ atque $s=\frac{cx^{a-b}}{b}\int\frac{x^{\frac{a-b}{b}}dx}{1-x}$

Ab hac ſerie iam ſubtraho hanc $\frac{fx^{m}}{g}+\frac{fx^{2m}}{g+b}+\frac{fx^{3m}}{g+2b}$ etc. cuius ſumma ſit t. Multiplicetur per $\frac{b}{m}x^{\frac{m(g-b)}{b}}$ erit $\frac{b}{m}x^{\frac{m(g-b)}{b}}t=\frac{fbx^{\frac{mg}{b}}}{mg}+\frac{fbx^{\frac{m(g+b)}{b}}}{m(g+b)}+\frac{fbx^{\frac{m(g+2b)}{b}}}{m(g+2b)}$ etc. Sumtisque differentialibus fiet $\frac{b}{m}D.\,x^{\frac{m(g-b)}{b}}t=dx(fx^{\frac{mg-b}{b}}+fx^{\frac{m(g+b)-b}{b}}+fx^{\frac{m(g+2b)-b}{b}}$ etc.$)=\frac{fx^{\frac{mg-b}{b}}dx}{1-x^{m}}$.

Quare habebitur $t=\frac{fm}{bx^{\frac{m(g-b)}{b}}}\int\frac{x^{\frac{m(g-b)}{b}}dx}{1-x^{m}}$. Ideoque $s-t=\frac{c}{bx^{\frac{a-b}{b}}}\int\frac{x^{\frac{a-b}{b}}}{1-x}-\frac{fm}{bx^{\frac{m(g-b)}{b}}}\int\frac{x^{\frac{mg-b}{b}}dx}{1-x^{m}}$. Subtractio vero ita debet fieri, vt a termino indicis m feriei s ſubtrahatur terminus primus ſeriei t, et a termino indicis $2m$ illius, terminus ſecundus huius ſeriei et ita porro.

§. 14. Quo noſtras ſeries logarithmicas eruamus, ſit $a=b$ et $g=b$. Quo facto erit $s=\frac{c}{b}\int\frac{dx}{1-x}=\frac{c}{b}l\frac{1}{1-x}$ et $t=\frac{f}{b}\int\frac{mx^{m-1}dx}{1-x^{m}}=\frac{f}{b}l\frac{1}{1-x^{m}}$. Ergo $s-t=$

$l(1-$

$l\frac{(1-x^m)^{\frac{f}{b}}}{(1-x)^{\frac{c}{b}}}$. Quo autem haec expressio fiat finita facto $x=1$, debebit esse $\frac{f}{b}=\frac{c}{b}$, hanc ob rem fiant omnes hae litterae $=1$, eritque $s-t=l\frac{1-x^m}{1-x}=l(1+x+x^2 - - - - - x^{m-1})$ Quae expressio dat differentiam inter has series $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}$ etc. et $\frac{x^m}{1}+\frac{x^{2m}}{2}+\frac{x^{3m}}{3}$ etc. Quare si $m=2$ erit $l(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+$ etc. si $m=3$, erit $l(1+x+x^2)=x+\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}-\frac{2x^6}{6}+$ etc. similique modo $l(1+x+x^2+x^3)=x+\frac{x^2}{2}+\frac{x^3}{3}-\frac{3x^4}{4}+$ etc. In his si fiat $x=1$, prodibunt eaedem series pro logarithmis numerorum naturalium, quas ante dedimus.

§. 15. Si $b=2g$, erit $t=\frac{fx^{\frac{m}{2}}}{b}\int\frac{mx^{\frac{m-2}{2}}dx}{1-x^m}$. Ponatur $x^m=y$, erit $t=\frac{f\sqrt{y}}{b}\int\frac{dy}{(1-y)\sqrt{y}}=\frac{f\sqrt{y}}{b}l\frac{1+\sqrt{y}}{1-\sqrt{y}}=\frac{fx^{\frac{m}{2}}}{b}l\frac{1+x^{\frac{m}{2}}}{1-x^{\frac{m}{2}}}$. Si praeterea sit $a=b$ erit $s=\frac{c}{b}l\frac{1}{1-x}$. At s est summa huius seriei $\frac{cx}{a}+\frac{cx^2}{2a}+\frac{cx^3}{3a}$ etc. atque $tx^{\frac{-m}{2}}=\frac{f}{b}l\frac{1+x^{\frac{m}{2}}}{1-x^{\frac{m}{2}}}$ dat hanc seriem $\frac{fx^{\frac{m}{2}}}{g}+\frac{fx^{\frac{3m}{2}}}{3g}+\frac{fx^{\frac{5m}{2}}}{5g}$ + etc. Sit $a=1$ et $g=1$ erit $s-tx^{\frac{-m}{2}}=cl\frac{1}{1-x}-$

$\frac{f}{2}$

$\frac{f}{2} l \frac{1 + x^{\frac{m}{2}}}{1 - x^{\frac{m}{2}}} = l \frac{(1 - x^{\frac{m}{2}})^{\frac{f}{2}}}{(1-x)^c (1 + x^{\frac{m}{2}})^{\frac{f}{2}}}$. Quae expreſſio quo fiat finita ſi $x = 1$ oportet ſit $\frac{f}{2} = c$ ſeu $f = 2c$. Sit igitur $c = 1$, et $m = 2n$ erit ſerierum $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$ etc. et $\frac{2x^n}{1} + \frac{2x^{3n}}{3} + \frac{2x^{5n}}{5}$ etc. differentia $= l \frac{1 - x^n}{(1-x)(1+x^n)}$. Ponatur $n = 2$ erit differentia haec $= l \frac{1+x}{1+x^2}$ factoque $x = 1$, erit ea $= 0$, quare haec ſeries $1 - \frac{3}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{3}{6} + \frac{1}{7}$ etc. erit $= 0$, vt iam ſupra inuenimus.

