

DE PROGRESSIONIBVS TRANS-
CENDENTIBVS, SEV QVARVM TERMINI
GENERALES ALGEBRAICE DARI
NEQVEVNT.

Auct. L. Euler.

x.

Cum nuper occasione eorum, quae Cel. Goldbach de seriebus cum Societate communicauerit, in expressionem quandam generalem inquirerem, quae huius Progressionis $1 + 1 + 1 + \dots + 1 + 2 + 1 + 2 \cdot 3 + 1 \cdot 2 \cdot 3 \cdot 4 + \dots$ terminos omnes daret; incidi considerans, quod ea in infinitum continuata tandem cum geometrica confundatur in sequentem expressionem, $\frac{1 \cdot 2^n \cdot 2^{n-1} \cdot 3^n \cdot 3^{n-1} \cdot 4^n \cdot 4^{n-1} \cdot 5^n}{1 + n \cdot 2 + n \cdot 3 + n \cdot 4 + n \cdot \dots}$ etc. quae dictae progressionis terminum ordine n exponit. Ea quidem in nullo casu abrumptitur; neque si n est numerus integer neque si fractus, sed ad quemvis terminum iaueniendum tantummodo approximationes suppeditat, nisi excipiatur casus $n=0$, et $n=1$, quibus ea actu abit in 1. Ponatur $n=2$, habebitur $\frac{1 \cdot 2^2 \cdot 2^1 \cdot 3^2 \cdot 3^1 \cdot 4^2 \cdot 4^1 \cdot 5^2 \cdot 5^1}{1+2+3+4+5+6}$ etc; termino secundo 2. Si $n=3$ habebitur $\frac{1 \cdot 2^3 \cdot 2^2 \cdot 3^3 \cdot 3^2 \cdot 4^3 \cdot 4^2 \cdot 5^3 \cdot 5^2 \cdot 6^3 \cdot 6^2}{1+3+6+10+15+20+25+30+35+40+45+50+55+60+65+70+75+80+85+90+95+100+105+110+115+120+125+130+135+140+145+150+155+160+165+170+175+180+185+190+195+200+205+210+215+220+225+230+235+240+245+250+255+260+265+270+275+280+285+290+295+300+305+310+315+320+325+330+335+340+345+350+355+360+365+370+375+380+385+390+395+400+405+410+415+420+425+430+435+440+445+450+455+460+465+470+475+480+485+490+495+500+505+510+515+520+525+530+535+540+545+550+555+560+565+570+575+580+585+590+595+600+605+610+615+620+625+630+635+640+645+650+655+660+665+670+675+680+685+690+695+700+705+710+715+720+725+730+735+740+745+750+755+760+765+770+775+780+785+790+795+800+805+810+815+820+825+830+835+840+845+850+855+860+865+870+875+880+885+890+895+900+905+910+915+920+925+930+935+940+945+950+955+960+965+970+975+980+985+990+995+1000+1005+1010+1015+1020+1025+1030+1035+1040+1045+1050+1055+1060+1065+1070+1075+1080+1085+1090+1095+1100+1105+1110+1115+1120+1125+1130+1135+1140+1145+1150+1155+1160+1165+1170+1175+1180+1185+1190+1195+1200+1205+1210+1215+1220+1225+1230+1235+1240+1245+1250+1255+1260+1265+1270+1275+1280+1285+1290+1295+1300+1305+1310+1315+1320+1325+1330+1335+1340+1345+1350+1355+1360+1365+1370+1375+1380+1385+1390+1395+1400+1405+1410+1415+1420+1425+1430+1435+1440+1445+1450+1455+1460+1465+1470+1475+1480+1485+1490+1495+1500+1505+1510+1515+1520+1525+1530+1535+1540+1545+1550+1555+1560+1565+1570+1575+1580+1585+1590+1595+1600+1605+1610+1615+1620+1625+1630+1635+1640+1645+1650+1655+1660+1665+1670+1675+1680+1685+1690+1695+1700+1705+1710+1715+1720+1725+1730+1735+1740+1745+1750+1755+1760+1765+1770+1775+1780+1785+1790+1795+1800+1805+1810+1815+1820+1825+1830+1835+1840+1845+1850+1855+1860+1865+1870+1875+1880+1885+1890+1895+1900+1905+1910+1915+1920+1925+1930+1935+1940+1945+1950+1955+1960+1965+1970+1975+1980+1985+1990+1995+2000+2005+2010+2015+2020+2025+2030+2035+2040+2045+2050+2055+2060+2065+2070+2075+2080+2085+2090+2095+2100+2105+2110+2115+2120+2125+2130+2135+2140+2145+2150+2155+2160+2165+2170+2175+2180+2185+2190+2195+2200+2205+2210+2215+2220+2225+2230+2235+2240+2245+2250+2255+2260+2265+2270+2275+2280+2285+2290+2295+2300+2305+2310+2315+2320+2325+2330+2335+2340+2345+2350+2355+2360+2365+2370+2375+2380+2385+2390+2395+2400+2405+2410+2415+2420+2425+2430+2435+2440+2445+2450+2455+2460+2465+2470+2475+2480+2485+2490+2495+2500+2505+2510+2515+2520+2525+2530+2535+2540+2545+2550+2555+2560+2565+2570+2575+2580+2585+2590+2595+2600+2605+2610+2615+2620+2625+2630+2635+2640+2645+2650+2655+2660+2665+2670+2675+2680+2685+2690+2695+2700+2705+2710+2715+2720+2725+2730+2735+2740+2745+2750+2755+2760+2765+2770+2775+2780+2785+2790+2795+2800+2805+2810+2815+2820+2825+2830+2835+2840+2845+2850+2855+2860+2865+2870+2875+2880+2885+2890+2895+2900+2905+2910+2915+2920+2925+2930+2935+2940+2945+2950+2955+2960+2965+2970+2975+2980+2985+2990+2995+3000+3005+3010+3015+3020+3025+3030+3035+3040+3045+3050+3055+3060+3065+3070+3075+3080+3085+3090+3095+3100+3105+3110+3115+3120+3125+3130+3135+3140+3145+3150+3155+3160+3165+3170+3175+3180+3185+3190+3195+3200+3205+3210+3215+3220+3225+3230+3235+3240+3245+3250+3255+3260+3265+3270+3275+3280+3285+3290+3295+3300+3305+3310+3315+3320+3325+3330+3335+3340+3345+3350+3355+3360+3365+3370+3375+3380+3385+3390+3395+3400+3405+3410+3415+3420+3425+3430+3435+3440+3445+3450+3455+3460+3465+3470+3475+3480+3485+3490+3495+3500+3505+3510+3515+3520+3525+3530+3535+3540+3545+3550+3555+3560+3565+3570+3575+3580+3585+3590+3595+3600+3605+3610+3615+3620+3625+3630+3635+3640+3645+3650+3655+3660+3665+3670+3675+3680+3685+3690+3695+3700+3705+3710+3715+3720+3725+3730+3735+3740+3745+3750+3755+3760+3765+3770+3775+3780+3785+3790+3795+3800+3805+3810+3815+3820+3825+3830+3835+3840+3845+3850+3855+3860+3865+3870+3875+3880+3885+3890+3895+3900+3905+3910+3915+3920+3925+3930+3935+3940+3945+3950+3955+3960+3965+3970+3975+3980+3985+3990+3995+4000+4005+4010+4015+4020+4025+4030+4035+4040+4045+4050+4055+4060+4065+4070+4075+4080+4085+4090+4095+4100+4105+4110+4115+4120+4125+4130+4135+4140+4145+4150+4155+4160+4165+4170+4175+4180+4185+4190+4195+4200+4205+4210+4215+4220+4225+4230+4235+4240+4245+4250+4255+4260+4265+4270+4275+4280+4285+4290+4295+4300+4305+4310+4315+4320+4325+4330+4335+4340+4345+4350+4355+4360+4365+4370+4375+4380+4385+4390+4395+4400+4405+4410+4415+4420+4425+4430+4435+4440+4445+4450+4455+4460+4465+4470+4475+4480+4485+4490+4495+4500+4505+4510+4515+4520+4525+4530+4535+4540+4545+4550+4555+4560+4565+4570+4575+4580+4585+4590+4595+4600+4605+4610+4615+4620+4625+4630+4635+4640+4645+4650+4655+4660+4665+4670+4675+4680+4685+4690+4695+4700+4705+4710+4715+4720+4725+4730+4735+4740+4745+4750+4755+4760+4765+4770+4775+4780+4785+4790+4795+4800+4805+4810+4815+4820+4825+4830+4835+4840+4845+4850+4855+4860+4865+4870+4875+4880+4885+4890+4895+4900+4905+4910+4915+4920+4925+4930+4935+4940+4945+4950+4955+4960+4965+4970+4975+4980+4985+4990+4995+5000+5005+5010+5015+5020+5025+5030+5035+5040+5045+5050+5055+5060+5065+5070+5075+5080+5085+5090+5095+5100+5105+5110+5115+5120+5125+5130+5135+5140+5145+5150+5155+5160+5165+5170+5175+5180+5185+5190+5195+5200+5205+5210+5215+5220+5225+5230+5235+5240+5245+5250+5255+5260+5265+5270+5275+5280+5285+5290+5295+5300+5305+5310