$$1 - A + B - C + D - \text{etc.} = \frac{1 + A'' + B'' + C'' + D'' + \text{etc.}}{1 + A + B + C + D + \text{etc.}}$$

atque

$$(1 + \alpha + \beta + \gamma + \text{etc.})\,(1 \quad \alpha + \beta - \gamma + \text{etc.}) =$$
$$1 - \alpha'' + \beta'' - \gamma'' + \delta'' - \text{etc.}$$

Ex his nunc, si pro serie $a + b + c + d + \text{etc.}$ substituatur haec $\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \text{etc.}$ secundum numeros primos procedens, sequentur omnia omnino theoremata, quae mecum communicare voluisti. Vale, V. C., ac favere perge Tui observantissimo

L. Eulero.

LETTRE XXIX.

Euler à Goldbach.

Sommaire. Application du calcul intégral à la sommation des séries.

(Sans date.)

Seriei, cujus terminus generalis est $\frac{1}{64x^2 - 64x + 15}$, vel $\frac{1}{2}\left(\frac{1}{8x-5} - \frac{1}{8x-3}\right)$ summa est $= \frac{1}{2}\int\frac{(zz - z^4)\,dz}{1 - z^8} = \frac{1}{2}\int\frac{zz\,dz}{(1+zz)(1+z^4)} = -\frac{1}{4}\int\frac{dz}{1+zz} + \frac{1}{4}\int\frac{(1+zz)\,dz}{1+z^4}$, si post integrationem ponatur $z = 1$. At seriei, cujus terminus generalis est $= \frac{3m}{64xx - 64x + 7} = \frac{m}{2}\left(\frac{1}{8x-7} - \frac{1}{8x-1}\right)$, summa est $= \frac{m}{2}\int\frac{(1-z^6)\,dz}{1-z^8} = \frac{m}{2}\int\frac{(1+zz+z^4)\,dz}{(1+zz)(1+z^4)} = \frac{m}{4}\int\frac{dz}{1+zz} + \frac{m}{4}\int\frac{(1+zz)\,dz}{1+z^4}$, posito post integrationem $z = 1$. Verum est $\int\frac{dz}{1+zz} = \frac{\pi}{4}$; $\int\frac{dz}{1+z^4} = \frac{\pi}{4\sqrt{2}} + \frac{1}{2\sqrt{2}}\,l\,(1 + \sqrt{2})$ et

$$\int \frac{zz\,dz}{1+z^4} = \frac{\pi}{4\sqrt{2}} - \frac{1}{2\sqrt{2}}\, l(1+\sqrt{2}).$$

Quare si a serie, cujus terminus generalis est $\frac{1}{(8x-5)(8x-3)}$, subtrahatur series, cujus terminus generalis $\frac{3m}{(8x-7)(8x-1)}$, seriei resultantis summa erit $= -\frac{(1+m)\pi}{16} + \frac{(1-m)\pi}{8\sqrt{2}}$. Vel seriei, cujus terminus generalis est =

$$\frac{3m}{64xx-64x+7} - \frac{1}{64xx-64x+15},$$

summa est $= \frac{(m+1)\pi}{16} + \frac{(m\ \ 1)\pi}{8\sqrt{2}}$. Quare si $m = 1$, summa erit $= \frac{\pi}{8}$; at ut summa sit $= 0$, oportet esse

$$m + 1 + m\sqrt{2} - \sqrt{2} = 0,$$

seu $m = \frac{\sqrt{2}-1}{\sqrt{2}+1}$.

II. Si in serie $\frac{1}{x(2x\ \ 1)(4x-1)}$ summa terminorum parium ab imparibus subtrahatur, prodit series

$$\frac{1}{1.1.3} - \frac{1}{2.3.7} + \frac{1}{3.5.11} - \frac{1}{4.7.15} + \text{etc.},$$

quae resolvitur in has tres:

$$+1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.} = \int \frac{dz}{1+z}$$

$$+\frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \frac{2}{9} - \frac{2}{11} + \text{etc.} = \int \frac{2\,dz}{1+zz}$$

$$-\frac{8}{3} + \frac{8}{7} - \frac{8}{11} + \frac{8}{15} - \frac{8}{19} + \frac{8}{23} - \text{etc.} = -\int \frac{8zz\,dz}{1+z^4}$$

et summa omnium est $= l2 + \frac{\pi}{2} - \pi\sqrt{2} + \frac{4}{\sqrt{2}}\, l(1+\sqrt{2})$, unde non video quomodo summa possit esse $= \pi - 4\, l2$. Sin autem res ita se haberet, foret $\pi = \frac{10\, l2 + 4\sqrt{2}\, l(1+\sqrt{2})}{2\sqrt{2}+1}$.

