LETTRE XXVI.

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Euler à Goldbach.

Sommaire. Considérations sur le théorème précédent.

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12 Nov. 1739 st v.

Si habeatur series quaecunque $a, b, c, d, e,$ etc. atque ponatur

$$P = a + b + c + d + e + \text{etc.}$$
$$Q = a^2 + b^2 + c^2 + d^2 + e^2 + \text{etc.}$$
$$R = a^3 + b^3 + c^3 + d^3 + e^3 + \text{etc.}$$
$$S = a^4 + b^4 + c^4 + d^4 + e^4 + \text{etc.}$$
etc.

ac praeterea ex terminis $a, b, c, d,$ etc. formentur

1. facta ex singulis, quorum summa sit $A = P$,
2. facta ex binis, quorum summa sit $= B$,
3. facta ex ternis, quorum summa sit $= C$,
4. facta ex quaternis, quorum summa sit $= D$,

etc.

His positis, si numerus, cujus logarithmus est $= 1$, denotetur littera e (quae ne confundatur cum termino e) erit

$$1 + A + B + C + D + \text{etc.} = e^{P + \frac{1}{2}Q + \frac{1}{3}R + \frac{1}{4}S + \text{etc.}}$$

sumendis vero terminis alternis, erit

$$1 + B + D + F + H + \text{etc.} = \frac{e^{P + \frac{1}{2}Q + \frac{1}{3}R + \frac{1}{4}S + \text{etc.}} + e^{-P + \frac{1}{2}Q - \frac{1}{3}R + \frac{1}{4}S - \text{etc.}}}{2}$$

atque

$$A + C + E + G + I + \text{etc.} = \frac{e^{P + \frac{1}{2}Q + \frac{1}{3}R + \frac{1}{4}S + \text{etc.}} - e^{-P + \frac{1}{2}Q - \frac{1}{3}R + \frac{1}{4}S - \text{etc.}}}{2}$$

Quod si nunc pro serie $a + b + c + d +$ etc. capiatur series potestatis cujuscunque numerorum primorum, ita ut sit

$$P = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \text{etc.}$$
$$Q = \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \text{etc.}$$
$$R = \frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \text{etc.}$$
$$S = \frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \text{etc.}$$
etc.

erit A ipsa series numerorum primorum P, B series factorum ex binis, C series factorum ex ternis et ita porro; unde fiet $1 + A + B + C + D +$ etc. series omnium numerorum puta

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{etc.} = \alpha\pi^n.$$

Quam ob rem erit

$$e^{P + \frac{1}{2}Q + \frac{1}{3}R + \frac{1}{4}S + \text{etc.}} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \text{etc.} = \alpha\pi^n$$

simili vero modo erit

$$e^{Q+\frac{1}{2}S+\frac{1}{3}V+\frac{1}{4}X+\text{etc.}} = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \text{etc.} = \beta\pi^{2n}$$

quae expressio per illam divisa dabit

$$e^{-P+\frac{1}{2}Q-\frac{1}{3}R+\frac{1}{4}S-\frac{1}{5}T+\frac{1}{6}V-\text{etc.}} = \frac{\beta\pi^n}{\alpha}$$

unde demonstrare potest egregia illa series

$$1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \text{etc.}$$

cujus summam Tu, V. C., demonstrasti esse $= \frac{\beta\pi^n}{\alpha}$.

His praemissis cum sit A series ipsorum numerorum primorum, B series factorum ex binis primis, C ex ternis et ita porro; scilicet

$$A = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \text{etc.}$$

$$B = \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{9^n} + \frac{1}{10^n} + \frac{1}{14^n} + \text{etc.}$$

$$C = \frac{1}{8^n} + \frac{1}{12^n} + \frac{1}{18^n} + \frac{1}{20^n} + \frac{1}{27^n} + \text{etc.}$$

$$D = \frac{1}{16^n} + \frac{1}{24^n} + \frac{1}{36^n} + \frac{1}{40^n} + \frac{1}{54^n} + \text{etc.}$$

$$E = \frac{1}{32^n} + \frac{1}{48^n} + \frac{1}{72^n} + \frac{1}{80^n} + \frac{1}{108^n} + \text{etc.}$$

etc.

sequitur fore $1 + A + B + C + D + \text{etc.} =$

$$e^{P+\frac{1}{2}Q+\frac{1}{3}R+\text{etc.}} = \alpha\pi^n$$

$1 + B + D + F + \text{etc.} =$

$$\frac{e^{P+\frac{1}{2}Q+\frac{1}{3}R+\text{etc.}} + e^{-P+\frac{1}{2}Q-\frac{1}{3}R+\text{etc.}}}{2} = \frac{1}{2}\left(\alpha + \frac{\beta}{\alpha}\right)\pi^n$$

$A + C + E + G + \text{etc.} =$

$$\frac{e^{P+\frac{1}{2}Q+\frac{1}{3}R+\text{etc.}} - e^{-P+\frac{1}{2}Q-\frac{1}{3}R+\text{etc.}}}{2} = \frac{1}{2}\left(\alpha - \frac{\beta}{\alpha}\right)\pi^n$$

hincque $1 - A + B - C + D - E + F - G + \text{etc.} = \frac{\beta}{\alpha}\pi^n$, quae est ipsa series a Te, V. C., primum inventa. Denique ex his constat fore summam seriei $1 + B + D + F + \text{etc.}$ in qua insunt producta ex numero pari numerorum primorum, ad summam seriei $A + C + E + G + \text{etc.}$, quae continet numeros primos ipsos et producta ex numero impari eorum, uti est $\alpha^2 + \beta$ ad $\alpha^2 - \beta$, quae est proportio, quam hodie mihi inveniendam proposuisti. Vale, V. C., mihique favere perge.

Euler.