+5315+5320+5325+5330+5335+5340+5345+5350+5355+5360+5365+5370+5375+5380+5385+5390+5395+5400+5405+5410+5415+5420+5425+5430+5435+5440+5445+5450+5455+5460+5465+5470+5475+5480+5485+5490+5495+5500+5505+5510+5515+5520+5525+5530+5535+5540+5545+5550+5555+5560+5565+5570+5575+5580+5585+5590+5595+5600+5605+5610+5615+5620+5625+5630+5635+5640+5645+5650+5655+5660+5665+5670+5675+5680+5685+5690+5695+5700+5705+5710+5715+5720+5725+5730+5735+5740+5745+5750+5755+5760+5765+5770+5775+5780+5785+5790+5795+5800+5805+5810+5815+5820+5825+5830+5835+5840+5845+5850+5855+5860+5865+5870+5875+5880+5885+5890+5895+5900+5905+5910+5915+5920+5925+5930+5935+5940+5945+5950+5955+5960+5965+5970+5975+5980+5985+5990+5995+6000+6005+6010+6015+6020+6025+6030+6035+6040+6045+6050+6055+6060+6065+6070+6075+6080+6085+6090+6095+6100+6105+6110+6115+6120+6125+6130+6135+6140+6145+6150+6155+6160+6165+6170+6175+6180+6185+6190+6195+6200+6205+6210+6215+6220+6225+6230+6235+6240+6245+6250+6255+6260+6265+6270+6275+6280+6285+6290+6295+6300+6305+6310+6315+6320+6325+6330+6335+6340+6345+6350+6355+6360+6365+6370+6375+6380+6385+6390+6395+6400+6405+6410+6415+6420+6425+6430+6435+6440+6445+6450+6455+6460+6465+6470+6475+6480+6485+6490+6495+6500+6505+6510+6515+6520+6525+6530+6535+6540+6545+6550+6555+6560+6565+6570+6575+6580+6585+6590+6595+6600+6605+6610+6615+6620+6625+6630+6635+6640+6645+6650+6655+6660+6665+6670+6675+6680+6685+6690+6695+6700+6705+6710+6715+6720+6725+6730+6735+6740+6745+6750+6755+6760+6765+6770+6775+6780+6785+6790+6795+6800+6805+6810+6815+6820+6825+6830+6835+6840+6845+6850+6855+6860+6865+6870+6875+6880+6885+6890+6895+6900+6905+6910+6915+6920+6925+6930+6935+6940+6945+6950+6955+6960+6965+6970+6975+6980+6985+6990+6995+7000+7005+7010+7015+7020+7025+7030+7035+7040+7045+7050+7055+7060+7065+7070+7075+7080+7085+7090+7095+7100+7105+7110+7115+7120+7125+7130+7135+7140+7145+7150+7155+7160+7165+7170+7175+7180+7185+7190+7195+7200+7205+7210+7215+7220+7225+7230+7235+7240+7245+7250+7255+7260+7265+7270+7275+7280+7285+7290+7295+7300+7305+7310+7315+7320+7325+7330+7335+7340+7345+7350+7355+7360+7365+7370+7375+7380+7385+7390+7395+7400+7405+7410+7415+7420+7425+7430+7435+7440+7445+7450+7455+7460+7465+7470+7475+7480+7485+7490+7495+7500+7505+7510+7515+7520+7525+7530+7535+7540+7545+7550+7555+7560+7565+7570+7575+7580+7585+7590+7595+7600+7605+7610+7615+7620+7625+7630+7635+7640+7645+7650+7655+7660+7665+7670+7675+7680+7685+7690+7695+7700+7705+7710+7715+7720+7725+7730+7735+7740+7745+7750+7755+7760+7765+7770+7775+7780+7785+7790+7795+7800+7805+7810+7815+7820+7825+7830+7835+7840+7845+7850+7855+7860+7865+7870+7875+7880+7885+7890+7895+7900+7905+7910+7915+7920+7925+7930+7935+7940+7945+7950+7955+7960+7965+7970+7975+7980+7985+7990+7995+8000+8005+8010+8015+8020+8025+8030+8035+8040+8045+8050+8055+8060+8065+8070+8075+8080+8085+8090+8095+8100+8105+8110+8115+8120+8125+8130+8135+8140+8145+8150+8155+8160+8165+8170+8175+8180+8185+8190+8195+8200+8205+8210+8215+8220+8225+8230+8235+8240+8245+8250+8255+8260+8265+8270+8275+8280+8285+8290+8295+8300+8305+8310+8315+8320+8325+8330+8335+8340+8345+8350+8355+8360+8365+8370+8375+8380+8385+8390+8395+8400+8405+8410+8415+8420+8425+8430+8435+8440+8445+8450+8455+8460+8465+8470+8475+8480+8485+8490+8495+8500+8505+8510+8515+8520+8525+8530+8535+8540+8545+8550+8555+8560+8565+8570+8575+8580+8585+8590+8595+8600+8605+8610+8615+8620+8625+8630+8635+8640+8645+8650+8655+8660+8665+8670+8675+8680+8685+8690+8695+8700+8705+8710+8715+8720+8725+8730+8735+8740+8745+8750+8755+8760+8765+8770+8775+8780+8785+8790+8795+8800+8805+8810+8815+8820+8825+8830+8835+8840+8845+8850+8855+8860+8865+8870+8875+8880+8885+8890+8895+8900+8905+8910+8915+8920+8925+8930+8935+8940+8945+8950+8955+8960+8965+8970+8975+8980+8985+8990+8995+9000+9005+9010+9015+9020+9025+9030+9035+9040+9045+9050+9055+9060+9065+9070+9075+9080+9085+9090+9095+9100+9105+9110+9115+9120+9125+9130+9135+9140+9145+9150+9155+9160+9165+9170+9175+9180+9185+9190+9195+9200+9205+9210+9215+9220+9225+9230+9235+9240+9245+9250+9255+9260+9265+9270+9275+9280+9285+9290+9295+9300+9305+9310+9315+9320+9325+9330+9335+9340+9345+9350+9355+9360+9365+9370+9375+9380+9385+9390+9395+9400+9405+9410+9415+9420+9425+9430+9435+9440+9445+9450+9455+9460+9465+9470+9475+9480+9485+9490+9495+9500+9505+9510+9515+9520+9525+9530+9535+9540+9545+9550+9555+9560+9565+9570+9575+9580+9585+9590+9595+9600+9605+9610+9615+9620+9625+9630+9635+9640+9645+9650+9655+9660+9665+9670+9675+9680+9685+9690+9695+9700+9705+9710+9715+9720+9725+9730+9735+9740+9745+9750+9755+9760+9765+9770+9775+9780+9785+9790+9795+9800+9805+9810+9815+9820+9825+9830+9835+9840+9845+9850+9855+9860+9865+9870+9875+9880+9885+9890+9895+9900+9905+9910+9915+9920+9925+9930+9935+9940+9945+9950+9955+9960+9965+9970+9975+9980+9985+9990+9995+10000+10005+10010+10015+10020+10025+10030+10035+10040+10045+10050+10055+10060+10065+10070+10075+10080+10085+10090+10095+10100+10105+10110+10115+10120+10125+10130+10135+10140+10145+10150+10155+10160+10165+10170+10175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norum; tamen ad interpolationem eius seriei, seu ad terminos, quorum indices sunt numeri fractiones, egregie accommodari potest. De hoc autem hinc explicare non constitui, cum infra magis idonea modi occurrant ad idem efficiendum. Id tantum de isto termino generali afferam, quod ad ea, quae sequuntur quasi manuducat. Quae si terminum cuius index $n = \frac{1}{2}$, seu qui aequaliter interiaceat inter primum 1, et praecedentem qui itidem est 1. Posito autem $n = \frac{1}{2}$, asecutus sum seriem $\sqrt{\frac{2 \cdot 4}{3 \cdot 3}}, \sqrt{\frac{4 \cdot 6}{5 \cdot 5}}, \sqrt{\frac{6 \cdot 8}{7 \cdot 7}}, \sqrt{\frac{8 \cdot 10}{9 \cdot 9}}$ etc. quae terminum quacumque exprimit. Haec autem series similis milii statim visa est eius, quam in *Wallisii* operibus pro area circulari vidisse memineram. Invenit enim *Wallisius* circulum esse ad quadratum diameter $\sqrt{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot \dots}$ etc. ad $3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot \dots$ etc. Si igitur fuerit diameter $= 1$, circumferentia area $= \sqrt{\frac{2 \cdot 4}{3 \cdot 3}}, \sqrt{\frac{4 \cdot 6}{5 \cdot 5}}, \sqrt{\frac{6 \cdot 8}{7 \cdot 7}}, \dots$ etc. Ex huius igitur cum mea convenientia concludere licet, terminum indicis $\frac{1}{2}$ esse aequalem radici quadratae ex circulo, cuius diameter $= 1$.