III. Seriei $(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}) - (\frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}) + \text{etc.}$

summa est $= \int \frac{dz(1-z+zz-z^3)}{1+z^4} =$

$$\int \frac{dz(1+zz)}{1+z^4} - \int \frac{z\,dz}{1+z^4} - \int \frac{z^3\,dz}{1+z^4} = \frac{\pi}{2\sqrt{2}} - \frac{\pi}{8} - \frac{l2}{4}.$$

IV. Si fuerit $\int d.\frac{xx\,dz}{dx} = \int \frac{dx}{1+x^3}$, erit utique

$$dz = \frac{dx}{xx} \int \frac{dx}{1+x^3} = \frac{dx}{xx}\left(x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \text{etc.}\right)$$

et

$$z = C + lx - \frac{x^3}{3.4} + \frac{x^6}{6.7} - \frac{x^9}{9.10} + \text{etc.}$$

Constans autem C, si z deberet simul cum x evanescere, foret infinita; sin autem C maneat indefinita, tum casu $x = 1$, quantitas z indefinitum, h. e. quemcunque valorem obtinebit.

V. Sinus ang. 18^0 est $= \frac{\sqrt{5}-1}{4}$, et sin. ang. 54^0 est $= \frac{\sqrt{5}+1}{4}$, unde erit $\frac{1}{\sin 18^0} - \frac{1}{\sin 54^0} = 2$, id quod etiam tabulae sinuum ostendunt; est enim $\frac{1}{\sin 18^0} = \sec.\ 72^0$ et $\frac{1}{\sin 54^0} = \sec.\ 36^0$.

VI. Serierum sequentium summae sunt

$$\left(\frac{1}{8x-5} - \frac{1}{8x-3}\right) = \int \frac{dz(zz-z^4)}{1-z^8} = \int \frac{zz\,dz}{(1+zz)(1+z^4)}$$

$$= \frac{\pi}{4\sqrt{2}} - \frac{\pi}{8} = \frac{\pi(\sqrt{2}-1)}{8}$$

$$\left(\frac{1}{8x-1} - \frac{1}{8x+1}\right) = \int \frac{dz(z^6-z^8)}{1-z^8} = \int \frac{z^6\,dz}{(1+zz)(1+z^4)}$$

$$= 1 - \int \frac{dz(1+zz+z^4)}{(1+zz)(1+z^4)} = 1 - \frac{1}{2}\int \frac{dz}{1+zz} - \frac{1}{2}\int \frac{dz(1+zz)}{1+z^4}$$

$$= 1 - \frac{\pi}{8} - \frac{\pi}{4\sqrt{2}} = 1 - \frac{\pi(1+\sqrt{2})}{8}$$

$$\left(\frac{1}{6x-5}-\frac{1}{6x-2}\right)=\int\frac{dz(1-z^3)}{1-z^6}=\int\frac{dz}{1+z^3}$$
$$=\frac{1}{3}\int\frac{dz}{1+z}+\frac{1}{3}\int\frac{2dz-zdz}{1-z+zz}$$
$$=\frac{1}{3}\int\frac{dz}{1+z}-\frac{1}{6}\int\frac{2zdz-dz}{1-z+zz}+\frac{1}{2}\int\frac{dz}{1-z+zz}$$
$$=\frac{l2}{3}+\frac{\pi}{3\sqrt{3}}.$$

VII. Seriei $1-\frac{1}{2^{2n-1}}+\frac{1}{3^{2n-1}}-$ etc. jamdudum quoque conjectavi summam esse $=p(l2)^{2n-1}$, at casu $n=2$ facile statim deprehendi valorem ipsius p nequidem rationaliter exhiberi posse.

Euler.

LETTRE XXX.

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Goldbach à Euler.

Sommaire. Théorèmes relatifs à la sommation des suites.

(Petrop.) d. 9 Dec. 1739.

Observavi heri denominatoribus 1, 1.2, 1.2.3, 1.2.3.4, etc. innumeris modis assignari posse numeratores algebraicos, ita ut series tota fiat summabilis, sic v. gr.

$$\frac{1}{1.2.3}+\frac{5}{1.2.3.4}+\frac{11}{1\ldots5}+\frac{19}{1\ldots6}+\frac{29}{1\ldots7}+\frac{41}{1\ldots8}+\frac{55}{1\ldots9}+\text{etc.}$$

est $=\frac{2}{1.2.3}+\frac{3}{1.2.3.4}+\frac{4}{1.2.3.4.5}+\frac{5}{1\ldots6}+\frac{6}{1\ldots7}+\frac{7}{1\ldots8}+\frac{8}{1\ldots9}+\text{etc.}=\frac{1}{2}$,

$$\frac{3}{1.2}-\frac{4}{1.2.3}+\frac{5}{1.2.3.4}-\frac{6}{1\ldots5}+\frac{7}{1\ldots6}-\frac{8}{1\ldots7}+\frac{9}{1\ldots8}-\frac{10}{1\ldots9}+\text{etc.}=1,$$