§. 16. Huiusmodi ſeries ſummam rationalem habentes nunc ex hac ipſa forma $l \frac{1+x}{1+xx}$, infinitae aliae poſſunt inueniri, aſſumendis aliis formis ſimilibus quae facto $x = 1$ euanescant. Ex hac enim forma $l \frac{1+x}{1+x^2}$ ſi per ſeries exprimatur ſtatim reſultat ſeries inuenta. Eſt nimirum $l(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} -$ etc. Atque $l(1+x^2) = \frac{x^2}{1} - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} -$ etc. Haec igitur ſeries á ſuperiore ſubtracta relinquit hanc $\frac{x}{1} - \frac{3x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \frac{3x^6}{6}$ etc. cuius ſumma erit $l \frac{1+x}{1+x^2}$. Similiter $l \frac{1+x}{1+x^3}$ dabit hanc ſeriem $\frac{x}{1} - \frac{x^2}{2} - \frac{2x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \frac{2x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} - \frac{2x^9}{9}$ etc. Ergo poſito $x = 1$ erit $0 = 1 - \frac{1}{2} - \frac{2}{3} - \frac{1}{4} + \frac{1}{5} + \frac{2}{6} + \frac{1}{7} - \frac{1}{8} - \frac{2}{9}$ quam eandem iam §. 9. inuenimus.

§. 17. Hac ratione omnium huiusmodi irregularium ſerierum, quae tamen ſecundum membra regulariter procedunt, ſummae poterunt inueniri, ſemper enim vt differentiae duarum ſerierum ſunt aeſtimandae. Vt ſit propoſita haec ſeries $1 - \frac{2}{2} + \frac{1}{3} + \frac{1}{4} - \frac{2}{5} + \frac{1}{6} + \frac{1}{7} - \frac{2}{8}$ etc. Haec eſt differentia harum ſerierum $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$

$+\frac{x^4}{4}+\frac{x^5}{5}$ etc. et $\frac{3x^2}{2}+\frac{3x^5}{5}+\frac{3x^8}{8}$ etc. facto $x=1$. Il-
lius autem ſumma eſt $l\frac{1}{1-x}$, huius vero ſumma eſt $\int\frac{3xdx}{1-x^3}$
ſeu $l\frac{1}{1-x}+\frac{1}{2}l(x^2+x+1)+\frac{\sqrt{-3}}{2}l\frac{2x+1-\sqrt{-3}}{2x+1+\sqrt{-3}}-\frac{\sqrt{-3}}{2}l\frac{1-\sqrt{-3}}{1+\sqrt{-3}}$.
Hac igitur ab illa ſubtracta factoque $x=1$ prodibit $-\frac{1}{2}l3$
$+\frac{\sqrt{-3}}{2}l\frac{3+\sqrt{-3}}{3-\sqrt{-3}}-\frac{\sqrt{-3}}{2}l\frac{1+\sqrt{-3}}{1-\sqrt{-3}}$, pro ſumma progreſſionis
propoſitae. Eſt vero $\frac{\sqrt{-3}}{2}l\frac{3+\sqrt{-3}}{3-\sqrt{-3}}$ peripheria circuli di-
uiſa per $\sqrt{3}$ poſito diametro $=1$ et $\frac{\sqrt{-3}}{2}l\frac{1-\sqrt{-3}}{1+\sqrt{-3}}$ huius
dimidium. Quare ſeriei ſumma quam proxime erit
0,3576.

§. 18. At ſi etiam ipſa membra non vniformiter incedunt difficilius ſumma aſſignatur. Sumamus hanc ſeriem $1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{2}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{3}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}-\frac{4}{14}$ etc. Haec eſt differentia inter has ſeries $1+\frac{1}{2}+\frac{1}{3}++\frac{1}{4}+\frac{1}{5}-------\frac{1}{i(\frac{i+3}{2})}$ et $\frac{2}{2}+\frac{3}{5}+\frac{4}{9}+\frac{5}{14}------\frac{i+1}{i(\frac{i+3}{2})}$ ita in infinitum continuatas, vt extremi termini eundem habeant denominatorem $i(\frac{i+3}{2})$. Harum ſerierum prioris ſumma eſt $C+li+l(i+3)-l2$, denotante C conſtantem §. 11. inuentam, nempe 0,577218. Altera ſeries ſubtrahenda in has duas reſoluitur $\frac{2}{3}(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}----\frac{1}{i})$ et $\frac{4}{3}(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}------\frac{1}{i+3})$. Illius ſumma eſt $\frac{2}{3}C+\frac{2}{3}li$; huius vero $\frac{4}{3}C-\frac{22}{9}+\frac{4}{3}l(i+3)$. Quae ambae ab illa ſumma $C+li+l(i+3)-l2$ ſubtractae relinquunt $-C+\frac{22}{9}-l2$ ſeu 1,173078 quam proxime pro ſumma ſeriei propoſitae.