S. 3. Arbitratus eram ante series 1, 2, 6, 24, etc. terminum generalem, si non algebraicum tamen exponentiali dari. Sed postquam intellectussem terminos quosdam intermedios a quadratura circuli pendere, neque algebraicas neque exponentialias quantitates ad eum exprimendum idoneas esse cognoui. Terminus enim generalis eius

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progressionis ita debet esse comparatus, vt tum quantitates algebraicas tum a quadratura circuli tum forte ab aliis quoque quadraturis pendentes comprehendat; id quod in nullam formulam nec algebraicam nec exponentiali competit.

§. 4. Cum autem considerassem, dari inter quantitates differentiales eiusmodi formulas, quae certis in casibus integrationem admittant et tum quantitates algebraicas praebeant, in aliis vero non admittant et tum quantitates a quadraturis curuarum pendentes exhibeant; animus subiit huiusmodi forte formulas ad progressionis memoriae aliarumque eius similius terminos generales suppeditando aptas esse. Progressiones vero, quae tales requirunt terminos generales, qui algebraice dari nequeunt, voco transcendentes; quemadmodum Geometrae omne id, quod vires communis Algebrae superat transcendens appellare solent.

§. 5. Id ergo meditatus sum, quomodo formulas differentiales ad progressionum terminos generales exprimendos accommodari maxime conveniat. Terminus autem generalis est formula, quam ingrediuntur tum quantitates constantes, tum alia quaepiam non constans vt π , quae ordinem terminorum seu indicem exponit: vt si tertius, terminus desideretur oporteat loco n pone 3. Sed in formula differentiali quantitatem quandam variabilem

riabilem inesse oportet. Pro qua non consultum est adhibere n , cum eius variabilitas non ad integrationem pertineat: sed postquam ea formula integrata est vel integrata esse ponitur, tum demum ad progressionem formandam inseruiat. In formula igitur differentiali insit oportet quantitas quaedam variabilis x , quae autem post integrationem alii ad progressionem spectanti aequalis ponenda est; et quod oritur, proprie est terminus, cuius index est n .

§. 6. Ut haec clarius concipientur, dicimus $\int p dx$ esse terminum generalem progressionis sequenti modo ex eo eruendae; denotet autem p functionem quamcunque ipsius x , et constantium in quarum numero adhuc ipsum n haberi debet. Concipiatur $\int p dx$ integratum tafique constante auctum, ut facto $x = 0$ totum integrale evanescat, tum ponatur x aequale quantitati cuidam cognitae. Quo facto in invento integrali non nisi quantitates ad progressionem pertinentes supererunt, et id exprimet terminum, cuius index $= n$. Seu integrale dicto modo determinatum erit proprius terminus generalis. Si quidem id haberi potest, non opus est formula differentiali, sed progressio inde formata habebit terminum generalem algebraicum; secus res se habet si integratio non succedit, nisi certis numeris loco n substitutis.

§. 7. Assumsi igitur plures huiusmodi formulas differentiales integrationem non admittentes

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nisi si ponatur loco n numerus integer affirmatus, vt seriei termini principales fiant algebraici: et inde progressiones formaui. Earum itaque termini generales in promptu erunt, et a quanam quadratura quique eius termini intermedii pendeant definire licebit. Hic quidem non plures eiusmodi formulas percurram; sed vnicam tantum aliquanto generaliorem pertractabo, quae valde late patet, et ad omnes progressiones, quarum quilibet termini sunt facta constantia ex numero factorum ab indice pendente accommodari potest; quiq; factores sunt fractiones, quarum numeratores et denominatores in progressione quacunque arithmeticā progrediuntur, vt: $\frac{2}{3} + \frac{2 \cdot 4}{3 \cdot 5} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} + \frac{2 \cdot 4 \cdot 6 \cdot 8}{3 \cdot 5 \cdot 7 \cdot 9} + \text{etc.}$

§. 8. Sit proposita haec formula $\int x^e dx (1-x)^n$ vicem termini generalis subiens, quae integrata ita, vt fiat $=0$, si sit $x=0$; et tum posito $x=1$, det terminum ordine n progressionis inde ortae. Videamus ergo qualem ea suppeditet progressionem. Est $(1-x)^n = 1 - \frac{n}{1}x + \frac{n(n-1)}{1 \cdot 2}x^2 - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3$ etc. Et propterea $x^e dx (1-x)^n = x^e dx - \frac{n}{1}x^{e+1} dx + \frac{n(n-1)}{1 \cdot 2}x^{e+2} dx - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{e+3} dx$ etc. Quare $\int x^e dx (1-x)^n = \frac{x^{e+1}}{e+1} - \frac{n \cdot x^{e+2}}{1 \cdot (e+2)} + \frac{n \cdot n-1 \cdot x^{e+1}}{1 \cdot 2 \cdot (e+3)}$ $- \frac{n \cdot n-1 \cdot n-2 \cdot x^{e+4}}{1 \cdot 2 \cdot 3 \cdot (e+4)}$ etc. Ponatur $x=1$, quia constantis additione non est opus, et habebitur

$$\frac{1}{e+1}$$

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$\frac{1}{e+1} - \frac{n}{1 \cdot (e+2)} + \frac{n \cdot n-1}{1 \cdot 2 \cdot (e+3)} - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3 \cdot (e+4)} + \text{etc.}$ terminus generalis seriei inuenienda. Quae talis erit, vt si $n=0$, prodeat terminus $= \frac{1}{e+1}$; si $n=1$ term. $= \frac{1}{(e+1)(e+2)}$, si $n=2$, term. $= \frac{1 \cdot 2}{(e+1)(e+2)(e+3)}$ si $n=3$. prodeat terminus $= \frac{1 \cdot 2 \cdot 3}{(e+1)(e+2)(e+3)(e+4)}$ lex qua hi termini progredivintur manifesta est.

§. 9. Hanc ergo affectus sum progressionem $\frac{1}{(e+1)(e+2)} + \frac{1 \cdot 2}{(e+1)(e+2)(e+3)} + \frac{1 \cdot 2 \cdot 3}{(e+1)(e+2)(e+3)(e+4)} + \text{etc.}$ cuius terminus generalis est $\int x^e dx (1-x)^n$. Termini vero ordine n ipsius haec erit forma $\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}{(e+1)(e+2) \cdots (e+n+1)}$. Haec quidem forma sufficit ad terminos indicum integrorum inueniendos, sed si indices non fuerint integri, ex ea ipsi termini inueniri nequeunt. His autem proximis inueniendis inseruit haec series $\frac{1}{e+1} - \frac{n}{1 \cdot (e+2)} + \frac{n \cdot n-1}{1 \cdot 2 \cdot (e+3)} - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3 \cdot (e+4)} + \text{etc.}$ Si $\int x^e dx (1-x)^n$ multiplicetur per $e+n+1$, habebitur progressio cuius terminus ordine n hanc formam habet $\frac{1 \cdot 2 \cdot 3 \cdots n}{(e+1)(e+2) \cdots (e+n)}$ cuius igitur verus terminus generalis erit $(e+n+1)$ $\int x^e dx (1-x)^n$. Hic obseruandum est, progressionem semper fieri algebraicam, quando loco assumatur numerus affirmatus. Ponatur e.g. $e=2$, progressionis terminus n^{mus} erit $\frac{1 \cdot 2 \cdot 3 \cdots n}{3 \cdot 4 \cdot 5 \cdots (n+2)}$ seu $\frac{1 \cdot 2 \cdot 3 \cdots n}{(n+2)!}$. Id quod ipse terminus generalis quoque indicat, qui erit $(n+3) \int x^e dx (1-x)^n$. Nam $\int x^e dx = \frac{x^{e+1}}{e+1}$ et $\int (1-x)^n dx = \frac{(1-x)^{n+1}}{n+1}$ et $\frac{d}{dx} (1-x)^n = -n(1-x)^{n-1}$ et $\frac{d}{dx} (1-x)^{n+1} = -(n+1)(1-x)^n$ et $\frac{d}{dx} (1-x)^{n+2} = -(n+2)(1-x)^{n+1}$ eius integrale est $\int (1-x)^{n+2} dx = \frac{(1-x)^{n+3}}{n+3} = \frac{(1-x)^{n+3}}{(n+3)!}$

cap. V.

F

$-(1-x)$

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$\frac{1}{n+3} \frac{(1-x)^{n+3}}{(n+3)} (n+3)$, vt hoc fiat $= o$ si $x=0$,

erit $C = \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}$. Ponatur $x=1$, erit
terminus generalis $\frac{n+3}{n+1} - \frac{2(n+3)}{n+2} + 1 = \frac{2}{(n+1)(n+2)}$.

§. 10. Vt igitur progressiones transcendentes adipiscamur, ponatur e aequale fractioni $\frac{f}{g}$. Erit progressionis terminus ordine n

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)(f+3g) \cdots (f+ng)} g^n \text{ siue } \frac{g \cdot 2g \cdot 3g \cdots ng}{(f+g)(f+2g)(f+3g) \cdots (f+ng)}$$

Terminus vero generalis erit $= \frac{(f+(n+1)g)}{g} \int x^g dx$
 $(1-x)^n$. Qui si dividatur per g^n , erit pro progressione $\frac{1}{f+g} + \frac{1 \cdot 2}{(f+g)(f+2g)} + \frac{1 \cdot 2 \cdot 3}{(f+g)(f+2g)(f+3g)} \text{ etc.}$
cuius terminus ordine n est $\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)(f+3g)}$. Eius progressionis igitur terminus generalis erit $\frac{(f+(n+1)g)}{g}$

$\int x^g dx (1-x)^n$. Vbi si fractio $\frac{f}{g}$ non sit numero integro aequalis, seu si f ad g non habuerit rationem multiplicem, progressio erit transcendens, et termini intermedii a quadraturis pendebunt.

§. 11. Exemplum quoddam in medium aferam, vt usus termini generalis clarius ob oculos ponatur. Sit in paragraphi praecedentis progressionе priore $f=1$, $g=2$, erit terminus ordine $n = \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n}{3 \cdot 5 \cdot 7 \cdot 9 \cdots (2n+1)}$, progressio vero ipsa haec $\frac{2}{3} + \frac{2 \cdot 4}{3 \cdot 5} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \text{ etc. cuius terminus generalis ergo erit } \frac{2n+3}{2} \int dx (1-x)^n \sqrt{x}$. Quaeratur terminus

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minus eius index $\frac{r}{2}$, fiet igitur $n = \frac{r}{2}$, s' et habetur terminus quae situs $= 2 \int dx \sqrt{(x-x^r)}$. Quod cum significet elementum areae circularis, per spicium est terminum quae situm esse aream circuli, cuius diameter $= 1$. Proposita porro sit haec series, $1 + \frac{r}{1} + \frac{r(r-1)}{1 \cdot 2} + \frac{r(r-1)(r-2)}{1 \cdot 2 \cdot 3}$ etc., quae est coefficientium binomii ad potestatem r eleuati. Terminus ordine n est ergo $\frac{r(r-1)(r-2)\dots(r-n+2)}{1 \cdot 2 \cdot 3 \dots (n-1)}$.

In §. praecedente habetur hic $\frac{(f+g)(f+2g)}{(f+g)(f+ng)}$. Hic, ut cum illo comparetur inuertendus est, ve habeatur $\frac{(f+g)(f+2g)}{2 \cdot 3 \cdot \dots \cdot (n-1)g}$. multiplicetur hic per $\frac{n}{(f+ng)}$, et erit is $\frac{(f+g)(f+2g)}{1 \cdot 2 \cdot \dots \cdot (n-1)}$. oportet igitur esse $f+g=r$ et $f+2g=r-1$, vnde fiet $g=r-1$, et $f=r+1$. Eodem modo tractetur terminus generalis $\frac{f+((n-1)r)}{g^{n-1}} \int x^{r-1} dx$.

$(1-x)^r$. Prohibit pro progressione proposita $1 + \frac{r(r-1)}{1 \cdot 2}$ etc., hic terminus generalis, $n(-1)^{n+1} \cdot \frac{(r-n+1)r}{(r-n+1)r} \int x^{r-1} dx (1-x)^n$. Sit $r=2$ erit huius progressionis $1, 2, 1, 0, 0, 0, \dots$ etc., terminus generalis $n(-1)^{n+1} \cdot \frac{((2-n)(3-n))}{(1-x)^n} \int x^{r-3} dx$. Hic autem notari debet, hunc casum et alios quibus $r+1$ fit numerus negatiuus, non posse ex generali deduci, quia tunc integrale non non fit $= 0$, si $x=0$. Pro his vero $\int x^r dx (1-x)^n$ peculiari modo integrari conuenit, post integrationem enim constans infinita est adiicienda; quando ve-

F 2

ro

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$r_0 e + r$ est numerus affirmativus, vt posui §. 8. constantis additione non est opus. Considerata autem progressionem, cuius terminus ordine n erat sequens, $\frac{r(r-1)(r-2)\dots(r-n+2)}{1 \cdot 2 \cdot 3 \cdots (n-1)}$, transmutari potest illa termini exponentis n forma in hanc $\frac{n(r-1)}{(1 \cdot 2 \cdot 3 \cdots (n-1))(1 \cdot 2 \cdots (r-n+1))}$. Sed per §. 14. est $r \cdot r-1 \cdots 1 = \int dx(-lx)^r$, et $1 \cdot 2 \cdot 3 \cdots (n-1)$ est $\int dx(-lx)^{n-1}$. et $1 \cdot 2 \cdots (r-n+1) = \int dx(-lx)^{r-n+1}$. Quamobrem ibi tractatae progressionis $1 + \frac{r}{1} + \frac{r \cdot r-1}{1 \cdot 2} + \frac{r \cdot r-1 \cdot r-2}{1 \cdot 2 \cdot 3} + \text{etc.}$ hic habetur terminus generalis $\frac{\int dx(-lx)^r}{\int dx(-lx)^{n-1} \int dx(-lx)^{r-n+1}}$. Si fuerit $r = 2$, erit terminus generalis

$$\frac{\int dx(-lx)^{n-1}}{\int dx(-lx)^{3-n}}, \text{ cui respondet haec progression} 1, 2, 1, 0, 0, 0, 0, 0, \text{ etc. Ut si quaeratur terminus indicis } \frac{2}{2}, \text{ erit is } = \frac{\int dx(-lx)^{\frac{1}{2}}}{\int dx(-lx)^{\frac{1}{2}} \int dx(-lx)^{\frac{3}{2}}}.$$

Dicta ergo area circuli $= A$, cuius diameter est $= 1$, quia est $\int dx(-lx)^{\frac{1}{2}} = \sqrt{A}$. et $\int dx(-lx)^{\frac{3}{2}} = \frac{2}{3}\sqrt{A}$; erit terminus medium interiacens inter duos primos terminos progressionis $1, 2, 1, 0, 0, 0, \text{ etc.}$ huius formae $\frac{4}{3}A$, hoc est $\frac{4}{3}$ quam proxime.

§. 12. Progredior nunc ad progressionem, de qua initio dixi, $1 + 1 \cdot 2 + 1 \cdot 2 \cdot 3$ etc. et in qua terminus ordine n est $1 \cdot 2 \cdot 3 \cdot 4 \cdots n$. Continetur haec progression in generali nostra, sed terminus ge-

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generalis peculiari modo inde deriuari debet. Hancenus scilicet terminum generalem habui si terminus ordine n est $\frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}{(f+g)(f+2g)\dots(f+ng)}$, qui si ponatur $f=1$ et $g=0$, abit in $1 \cdot 2 \cdot 3 \cdot \dots \cdot n$, cuius terminus generalis quaeritur: substituantur ergo in termino generali $\frac{f+(n+1)g}{g^{n+1}} \int x^{\frac{f}{g}} dx$.

Hic, huiusmodi loco f et g , erit terminus ge-

neralis quae situs $\int \frac{x^{\frac{1}{g}} dx (1-x)^n}{0^{n+1}}$. Qui vero huius expressionis sit. valor, sequenti modo inuestigo.

¶. 13. Ex conditione, qua huiusmodi termini generales usui accommodari debent, intellegitur loco x alias functiones ipsius x posse subrogari, dummodo eae tales fuerint, ut sint $=0$ si $x=0$ et $=1$ si $x=1$. Huiusmodi enim functiones si loco x substituantur, terminus generalis perinde satisfaciet ac ante. Ponatur igitur $f=1$ et $g=0$; loco x et consequenter $\frac{f+(n+1)g}{g^{n+1}} \int x^{\frac{f}{g}} dx$ loco dx ; quo factum habebitur $\int g^n dx (1-x)^n$.

Iam hinc ponitur $f=1$, et $g=0$, habebitur $\int x^{\frac{1}{0}} dx (1-x)^n$. Cum autem sit $x^{\frac{1}{0}} = 1$; habemus hic casum, quo numerator et denominator euaneantur ($1-x^0$) et 0^n . Per regulam igitur cognitam quae-

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ramus valorem fractionis $\frac{1-x^0}{0}$. Id quod fiet quae-
rendo valorem fractionis $\frac{x-x^z}{z}$ tum, cum z eu-
nescit, differentietur igitur et numerator et de-
nominator sola z variabili posita; habebitur $\frac{-x^z dz/lx}{dz}$
seu $-x^z/lx$, si iam ponatur $z=0$, prodibit $-lx$. Est
itaque $\frac{1-x^0}{0} = -lx$.

§. 14. Cum igitur sit $\frac{1-x^0}{0} = -lx$, erit
 $\frac{(1-x^0)^n}{0^n} = (-lx)^n$, et propterea terminus genera-
lis quaeitus $\int \frac{dx(1-x^0)^n}{0^n}$ transmutatus est in $\int dx$
 $(-lx)^n$. Cuius valor inueniri per quadraturas pot-
est. Quamobrem huius progressionis 1, 2, 6, 24,
120, 720, etc. terminus generalis est $\int dx(-lx)^n$,
codem modo adhibendus; quo supra praeceptum
est. Hunc autem esse terminum generalem pro-
gressionis propositae ex eo quoque cognoscitur,
quod terminos, quorum indices sunt numeri in-
tegri affirmatiui, reuera praebeat, sit v. g. $n=3$,
erit $\int dx(-lx)^3 = \int -dx(lx)^3 = -x(lx)^3 + 3(xlx)^2$
 $-6x^2 lx + 6x$ constantis additione opus non est,
cum facto $x=0$ omnia euanscant, ponatur igitur
 $x=1$, quia $1^3=1$, omnes termini logarithmis af-
fe*cti*

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fecti evanescant et restabit 6, qui est terminus tertius.

(§. 15.) Verum quidem est, hanc methodum terminorum istius seriei inueniendorum nimis esse operosam, eorum nimirum quorum indices sunt numeri integri, qui utique facilius continuanda progressione obtinentur. Verum tamen ad terminos indicum fractorum inueniendos per quam est idonea, quippe qui adhuc ne operosissima quidem methodo definiti potuerunt. Si ponatur $n = \frac{a}{2}$ habebitur respondens terminus $\int dx \sqrt{-1x}$ cuius valor per quadraturas datur. Sed initio ostendi hunc terminum esse aequalem radici quadratae ex circulo cuius diameter est 1. Hinc quidem idem concludere non licet, ob defectum analysis; infra autem sequetur methodus eosdem terminos intermedios ad algebraicarum curuarum quadraturas reducendi. Ex cuius cum hac comparatione forte nonnihil ad amplificationem analysis derivari poterit.

§. 16. Progressionis cuius terminus ordine n indicatur per $\frac{1}{(f+g)(f+2g)(f+3g)} \dots \frac{n}{(f+ng)}$ terminus generalis est per §. 10.

$\frac{f+(n+1)g}{g^{n+1}} \int dx$

($-1x$)ⁿ. Si autem terminus ordine n fuerit 1. 2.

3. — — — n , tum est terminus generalis $\int dx$

($-1x$)ⁿ. Quae formula, si loco 1. 2. 3. — — — n sub-

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substituatur, habebitur $\frac{\int dx (-lx)^n}{(f+g)(f+2g) \dots (f+ng)} =$
 $\frac{f+(n+1)g}{g^{n+1}} \int x^{\frac{f}{g}} dx (1-x)^n$. Ex quo efficitur $(f+g)$
 $(f+2g) \dots (f+ng) = \frac{g^{n+1} \int dx (-lx)^n}{(f+(n+1)g) \int x^{\frac{f}{g}} dx (1-x)^n}$

Quae expressio igitur est terminus generalis huius generalis progressionis $f+g$, $(f+g)(f+2g)$, $(f+g)(f+2g)(f+3g)$ etc. Huiusmodi igitur progressionum omnium opem termini generalis omnes termini cuiuscunque indicis definiuntur. Quae infra sequentur de reductione $\int dx (-lx)^n$ ad quadraturas notiores seu curuarum algebraicarum, etiam hic vsum habebunt.

§. 17. Sit $f+g=1$, et $f+2g=3$, erit $g=2$ et $f=-1$. Vnde oriatur haec progressionis particularis $1, 1 \cdot 3, 1 \cdot 3 \cdot 5, 1 \cdot 3 \cdot 5 \cdot 7$, etc. Cuius igitur terminus generalis est $\frac{2^{n+1} \int dx (-lx)^n}{(2n+1) \int x^{\frac{-1}{2}} dx (1-x)^n}$.

Quanquam hic exponens ipsius x sit negatiuus, tamen id incommodum, de quo supra dictum, hic locum non habet, cum sit unitate minor. Ponatur $n=\frac{1}{2}$ ut inueniatur terminus ordine $\frac{1}{2}$, erit

$$is = \frac{2^{\frac{3}{2}} \int dx \sqrt{-lx}}{2 \int x^{\frac{-1}{2}} dx \sqrt{1-x}} = \frac{\sqrt{2} \int dx \sqrt{-lx}}{\int \frac{dx}{\sqrt{x-x^2}}}. \text{ Per §. 15.}$$

autem constat dare $\int dx \sqrt{-lx}$ radicem quadratam

ex

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ex circulo, secus diameter $= \pi$, sit peripheria eius circuli π , herit area $= \frac{1}{2}\pi$, adeoque $\int dx \sqrt{1-x^2}$ dat $\frac{1}{2}\pi p$. Deinde $\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dx}{\sqrt{1-x^2}} + \sqrt{(x+x^2)}$ dat arcum circuli, cuius sinus versus est x . Posito itaque $x=1$, proneniet $\frac{\pi}{2}p$. Quamobrem terminus quae situs erit $= \sqrt{\frac{2}{p}}$.

S. 18. Cum progressionis, cuius terminus ordine n indicatur per $(f+g)(f+2g)\dots(f+ng)$, terminus generalis per §. 16. sit $\frac{f}{g^{n+1}} \int dx (-lx)^n$. similiter si terminus ordine n fuerit $(b+k)(b+2k)\dots(b+nk)$, erit terminus generalis $\frac{1}{k^{n+1}} \int dx (-lx)^n$. Dividatur illa pro-

$(b+(n+1)k) \int x^k dx (1-x)^n$ gressio per hanc, nempe terminus primus per primum, secundas per secundum et ita perro: devicietur ad nouam progressionem, cuius terminus ordine n erit $\frac{(f+g)(f+2g)\dots(f+ng)}{(b+k)(b+2k)\dots(b+nk)}$. Et terminus generalis huius progressionis ex illis duabus, compositus erit $\frac{g^{n+1}(b+(n+1)k) \int x^k dx (1-x)^n}{k^{n+1}(f+(n+1)g) \int x^k dx (1-x)^n}$. Qui vacuus est ab integrali logarithmico $\int dx (-lx)^n$.

S. 19. In omnibus huiusmodi terminis generalibus hoc maxime notandum est, non quidem

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dem loco f, g, b, k numeros constantes ponit oportere, sed eos quomodoconque ab n pendentes quoque assumi posse. In integratione enim eae literae perinde atque n tractantur, omnes tanquam constantes. Sit terminus ordine n hic $(f+g)$
 $(f+2g) \dots (f+ng)$, ponatur $g=1$, sed $f=\frac{nn-n}{2}$. Quia progressio ipsa est $(f+g), (f+g)$
 $(f+2g), (f+g)(+2g)(f+3g)$ etc. ponatur
 ubique 1 loco g , erit ea, $f+1, (f+1)(f+2)$,
 $(f+1)(f+2)(f+3)$. Sed loco f scribi debet in
 termino primo 0, in secundo 1, in tertio 3, in
 quarto 6 et ita porro, prodibit haec progressio
 $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, etc. cuius igitur ter-
 minus generalis $\frac{2 \int dx (-1/x)^n}{(nn+n+2) \int x^{\frac{nn-n}{2}} dx (1-x)}$ — — —
 $\frac{2 \int dx (-1/x)^n}{(nn+n+2) \int dx (x^{\frac{n-1}{2}} - x^{\frac{n+1}{2}})^n}$

§. 20. Accedo nunc ad eas progressiones, unde compendium illud in definiendis terminis intermediis huius progressionis 1, 2, 6, 24, 120, etc. nactus sum. Id enim latius patet quam ad haec solam progressionem, quoniam eius terminus generalis $\int dx (-1/x)^n$ etiam in infinitarum aliarum progressionum terminos generales ingreditur. Assumo hunc terminum generalem $\frac{f+(n+1)g}{g^{n+1}} \int x^{\frac{f}{g}} dx (x-x)^n$, cui respondet terminus ordine n hic

E. 2. 3.

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Pono hic $f = n$, $g = 1$,
 $\frac{d}{dx}(f+ng)(f+ng) \dots (f+ng)$. Pono hic $f = n$, $g = 1$,
 erit terminus generalis $(2n+1)/x^n dx(x-x)^n$ vel
 $(2n+1)/\int dx(x-x)^n$ et forma eius ordine n ,
 $\frac{(2n+1)(2n+3)\dots(2n+n)}{2^n n!}$. Progressio vero ipsa
 $\frac{(2n+1)(2n+3)\dots(2n+3)}{2^n n!}, \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 6\cdot 8\cdot 10}, \dots$, etc. vel haec $\frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 2}, \frac{1\cdot 2\cdot 3\cdot 5}{1\cdot 2\cdot 3\cdot 4}$,
 $\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{1\cdot 2\cdot 4\cdot 6}, \dots$ etc. In qua numeratores sunt quadra-
 ta progressionis $1, 2, 6, 24$. inter denominatores
 vero duos proximos aequidistant facile inuenitur.
 Sit in² progressionе $1, 2, 6, 24$ etc. terminus,
 cuius index $\frac{1}{2}, A$, erit progressionis illius termi-
 nus ordine $\frac{1}{2} = \frac{A}{1}$.

S. 21. Ponatur in termino generali $(2n+1)$
 $x^n dx(x-x)^n$, $n = \frac{1}{2}$ erit terminus huius expo-
 nentis $= 2 \int dx \sqrt{x-x^2} = \frac{A}{1}$, vnde $A = \sqrt{1 \cdot 2}$, $\int dx$
 $\sqrt{x-x^2}$ = termino progressionis $1, 2, 6, 24$, etc.
 cuius index est $\frac{1}{2}$, qui ergo ut ex eo elucet est
 radix quadrata ex circulo diametri 1 . Dicatur
 nunc terminus huius progressionis ordine $\frac{3}{2}, A$,
 erit respondens in assumta progressionе $= \frac{AA}{1 \cdot 2 \cdot 3} = \frac{3}{2}$.

$4 \int dx(x-x^2)^{\frac{3}{2}}$ ergo $A = \sqrt{1 \cdot 2 \cdot 3 \cdot 4} \int dx(x-x^2)^{\frac{3}{2}}$
 Simili modo inuenitur terminus ordine $\frac{5}{2} = \sqrt{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$.

$2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \int dx(x-x^2)^{\frac{5}{2}}$. Ex quibus generaliter
 concluso terminum ordine $\frac{p}{2}$ fore $= \sqrt{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot (p+1)} \int dx(x-x^2)^{\frac{p}{2}}$. Hoc igitur modo
 inveniuntur omnes termini progressionis, $1, 2, 6, 24$, etc. quorum indices sunt fractiones denomi-
 natore existente 2 .

G. 2

S. 22.

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§. 22. Porro in termino generali $\frac{f+(n+1)g}{g^{n+1}}$

$\int x^{\frac{f}{g}} dx (1-x)^n$, pono $f=2n$ manente $g=1$, prodibit $(3n+1) \int dx (xx-x^3)^n$ terminus generalis huius progressionis $\frac{1}{3}, \frac{1 \cdot 2}{5 \cdot 6}, \frac{1 \cdot 2 \cdot 3}{7 \cdot 8 \cdot 9}$ etc. Multiplacetur ille per praecedentem $(2n+1) \int dx (x-xx)^n \int dx (xx-x^3)^n$. Qui dabit hanc progressionem $\frac{1 \cdot 1 \cdot 1}{1 \cdot 2 \cdot 3}, \frac{1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$ etc. ubi numeratores sunt cubi terminorum respondentium progressionis 1. 2. 6. etc. Huius progressionis terminus ordine $\frac{1}{3}$ fit A, erit respondens illius $\frac{A^3}{1} = 2 (\frac{2}{3} + 1) \int dx (x-xx)^{\frac{1}{3}} \int dx (xx-x^3)^{\frac{1}{3}}$, ergo terminus ordine $\frac{1}{3}$ est $\sqrt[3]{1 \cdot 2 \cdot \frac{5}{3}} \int dx (x-xx)^{\frac{1}{3}} \int dx (xx-x^3)^{\frac{1}{3}}$. Similiter term. ordine $\frac{2}{3}$ est $\sqrt[3]{1 \cdot 2 \cdot 3 \cdot \frac{7}{3}} \int dx (x-xx)^{\frac{2}{3}} \int dx (xx-x^3)^{\frac{2}{3}}$. Atque terminus ordine $\frac{4}{3}$ est $\sqrt[3]{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \frac{11}{3}} \int dx (x-xx)^{\frac{4}{3}} \int dx (xx-x^3)^{\frac{4}{3}}$ et generaliter terminus ordine $\frac{p}{3}$ est $\sqrt[3]{1 \cdot 2 \cdot \dots \cdot p \cdot (\frac{2p+1}{3})} (\frac{p}{3} + 1) \int dx (x-xx)^{\frac{p}{3}} \int dx (xx-x^3)^{\frac{p}{3}}$.

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§. 23. Si ulterius progredi velimus, ponendo $f=3n$, oportebit terminum generalem $(4n+1) \int dx (x^2-x^4)^n$ in praecedentes multiplicare, unde habetur $(2n+1)(3n+1)(4n+1) \int dx (x-xx)^n \int dx (x^2-x^4)^n \int dx (x^3-x^4)^n$ qui est pro hac serie $\frac{1 \cdot 1 \cdot 1 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} \frac{1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}$ etc. Ex qua definitur termini

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progressionis 1, 2, 6, 24, etc. quorum indicet sunt fractiones denominatorem 4 habentes. Nimirum terminus cuius index est $\frac{p}{q}$ inuenietur $= \sqrt[q]{1.2.3 \dots p}$.

$\dots - p. (\frac{2p}{q} + 1) (\frac{3p}{q} + 1) (\frac{4p}{q} + 1) \dots (p+1) \int dx (x-xx)^{\frac{p}{q}} \int dx (xx-x^3)^{\frac{p}{q}} \int dx (x^3-x^4)^{\frac{p}{q}} \dots$. Hinc generaliter concludere licet terminum ordine $\frac{p}{q}$ esse $= \sqrt[q]{(1.2.3 \dots p) (\frac{2p}{q} + 1) (\frac{3p}{q} + 1) (\frac{4p}{q} + 1) \dots (p+1)} (\int dx (x-xx)^{\frac{p}{q}} \int dx (x^2-x^3)^{\frac{p}{q}} \int dx (x^3-x^4)^{\frac{p}{q}} \dots \int dx (x^q-x^{q+1})^{\frac{p}{q}})$. Ex hac igitur formula termini cuiusunque indicis fracti inuenientur per quadraturas curvarum algebraicarum: ad id autem requiritur $(1.2.3 \dots p)$ terminus cuius index est numerator fractionis propositae.

9.24. Eodem modo vterius progreedi licet ad progressiones magis compositas, assumendis terminis generalibus magis compositis, sed ea longius non persequor. Possunt etiam signa integralia multiplicari, vt terminus generalis sit $\int q dx / \int p dx$, nimirum integrale ipsius $p dx$ debet multiplicari per $q dx$, et quod resultat denuo integrari, id quod demum dabit factio $x=1$ terminum serier. In ytrage autem integratione, vt sit determinata, oportet addenda constanter efficere; ve posito $x=0$, integrale fiat itidem $=0$. Similiter tractandi sunt termini generales,

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qui pluribus signis integralibus continentur ut $\int r dx$
 $\int q dx \int p dx$. Attamen semper loco p, q, r etc. tales sunt sumenda functiones, ut, quoties n fuerit numerus integer affirmatiuus, prodeant termini ad minimum algebraici.

§. 25. Sit terminus generalis $\int \frac{dx}{x} \int x^e dx (1-x)$
 hic in seriem conuersus dat $\frac{x^{e+1}}{(e+1)^2} - \frac{n \cdot x^{e+2}}{1 \cdot (e+2)^2}$
 $+ \frac{n(n-1) \cdot x^{e+3}}{1 \cdot 2 \cdot (e+3)^2}$ etc. Posito $x=1$, habebitur terminus ordine n per hanc seriem $\frac{1}{(e+1)^2} - \frac{n}{1 \cdot (e+2)^2}$
 $+ \frac{n(n-1)}{1 \cdot 2 \cdot (e+3)^2}$ etc. Progressio vero ipsa haec erit a termino cuius index est 0 incipiens $\frac{1}{(e+1)^2}$,
 $\frac{(e+2)^2 \cdot (e+1)^2}{(e+2)^2 \cdot (e+1)^2}$, $\frac{(e+3)^2 \cdot (e+2)^2 - 2(e+3)^2 \cdot (e+1)^2 + (e+2)^2 \cdot (e+1)^2}{(e+3)^2 \cdot (e+2)^2 \cdot (e+1)^2}$,
 $\frac{(e+4)^2 \cdot (e+3)^2 \cdot (e+2)^2 - 3(e+4)^2 \cdot (e+3)^2 \cdot (e+1)^2 + 3(e+4)^2 \cdot (e+2)^2 \cdot (e+1)^2}{(e+4)^2 \cdot (e+3)^2 \cdot (e+2)^2 \cdot (e+1)^2}$ etc. Lex huins progressionis manifesta est, et non indiget explicatione. Sit $e=0$, erit $\int dx (1-x)^n = \frac{1-(1-x)^{n+1}}{n+1}$ ergo terminus generalis est $\int \frac{dx - dx(1-x)^{n+1}}{(n+1)x}$, progressio vero haec erit $\frac{1}{1}, \frac{4-1}{4 \cdot 1}, \frac{9-4-2 \cdot 9 \cdot 1+4 \cdot 1}{9 \cdot 4 \cdot 1}, \frac{16 \cdot 9 \cdot 4-3 \cdot 16 \cdot 9 \cdot 1+2 \cdot 16 \cdot 4 \cdot 1-9 \cdot 4 \cdot 1}{16 \cdot 9 \cdot 4 \cdot 1}$ etc. Huius differentiae hanc constituent progressionem, $\frac{-1}{4 \cdot 1}, \frac{-9+4}{9 \cdot 4 \cdot 1}, \frac{-16 \cdot 9+2 \cdot 16 \cdot 9 \cdot 4}{16 \cdot 9 \cdot 4 \cdot 1}$ etc.

§. 26. In hac dissertatione ergo id, quod praecipue intendi, assecutus sum; nempe ut terminos

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minos generales inuenirem omnium progressionum, quoru[m] singuli termini sunt facta ex factoribus in progressione arithmeticā progradientibus, in quibusque numeris factorum vt libuerit ab indicibus terminorum pendeat. Quanquam autem hic semper numerus factorum indici aequalis positus sit, tamen si is alio modo inde pendens desideretur, res nihil habet difficultatis. Index denotatus est litera n , si iam quis requirat vt numerus factorum sit $\frac{m+n}{2}$, alia operatione opus non est, nisi vt ubique loco n substituatur $\frac{m+n}{2}$.

§. 27. Coronid[s] loco adhuc aliiquid, curiosum id quidem magis quam utile adiungam. Notum est per $d^n x$ intelligi differentiale ordinis n ipsius x , et d^np , si p denotet functionem quamplam ipsius x , ponaturque dx constans, esse homogeneum cum dx^n , semper autem, quando n est numerus integer affirmatius, ratio quam habet d^np ad dx^n algebraice potest exprimi, vt si $n=2$ et $p=x^3$, erit $d^2(x^3)$ ad dx^2 vt $6x$ ad 1 . Quae ritur nunc si n sit numerus fractus, qualis tum futura sit ratio. Difficultas in his casibus facile intelligitur, nam si n est numerus integer affirmatius, d^n continuata differentiatione inuenitur, talis autem via non patet, si n est numerus fractus. Sed tamen ope interpolationum progressionum, de quibus in hac dissertatione explicauimus expedire licebit.

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§. 28. Sit iuuenienda ratio inter $d^n(z^e)$ et dz^n posito dz constante, seu requiritur valor fractionis $\frac{d^n(z^e)}{dz^n}$. Videamus primo qui sint eius valores si n est numerus integer, vt postmodum generaliter illatio fieri possit. Si $n=1$, erit eius valor $e z^{e-1} = \frac{1 \cdot 2 \cdot 3 \cdots e}{1 \cdot 2 \cdot 3 \cdots (e-1)} z^{e-1}$, hoc modo e exprimo, vt facilius poslea ea quae tradita sunt huc referantur. Si $n=2$, erit valor $e(e-1) z^{e-2} = \frac{1 \cdot 2 \cdot 3 \cdots e}{1 \cdot 2 \cdot 3 \cdots (e-2)} z^{e-2}$. Si $n=3$, habebitur $e(e-1)(e-2)z^{e-3} = \frac{1 \cdot 2 \cdot 3 \cdots e}{1 \cdot 2 \cdot 3 \cdots (e-3)} z^{e-3}$. Hinc generaliter infero quicquid sit n fore semper $\frac{d^n(z^e)}{dz^n} = \frac{1 \cdot 2 \cdot 3 \cdots e}{1 \cdot 2 \cdot 3 \cdots (e-n)} z^{e-n}$. Est autem per §. 14, $1 \cdot 2 \cdot 3 \cdots e = \int dx (-lx)^e$ et $1 \cdot 2 \cdot 3 \cdots (e-n) = \int dx (-lx)^{e-n}$. Quare habetur $\frac{d^n(z^e)}{dz^n} = z^{e-n} \frac{\int dx (-lx)^e}{\int dx (-lx)^{e-n}}$ vel $d^n(z^e) = z^{e-n} dz^n \frac{\int dx (-lx)^e}{\int dx (-lx)^{e-n}}$. Ponitur hic dz constans et $\int dx (-lx)^e$ vt et $\int dx (-lx)^{e-n}$ ita debent integrari, vt supra praeceptum est, et tum ponere oportet $x=1$.

§. 29. Non necesse est, quomodo verum eliciatur, ostendere, apparebit id ponendo loco n numerum integrum affirmatiuum quemcunq; Quaeratur autem quid sit $d^{\frac{1}{2}}z$, si sit dz constans. Erit ergo $e=1$ et $n=\frac{1}{2}$. Habebitur itaque $d^{\frac{1}{2}}z = \frac{\int dx (-lx)}{\sqrt{-lx}} Vz dz$

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\sqrt{zdz} . Est autem $\int dx(-lx) = 1$ et dicta area circuli A, cuius diameter est 1, erit $\int dx \sqrt{-lx} = VA$, unde $d^{\frac{1}{2}} z = \sqrt{\frac{zdz}{A}}$. Proposita igitur sit. haec aequatio ad quampliam curuam $y^{\frac{1}{2}} z = z\sqrt{dy}$, ubi dz ponitur constans, et quaeratur qualis ea sit curua.

Cum sit $d^{\frac{1}{2}} z = \sqrt{\frac{zdz}{A}}$ abibit ea aequatio in hanc $y^{\frac{1}{2}} \sqrt{\frac{zdz}{A}} = z\sqrt{dy}$, quae quadrata dat $\frac{yydz}{A} = zdy$: unde inuenitur $\frac{1}{A} dz = c - \frac{1}{y}$, vel $yz = cAy - A$, quae est aequatio ad curuam quaesitam.

PROBLEMATIS DE STATIONIBVS PLANETARVM CASVS ALTER. AVCTORE F. C. Maiero.

Sint datae duorum Planetarum orbitae ad eclipticam planum reductae, earumque positio; Fig. 2. sit quoque data vnius distantia a Sole: Quae-
ritur distantia alterius stationaria.

Maioris orbitae axis AB sit = A.

eiusdem eccentricitas dupla OF sit = F.

eiusdem parameter — — = P.

Tom. V.

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